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Dedicated to Hidenori Fujiwara on the occasion of his 65-th birthday

Rayleigh theorem, projection of orbital measures, and spline functions

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Abstract

We consider a random matrix X uniformly distributed on an orbit $\mathcal{O}$ for the action of the orthogonal group on the space of real symmetric matrices, or of the unitary group on the space of Hermitian matrices. The problem is to evaluate the distribution of the eigenvalues of a compression of X. We give a survey about this question and present some new results. Baryshnikov formula and Olshanski determinantal formula are revisited, and a Markov-Krein type formula is established.

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Introduction. — A general problem in random matrix theory is to determine the distribution of the eigenvalues of a random matrix. Let X be a real symmetric or Hermitian random matrix, and P an orthogonal projection. The question we consider is the evaluation of the distribution of the eigenvalues of the compression PXP.

For a real symmetric matrix $X \in \mathcal{H}_n(\mathbb{R}) := Sym(n,\mathbb{R})$, or a Hermitian matrix $X \in \mathcal{H}_n(\mathbb{C}) := Herm(n,\mathbb{C})$, the classical spectral theorem says that the eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ are real and the corresponding eigenspaces orthogonal. Let p denote the projection of $\mathcal{H}_n(\mathbb{F})$ onto $\mathcal{H}_{n-1}(\mathbb{F})$ which maps the matrix X to the $(n-1)\times(n-1)$ upper left corner of X. The theorem of Rayleigh says that the sequence of the eigenvalues $\mu_1 \leq \cdots \leq \mu_{n-1}$ of the matrix $Y = p(X)$ interlace the sequence of the eigenvalues of X:

$$\lambda_1 \leq \mu_1 \leq \lambda_2 \leq \cdots \leq \mu_{n-1} \leq \lambda_n.$$

This interlacing property will be written $\mu \preceq \lambda$. Let $U_n(\mathbb{F})$ denote the orthogonal group $O(n)$ if $\mathbb{F} = \mathbb{R}$, or the unitary group if $\mathbb{F} = \mathbb{C}$. The group $U_n(\mathbb{F})$ acts on $\mathcal{H}_n(\mathbb{F})$ by the transformations $X \mapsto uXu^*$ $(u \in U_n(\mathbb{F}))$. Let $\mathcal{O}_A$ denote the orbit of $A = \mathrm{diag}(a_1, \ldots, a_n)$, with $a_1 \leq \ldots \leq a_n$,

$$\mathcal{O}_A = \{X = uAu^* \mid u \in U_n(\mathbb{F})\}.$$

By the spectral theorem

$$\mathcal{O}_A = \{X \in \mathcal{H}_n(\mathbb{F}) \mid \mathrm{spectrum}(X) = \{a_1, \ldots, a_n\}\}.$$

Let $\Lambda^{(n)}$ denote the map from $\mathcal{H}_n(\mathbb{F})$ to $(\mathbb{R})_+ = \{x \in \mathbb{R}^n \mid x_1 \leq \ldots \leq x_n\}$ which maps a matrix to the sequence of its eigenvalues. One can add to the theorem of Rayleigh that

$$\Lambda^{(n-1)}\big(p(\mathcal{O}_A)\big) = \{(\mu_1, \ldots, \mu_{n-1}) \in (\mathbb{R}^{n-1})_+ \mid \mu \preceq a\}.$$

The orbit $\mathcal{O}_A$ carries a natural probability measure, the orbital measure μ_A, which is the image under the map $u \mapsto uAu^*$ of the normalized Haar measure α_n on the compact group $U_n(\mathbb{F})$: for a function f on $\mathcal{H}_n(\mathbb{F})$,

$$\int_{\mathcal{H}_n(\mathbb{F})} f(X)\mu_A(dX) = \int_{U_n(\mathbb{F})} f(uAu^*)\alpha_n(du).$$

Let us recall the definition of the radial part of a measure $\mu \in \mathcal{H}_n(\mathbb{F})$ which is $U_n(\mathbb{F})$-invariant. The integral of a function f can be written

$$\begin{aligned}&\int_{\mathcal{H}_n(\mathbb{F})} f(X)\mu(dX)\\ &= \int_{(\mathbb{R}^n)_+}\Big(\int_{U_n(\mathbb{F})} f\big(u\,\mathrm{diag}(t_1, \ldots, t_n)\,u^*\big)\alpha_n(du)\Big)\nu(dt). \qquad (*)\end{aligned}$$

The measure ν is called the radial part of μ. If μ is a probability measure, then ν is the joint distribution of the eigenvalues of the random matrix X.

More generally consider the projection p_k from $\mathcal{H}_n(\mathbb{F})$ onto $\mathcal{H}_k(\mathbb{F})$ which maps a matrix X to the $k \times k$ upper left corner. For $A = \mathrm{diag}(a_1, \ldots, a_n)$ let $\mu_A^{(k)}$ denote the image of μ_A by the projection p_k: for a function f on $\mathcal{H}_k(\mathbb{F})$,

$$\int_{\mathcal{H}_k(\mathbb{F})} f(Y)\mu_A^{(k)}(dY) = \int_{\mathcal{H}_n(\mathbb{F})} f\big(p_k(X)\big)\mu_A(dX),$$

and $\nu_A^{(k)}$ the radial part of $\mu_A^{(k)}$. Let $\lambda_1^{(k)} \leq \ldots \leq \lambda_k^{(k)}$ denote the eigenvalues of $X^{(k)} = p_k(X)$. The system $(\lambda_i^{(k)})$ satisfies the interlacing property

$$\lambda^{(n)} \succeq \lambda^{(n-1)} \succeq \cdots \succeq \lambda^{(1)}.$$

Observe that $\lambda_1^{(1)}$ equals the entry X_{11}. For $X \in \mathcal{O}_A$ the eigenvalues $\lambda_i^{(n)}$ of X agree with the one of A: $\lambda_i^{(n)} = a_i$. If X is distributed on the orbit $\mathcal{O}_A$ according to the orbital measure μ_A, then the joint distribution of the eigenvalues $\lambda_1^{(k)}, \ldots \lambda_k^{(k)}$ of $X^{(k)} = p_k(X)$ is the radial part $\nu_A^{(k)}$ of $\mu_A^{(k)}$.

In this paper we will give a survey of some results about the measures $\nu_A^{(k)}$, and present some new results. In case of $\mathbb{F} = \mathbb{C}$, It has been observed by Okounkov that the density of the measure $\mu_A^{(1)} = \nu_A^{(1)}$ on $\mathbb{R}$ is a spline function (see [Olshanski-Vershik,1996], p.170). The measure $\nu_A^{(n-1)}$ has been determined by Baryshnikov [2001]. A proof of Baryshnikov formula will be given in Section 1. For general k Olshanski has established a formula for the density of $\nu_A^{(k)}$, which involves a determinant of spline functions. In Section 6 we will give a proof of Olshanski's formula which is slightly different from the original one. In both proofs the starting observation is that the Fourier-Laplace transform of the projection $\mu_A^{(k)}$ is equal to the restriction of the Fourier-Laplace transform of μ_A to the subspace $\mathcal{H}_k(\mathbb{C})$.

In case of $\mathbb{F} = \mathbb{R}$, much less is known. An explicit formula for $\mu_A^{(1)}$ has been obtained by Fourati [2011]. A formula for $\nu_A^{(n-1)}$ is given in Section 1. For general k a Markov-Krein type formula is established in Section 9 for $\mu_A^{(k)}$.

Similar situations have been considered. If one considers the projection of the orbital measure μ_A onto the space of diagonal matrices one gets a measure on $\mathbb{R}^n$ whose Fourier-Laplace transform is given by the Itzykson-Zuber-Harish-Chandra integral. More generally one considers the action of a compact Lie group on its Lie algebra by the adjoint representation, and the projection of an orbital measure onto a Cartan subalgebra. In

[Heckman, 1982] one considers the coadjoint representation of a compact Lie group on the dual of its Lie algebra, and the projection of an orbital measure onto the dual of the Lie algebra of a closed subgroup.

In [Defosseux,2011], the author considers the projection of orbital measures for the adjoint representation of a compact simple Lie group on its Lie algebra. Notice that the action of $U_n(\mathbb{C})$ on $\mathcal{H}_n(\mathbb{C})$ is a special case. The method uses a theorem of Heckman about asymptotic properties in branching problems. In particular the author obtains a generalization of Baryshnikov results. Lemma 3.8 in [Defosseux, 2011] generalizes part a) in Theorem 1.3 of the present paper, and Theorem 3.4 generalizes Proposition 2.3.

In [Neretin,2003] various integrals related to the so-called Rayleigh triangles are studied and evaluated, in the same flavor as of Baryshnikov's formula.

1. Rayleigh theorem and Baryshnikov formula. — Consider an $n\times n$ matrix A in $\mathcal{H}_n(\mathbb{F})$ with eigenvalues $\lambda_1,\dots,\lambda_n$, and let $v_1,\dots,v_n$ be corresponding unit eigenvectors. By the spectral theorem for Hermitian matrices,

$$Ax=\sum_{i=1}^n \lambda_i(x|v_i)v_i.$$

Let $\mathcal{Y}$ be a subspace of dimension $n-1$ in $\mathbb{F}^n$, and $\mu_1\leq\cdots\leq\mu_{n-1}$ the eigenvalues of $B=pAp$, where p is the orthogonal projection onto $\mathcal{Y}$, and B is restricted to $\mathcal{Y}$.

THEOREM 1.1 (RAYLEIGH). — *Assume the eigenvalues $\lambda_1,\dots,\lambda_n$ to be distinct, $\lambda_1<\cdots<\lambda_n$, and that no one of the eigenvectors v_i belongs to $\mathcal{Y}$. Then the eigenvalues of B interlace those of A:*

$$\lambda_1<\mu_1<\lambda_2<\cdots<\mu_{n-1}<\lambda_n.$$

Proof.

We can assume that $\mathcal{Y}=\{x\in\mathbb{F}^n\mid x_n=0\}$. If a_{ij} are the entries of A $(1\leq i,j\leq n)$, then the matrix B is the $(n-1)\times(n-1)$ left corner of A: $b_{ij}=a_{ij}$ $(1\leq i,j\leq n-1)$.

One computes in two ways the following rational function:

$$f(z)=\big((zI-A)^{-1}e_n|e_n\big).$$

On one hand, by Cramer's formulas,

$$f(z)=\frac{\det^{(n-1)}(zI_{n-1}-B)}{\det^{(n)}(zI_n-A)}.$$

On the other hand, by using the spectral decomposition of A,

$$f(z) = \sum_1^n \frac{1}{z-\lambda_i}|(e_n|v_i)|^2 = \sum_{i=1}^n \frac{w_i}{z-\lambda_i},$$

with

$$w_i = |(e_n|v_i)|^2.$$

Hence

$$\frac{\prod_{j=1}^{n-1}(z-\mu_i)}{\prod_{i=1}^{n}(z-\lambda_i)} = \sum_{i=1}^n \frac{w_i}{z-\lambda_i}.$$

The function f is decreasing from $+\infty$ to $-\infty$ in each interval $]\lambda_i, \lambda_{i+1}[$ $(i = 1, \dots, n-1)$. Therefore each of these intervals contains one and only one zero of f. ∎

If we drop the condition that no one of the eigenvectors v_i is orthogonal to e_n, then one gets only

$$\lambda_1 \leq \mu_1 \leq \lambda_2 \leq \cdots \leq \mu_{n-1} \leq \lambda_n.$$

Note that the vector $w = (w_1, \dots, w_n)$ belongs to the simplex

$$\Delta_{n-1} = \{w \in \mathbb{R}^n \mid w_i \geq 0, \ \sum_{i=1}^n w_i = 1\}.$$

The residue w_i of f at the pole λ_i is given by

$$w_i = \frac{\prod_{j=1}^{n-1}(\lambda_i - \mu_j)}{\prod_{j=1, j\neq i}^{n}(\lambda_i - \lambda_j)}.$$

The map

$$\Phi : (\mu_1, \dots, \mu_{n-1}) \mapsto (w_1, \dots, w_n)$$

is a diffeomorphism from $\{\lambda_1 < \mu_1 < \lambda < \cdots < \mu_{n-1} < \lambda_n\}$ onto $\{w_i > 0, \ w_1 + \cdots + w_n = 1\}$, and a homeomorphism from $\{\lambda_1 \leq \mu_1 \leq \lambda_2 \leq \cdots \mu_{n-1} \leq \lambda_n\}$ onto Δ_{n-1}.

Proposiiton 1.2. — *Consider the differential form ω of degree $n-1$,*

$$\omega = dw_1 \wedge dw_2 \wedge \cdots \wedge dw_{n-1}.$$

Then

$$\Phi^*(\omega) = \frac{V_{n-1}(\mu_1, \dots, \mu_{n-1})}{V_n(\lambda_1, \dots, \lambda_n)} d\mu_1 \wedge d\mu_2 \wedge \cdots \wedge d\mu_{n-1},$$

where V_n is the Vandermonde polynomial in n variables:

$$V_n(z_1, \dots, z_n) = \prod_{1 \le i < j \le n} (z_j - z_i).$$

Proof.

Let us compute the differential of Φ:

$$\frac{\partial w_i}{\partial \mu_j} = -\frac{\prod_{k=1}^{n-1}(\lambda_i - \mu_k)}{\prod_{k=1,k\neq i}^{n}(\lambda_i - \lambda_k)} \frac{1}{\lambda_i - \lambda_j},$$

and its Jacobian

$$\det\Big(\frac{\partial w_i}{\partial \mu_j}\Big)_{1\le i,j\le n-1}$$
$$= \pm \prod_{i=1}^{n-1} \Bigg(\frac{\prod_{k=1}^{n-1}(\lambda_i - \mu_k)}{\prod_{k=1,k\neq i}^{n-1}(\lambda_i - \lambda_k)} \Bigg) \det\Big(\frac{1}{\lambda_i - \mu_j}\Big)_{1\le i,j\le n-1}.$$

We use now the Cauchy formula:

$$\det\Big(\frac{1}{\lambda_i - \mu_j}\Big)_{1\le i,j\le n-1}$$
$$= V_{n-1}(\lambda_1, \dots, \lambda_{n-1}) V_{n-1}(\mu_1, \dots, \mu_{n-1}) \prod_{i,j=1}^{n-1} \frac{1}{\lambda_i - \mu_j}.$$

Finally one checks that

$$\prod_{i=1}^{n-1} \prod_{k=1,k\neq i} (\lambda_i - \lambda_k) = V_n(\lambda_1, \dots, \lambda_n) V_{n-1}(\lambda_1, \dots, \lambda_{n-1}). \qquad \square$$

We fix now $A = \mathrm{diag}(a_1, \dots, a_n)$, and consider the random matrix $X = uAu^*$ with $u \in U(n, \mathbb{F})$ ($\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, $U(n, \mathbb{F}) = O(n)$ or $U(n)$). The eigenvalues of uAu^* are $a_1, \dots, a_n$ with eigenvectors $ue_1, \dots, ue_n$. We fix the subspace $\mathcal{Y} \simeq Herm(n-1, \mathbb{F})$:

$$w_i = |(e_n|ue_i)|^2 = |u_{ni}|^2.$$

Let $\mu_1 \le \dots \le \mu_{n-1}$ denote the eigenvalues of the matrix $B = p(uAu^*)p$. We assume that $u \in U(n, \mathbb{F})$ is uniformly distributed, i.e. that its law is the normalized Haar measure α_n of $U(n, \mathbb{F})$.

THEOREM 1.3. — *The joint distribution of the eigenvalues* $(\mu_1, \ldots, \mu_{n-1})$ *is a probability measure on* $\mathbb{R}^{n-1}$ *supported by*

$$\{\mu \in \mathbb{R}^{n-1} \mid a_1 \le \mu_1 \le a_2 \le \cdots \le \mu_{n-1} \le a_n\}.$$

a) If $\mathbb{F} = \mathbb{C}$, *its density is given by the Barychnikov formula*

$$(n-1)! \frac{V_{n-1}(\mu_1, \ldots, \mu_{n-1})}{V_n(a_1, \ldots, a_n)}.$$

This means that, for a function f *defined on* $(\mathbb{R}^{n-1})_+$,

$$\int_{(\mathbb{R}^{n-1})_+} f(t) \nu_A^{(n-1)}(dt)$$
$$= \frac{(n-1)!}{V_n(a)} \int_{a_1}^{a_2} dt_1 \int_{a_2}^{a_3} dt_2 \ldots \int_{a_{n-1}}^{a_n} dt_{n-1} V_{n-1}(t) f(t).$$

b) If $\mathbb{F} = \mathbb{R}$, *the density is given by*

$$\frac{\Gamma\left(\frac{n}{2}\right)}{\pi^{\frac{n}{2}}} \frac{V_{n-1}(\mu_1, \ldots, \mu_{n-1})}{\prod_{i=1}^{n} \prod_{j=1}^{n-1} \sqrt{|a_i - \mu_j|}},$$

([Baryshnikov,2001], Proposition 4.2, for $\mathbb{F} = \mathbb{C}$.)

Both formulas for the density can be written

$$\frac{\Gamma\left(n\frac{d}{2}\right)}{\Gamma\left(\frac{d}{2}\right)^n} \frac{V_{n-1}(\mu_1, \ldots, \mu_{n-1})}{V_n(a_1, \ldots, a_n)^{d-1}} \prod_{i=1}^{n} \prod_{j=1}^{n-1} |a_i - \mu_j|^{\frac{d}{2}-1},$$

with $d = 1$ if $\mathbb{F} = \mathbb{R}$, $d = 2$ if $\mathbb{F} = \mathbb{C}$.

Proof.
Consider the map

$$\psi : U(n, \mathbb{F}) \to \Delta_{n-1}, \quad u \mapsto (|u_{n,i}|^2).$$

The image under ψ of the normalized Haar measure of $U(n, \mathbb{F})$ is the Dirichlet distribution on Δ_{n-1} defined by the differential form

$$\frac{\Gamma\left(n\frac{d}{2}\right)}{\Gamma\left(\frac{d}{2}\right)^n} (w_1 \ldots w_n)^{\frac{d}{2}-1} dw_1 \wedge \cdots \wedge dw_{n-1}.$$

Furthermore

$$w_1 \dots w_n = \frac{\prod_{i=1}^n \prod_{j=1}^{n-1} (a_i - \mu_j)}{V_n(a_1, \dots, a_n)^2}.$$

Therefore Theorem 1.3 follows from Proposition 1.2

∎

The method of proof is taken from the evaluation of the spherical functions for the symmetric space $GL(n, \mathbb{C})/U_n(\mathbb{C})$ by Gel'fand and Naimark ([1957], II.9.3, pp. 44-49). See next Section.

2. Applications of Baryshnikov formula. — We will give three examples of applications of Baryshnikov formula (Theorem 1.3).

1. Evaluation of the spherical functions of the symmetric space $GL_n(\mathbb{C})/U_n(\mathbb{C})$. Method of Gel'fand and Naimark.

For $X \in \mathcal{H}_n(\mathbb{F})$ the principal minor $\Delta_k(X)$ is the determinant of the $k \times k$ upper left corner of X. For $\mathbf{s} = (s_1, \dots, s_n) \in \mathbb{C}^n$, the power function $\Delta_{\mathbf{s}}$ is defined by

$$\Delta_{\mathbf{s}}(X) = \prod_{k=1}^n \Delta_k(X)^{s_k - s_{k+1}} \quad (s_{n+1} = 0).$$

If X is positive definite, then the principal minors are > 0, and $\Delta_{\mathbf{s}}(X)$ is well defined. If the exponents s_k are positive integers, then $\Delta_{\mathbf{s}}(X)$ is well defined for all X. The spherical function $\Phi_{\mathbf{s}}^{(n)}$ of the symmetric space $GL(n, \mathbb{F})/U_n(\mathbb{F})$ can be seen as a $U_n(\mathbb{F})$-invariant function on $\Omega_n(\mathbb{F})$, the cone of positive definite matrices in $\mathcal{H}_n(\mathbb{F})$, and admits the following integral representation:

$$\Phi_{\mathbf{s}}^{(n)}(X) = \int_{U_n(\mathbb{F})} \Delta_{\mathbf{s}}(uXu^*) \alpha_n(du).$$

It can be written

$$\Phi_{\mathbf{s}}^{(n)}(X) = \Delta_n(X)^{s_n} \int_{\mathcal{O}_X} \prod_{k=1}^{n-1} \Delta_k(Y)^{s_k - s_{k+1}} \mu_X(dY).$$

The integrant only depends on the projection Y' of Y onto $\mathcal{H}_{n-1}(\mathbb{F})$. Therefore

$$\Phi_{\mathbf{s}}^{(n)}(X) = \Delta_n(X)^{s_n} \int_{\mathcal{H}_{n-1}(\mathbb{F})} \prod_{k=1}^{n-1} \Delta_k(Y')^{s_k - s_{k+1}} \mu_X^{(n-1)}(dY').$$

Since the projection $\mu_X^{(n-1)}$ is $U_{n-1}(\mathbb{F})$-invariant,

$$\Phi_{\mathbf{s}}^{(n)}(X) = \Delta_n(X)^{s_n} \int_{(\mathbb{R}^{n-1})_+} \Phi_{\mathbf{s}'}^{(n-1)}(t)\nu_X^{(n-1)}(dt),$$

with $\mathbf{s}' = (s_1, \ldots, s_{n-1})$.

PROPOSITION 2.1. — *Assume $\mathbb{F} = \mathbb{C}$. Let $X \in \mathcal{H}_n(\mathbb{C})$ be positive definite, with eigenvalues $x_1, \ldots, x_n$. Then*

$$\Phi_{\mathbf{s}}^{(n)}(X) = \frac{\delta_n!}{V_n(\lambda)V_n(x)} \det(x_i^{\lambda_j})_{1\leq i,j\leq n},$$

with $\lambda_j = s_j + n - j$, $\delta_n = (0, 1, 2, \ldots, n-1)$, $\delta_n! = 2!3!\ldots(n-1)!$. [Gelfand-Naimark,1957]

Proof. We will prove the formula by recursion on n. For $n = 1$ there is nothing to prove. Let us assume that the formula holds for $n - 1$. Then, by Theorem 1.3,

$$\Phi_{\mathbf{s}}^{(n)}(X) = \frac{(n-1)!(n-2)!\ldots 2!}{V_{n-1}(\lambda')V_n(x)}(x_1 \ldots x_n)^{s_n}$$
$$\int_{x_1}^{x_2} dt_1 \int_{x_2}^{x_3} dt_2 \cdots \int_{x_{n-1}}^{x_n} dt_{n-1} \det(t_i^{\lambda'_j})_{1\leq i,j\leq n-1},$$

with $\lambda'_j = \lambda_j - \lambda_n - 1$. Let us compute the integral:

$$\int_{x_1}^{x_2} dt_1 \cdots \int_{x_{n-1}}^{x_n} dt_{n-1} \det(t_i^{\lambda_j - \lambda_n - 1})_{1\leq i,j\leq n-1}$$
$$= \det\Big(\int_{x_i}^{x_{i+1}} t^{\lambda_j-\lambda_n-1}dt\Big)$$
$$= \frac{1}{\prod_{j=1}^{n-1}(\lambda_j - \lambda_n)} \det\Big((x_{i+1})^{\lambda_j-\lambda_n} - (x_i)^{\lambda_j-\lambda_n}\Big)_{1\leq i,j\leq n}.$$

It remains to prove the identity:

$$D := \det\big(x_i^{\lambda_j}\big)_{1\leq i,j\leq n} = (x_1 \ldots x_n)^{\lambda_n} \det\big(x_i^{\lambda_j-\lambda_n} - x_{i+1}^{\lambda_j-\lambda_n}\big)_{1\leq i,j\leq n-1}.$$

One divides the i-th row of D by $x_i^{\lambda_n}$:

$$D = (x_1, \ldots x_n)^{\lambda_n} \begin{vmatrix} x_1^{\lambda_1-\lambda_n} & x_1^{\lambda_2-\lambda_n} & \ldots & x_1^{\lambda_{n-1}-\lambda_n} & 1 \\ x_2^{\lambda_1-\lambda_n} & x_2^{\lambda_2-\lambda_n} & \ldots & x_2^{\lambda_{n-1}-\lambda_n} & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ x_n^{\lambda_1-\lambda_n} & x_n^{\lambda_2-\lambda_n} & \ldots & x_n^{\lambda_{n-1}-\lambda_n} & 1 \end{vmatrix}.$$

In this determinant one substracts the second row from the first, the third from the second, and so on. We are done. ∎

2. The Harish-Chandra-Itzykson-Zuber integral

We consider the integral

$$\mathcal{E}_n(Z,X)=\int_{U_n(\mathbb{F})} e^{\operatorname{tr}(ZuXu^*)}\alpha_n(du)=\int_{\mathcal{O}_X} e^{\operatorname{tr}(ZY)}\mu_X(dY).$$

Take $Z=\operatorname{diag}(z_1,\dots,z_n)$, and let Y' denote the projection of Y onto $\mathcal{H}_n(\mathbb{F})$. Observing that, for $Y\in\mathcal{O}_X$, $\operatorname{tr}Y=\operatorname{tr}X$, we get

$$\operatorname{tr}(ZY)=\operatorname{tr}(Z'Y')+z_n(\operatorname{tr}X-\operatorname{tr}Y'),$$

with $Z'=\operatorname{diag}(z_1,\dots,z_{n-1})$, and

$$\mathcal{E}_n(Z,X)=e^{z_n\operatorname{tr}X}\int_{\mathcal{O}_X} e^{-z_n\operatorname{tr}(Y')}e^{\operatorname{tr}(Z'Y')}\mu_X(dY).$$

The integrant only depends on the projection Y' of Y on $\mathcal{H}_{n-1}(\mathbb{F})$. therefore

$$\mathcal{E}_n(Z,X)=e^{z_n\operatorname{tr}(X)}\int_{\mathcal{H}_{n-1}(\mathbb{F})} e^{-z_n\operatorname{tr}(Y')}e^{\operatorname{tr}(Z'Y')}\mu_X^{(n-1)}(dY').$$

By using the integral formula (*) we get

$$\mathcal{E}_n(Z,X)$$
$$=\int_{(\mathbb{R}^{n-1})_+} e^{-z_n(t_1+\cdots+t_{n-1})}\Big(\int_{U_{n-1}(\mathbb{F})} e^{\operatorname{tr}(Z'vTv^*)}\alpha_{n-1}(dv)\Big)\nu_X^{(n-1)}(dt),$$

where $T=\operatorname{diag}(t_1,\dots,t_{n-1})$. Hence we have obtained the following recursion formula:

$$\mathcal{E}_n(Z,X)=e^{z_n\operatorname{tr}X}\int_{(\mathbb{F}^{n-1})_+}\mathcal{E}_{n-1}(z',t)e^{-(t_1+\cdots+t_{n-1})z_n}\nu_X^{(n-1)}(dt).$$

In case of $\mathbb{F}=\mathbb{C}$ we get an alternative proof of Harish-Chandra-Itzykson-Zuber formula:

Proposition 2.2. — *Assume $\mathbb{F}=\mathbb{C}$. Then*

$$\mathcal{E}_n(z,x)=\frac{\delta_n!}{V_n(z)V_n(x)}\det\big(e^{z_ix_j}\big)_{1\le i,j\le n}.$$

Proof. The scheme is the same as for the proof of Proposition 2.1. Assume that the formula holds for $n-1$. Then, by Theorem 1.3,

$$\mathcal{E}_n(z,x) = \frac{(n-1)!(n-2)!\dots 2!}{V_{n-1}(z')V_n(x)} e^{z_n(x_1+\dots+x_n)}$$
$$\int_{x_1}^{x_2} dt_1 \int_{x_2}^{x_2} dt_2 \dots \int_{x_{n-1}}^{x_n} dt_{n-1} \det\big(e^{(z_j-z_n)t_i}\big)_{1\le i,j\le n-1}.$$

Let us compute the integral

$$\int_{x_1}^{x_2} dt_1 \int_{x_2}^{x_2} dt_2 \dots \int_{x_{n-1}}^{x_n} dt_{n-1} \det\big(e^{(z_j-z_n)t_i}\big)_{1\le i,j\le n-1}$$
$$= \det\Big(\int_{x_i}^{x_{i+1}} e^{(z_j-z_n)t_i} dt_i\Big)_{1\le i,j\le n-1}$$
$$= \frac{1}{\prod_{i=1}^{n-1}(z_i-z_n)} \det\Big(e^{(z_j-z_n)x_{i+1}} - e^{(z_j-z_n)x_i}\Big)_{1\le i,j\le n-1}.$$

One finishes the computation as in the proof of Proposition 2.1. ∎

Both proofs are very similar. This can be explained by the following Mehler-Heine type formula:

$$\lim_{\tau\to\infty} \Phi^{(n)}_{\tau\mathbf{s}}\Big(I + \frac{1}{\tau}X\Big) = \mathcal{E}_n\big(\mathrm{diag}(s_1,\dots,s_n), X\big).$$

The proof of this formula uses

$$\lim_{\tau\to\infty} \det\Big(I+\frac{1}{\tau}X\Big)^{\tau\alpha} = \exp(\alpha\,\mathrm{tr}\,X).$$

3. Gelfand-Tsetlin polytope and invariant measure

We will denote by Λ^n the map from $\mathcal{H}_n(\mathbb{F})$ onto $(\mathbb{R}^n)_+$ which assigns to a Hermitian matrix X its eigenvalues $\lambda_1 \le \dots \le \lambda_n$. Let $\lambda_1^{(k)} \le \dots \le \lambda_k^{(k)}$ be the eigenvalues of the projection $p_k^n(X)$ of X on $\mathcal{H}_k(\mathbb{F})$. By the Rayleigh theorem (Theorem 1.1)

$$\lambda^{(n)}(X) \succeq \lambda^{(n-1)}\big(p^n_{n-1}(X)\big) \succeq \dots \succeq \lambda^{(1)}\big(p^n_1(X)\big).$$

For $a \in (\mathbb{R}^n)_+$ the Gelfand-Tsetlin polytope P_a is the convex compact set in

$$(\mathbb{R}^{n-1})_+ \times (\mathbb{R}^{n-2})_+ \times \dots \times (\mathbb{R}^2)_+ \times \mathbb{R}$$

defined by the interlacing conditions:

$$P_a = \{(x^{(n-1)}, \dots, x^{(1)}) \mid a \succeq x^{(n-1)} \succeq \dots \succ x^{(1)}\}.$$

Proposition 2.3. — *Assume $\mathbb{F} = \mathbb{C}$. Let X run over the orbit $\mathcal{O}_A$. Considering the system of the eigenvalues $(\lambda_i^{(k)})$ $(k = 1, \dots, n-1,$ $i = 1, \dots, k)$ we get a map from $\mathcal{O}_A$ into P_a. Then this map is onto P_a, and the image of the orbital measure μ_A is the normalized Lebesgue measure of P_a. Furthermore the volume of P_a (with respect for the Lebesgue measure) is given by*

$$\mathrm{vol}(P_a) = \frac{V_n(a)}{(n-1)!(n-2)!\dots 2!}.$$

[Baryshnikov,2001], Proposition 4.7.

Proof. The map we consider from $\mathcal{O}_A$ into P_a is given by

$$X \mapsto \Big(\Lambda^{(n-1)}\big(p^n_{n-1}(X)\big), \Lambda^{(n-2)}\big(p^n_{n-2}(X)\big), \dots, p^n_1(X)\Big),$$

We will evaluate the following integral: for a function f defined on P_a,

$$I(f) = \int_{\mathcal{O}_A} f\Big(\Lambda^{(n-1)}\big(p^n_{n-1}(X)\big), \Lambda^{(n-2)}\big(p^n_{n-2}(X)\big), \dots, p^n_1(X)\Big)\mu_A(dX).$$

This integral can also be written

$$I(f) = \int_{U_n(\mathbb{C})}$$
$$f\Big(\Lambda^{(n-1)}\big(p^n_{n-1}(uAu^*)\big), \Lambda^{(n-2)}\big(p^n_{n-2}(uAu^*)\big), \dots, p^n_1(uAu^*)\Big)\alpha_n(du).$$

We will show that

$$I(f) = \frac{\delta_n\,!}{V_n(a)} \int_{P_a} f(t^{(n-1)}, t^{(n-2)}, \dots, t^{(1)}) \prod_{k=1}^{n-1}\prod_{i=1}^{k} dt_i^{(k)}.$$

We will prove this formula by induction on n. For $n = 2$ it is simply

$$\int_{U_2(\mathbb{C})} f\big((uAu^*)_{11}\big)\alpha_2(du) = \frac{1}{a_2 - a_1}\int_{a_1}^{a_2} f(t)dt,$$

which is a special case of Theorem 1.3. Assume that the formula holds for $n-1$. The idea is to use the following integration formula

$$\int_{U_n(\mathbb{C})} \varphi(u)\alpha_n(du) = \int_{U_{n-1}(\mathbb{C})\backslash U_n(\mathbb{C})}\Big(\int_{U_{n-1}(\mathbb{C})} \varphi(vu)\alpha_{n-1}(dv)\Big)\dot{\alpha}_n(d\dot{u}).$$

Consider the following integral over $U_{n-1}(\mathbb{C})$:

$$J(X) = \int_{U_{n-1}(\mathbb{C})} f\Big(\Lambda^{(n-1)}\big(p^n_{n-1}(vXv^*)\big), \dots, p^n_1(vXv^*)\Big)\alpha_{n-1}(dv).$$

We observe that, for $k \le n-1$,

$$\begin{aligned} &p^n_k = p^{n-1}_k \circ p^n_{n-1}, \\ &p^n_k(vXv^*) = p^{n-1}_k \circ p^n_{n-1}(vXv^*) = p^{n-1}_k\big(vp^n_{n-1}(X)v^*\big), \\ &\Lambda^{(n-1)}\big(vp^n_{n-1}(X)v^*\big) = \Lambda^{(n-1)}\big(p^n_{n-1}(X)\big). \end{aligned}$$

Therefore, with $Y = p^{n-1}_n(X) \in \mathcal{H}_{n-1}(\mathbb{C})$,

$$J(X) = \\ \int_{U_{n-1}(\mathbb{C})} f\Big(\Lambda^{(n-1)}(Y), \Lambda^{(n-2)}\big(p^{n-1}_{n-2}(vYv^*)\big), \dots, p^{n-1}_1(vYv^*)\Big)\alpha_{n-1}(dv).$$

By the induction hypothesis,

$$J(X) = \frac{(\delta_{n-1})!}{V_{n-1}(y)} \int_{P_y} f(y, t^{(n-2)}, \dots t^{(1)}) \prod_{k=1}^{n-2}\prod_{i=1}^{k} dt_i^{(k)}.$$

Observe that the formula in Theorem 1.3 can be written, for a function F defined on $(\mathbb{R}^{n-1})_+$,

$$\begin{aligned} &\int_{\mathcal{O}_A} F\Big(\Lambda^{(n-1)}\big(p^n_{n-1}(X)\big)\Big)\mu_A(dX) \\ &= \frac{(n-1)!}{V_n(a)} \int_{a\succ y} F(y)V_{n-1}(y)dy_1\dots dy_{n-1}. \end{aligned}$$

Taking

$$F(y) = \frac{(\delta_{n-1})!}{V_{n-1}(y)} \int_{P_y} f(y, t^{(n-2)}, \dots, t^{(1)}) \prod_{k=1}^{n-2}\prod_{i=1}^{k} dt_i^{(k)},$$

we obtain

$$\begin{aligned} &\int_{\mathcal{O}_A} J(X)\mu_A(dX) = \frac{(\delta_{n-1})!\,(n-1)!}{V_n(a)} \\ &\int_{P_a} f(y, t^{(n-2)}, \dots, t^{(1)}) \prod_{k=1}^{n-2}\prod_{i=1}^{k} dt_i^{(k)} \prod_{i=1}^{n-1} dy_i. \end{aligned}$$

This shows that the formula holds for n since the left handside equals $I(f)$. ∎

3. Fourier-Laplace transform of invariant measures. — The Fourier-Laplace transform $\widehat{\mu}$ of a bounded measure μ on $\mathcal{H}_n(\mathbb{F})$ is defined, for $Z \in i\mathcal{H}_n(\mathbb{F})$, by

$$\widehat{\mu}(Z) = \int_{\mathcal{H}_n(\mathbb{F})} e^{\mathrm{tr}(ZX)} \mu(dX).$$

If the support of μ is compact, it is defined for Z in the complexified space of $\mathcal{H}_n(\mathbb{F})$, i.e. $Sym(n,\mathbb{C})$ if $\mathbb{F}=\mathbb{R}$, $M(n,\mathbb{C})$ if $\mathbb{F}=\mathbb{C}$. The Fourier-Laplace transform $\widehat{\mu_A}$ of the orbital measure μ_A is given by

$$\widehat{\mu_A}(Z) = \int_{\mathcal{O}_A} e^{\mathrm{tr}(ZX)} \mu_A(dX) = \int_{U_n\mathbb{F})} e^{\mathrm{tr}(ZuAu^*)} \alpha(du).$$

This function can be written

$$\widehat{\mu_A}(Z) = \mathcal{E}_n(z;a),$$

if $Z = \mathrm{diag}(z_1,\ldots,z_n)$, $z=(z_1,\ldots,z_n)$, $a=(a_1,\ldots,a_n)$, and $a_1,\ldots,a_n$ are the eigenvalues of A, where $\mathcal{E}_n$ is an analytic function on $\mathbb{C}^n \times \mathbb{C}^n$, biinvariant under the permutation group $\mathfrak{S}_n$.

If the measure μ is $U_n(\mathbb{F})$-invariant, with radial part ν, then, for $Z = \mathrm{diag}(z_1,\ldots,z_n)$,

$$\widehat{\mu}(Z) = \int_{(\mathbb{R}^n)_+} \mathcal{E}_n(z,t)\nu(dt).$$

Recall the celebrated Harish-Chandra-Itzykson-Zuber integral

Proposition 3.1. — *Assume $\mathbb{F}=\mathbb{C}$. For $Z = \mathrm{diag}(z_1,\ldots,z_n)$,*

$$\widehat{\mu_A}(Z) = \mathcal{E}_n(z,a) = \delta_n! \frac{1}{V_n(a)V_n(z)} \det\big(e^{z_i a_j}\big)_{1\le i,j\le n},$$

where $V_n(z)$ is the Vandermonde polynomial

$$V_n(z) = \prod_{i<j} (z_j - z_i),$$

and

$$\delta_n = (n-1, n-2, \ldots, 1, 0), \quad \delta_n! = (n-1)!(n-2)!\ldots 2!$$

See Proposition 2.2 for a proof using Baryshnikov formula. In case of $\mathbb{F} = \mathbb{R}$ there is no such simple formula for the Fourier-Laplace transform of an orbital measure.

Let μ be a bounded measure on $\mathcal{H}_n(\mathbb{F})$, and $\mu^{(k)}$ the projection of μ on the subspace $\mathcal{H}_k(\mathbb{F})$. Then the Fourier-Laplace transform of $\mu^{(k)}$ equals the restriction to $\mathcal{H}_k(\mathbb{F})$ of the Fourier-Laplace transform of μ. By using this fact Olshanski establishes a formula for the projection $\mu_A^{(k)}$, more precisely for its radial part $\nu_A^{(k)}$. We will give a slightly different proof in next sections.

We will consider the following type of functions: for an analytic function f defined near 0 in $\mathbb{C}$ we put, for $x, y \in \mathbb{C}^n$,

$$\mathcal{D}_n(f; x; y) = \frac{1}{V_n(x)V_n(y)} \det\big(f(x_i y_j)\big)_{1 \leq i,j \leq n}.$$

This function is defined for $x_i \neq x_j$, $y_i \neq y_j$ $(i \neq j)$, and extends as an analytic function in a neighborhood of 0 in $\mathbb{C}^n \times \mathbb{C}^n$ (see [Hua,1963], Theorem 1.2.2, or [Faraut,2008], Proposition 12.3.3). For $\mathbb{F} = \mathbb{C}$, by Proposition 3.1, $\mathcal{E}_n(z; a) = \delta_n! \mathcal{D}_n(f; z; a)$ with $f(t) = e^t$. With $f(t) = \frac{1}{1-t}$, the classical Cauchy formula can be written

$$\mathcal{D}_n(f; x; y) := \frac{1}{V_n(x)V_n(y)} \det\Big(\frac{1}{1 - x_i y_j}\Big)_{1 \leq i,j \leq n} = \prod_{i,j=1}^{n} \frac{1}{1 - x_i y_j}.$$

4. Restriction of functions defined by determinantal formulas. Let $f_1, \ldots, f_n$ be n analytic functions defined in a neighborhood of 0 in $\mathbb{C}$, and F the function defined in a neiborhood of 0 in $\mathbb{C}^n$, for $z_i \neq z_j$ $(i \neq j)$, by

$$F(z_1, \ldots, z_n) = \frac{1}{V_n(z)} \det\big(f_j(z_i)\big)_{1 \leq i,j \leq n}.$$

The function F extends as an analytic function in a neighborhood of 0 in $\mathbb{C}^n$.

Theorem 4.1. — *For $0 \leq k \leq n-1$,*

$$F(z_1, \dots, z_k, 0, \dots, 0) = \frac{(-1)^{\varepsilon(n,k)}}{1!2!\dots(n-k-1)!}$$

$$\frac{1}{V_k(z_1,\dots,z_k)(z_1\dots z_k)^{n-k}} \begin{vmatrix} f_1(z_1) & \dots & f_n(z_1) \\ \vdots & & \vdots \\ f_1(z_k) & \dots & f_n(z_k) \\ f_1^{(n-k-1)}(0) & \dots & f_n^{(n-k-1)}(0) \\ \vdots & & \vdots \\ f'(0) & \dots & f_n'(0) \\ f_1(0) & \dots & f_n(0) \end{vmatrix},$$

with

$$\varepsilon(n,k) = \sum_{j=k}^{n-1} j = \frac{1}{2}(n-k)(n+k-1).$$

Proof.

We prove the formula by recursion downwards on k, starting from $k = n$. Assume that the formula holds for k. We replace the entries of the k-th line by

$$f_j(z_k) - \big(f_j(0) + z_k f_j'(0) + \frac{1}{2} z_k^2 f_j''(0) + \dots + \frac{1}{(n-k-1)!} z_k^{n-k-1} f_j^{(n-k-1)}(0)\big),$$

and observe that

$$\lim_{z_k \to 0} \frac{1}{z_k^{n-k}} \Big(f_j(z_k) - \big(f_j(0) + z_k f_j'(0) + \frac{1}{2} z_k^2 f_j''(0) + \cdots + \frac{1}{(n-k-1)!} z_k^{n-k-1} f_j^{(n-k-1)}(0)\big)\Big) = \frac{1}{(n-k)!} f_j^{(n-k)}(0).$$

Furthermore

$$V_k(z_1,\dots,z_k)(z_1\dots z_{k-1})^{n-k}\big|_{z_k=0} = (-1)^{k-1} V_{k-1}(z_1,\dots,z_{k-1})(z_1\dots z_{k-1})^{n-k+1}.$$

We obtain

$$F(z_1,\dots,z_k,0,\dots,0)=\frac{(-1)^{\varepsilon(n,k-1)}}{1!2!\dots(n-k)!}$$
$$\frac{1}{V_{k-1}(z_1,\dots,z_{k-1})(z_1\dots z_{k-1})^{n-k+1}}\begin{vmatrix} f_1(z_1) & \dots & f_n(z_1)\\ \vdots & & \vdots\\ f_1(z_{k-1}) & \dots & f_n(z_{k-1})\\ f_1^{(n-k)}(0) & \dots & f_n^{(n-k)}(0)\\ \vdots & & \vdots\\ f_1'(0) & \dots & f_n'(0)\\ f_1(0) & \dots & f_n(0)\end{vmatrix}.$$

∎

Recall the notation we introduced at the end of Section 3: for a function f defined near 0 in $\mathbb{C}$, and for $x,y\in\mathbb{C}^n$,

$$\mathcal{D}_n(f;x;y)=\frac{1}{V_n(x)V_n(y)}\det\bigl(f(x_iy_j)\bigr)_{1\le i,j\le n}.$$

We will also use the notation

$$a_0^{(n)}(f)=\mathcal{D}_n(f;0;0).$$

Corollary 4.2. — *Let the function f be analytic in a neighborhood of 0. Then,*

$$a_0^{(n)}(f)=\prod_{j=0}^{n-1}\frac{f^{(j)}(0)}{j!}.$$

Let $a_1,\dots,a_n$ be distinct complex numbers, and $0\le k\le n$. Then

$$\mathcal{D}_n(f;a;z_1,\dots,z_k,0,\dots,0)$$
$$=\frac{a_0^{(n-k)}(f)}{V_n(a)V_k(z)(z_1\dots z_k)^{n-k}}\begin{vmatrix} 1 & \dots & 1\\ a_1 & \dots & a_n\\ \vdots & & \vdots\\ a_1^{n-k-1} & \dots & a_n^{n-k-1}\\ f(a_1z_1) & \dots & f(a_nz_1)\\ \vdots & & \vdots\\ f(a_1z_k) & \dots & f(a_nz_k)\end{vmatrix}.$$

In particular, for all a,

$$\mathcal{D}_n(f;a;0) = a_0^{(n)}(f).$$

5. Spline functions and divided differences. — Given n real numbers $a_1 < a_2 < \cdots < a_n$ there is a unique function f on $\mathbb{R}$ such that

- $f(t) \geq 0$, $\mathrm{supp}(f) = [a_1, a_n]$,
- f is of class $\mathcal{C}^{n-3}$,
- The restriction of f to each interval $[a_i, a_{i+1}]$ is a polynomial of degree $\leq n-2$.
- $\int f(t)dt = 1$.

This function is denoted by $M_n(a_1, \ldots, a_n; t)$ and called fundamental spline function. The numbers $a_1, \ldots, a_n$ are called the knots of the spline function.

Consider the simplex

$$\Delta_{n-1} = \{u \in \mathbb{R}^n \mid u_i \geq 0,\ u_1 + \cdots + u_n = 1\}.$$

The image of the normalized Lebesgue measure β on Δ_{n-1} under the map

$$\Delta_{n-1} \to \mathbb{R},\ u \mapsto a_1 u_1 + \cdots + a_n u_n$$

is equal to $M(a_1, \ldots, a_n; t)dt$. This means that, for a function f on $\mathbb{R}$,

$$\int_{\mathbb{R}} f(t) M_n(a_1, \ldots, a_n; t)dt = \int_{\Delta_{n-1}} f(a_1 u_1 + \cdots + a_n u_n)\beta(du).$$

Recall the definition of the divided differences. For a function f on $\mathbb{R}$, and real numbers a_i,

$$f[a_1, a_2] = \frac{f(a_2) - f(a_1)}{a_2 - a_1},$$

and

$$f[a_1, \ldots, a_n] = \frac{f[a_2, \ldots, a_n] - f[a_1, \ldots, a_{n-1}]}{a_n - a_1}.$$

Let us recall recall two basic facts.

Proposition 5.1. — *The divided difference $f[a_1, \ldots, a_k]$ can be expressed as a determinant divided by the Vandermonde polynomial:*

$$f[a_1, \ldots, a_n] = \frac{1}{V_n(a_1, \ldots, a_n)} \begin{vmatrix} 1 & 1 & \ldots & 1 \\ a_1 & a_2 & \ldots & a_n \\ \vdots & \vdots & & \vdots \\ a_1^{n-2} & a_2^{n-2} & \ldots & a_n^{n-2} \\ f(a_1) & f(a_2) & \ldots & f(a_n) \end{vmatrix}.$$

The connection between the spline functions and the divided differences is the following

Proposition 5.2 (Hermite-Genocchi). — *For a function f of class $\mathcal{C}^{n-1}$,*

$$f[a_1,\dots,a_n] = \frac{1}{(n-1)!}\int_{\mathbb{R}} f^{(n-1)}(t)M_n(a_1,\dots,a_n;t)dt.$$

Observe that for $n=2$ it is nothing but an elementary formula:

$$\int_{\mathbb{R}} f'(t)M_2(a_1,a_2;t)dt = \frac{1}{a_2-a_1}\int_{a_1}^{a_2} f'(t)dt$$
$$= \frac{f(a_2)-f(a_1)}{a_2-a_1} = f[a_1,a_2].$$

Notice that, from Proposition 5.2, one gets the following recurrence formula:

$$M_n(a_1,\dots,a_n;t)$$
$$= \frac{n-1}{a_n-a_1}\int_{-\infty}^{t}\big(M_{n-1}(a_2,\dots,a_n;s) - M_{n-1}(a_1,\dots,a_{n-1};s)\big)ds.$$

(For these basic results about fundamental spline functions, see, for instance [Risler,1991], or [Faraut,2005].)

We will use a generalization of Proposition 5.1:

Proposition 5.3. — *Let $f_1,\dots,f_k$ be functions defined on $\mathbb{R}$.*

$$\begin{vmatrix} 1 & 1 & \dots & 1 \\ a_1 & a_2 & \dots & a_n \\ \vdots & \vdots & & \vdots \\ a_1^{n-k-1} & a_2^{n-k-1} & \dots & a_n^{n-k-1} \\ f_1(a_1) & f_1(a_2) & \dots & f_1(a_n) \\ \vdots & \vdots & & \vdots \\ f_k(a_1) & f_k(a_2) & \dots & f_k(a_n) \end{vmatrix}$$
$$= \Big(\prod_{0<j-i\le n-k}(a_j-a_i)\Big)\det\Big(f_i[a_j,\dots,a_{j+n-k}]\Big)_{1\le i,j\le k}.$$

Proof.

Put $\varphi_k(t) = t^k$, and observe that, for $b_i \in \mathbb{R}$,

$$\varphi_k[b_1, \ldots, b_{k+1}] = 1, \quad \varphi_k[b_1, \ldots, b_\ell] = 0 \text{ for } \ell > k+1.$$

Let D denote the left handside. It can be written

$$D = \begin{vmatrix} \varphi_0(a_1) & \varphi_0(a_2) & \ldots & \varphi_0(a_n) \\ \varphi_1(a_1) & \varphi_1(a_2) & \ldots & \varphi_1(a_n) \\ \vdots & \vdots & & \vdots \\ \varphi_{n-k-1}(a_1) & \varphi_{n-k-1}(a_2) & \ldots & \varphi_{n-k-1}(a_n) \\ f_1(a_1) & f_1(a_2) & \ldots & f_1(a_n) \\ \vdots & \vdots & & \vdots \\ f_k(a_1) & f_k(a_2) & \ldots & f_k(a_n) \end{vmatrix}$$

One substracts the first column from the second, the second from the third, and so on:

$$C_n \leftarrow C_n - C_{n-1},\ C_{n-1} \leftarrow C_{n-1} - C_{n-2}, \ldots, C_2 \leftarrow C_2 - C_1.$$

Then one gets

$$D = (a_2 - a_1)(a_3 - a_2) \ldots (a_n - a_{n-1}) \begin{vmatrix} \varphi_1[a_1, a_2] & \varphi_1[a_2, a_3] & \ldots & \varphi_1[a_{n-1}, a_n] \\ \vdots & \vdots & & \vdots \\ \varphi_{n-k-1}[a_1, a_2] & \varphi_{n-k-1}[a_2, a_3] & \ldots & \varphi_{n-k-1}[a_{n-1}, a_n] \\ f_1[a_1, a_2] & f_1[a_2, a_3] & \ldots & f_1[a_{n-1}, a_n] \\ \vdots & \vdots & & \vdots \\ f_k[a_1, a_2] & f_k[a_2, a_3] & \ldots & f_k[a_{n-1}, a_n] \end{vmatrix}$$

Then we repeat the operation:

$$D = \big((a_2 - a_1) \ldots (a_n - a_{n-1})\big)\big((a_3 - a_1) \ldots (a_n - a_{n-2})\big) \begin{vmatrix} \varphi_2[a_1, a_2, a_3] & \ldots & \varphi_2[a_{n-2}, a_{n-1}, a_n] \\ \vdots & & \vdots \\ \varphi_{n-k-1}[a_1, a_2, a_3] & \ldots & \varphi_{n-k-1}[a_{n-2}, a_{n-1}, a_n] \\ f_1[a_1, a_2, a_3] & \ldots & f_1[a_{n-2}, a_{n-1}, a_n] \\ \vdots & & \vdots \\ f_k[a_1, a_2, a_3] & \ldots & f_k[a_{n-2}, a_{n-1}, a_n] \end{vmatrix}$$

After $n - k$ such operations we get the formula in Proposition 5.3. ∎

6. Olshanski's determinantal formula. — We will give in this section a proof of Olshanski's determinantal formula for the density of the radial part $\nu_A^{(k)}$ of the projection $\mu_A^{(k)}$, slightly different from the original one [Olshanski,2013]. As it has been said in Section 3 the measure $\mu_A^{(k)}$ and its radial part $\nu_A^{(k)}$ will be determined by using the fact that the Fourier-Laplace transform of $\mu_A^{(k)}$ is equal to the restriction to $\mathcal{H}_k(\mathbb{F})$ of the Fourier-Laplace transform of μ_A. Recall that, if $Z = \operatorname{diag}(z_1, \ldots, z_n)$,

$$\widehat{\mu_A}(Z) = \mathcal{E}(a; z) = \delta_n! \mathcal{D}_n(f; a; z), \text{ with } f(t) = e^t.$$

We will see that, with the notation of Section 3 and 4, $\mathcal{D}_n(f; a; z_1, \ldots, z_k, 0 \ldots, 0)$ can be expressed as an integral on $\mathbb{R}^k$ involving $\mathcal{D}_k(f^{(n-k)}; z; t)$. For k functions $f_1, \ldots, f_k$ we have put

$$D = \begin{vmatrix} \varphi_0(a_1) & \varphi_0(a_2) & \ldots & \varphi_0(a_n) \\ \varphi_1(a_1) & \varphi_1(a_2) & \ldots & \varphi_1(a_n) \\ \vdots & \vdots & & \vdots \\ \varphi_{n-k-1}(a_1) & \varphi_{n-k-1}(a_2) & \ldots & \varphi_{n-k-1}(a_n) \\ f_1(a_1) & f_1(a_2) & \ldots & f_1(a_n) \\ \vdots & \vdots & & \vdots \\ f_k(a_1) & f_k(a_2) & \ldots & f_k(a_n) \end{vmatrix}$$

By Proposition 5.3,

$$D = \Big(\prod_{0<j-i\leq n-k} (a_j - a_i) \Big) \det\big(f_i[a_j, \ldots, a_{j+n-k}]\big)_{1\leq i,j\leq k}.$$

By the Hermite-Genocchi formula (Proposition 5.2),

$$f_i[a_j, \ldots a_{j+n-k}] = \frac{1}{(n-k)!} \int_{\mathbb{R}} f_i^{(n-k)}(t) M_{n-k+1}(a_j, \ldots, a_{j+n-k}; t) dt,$$

and

$$\begin{aligned} &\det\big(f_i[a_j, \ldots, a_{j+n-k}]\big)_{1\leq i,j\leq k} \\ &= \Big(\frac{1}{(n-k)!}\Big)^k \det\Big(\int_{\mathbb{R}} f_i^{(n-k)}(t) M_{n-k+1}(a_j, \ldots, a_{j+n-k}; t) dt\Big)_{1\leq i,j\leq k}. \end{aligned}$$

We use now the integral form of the Cauchy-Binet formula: for $2k$ functions u_i, v_i,

$$\begin{aligned} &\int_{\mathbb{R}^k} \det\big(u_j(t_i)\big)_{1\leq i,j\leq k} \det\big(v_j(t_i)\big)_{1\leq i,j\leq k} dt_1 \ldots dt_k \\ &= k! \det\Big(\int_{\mathbb{R}} u_i(t) v_j(t) dt\Big). \end{aligned}$$

Then we get

$$\det\big(f_i[a_j,\dots,a_{j+n-k}]\big)_{1\le i,j\le k} = \Big(\frac{1}{(n-k)!}\Big)^k$$
$$\int_{(\mathbb{R}^k)_+} \det\big(f_j^{(n-k)}(t_i)\big)\det\big(M_{n-k+1}(a_j,\dots,a_{j+n-k};t_i)\big)dt_1\dots dt_k.$$

We define now, for a function f, the k functions $f_j(t)=f(z_jt)$. Then

$$f_j^{(n-k)}(t) = z_j^{n-k}f^{(n-k)}(z_jt),$$

and

$$\det\big(f_j^{(n-k)}(t_i)\big) = V_k(z)(z_1\dots z_k)^{n-k}V_k(t)\mathcal{D}_k(f^{(n-k)};z;t).$$

By Corollary 4.2,

$$\mathcal{D}_n(f;a;z_1,\dots,z_k,0,\dots,0) = \frac{a_0^{(n-k)}(f)}{V_n(a)V_k(z)(z_1\dots z_k)^{n-k}}\cdot D.$$

We get finally:

THEOREM 6.1. — *Let f be an analytic function defined near 0 in $\mathbb{C}$. For $1\le k\le n$,*

$$\mathcal{D}_n(f;a;z_1,\dots,z_k,0,\dots,0) = \frac{a_0^{(n-k)}(f)}{\prod_{j-i\ge n-k+1}(a_j-a_i)}\Big(\frac{1}{(n-k)!}\Big)^k$$
$$\int_{(\mathbb{R}^k)_+}\mathcal{D}_k(f^{(n-k)};z;t)\det\big(M_{n-k+1}(a_j,\dots,a_{j+n-k};t_i)\big)V_k(t)dt_1\dots dt_k.$$

We specialize now the previous formula by taking $f(t)=e^t$. Then

$$f^{(n-k)}(t)=e^t,\quad a_0^{(k)}(f)=\frac{1}{2!3!\dots(k-1)!}.$$

By using the relations

$$\widehat{\mu_a}\big(\mathrm{diag}(z1,\dots,z_n)\big) = 2!\dots(n-1)!\mathcal{D}_n(f;a;z),$$
$$\widehat{\mu_A^{(k)}}\big(\mathrm{diag}(z_1,\dots,z_k)\big) = 2!\dots(n-1)!\mathcal{D}_n(f;a;z_1,\dots,z_k,0,\dots,0),$$
$$\mathcal{E}_k(z_1,\dots,z_k;t) = 2!\dots(k-1)!\mathcal{D}_k(f;z;t),$$

we get

$$\widehat{\mu_A^{(k)}}\big(\mathrm{diag}(z_1,\ldots,z_k)\big) = \frac{C(n,k)}{\prod_{j-i\geq n-k+1}(a_j-a_i)}$$
$$\int_{(\mathbb{R}^k)_+} \mathcal{E}_k(z_1,\ldots,z_k;t)\det\big(M_{n-k+1}(a_j,\ldots,a_{j+n-k};t_i)V_k(t)dt_1\ldots dt_k,$$

with

$$C(n,k)=\prod_{i=1}^{k-1}\binom{n-k+i}{i}.$$

Olshanski's formula follows:

Theorem 6.2 (Olshanski,2013). — *The radial part $\nu_A^{(k)}$ of the projection $\mu_A^{(k)}$ on the subspace $\mathcal{H}_k(\mathbb{C})$ of the orbital measure μ_A is given by*

$$\nu_A^{(k)}(dt) = \frac{C(n,k)}{\prod_{j-i\geq n-k+1}(a_j-a_i)}$$
$$\det\big(M_{n-k+1}(a_j,\ldots,a_{j+n-k};t_i)\big)_{1\leq i,j\leq k}V_k(t)dt_1\ldots dt_k.$$

Special cases

a) If $k=1$, then $C(n,k)=1$ and

$$\nu_A^{(1)}(dt) = M_n(a_1,\ldots,a_n;t)dt.$$

This was observed by Okounkov ([Olshanski-Vershik,1996] p.170. See also [Faraut,2005]).

b) If $k=n-1$, then $C(n,k)=(n-1)!$, and

$$\prod_{j-i\geq 2}(a_j-a_i) = \frac{V_n(a)}{\prod_{j-i=2}(a_j-a_i)}.$$

Since $a_1<a_2<\cdots<a_n$, the determinant

$$\det M_2(a_j,a_{j+1};t_i)_{1\leq i,j\leq n-1}$$

vanishes unless $t\preceq a$, and then equals

$$M_2(a_1,a_2;t_1)M_2(a_2,a_3;t_2)\ldots M_2(a_{n-1},a_n;t_n).$$

One gets Baryshnikov formula (Theorem 1.3.a).

Remark

It is possible to evaluate directly the integral:

$$Z(a) = \int_{(\mathbb{R}^k)_+} \det\big(M_{n-k+1}(a_j,\ldots,a_{j+n-k};t_i)V_k(t)dt_1\ldots dt_k.$$

By the Cauchy-Binet formula,

$$Z(a) = \det\Big(\int_{\mathbb{R}} M_{n-k+1}(a_j,\ldots,a_{j+n-k};t)t^{i-1}dt\Big).$$

The moments of the spline functions are known:

$$\int_{\mathbb{R}} M_n(a_1,\ldots,a_n;t)t^m dt = \frac{m!(n-1)!}{(m+n-1)!}h_m(a_1,\ldots,a_n),$$

where h_m is the complete symmetric function of degree m. Hence

$$Z(a) = \det\Big(\frac{(i-1)!(n-k)!}{(n-k+i-1)!}h_{i-1}(a_j,\ldots,a_{j+n-k})\Big).$$

Using the identity

$$h_m(a_2,\ldots,a_n) - h_m(a_1,\ldots,a_{n-1}) = (a_1-a_n)h_{m-1}(a_1,\ldots,a_n),$$

which can be obtained from the generating formula:

$$\sum_{m=0}^{\infty} h_m(a_1,\ldots,a_n)z^m = \prod_{i=1}^{n}\frac{1}{1-a_i z},$$

one gets

$$\det\big(h_{i-1}(a_j,\ldots,a_{j+n-k})\big) = \prod_{j-i\geq n-k+1}(a_j-a_i).$$

Finally

$$Z(a) = \frac{1}{C(n,k)}\prod_{j-i\geq n-k+1}(a_j-a_i).$$

The following corollary can be seen as a multivariate analogue of Hermite-Genocchi formula (Proposition 5.2). It will be used in next Section.

Corollary 6.3. — *For a function f analytic near 0 in $\mathbb{C}$,*

$$\mathcal{D}_n(f;a;z_1,\dots,z_k,0,\dots,0)$$
$$= \Big(\prod_{i=0}^{k-1}\frac{j!}{(n-k+j)!}\Big)a_0^{(n-k)}(f)\int_{(\mathbb{R}^k)_+}\mathcal{D}_k(f^{(n-k)};z;t)\nu_A^{(k)}(dt).$$

7. A multivariate Markov-Krein formula for the projection of an orbital measure ($\mathbb{F}=\mathbb{C}$). — For $\gamma\in\mathbb{C}$ and $Z\in M_n(\mathbb{C})$ one defines

$${}_1F_0^{(n)}(\gamma;Z)=\det{}^{(n)}(I_n-Z)^{-\gamma},$$

and, for $X,Y\in M_n(\mathbb{C})$,

$${}_1\mathcal{F}_0^{(n)}(\gamma;X,Y)=\int_{U_n(\mathbb{C})}\det{}^{(n)}(I_n-XuYu^*)^{-\gamma}\alpha_n(du).$$

Proposition 7.1. — *The function ${}_1\mathcal{F}_0^{(n)}(\gamma;X,Y)$ admits a determinantal formula: if $X=\operatorname{diag}(x_1,\dots,x_n)$, $Y=\operatorname{diag}(y_1,\dots,y_n)$, then*

$$\frac{1}{V_n(x)V_n(y)}\det\big((1-x_iy_j)^{-\gamma}\big)_{1\leq i,j\leq n}=\Big(\prod_{k=1}^{n-1}\frac{(\gamma)_k}{k!}\Big){}_1\mathcal{F}_0^{(n)}(\gamma+n-1;X,Y).$$

In the special case $\gamma=1$, by the Cauchy formula,

$${}_1\mathcal{F}_0(n;X,Y)=\prod_{i,j=1}^{n}\frac{1}{1-x_iy_j}.$$

With the notation at the end of Section 3, the left handside of the formula in Proposition 7.1 can be written $\mathcal{D}_n(f;x,y)$ with $f(t)=(1-t)^{-\gamma}$, and

$$\prod_{k=1}^{n-1}\frac{(\gamma)_k}{k!}=a_0^{(n)}(f).$$

Proof.

The function ${}_1\mathcal{F}_0^{(n)}$ admits a Schur expansion:

$${}_1\mathcal{F}_0^{(n)}(\gamma;X,Y)=\sum_{\mathbf{m}}\frac{(\gamma)_{\mathbf{m}}}{(n)_{\mathbf{m}}}s_{\mathbf{m}}(x)s_{\mathbf{m}}(y),$$

where $s_{\mathbf{m}}$ is the Schur function associated to the partition $\mathbf{m} = (m_1, \ldots, m_n)$. Recall also the definition of the generalized Pochammer symbol

$$(\alpha)_{\mathbf{m}} = \prod_{i=1}^{n} (\alpha - i + 1)_{m_i}.$$

We apply Proposition 12.3.3 in [Faraut,2008] (see also [Hua,1963], §I.1.2) to the one variable function

$$f(t) = (1-t)^{-\gamma} = \sum_{m=1}^{\infty} \frac{(\gamma)_m}{m!} t^m.$$

Let us put

$$c_m = \frac{(\alpha)_m}{m!},$$

and compute

$$a_{\mathbf{m}} = c_{m_1+\delta_1} c_{m_2+\delta_2} \ldots c_{m_n+\delta_n} \quad (\delta_i = n - i).$$

We get

$$\prod_{i=1}^{n} (\gamma)_{m_i+\delta_i} = \Big(\prod_{k=1}^{n-1} (\gamma)_k \Big) (\gamma + n - 1)_{\mathbf{m}}.$$

Similarly, since $m! = (1)_m$,

$$\prod_{i=1}^{n} (m_i + \delta_i)! = \Big(\prod_{k=1}^{n-1} k! \Big) (n)_{\mathbf{m}}.$$

Hence

$$a_{\mathbf{m}} = \frac{(\gamma + n - 1)_{\mathbf{m}}}{(n)_{\mathbf{m}}}.$$

By Proposition 12.3.3 in [Faraut,2008],

$$\frac{1}{V_n(x)V_n(y)} \det\Big(\frac{1}{(1 - x_i y_j)^{\gamma}} \Big)_{1 \le i,j \le n} = \Big(\prod_{k=1}^{n-1} \frac{(\gamma)_k}{k!} \Big) {}_1\mathcal{F}_0^{(n)}(\gamma + n - 1; X, Y).$$

□

THEOREM 7.2. — *If the eigenvalues of $Z \in M_k(\mathbb{C})$ are not real, then*

$$\int_{\mathcal{H}_k} \det{}^{(k)}(Z - X)^{-n} \mu_A^{(k)}(dX) = \prod_{i=1}^{n} \det{}^{(k)}(Z - a_i I_k)^{-1}.$$

Equivalently, if $Z = \mathrm{diag}(z_1, \ldots, z_k)$ *with* $z_i \in \mathbb{C} \setminus \mathbb{R}$*, then*

$$\int_{\mathcal{H}_k} \det{}^{(k)}(Z - X)^{-n} \mu_A^{(k)}(dX) = \prod_{i=1}^{n} \prod_{j=1}^{k} \frac{1}{z_j - a_i}.$$

Proof.

The second formula in Theorem 7.2 is equivalent to

$$I(Z) := \int_{\mathcal{H}_k} \det{}^{(k)}(I_k - ZX)^{-n} \mu_A^{(k)}(dX) = \prod_{i=1}^{n} \prod_{j=1}^{k} \frac{1}{1 - a_i z_j}.$$

We will evaluate $I(Z)$ by using the formula (*) in the introduction:

$$I(Z) = \int_{(\mathbb{R}^k)_+} \Big(\int_{U_k(\mathbb{C})} \det{}^{(k)}(I_k - ZuTu^*)^{-n} \alpha_k(du) \Big) \nu_A^{(k)}(dt),$$

where $T = \mathrm{diag}(t_1, \ldots, t_k)$. The integral over $U_k(\mathbb{C})$ equals ${}_1\mathcal{F}_0^{(k)}(n; Z, T)$. By Proposition 5.1,

$${}_1\mathcal{F}_0^{(k)}(n; Z, T) = \frac{1}{a_0^{(k)}(f)} \mathcal{D}_k(f; ; z, t),$$

with $f(t) = (1-t)^{-(n-k+1)}$. Hence

$$I(Z) = \frac{1}{a_0^{(k)}(f)} \int_{\mathbb{R}^k} \mathcal{D}_k(f; z, t) \nu_A^{(k)}(dt).$$

Observe that

$$\Big(\frac{d}{dt}\Big)^{n-k} \frac{1}{1-t} = (n-k)! \frac{1}{(1-t)^{n-k+1}}.$$

By Corollary 6.3,

$$\int_{\mathbb{R}^k} \mathcal{D}_k(f; z, t) \nu_A^{(k)}(dt) = \frac{a_0^{(k)}(f)}{a_0^{(n)}(f_0)} \mathcal{D}_n(f_0; a; z_1, \ldots, z_k, 0, \ldots, 0),$$

with $f_0(t) = (1-t)^{-1}$. We obtain finally

$$I(Z) = {}_1\mathcal{F}_0^{(n)}(n; a; z_1, \ldots, z_k, 0, \ldots, 0) = \prod_{i=1}^{n} \prod_{j=1}^{k} \frac{1}{1 - a_i z_j}. \qquad \square$$

8. Generalized binomial formula ($\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$). — In case of $\mathcal{H}_n(\mathbb{R})$ we don't know any determinantal formula for the radial part $\nu_A^{(k)}$ of the projection $\mu_A^{(k)}$ of the orbiatal measure μ_A. However we will see in next Section that there is a multivariate Markov-Krein type formula for the projection $\mu_A^{(k)}$.

We assume that $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. We consider the multivariate binomial formula

$$ {}_1F_0^{(n)}(\gamma; Z) := \det{}^{(n)}(I_n - Z)^{-\gamma} = \sum_{\mathbf{m}} d_{\mathbf{m}} \frac{(\alpha)_{\mathbf{m}}}{\left(\frac{N}{n}\right)_{\mathbf{m}}} \Phi_{\mathbf{m}}^{(n)}(Z), $$

where $\Phi_{\mathbf{m}}$ is the spherical polynomial associated to the partition $\mathbf{m} = (m_1, \ldots, m_n)$. Furthermore

$$ N = \dim \mathcal{H}_n(\mathbb{F}) = n + \frac{d}{2} n(n-1), $$

with $d = 1$ if $\mathbb{F} = \mathbb{R}$, and $d = 2$ if $\mathbb{F} = \mathbb{C}$, and $d_{\mathbf{m}}$ is the degree of the irreducible representation of $GL(n, \mathbb{F})$ associated to the partition $\mathbf{m}$. The generalized Pochhammer coefficients are given by

$$ (\gamma)_{\mathbf{m}} = \prod_{i=1}^{n} \left(\gamma - (i-1)\frac{d}{2}\right). $$

The series converges for Z for $\|Z\|_{\mathrm{op}} < 1$. (See [Faraut-Korányi,1994], Proposition XII.1.3.)

In case of $\mathbb{F} = \mathbb{C}$, the spherical polynomial is the normalized Schur function:

$$ \Phi_{\mathbf{m}}^{(n)}(Z) = \frac{s_{\mathbf{m}}(Z)}{s_{\mathbf{m}}(I_n)}. $$

One defines also the two variables function ${}_1\mathcal{F}_0^{(n)}(\gamma; X, Y)$, for $X, Y \in \mathcal{H}_n$, by

$$ {}_1\mathcal{F}_0^{(n)}(\gamma; X, Y) = \int_{U_n(\mathbb{F})} {}_1F_0^{(n)}(\gamma; guYu^*g^*)\,du, $$

if $X = gg^*$, with $g \in GL(n, \mathbb{F})$. It admits the following spherical Taylor expansion

$$ {}_1\mathcal{F}_0^{(n)}(\gamma; X, Y) = \sum_{\mathbf{m}} d_{\mathbf{m}} \frac{(\gamma)_{\mathbf{m}}}{\left(\frac{N}{n}\right)_{\mathbf{m}}} \Phi_{\mathbf{m}}^{(n)}(X) \Phi_{\mathbf{m}}^{(n)}(Y). $$

For $\alpha = n\frac{d}{2}$ this function has a simple evaluation:

PROPOSITION 8.1. — *If* $X = \mathrm{diag}(x_1, \dots, x_n)$, $Y = \mathrm{diag}(y_1, \dots, y_n)$,

$$ {}_1\mathcal{F}_0^{(n)}\big(n\frac{d}{2}; X, Y\big) = \prod_{i,j=1}^{n} (1 - x_i y_j)^{-\frac{d}{2}}. $$

[Bouali,2006], Proposition 1.4.

Proof.
Let μ be a measure on $M_n(\mathbb{F})$ which is biinvariant under $U_n(\mathbb{F})$. For $X, Y \in \mathcal{H}_n(\mathbb{F})$ define

$$ \mathcal{F}(X, Y) = \int_{M_n(\mathbb{F})} e^{\mathrm{tr}(X\xi Y\xi^*)} \mu(d\xi). $$

a) We will first prove that

$$ \mathcal{F}(X, Y) = \sum_{\mathbf{m}} \frac{d_{\mathbf{m}}}{\left(\frac{N}{n}\right)_{\mathbf{m}}} \gamma_{\mathbf{m}} \Phi_{\mathbf{m}}^{(n)}(X) \Phi_{\mathbf{m}}^{(n)}(Y), $$

with

$$ \gamma_{\mathbf{m}} = \int_{M_n(\mathbb{F})} \Phi_{\mathbf{m}}^{(n)}(\xi\xi^*) \mu(d\xi). $$

If X is real symmetric or Hermitian and positive definite we can write

$$ \mathcal{F}(X, Y) = \int_{M_n(\mathbb{F})} e^{\mathrm{tr}(X^{\frac{1}{2}}\xi Y \xi^* X^{\frac{1}{2}})} \mu(d\xi). $$

From the spherical expansion

$$ e^{\mathrm{tr}(Z)} = \sum_{\mathbf{m}} \frac{d_{\mathbf{m}}}{\left(\frac{N}{n}\right)_{\mathbf{m}}} \Phi_{\mathbf{m}}^{(n)}(Z) $$

([Faraut-Korányi,1994], Proposition XII.1.3), it follows that

$$ \mathcal{F}(X, Y) = \sum_{\mathbf{m}} \frac{d_{\mathbf{m}}}{\left(\frac{N}{n}\right)_{\mathbf{m}}} \int_{M_n(\mathbb{F})} \Phi_{\mathbf{m}}^{(n)}(X^{\frac{1}{2}}\xi Y \xi^* X^{\frac{1}{2}}) \mu(d\xi). $$

The product formula for the spherical polynomials can be written, for $g \in M_n(\mathbb{F})$

$$ \int_{U_n(\mathbb{F})} \Phi_{\mathbf{m}}^{(n)}(guYu^*g^*) \alpha_n(du) = \Phi_{\mathbf{m}}^{(n)}(gg^*) \Phi_{\mathbf{m}}^{(n)}(X). $$

By using the right $U_n(\mathbb{F})$-invariance of the measure μ, we obtain

$$\begin{aligned}&\int_{M_n(\mathbb{F})} \Phi^{(n)}_{\mathbf{m}}(X^{\frac12}\xi Y\xi^* X^{\frac12})\mu(d\xi)\\ &= \int_{M_n(\mathbb{F})}\Big(\int_{U_n(\mathbb{F})} \Phi^{(n)}_{\mathbf{m}}(X^{\frac12}\xi uYu^*\xi^* X^{\frac12})\alpha_n(du)\Big)\mu(d\xi)\\ &= \Phi^{(n)}_{\mathbf{m}}(Y)\int_{M_n(\mathbb{F})} \Phi^{(n)}_{\mathbf{m}}(X^{\frac12}\xi\xi^* X^{\frac12})\mu(d\xi).\end{aligned}$$

Then, by using the left $U_n(\mathbb{F})$-invariance of μ, we get

$$\begin{aligned}&= \Phi^{(n)}_{\mathbf{m}}(Y)\int_{M_n(\mathbb{F})}\Big(\int_{U_n(\mathbb{F})} \Phi^{(n)}_{\mathbf{m}}(X^{\frac12}u\xi\xi^*u^* X^{\frac12})\alpha_n(du)\Big)\mu(d\xi)\\ &= \Phi^{(n)}_{\mathbf{m}}(Y)\Phi^{(n)}_{\mathbf{m}}(X)\int_{M_n(\mathbb{F})} \Phi^{(n)}_{\mathbf{m}}(\xi\xi^*)\mu(d\xi).\end{aligned}$$

Therefore

$$\mathcal{F}(X,Y) = \sum_{\mathbf{m}} \frac{d_{\mathbf{m}}}{\left(\frac{N}{n}\right)_{\mathbf{m}}}\gamma_{\mathbf{m}}\Phi^{(n)}_{\mathbf{m}}(X)\Phi^{(n)}_{\mathbf{m}}(Y).$$

b) We take now for μ a Gaussian measure on $M_n(\mathbb{F})$:

$$\mu(d\xi) = \pi^{-\frac{d}{2}n^2}e^{-\operatorname{tr}(\xi\xi^*)}m(d\xi).$$

The image of the measure μ by the map

$$M(n,\mathbb{F}) \to \overline{\Omega_n(\mathbb{F})}, \quad \xi \mapsto \xi\xi^*$$

($\Omega_n(\mathbb{F}) \subset \mathcal{H}_n(\mathbb{F})$ denotes the cone of positive definite matrices) is the Wishart distribution

$$W(d\eta) = \frac{1}{\Gamma_n\left(n\frac{d}{2}\right)}e^{-\operatorname{tr}\eta}\det(\eta)^{n\frac{d}{2}-\frac{N}{n}}m(d\eta),$$

where m is the Lebesgue measure on $\Omega_n(\mathbb{F})$ (Proposition VI.11 in [Faraut-Korányi,1994]. We get

$$\begin{aligned}\gamma_{\mathbf{m}} &= \frac{1}{\Gamma_n\left(n\frac{d}{2}\right)}\int_{\Omega_n(\mathbb{F})} \Phi^{(n)}_{\mathbf{m}}(\eta)e^{-\operatorname{tr}\eta}\det(\eta)^{n\frac{d}{2}-\frac{N}{n}}m(d\eta)\\ &= \frac{\Gamma_n\left(n\frac{d}{2}+\mathbf{m}\right)}{\Gamma_n\left(n\frac{d}{2}\right)} = \left(n\frac{d}{2}\right)_{\mathbf{m}}.\end{aligned}$$

Therefore

$$\mathcal{F}(X,Y) = {}_1\mathcal{F}_0^{(n)}(n\frac{d}{2}; X, Y).$$

c) For X and Y diagonal,

$$X = \operatorname{diag}(x_1, \dots, x_n), \ y = \operatorname{diag}(y_1, \dots, y_n),$$

$$\begin{aligned}\mathcal{F}(X,Y) &= \pi^{-\frac{d}{2}n^2} \int_{M(n,\mathbb{F})} e^{\sum_{i,j=1}^n x_i y_j |\xi_{ij}|^2} e^{-\sum_{i,j=1}^n |\xi_{ij}|^2} \prod_{i,j=1}^n d\xi_{ij} \\ &= \prod_{i,j=1}^n \pi^{-\frac{d}{2}} \int_{\mathbb{F}} e^{-(1-x_i y_j)|\xi_{ij}|^2} d\xi_{ij} \\ &= \prod_{i,j=1}^n (1 - x_i y_j)^{-\frac{d}{2}}.\end{aligned}$$

∎

9. A multivariate Markov-Krein formula for projections of orbital measures ($\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$). — Recall that the Markov-Krein transform maps a bounded positive measure ν on $\mathbb{R}$ to a probability measure μ on $\mathbb{R}$ such that, for $z \in \mathbb{C} \setminus \mathbb{R}$,

$$\int_{\mathbb{R}} (z-t)^{-\theta} \mu(dt) = \exp\Big(-\int_{\mathbb{R}} \log(z-t)\nu(dt)\Big),$$

where $\theta = \nu(\mathbb{R})$. If ν is a discrete measure,

$$\nu = \sum_{i=1}^n \tau_i \delta_{a_i},$$

then this can be written:

$$\int_{\mathbb{R}} (z-t)^{-\theta} \mu(dt) = \prod_{i=1}^n (z-a_i)^{-\tau_i}, \quad \theta = \sum_{i=1}^n \tau_i.$$

(See [Fourati, 2001], and [Faraut-Fourati, 2014].) In this section we will establish a multivariate analogue of such a relation for the projections $\mu_A^{(k)}$ of an orbital measure μ_A.

Let F be a $U_n(\mathbb{F})$-invariant continuous function defined in an open ball $\mathcal{D}$ in $\mathcal{H}_n(\mathbb{F})$. There is a unique kernel $F(X,Y)$ defined on $\mathcal{D} \times \mathcal{D}$ such that

$$F(X, I_n) = F(X),$$

and, for every $h \in M_n(\mathbb{F})$,

$$F(hXh^*, Y) = F(X, h^*Yh).$$

One defines the associated kernel $\mathcal{F}(X, Y)$ by

$$\mathcal{F}(X, Y) = \int_{U_n(\mathbb{F})} F(uXu^*, Y)\alpha_n(u).$$

Then $\mathcal{F}(X, Y)$ is $U_n(\mathbb{F})$-invariant with respect to both variables X, Y, and is determined by its restriction to diagonal matrices. Hence it can be seen as a function on $\mathbb{R}^n \times \mathbb{R}^n$.

If F admits a spherical Taylor expansion

$$F(X) = \sum_{\mathbf{m}} a_{\mathbf{m}} \Phi_{\mathbf{m}}(X),$$

then

$$\mathcal{F}(X, Y) = \sum_{\mathbf{m}} a_{\mathbf{m}} \Phi_{\mathbf{m}}^{(n)}(X) \Phi_{\mathbf{m}}^{(n)}(Y).$$

Recall that $\mu_A^{(k)}$ denotes the projection of the orbital measure μ_A on $\mathcal{H}_k(\mathbb{F})$.

Proposition 9.1. — *With the previous notation*

$$\int_{\mathcal{H}_k(\mathbb{F})} F(X, Y)\mu_A^{(k)}(dY) = \mathcal{F}(x_1, \ldots, x_k, 0, \ldots, 0; a_1, \ldots, a_n)$$

Proof.

Let p_k denotes the projection onto $\mathcal{H}_k(\mathbb{F})$. It can be written $p_k(X) = I_k X I_k$. The measure $\mu_A^{(k)}$ is the image of μ_A under p_k: if f is a continuous function on $\mathcal{H}_n(\mathbb{F})$,

$$\int_{\mathcal{H}_k(\mathbb{F})} f(X)\mu_A^{(k)}(dX) = \int_{\mathcal{H}_n(\mathbb{F})} f\big(p_k(X)\big)\mu_A(dX),$$

and

$$\int_{\mathcal{H}_k(\mathbb{F})} F(X, Y)\mu_A^{(k)}(dY) = \int_{\mathcal{H}_n(\mathbb{F})} F\big(X, p_k(Y)\big)\mu_A(dY).$$

By the invariance property of the kernel F we get $F\big(X, p_k(Y)\big) = F\big(p_k(X), Y\big)$ and

$$\int_{\mathcal{H}_n(\mathbb{F})} F\big((p_k(X), Y\big)\mu_A(dY) = \int_{U_n(\mathbb{F})} F\big(p_k(X), uAu^*\big)\alpha(du)$$
$$= \mathcal{F}\big(p_k(X), A\big) = \mathcal{F}(x_1, \ldots, x_k, 0, \ldots 0; a_1, \ldots, a_n).$$

□

For instance, if $F(X) = e^{\operatorname{tr} X}$, then $F(X,Y) = e^{\operatorname{tr}(XY)}$ and $\mathcal{F}(X,Y) = \mathcal{E}_n(X,Y)$ with the notation of Section 1. In this case $\int F(X,Y)\mu_A^{(k)}$ is the Fourier-Laplace transform of $\mu_A^{(k)}$ and Proposition 7.1 simply means that the Fourier-Laplace transform of $\mu_A^{(k)}$ is the restriction to $\mathcal{H}_k(\mathbb{F})$ of the Fourier-Laplace transform of μ_A:

$$\int_{\mathcal{H}_k(\mathbb{F})} F(X,Y)\mu_A^{(k)}(dY) = \int_{\mathcal{H}_k(\mathbb{F})} e^{\operatorname{tr}(XY)}\mu_A^{(k)}(dY)$$
$$= \widehat{\mu_A^{(k)}}(X) = \mathcal{E}_n(x_1,\dots,x_k,0,\dots,0;a_1,\dots,a_n).$$

Consider now the case of a spherical polynomial: $F(X) = \Phi_{\mathbf{m}}^{(n)}(X)$. Then, if X, Y are positive definite,

$$F(X,Y) = \Phi_{\mathbf{m}}^{(n)}(Y^{\frac{1}{2}}XY^{\frac{1}{2}}) = \Phi_{\mathbf{m}}^{(n)}(X^{\frac{1}{2}}YX^{\frac{1}{2}}),$$

and, by the product formula for the spherical polynomials,

$$\mathcal{F}(X,Y) = \Phi_{\mathbf{m}}^{(n)}(X)\Phi_{\mathbf{m}}^{(n)}(Y).$$

We obtain

$$\int_{\mathcal{H}_k(\mathbb{F})} F(X,Y)\mu_A^{(k)}(dY) = \Phi_{\mathbf{m}}^{(n)}(x_1,\dots,x_k,0,\dots,0)\Phi_{\mathbf{m}}^{(n)}(a_1,\dots,a_n).$$

This vanishes if $m_{k+1} > 0$. Taking $X = I_k$ we get

$$\int_{\mathcal{H}_k(\mathbb{F})} \Phi_{\mathbf{m}}^{(n)}(I_kYI_k)\mu_A^{(k)}(dY) = \Phi_{\mathbf{m}}^{(n)}(I_k)\Phi_{\mathbf{m}}^{(n)}(a_1,\dots,a_n).$$

Using the fact that the restriction to $\mathcal{H}_k(\mathbb{F})$ of $\Phi_{\mathbf{m}}^{(n)}$ with $\mathbf{m} = (m_1,\dots,m_k,0,\dots,0)$ is proportional to $\Phi_{\mathbf{m}'}^{(k)}$ with $\mathbf{m}' = (m_1,\dots,m_k)$: for $X \in \mathcal{H}_k(\mathbb{F})$

$$\Phi_{\mathbf{m}'}^{(k)}(X) = \frac{\Phi_{\mathbf{m}}^{(n)}(X)}{\Phi_{\mathbf{m}}^{(n)}(I_k)},$$

we obtain

PROPOSITION 9.2. — *The spherical moments of the measure $\mu_A^{(k)}$ are given by*

$$\int_{\mathcal{H}_k(\mathbb{F})} \Phi_{\mathbf{m}}^{(k)}(X)\mu_A^{(k)}(dX) = \Phi_{\mathbf{m}}^{(n)}(A).$$

As a third example we consider the function

$$F(X) = {}_1F_0^{(n)}(\gamma; X) = \det{}^{(n)}(I_n - X)^{-\gamma}.$$

Then

$$F(X,Y) = \det{}^{(n)}(I_n - XY)^{-\gamma},$$

and

$$\mathcal{F}(X,Y) = {}_1\mathcal{F}_0^{(n)}(\gamma; X, Y).$$

For $\gamma = n\frac{d}{2}$, by Proposition 8.1,

$${}_1\mathcal{F}_0^{(n)}(n\frac{d}{2}; X, Y) = \prod_{i,j}^{n}(1 - x_i y_j)^{-\frac{d}{2}}.$$

By Proposition 9.1, we get in this case, for $X \in \mathcal{H}_k$,

$$\begin{aligned}\int_{\mathcal{H}_k(\mathbb{F})} \det{}^{(k)}(I_k - XY)^{-\frac{d}{2}}\mu_A^{(k)}(dY) &= \prod_{i=1}^{n}\prod_{j=1}^{k}(1 - a_i x_j)^{-\frac{d}{2}} \\ &= \prod_{i=1}^{n}\det{}^{(k)}(I_k - a_i X)^{-\frac{d}{2}}\end{aligned}.$$

By changing X into Z^{-1}, and by analytic continuation we obtain

THEOREM 9.3. — *For $Z \in \mathcal{H}_k(\mathbb{F}) \pm i\Omega_n(\mathbb{F})$,*

$$\int_{\mathcal{H}_k(\mathbb{F})} \det{}^{(k)}(Z - X)^{-\frac{nd}{2}}\mu_A^{(k)}(dY) = \prod_{i=1}^{n}\det{}^{(k)}(Z - a_i I_k)^{-\frac{d}{2}}.$$

For $k = 1$, it says that, for $z \in \mathbb{C} \setminus \mathbb{R}$,

$$\int_{\mathbb{R}} (z - t)^{-n\frac{d}{2}}\mu_A^{(1)}(dt) = \prod_{i=1}^{n}(z - a_i)^{-\frac{d}{2}}.$$

If $\mathbb{F} = \mathbb{C}$, it follows that the density of $\mu_A^{(1)}$ is a spline function. In fact, for $z \in \mathbb{C} \setminus \mathbb{R}$,

$$\int_{\mathbb{R}} (z - t)^{-n} M_n(a_1, \ldots, a_n; t)dt = \prod_{i=1}^{n}\frac{1}{z - a_i}.$$

If $\mathbb{F} = \mathbb{R}$, this relation has been used by Fourati ([2011]) to establish an explicit formula for the density of $\mu_A^{(1)}$. The proof uses boundary values of holomorphic functions. We don't know whether this method can be extended for $k > 1$.

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