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ORBITAL MEASURES AND SPLINE FUNCTIONS

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1 Projection of orbital measures

We note $\mathcal{H}_n(\mathbb{R}) = Sym(n,\mathbb{R})$, $\mathcal{H}_n(\mathbb{C}) = Herm(n,\mathbb{C})$. For a matrix $X \in \mathcal{H}_n(\mathbb{F})$ ($\mathbb{F} = \mathbb{R}$ ou $\mathbb{C}$), the classical spectral theorem says that the eigenvalues $\lambda_1, \ldots, \lambda_n$ are real and the corresponding eigenvectors are orthogonal. Note $\Lambda^{(n)}$ the map $\mathcal{H}_n(\mathbb{F})$ onto $(\mathbb{R}^n)_+$,

$$(\mathbb{R}^n)_+ := \{t \in \mathbb{R}^n \mid t_1 \leq t_2 \leq \cdots \leq t_n\},$$

which, to a matrix X, associate the sequence of the eigenvalues in the increasing order:

$$\Lambda^{(n)}(X) = (\lambda_1, \ldots, \lambda_n).$$

Note $U_n(\mathbb{F}) = O(n)$, the orthogonal group, if $\mathbb{F} = \mathbb{R}$, and $U_n(\mathbb{F}) = U(n)$, the unitary group, if $\mathbb{F} = \mathbb{C}$. The group $U_n(\mathbb{F})$ acts on the space $\mathcal{H}_n(\mathbb{F})$ by the transformations $X \mapsto uXu^*$ ($u \in U_n(\mathbb{F})$). Note $\mathcal{O}_A$ the orbit of the diagonal matrix $A = \mathrm{diag}(a_1, \ldots, a_n)$ ($a_i \in \mathbb{R}$, $a_1 \leq \cdots \leq a_n$) :

$$\mathcal{O}_A = \{uAu^* \mid u \in U_n(\mathbb{F})\}.$$

From the spectral theorem it follows that

$$\mathcal{O}_A = \big\{X \in \mathcal{H}_n(\mathbb{F}) \mid \mathrm{spectrum}(X) = \{a_1, \ldots, a_n\}\big\}.$$

The orbit $\mathcal{O}_A$ carries a natural probability measure: the orbital measure μ_A, image of the normalized Haar measure α of the compact group $U_n(\mathbb{F})$ under the map

$$U_n(\mathbb{F}) \to \mathcal{H}_n(\mathbb{F}), \quad u \mapsto uAu^*.$$

For a continuous function f defined on $\mathcal{H}_n(\mathbb{F})$,

$$\int_{\mathcal{H}_n(\mathbb{F})} f(X)\mu_A(dX) = \int_{U_n(\mathbb{F})} f(uAu^*)\alpha(du).$$

Note p_k the projection of $\mathcal{H}_n(\mathbb{F})$ onto $\mathcal{H}_k(\mathbb{F})$ which maps a matrix $X \in \mathcal{H}_n(\mathbb{F})$ to the matrix $Y = p_k(X) \in \mathcal{H}_k(\mathbb{F})$ of the k first rows and the k first columns of X. We will study the image $\mu_A^{(k)}$ of the orbital measure μ_A under the projection p_k.

Let μ be a measure on $\mathcal{H}_n(\mathbb{F})$ which is invariant under $U_n(\mathbb{F})$. The integral of a function f is written as follows

$$\int_{\mathcal{H}_n(\mathbb{F})} f(X)\mu(dX) = \int_{(\mathbb{R}^n)_+} \Big(\int_{U_n(\mathbb{F})} f(u\mathrm{diag}(t_1, \ldots, t_n)u^*)\alpha(du)\Big)\nu(dt),$$

where ν is a measure on $(\mathbb{R}^n)_+$, called the radial part of μ. If μ is a probability measure on $\mathcal{H}_n(\mathbb{F})$ which is $U_n(\mathbb{F})$-invariant, its radial part ν is also the joint distribution of the eigenvalues for a random matrix whose distribution is μ. We will note $\nu_A^{(k)}$ the radial part of $\mu_A^{(k)}$.

Assume now $\mathbb{F} = \mathbb{C}$. We will start with the simplest case $k = 1$, and will see that the projection $\mu_A^{(1)}$ involves spline functions (Okounkov, 1996).

If $k = n - 1$ this question is related to an interlacing property of the eigenvalues. The measure $\nu_A^{(n-1)}$ is given by a formula due to Baryshnikov (2001).

We will study the general case, $1 \leq k \leq n - 1$, by using the Fourier transform. The radial part $\nu_A^{(k)}$ has a density which can be written as a determinant of spline functions. This is the Olshanski's determinantal formula (2013).

In last two Sections, we consider the projection onto the subspace $\mathcal{D}_n$ of diagonal matrices. Horn's Theorem describes the image of the orbit $\mathcal{O}_A$: it is the convex hull of points $\sigma(a)$, with $\sigma \in \mathfrak{S}_n$, the symmetric group. The image of the orbital measure is given as a special case of Heckman's formula.

2 Projection of the orbital measure μ_A and Peano measure

We assume in this section that $\mathbb{F} = \mathbb{C}$. We consider the projection $M_A := \mu_A^{(1)} = \nu_A^{(1)}$ of the orbital measure μ_A on $\mathcal{H}_1(\mathbb{C}) = \mathbb{R}E_{11} \simeq \mathbb{R}$: if f is a continuous function on $\mathbb{R}$,

$$\int_{\mathbb{R}} f(t) M_A(dt) = \int_{U(n)} f\big((uAu^*)_{11}\big)\alpha_n(du).$$

One establishes easily

$$(uAu^*)_{11} = a_1|u_{11}|^2 + \cdots + a_n|u_{1n}|^2.$$

Proposition 2.1. *Consider the map*

$$\Phi : U(n) \to S = S(\mathbb{C}^n),$$

the unit sphere in $\mathbb{C}^n$, which maps the matrix $u \in U(n)$ to the first row:

$$u \mapsto (u_{11}, \ldots, u_{1n}).$$

The image under Φ of the Haar measure α is the normalized uniform measure σ on S.

Proof. The image under Φ of the Haar measure α is a measure on S which is $U(n)$-invariant. ∎

Note Δ_n the simplex defined by

$$\Delta_n = \{\tau = (\tau_1, \dots, \tau_{n+1}) \in \mathbb{R}^{n+1} \mid \tau_i \geq 0,\ \tau_1 + \dots + \tau_{n+1} = 1\},$$

and let β_n be the normalized uniform measure on Δ_n, i.e. the restriction to Δ_n of the Lebesgue measure on the hyperplane with equation $\tau_1 + \dots + \tau_{n+1} = 1$, normalized in such a way that the total measure is equal to 1. Note also D_n the closed set of $\mathbb{R}^n$ defined by

$$D_n = \{\tau = (\tau_1, \dots, \tau_n) \in \mathbb{R}^n \mid \tau_i \geq 0,\ \tau_1 + \dots + \tau_n \leq 1\}.$$

This is the projection of Δ_n on the horizontal hyperplane with equation $\tau_{n+1} = 0$. The volume of D_n is equal to

$$\mathrm{vol}^{(n)}(D_n) = \frac{1}{n!}.$$

The integral of a function f defined on Δ_n with respect to the measure β_n can be given as an integral on D_n as follows:

$$\int_{\Delta_n} f(\tau)\beta_n(d\tau) = n! \int_{D_n} f(\tau_1, \dots, \tau_n, 1 - \tau_1 - \dots - \tau_n) d\tau_1 \dots d\tau_n.$$

Proposition 2.2. *Consider the map*

$$\Psi : S(\mathbb{C}^n) \to \Delta_{n-1}, \quad u = (u_1, \dots, u_n) \mapsto \tau = (|u_1|^2, \dots, |u_n|^2).$$

The image under Ψ of the measure σ is equal to the measure $\beta = \beta_{n-1}$: if f is a continuous function on Δ_{n-1},

$$\int_{S(\mathbb{C}^n)} f(|u_1|^2, \dots, |u_n|^2)\sigma(du) = \int_{\Delta_{n-1}} f(\tau_1, \dots, \tau_n)\beta(d\tau).$$

Proof.

Observe first that, if F is a function defined on $(\mathbb{R}_+)^n$ which is integrable with respect to the Lebesgue measure,

$$\begin{aligned}\int_{(\mathbb{R}_+)^n} & F(x_1,\dots,x_n)dx_1\dots dx_n \\ &= \frac{1}{(n-1)!}\int_0^\infty \Big(\int_{\Delta_{n-1}} F(\rho\tau_1,\dots,\rho\tau_n)\beta(d\tau)\Big)\rho^{n-1}d\rho.\end{aligned}$$

Let f be a function defined on Δ_{n-1}, integrable with respect to the measure β, and let f_0 be a function defined on $\mathbb{R}_+$ integrable with respect to the measure $\rho^{n-1}d\rho$. We associate to the functions f and f_0 the function F_1 defined on $(\mathbb{R}_+)^n$ by puting

$$F_1(x) = f_0(\rho)f(\tau_1,\dots,\tau_n), \text{ if } x = (\rho\tau_1,\dots,\rho\tau_n),\ \tau = (\tau_1,\dots,\tau_n) \in \Delta_{n-1}$$

Then

$$\begin{aligned}\int_{(\mathbb{R}_+)^n} & F_1(x_1,\dots,x_n)dx_1\dots dx_n \\ &= \frac{1}{(n-1)!}\int_0^\infty f_0(\rho)\rho^{n-1}d\rho_i \int_{\Delta_{n-1}} f(\tau_1,\dots,\tau_n)\beta(d\tau).\end{aligned}$$

We consider also the function F_2 defined on $\mathbb{C}^n$ by puting

$$F_2(z) = f_0(r^2)f(|u_1|^2,\dots,|u_n|^2),$$

if $z = (ru_1,\dots,ru_n)$, $u = (u_1,\dots,u_n) \in S$. Then, on one hand, since $F_2(z) = F_1(|z_1|^2,\dots,|z_n|^2)$,

$$\begin{aligned}\int_{\mathbb{C}^n} F_2(z)m(dz) &= (2\pi)^n \int_{(\mathbb{R}_+)^n} F_1(r_1^2,\dots,r_n^2)r_1dr_1\dots r_ndr_n \\ &= \pi^n \int_{(\mathbb{R}_+)^n} F(t_1,\dots,t_n)dt_1\dots dt_n \\ &= \frac{\pi^n}{(n-1)!}\int_0^\infty f_0(\rho)\rho^{n-1}d\rho \int_{\Delta_{n-1}} f(\tau_1,\dots,\tau_n)\beta(d\tau),\end{aligned}$$

where m denotes the Lebesgue measure on $\mathbb{C}^n \simeq \mathbb{R}^{2n}$. And, on the other hand,

$$\int_{\mathbb{C}^n} F_2(z)m(dz)$$

$$\begin{aligned} &= 2\frac{\pi^n}{(n-1)!}\int_0^\infty f_0(r^2)r^{2n-1}dr\int_S f(|u_1|^2,\dots,|u_n|^2)\sigma(du) \\ &= \frac{\pi^n}{(n-1)!}\int_0^\infty f_0(\rho)\rho^{n-1}d\rho\int_S f(|u_1|^2,\dots,|u_n|^2)\sigma(du). \end{aligned}$$

By comparing these equalities one gets the statement $\square$

To a point $a = (a_1, \dots, a_n) \in \mathbb{R}^n$ one associates the measure $M_n(a_1, \dots, a_n; dt)$ on $\mathbb{R}$, image of the measure β by the map

$$\Theta : \Delta_{n-1} \to \mathbb{R},\ \tau \mapsto a_1\tau_1 + \cdots + a_n\tau_n.$$

That is, if f is a continuous function on $\mathbb{R}$,

$$\int_{\mathbb{R}} f(t)M_n(a_1, \dots, a_n; dt) = \int_{\Delta_{n-1}} f(a_1\tau_1 + \cdots + a_n\tau_n)\beta(d\tau).$$

$M_n(a_1, \dots, a_n; dt)$ is a probability measure on $\mathbb{R}$ with support $[\min a_i, \max a_i]$. We call it *Peano measure.* For $n = 2$,

$$\int_{\mathbb{R}} f(t)M_2(a_1, a_2; dt) = \int_0^1 f\big(a_1\tau + (1-\tau)a_2\big)d\tau = \frac{1}{a_2 - a_1}\int_{a_1}^{a_2} f(t)dt.$$

Theorem 2.3. (Okounkov) *The projection $M_A(dt)$ on the line $\mathbb{R}E_{11}$ of the orbital measure μ_A is equal to the Peano measure $M_n(a_1, \dots, a_n; dt)$,*

$$M_A(dt) = M_n(a_1, \dots, a_n; dt).$$

Proof. The map

$$U(n) \to \mathbb{R},\ u \mapsto (uAu^*)_{11}$$

can be factorized as follows

$$U(n) \xrightarrow{\Phi} S(\mathbb{C}^n) \xrightarrow{\Psi} \Delta_{n-1} \xrightarrow{\Theta} \mathbb{R}$$

$$u \mapsto \xi = (u_{11}, \dots, u_{1n}) \mapsto \tau = (|\xi_1|^2, \dots, |\xi_n|^2) \mapsto t = a_1\tau_1 + \cdots + a_n\tau_n.$$

By Proposition 2.1,

$$\int_{\mathbb{R}} f(t)M_A(dt) = \int_{S(\mathbb{C}^n)} f(a_1|u_1|^2 + \cdots + a_n|u_n|^2)\sigma(du).$$

and, by Proposition 2.2 and the definition of the Peano measure,

$$\int_{S(\mathbb{C}^n)} f(a_1|u_1|^2 + \cdots + a_n|u_n|^2)\sigma(du)$$
$$= \int_{\Delta_{n-1}} f(a_1\tau_1 + \cdots + a_n\tau_n)\beta(d\tau) = \int_{\mathbb{R}} f(t)M_n(a_1, \ldots, a_n; dt).$$

∎

3 Divided differences, Peano measures and spline functions

Let f be a function defined on $\mathbb{R}$. If the real numbers a_i are distinct, the divided differences of f are defined as follows

$$\begin{aligned} f[a_1, a_2] &= \frac{f(a_2) - f(a_1)}{a_2 - a_1}, \\ f[a_1, a_2, \ldots, a_n] &= \frac{f[a_2, \ldots, a_n] - f[a_1, \ldots, a_{n-1}]}{a_n - a_1}. \end{aligned}$$

If f is of class $\mathcal{C}^{n-1}$ the divided differences $f[a_1, \ldots, a_k]$ $(k \leq n)$ are defined for every numbers a_i, distinct or not, by going to the limit. In particular, if $a_1 = a_2 = \cdots = a_k = a$, then

$$f[a, \ldots, a] = \frac{1}{(k-1)!} f^{(k-1)}(a).$$

Assume the numbers $a_1, \ldots, a_n$ to be distinct. Let p be the interpolation polynomial of the function f with respect to the points $a_1, \ldots, a_n$: p is the polynomial of degree $\leq n-1$ such that $p(a_i) = f(a_i)$ $(i = 1, \ldots, n)$. Recall the following Newton formula: the interpolation polynomial p can be written

$$p(t) = \sum_{k=1}^{n} f[a_1, \ldots, a_k](t - a_1) \cdots (t - a_{k-1}).$$

Let $c_0, \ldots, c_{n-1}$ be the coeficients of the interpolation polynomial:

$$p(t) = c_0 + \cdots + c_{n-1}t^{n-1}.$$

By the Newton formula $c_{n-1} = f[a_1, \ldots, a_n]$. Les coefficients c_k are solutions of the system

$$\begin{aligned} c_0 + c_1 a_1 + \cdots + c_{n-1} a_1^{n-1} &= f(a_1), \\ &\vdots \\ c_0 + c_1 a_n + \cdots + c_{n-1} a_n^{n-1} &= f(a_n). \end{aligned}$$

From Cramer's formulas one gets:

Proposition 3.1. *The divided differences admit the following determinantal representation:*

$$f[a_1, \ldots, a_n] = \frac{1}{V_n(a_1, \ldots, a_n)} \begin{vmatrix} 1 & 1 & \ldots & 1 \\ a_1 & a_2 & \ldots & a_n \\ \vdots & \vdots & & \vdots \\ a_1^{n-2} & a_2^{n-2} & \ldots & a_n^{n-2} \\ f(a_1) & f(a_2) & \ldots & f(a_n) \end{vmatrix}$$

where V_n is the Vandermonde polynomial in n variables,

$$V_n(a_1, \ldots, a_n) = \begin{vmatrix} 1 & 1 & \ldots & 1 \\ a_1 & a_2 & \ldots & a_n \\ \vdots & \vdots & & \vdots \\ a_1^{n-2} & a_2^{n-2} & \ldots & a_n^{n-2} \\ a_1^{n-1} & a_2^{n-1} & \ldots & a_n^{n-1} \end{vmatrix} = \prod_{1 \le i < j \le n} (a_j - a_i).$$

It follows that

$$f[a_1, \ldots, a_n] = \sum_{i=1}^{n} \gamma_i f(a_i),$$

where

$$\gamma_i = \gamma_i(a_1, \ldots, a_n) = \frac{1}{\prod_{j \neq i} (a_i - a_j)}.$$

Next Theorem expresses the relation between divided differences and Peano measures.

Theorem 3.2. (Hermite-Genocchi) *If f is a function of class $\mathcal{C}^{n-1}$ on $\mathbb{R}$, then*

$$f[a_1, \ldots, a_n] = \frac{1}{(n-1)!} \int_{\mathfrak{R}} f^{(n-1)}(t) M_n(a_1, \ldots, a_n; dt).$$

Proof.

The proof uses a recursion. For $n = 2$, if f is a function of class $\mathcal{C}^1$,

$$\begin{aligned}\int_0^1 f'\big(a_1\tau + a_2(1-\tau)\big)d\tau &= \frac{1}{a_1 - a_2}\Big[f\big((a_1-a_2)\tau + a_2\big)\Big]_0^1 \\ &= \frac{1}{a_1-a_2}\big(f(a_1) - f(a_2)\big) = f[a_1, a_2].\end{aligned}$$

Assume that the relation holds for n, and let us prove it for $n+1$. If f is of class $\mathcal{C}^n$, by a partial integration one gets

$$\begin{aligned}&\frac{1}{n!}\int_{\Delta_n} f^{(n)}(a_1\tau_1 + \cdots + a_{n+1}\tau_{n+1})\beta_n(d\tau) \\ = &\int_{D_n} f^{(n)}\big(a_1\tau_1 + \cdots + a_n\tau_n + a_{n+1}(1-\tau_1-\cdots-\tau_n)\big)d\tau_1\ldots d\tau_n \\ = &\int_{D_n} f^{(n)}\big((a_1-a_{n+1})\tau_1 + \cdots + (a_n - a_{n+1})\tau_n + a_{n+1}\big)d\tau_1\ldots d\tau_n \\ = &\int_{D_{n-1}}\Big(\int_0^{1-\tau_2-\cdots-\tau_n} f^{(n)}\big((a_1-a_{n+1})\tau_1 + (a_2-a_{n+1})\tau 2 + \cdots \\ &+(a_n - a_{n+1})\tau_n + a_{n+1}\big)d\tau_1\Big)d\tau_2\ldots d\tau_n.\end{aligned}$$

The integral with respect to τ_1 gives

$$\begin{aligned}&\frac{1}{a_1-a_{n+1}}\Big(f^{(n-1)}\big((a_1-a_{n+1})(1-\tau_2-\cdots-\tau_n) \\ &+(a_2-a_{n+1})\tau_2 + \cdots + (a_n-a_{n+1})\tau_n + a_{n+1}\big) \\ &-f^{(n-1)}\big((a_2-a_{n+1})\tau_2 + \cdots + (a_n - a_{n+1})\tau_n + a_{n+1}\big)\Big) \\ = &\frac{1}{a_1-a_{n+1}}\Big(f^{(n-1)}\big(a_1(1-\tau_2-\cdots-\tau_n) + a_2\tau_2 + \cdots + a_n\tau_n\big) \\ &-f^{(n-1)}\big(a_2\tau_2 + \cdots + a_n\tau_n + a_{n+1}(1-\tau_2-\cdots-\tau_n)\big)\Big)\end{aligned}$$

We get finally

$$\begin{aligned}&\frac{1}{n!}\int_{\Delta_n} f^{(n)}(a_1\tau_1 + \cdots + a_{n+1}\tau_{n+1})\beta_n(d\tau) = \frac{1}{a_{n+1}-a_1}\times \\ &\Big(\frac{1}{(n-1)!}\int_{\Delta_{n-1}} f^{(n-1)}(a_2\tau_2 + \cdots + a_n\tau_n + a_{n+1}\tau_{n+1})\beta_{n-1}(d\tau)\end{aligned}$$

$$
\begin{aligned}
& -\frac{1}{(n-1)!}\int_{\Delta_{n-1}} f^{(n-1)}(a_1\tau_1+\cdots+a_n\tau_n)\beta_{n-1}(d\tau)\Big) \\
= & \ \frac{1}{a_{n+1}-a_1}\big(f[a_2,\ldots,a_{n+1}]-f[a_1,\ldots,a_n]\big) = f[a_1,\ldots,a_{n+1}].
\end{aligned}
$$

∎

Taking $f(t) = e^{zt}$ one gets the Fourier-Laplace transform of the Peano measure:

$$
\begin{aligned}
\widehat{M_n}(a_1,\ldots,a_n;z) &= \int_{\mathfrak{R}} e^{zt} M_n(a_1,\ldots,a_n;dt) \\
&= \frac{(n-1)!}{V_n(a_1,\ldots,a_n)}\ \frac{1}{z^{n-1}} \begin{vmatrix} 1 & 1 & \ldots & 1 \\ a_1 & a_2 & \ldots & a_n \\ \vdots & \vdots & & \vdots \\ a_1^{n-2} & a_2^{n-2} & \ldots & a_n^{n-2} \\ e^{a_1 z} & e^{a_2 z} & \ldots & e^{a_n z} \end{vmatrix}.
\end{aligned}
$$

Corollary 3.3. *In the distribution sense, if the numbers a_i are distinct,*

$$\frac{1}{(n-1)!}\Big(-\frac{d}{dt}\Big)^{n-1} M_n(a_1,\ldots,a_n;dt) = \sum_{i=1}^{n} \gamma_i \delta_{a_i}.$$

Hence, if the numbers a_i are distinct, and if $n \geq 2$, the Peano measure $M_n(a_1,\ldots,a_n;dt)$ is absolutely continuous with respect to the Lebesgue measure,

$$M_n(a_1,\ldots,a_n;dt) = M_n(a_1,\ldots,a_n;t)dt.$$

Corollary 3.4. *Assume $a_1 < \cdots < a_n$. The Peano function admits the following representation*

$$M_n(a_1,\ldots,a_n;t) = (-1)^{n-1}(n-1)\sum_{i=1}^{n} \gamma_i (t-a_i)_+^{n-2}.$$

Proof. Consider the function

$$E(t) = \frac{1}{(n-2)!} t_+^{n-2}.$$

In the distribution sense
$$\Big(\frac{d}{dt}\Big)^{n-1}E=\delta_0,$$
and this can be written $E*\delta_0^{(n-1)}=\delta_0$. We get from Corollary 8.4
$$E*\big(\delta_0^{(n-1)}*M_n\big)=(-1)^{n-1}(n-1)!\sum_{i=1}^n\gamma_iE*\delta_{a_i}.$$
since the support of M_n is compact, the convolution product is associative in that case. The left handside is equal to
$$(E*\delta_0^{(n-1)})*M_n=M_n.$$
therefore
$$M_n(a_1,\dots,a_n;dt=(-1)^{n-1}(n-1)\sum_{i=1}^n(t-a_i)_+^{n-1}.$$

∎

If the numbers a_i are distinct, the function $M_n(a_1,\dots,a_n;t)$ is of class $\mathcal{C}^{n-3}$, and, if $a_1<a_2<\cdots<a_n$, the restriction of the function $M_n(a_1,\dots,a_n;t)$ to each of the intervals $]a_i,a_{i+1}[$ is a polynomial of degree $\leq n-2$. These properties express that $M_n(a_1,\dots,a_n;t)$ is a spline function of degree $n-2$ whose knots are the numbers $a_1,\dots,a_n$.

Proposition 3.5. *Assume the numbers a_i to be distinct, $a_1<\cdots<a_n$. The function $f(t)=M_n(a_1,\dots,a_n;t)$ is characterized by the following properties :*

(1) $\mathrm{supp}(f)=[a_1,a_n]$,

(2) The restriction of f to each of the intervals $[a_i,a_{i+1}[$ $(i=1,\dots n-1)$ is a polynomial of degree $\leq n-2$.

(3) If $n\geq 3$, then f is of class $\mathcal{C}^{(n-3)}$.

(4)
$$\int_{\mathbb{R}}f(t)dt=1.$$

Proof.

Note $\mathcal{E}_n(a_1,\dots,a_n)$ the space of functions on $\mathbb{R}$ satisfying (1) et (2). Its dimension is given by
$$\dim\,\mathcal{E}_n(a_1,\dots,a_n)=(n-1)^2.$$

Consider the $n(n-2)$ linear forms on $\mathcal{E}_n(a_1,\ldots,a_n)$

$$L_{ij}(f) = f^{(j)}(a_i+) - f^{(j)}(a_i-) \quad (i=1,\ldots,n,\ j=0,\ldots,n-3).$$

The linear forms L_{ij} are linearly independant. They express Condition (3). By the rank theorem the functions in $\mathcal{E}_n(a_1,\ldots,a_n)$ satisfying Condition (3) form a vector subspace of dimension 1. In fact

$$(n-1)^2 - n(n-2) = 1.$$

Hence these functions are proportinnal to $M_n(a_1,\ldots,a_n;t)$. Therefore $M_n(a_1,\ldots,a_n;t)$ is the unique function satisfying Conditions (1), (2), (3), (4). ∎

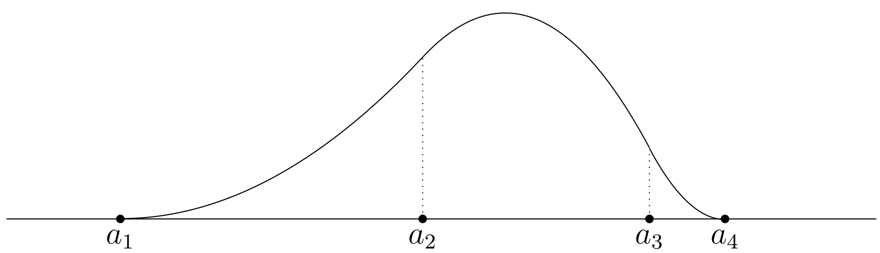


Figure 1. Graph of $M_4(a_1,a_2,a_3,a_4)$ $(a_1=-4,\ a_2=0,\ a_3=3,\ a_4=4)$.

The Peano measure possesses a remarkable geometric meaning: Let $A_1, A_2, \ldots A_n$ be the n vertices of a simplex Q in $\mathbb{R}^{n-1}$. The simplex Q is the set of convex combinations of the points $A_1,\ldots,A_n$:

$$Q = \Big\{ x = \sum_{i=1}^{n} t_i A_i \mid t_i \geq 0,\ \sum_{i=1}^{n} t_i = 1 \Big\}.$$

Let $a_1,\ldots,a_n$ denote the abscisses of the projections of $A_1,\ldots,A_n$ on the first coordinate axis.

Proposition 3.6. *Let Q_t be the intersection of the simplex Q by the hyperplane with equation $x_1 = t$. We assume that the volume $\mathrm{vol}^{(n-1)}(Q)$ equals 1. If the numbers $a_1,\ldots,a_n$ are distinct, then*

$$\mathrm{vol}^{(n-2)}(Q_t) = M_n(a_1,\ldots,a_n;t).$$

Proof. If f is a continuous function on $\mathbb{R}$, then

$$\int_Q f(x_1)dx_1 \dots dx_{n-1} = \int_{\mathfrak{R}} f(t)\mathrm{vol}^{(n-2)}(Q_t)dt.$$

Define the map

$$\Phi : \Delta_{n-1} \to Q, \quad t \mapsto x = \sum_{i=1}^{n} t_i A_i.$$

The image under Φ of the measure β is the restriction to Q of the Lebesgue measure on $\mathbb{R}^{n-1}$. Hence

$$\int_Q f(x_1)dx_1 \dots dx_{n-1} = \int_{\Delta_{n-1}} f(t_1a_1 + t_2a_2 + \dots + t_na_n)\beta(dt).$$

Since the relation holds for every function f on $\mathbb{R}$, it follows that

$$\mathrm{vol}^{n-2}(Q_t) = M_n(a_1, \dots, a_n; t).$$

∎

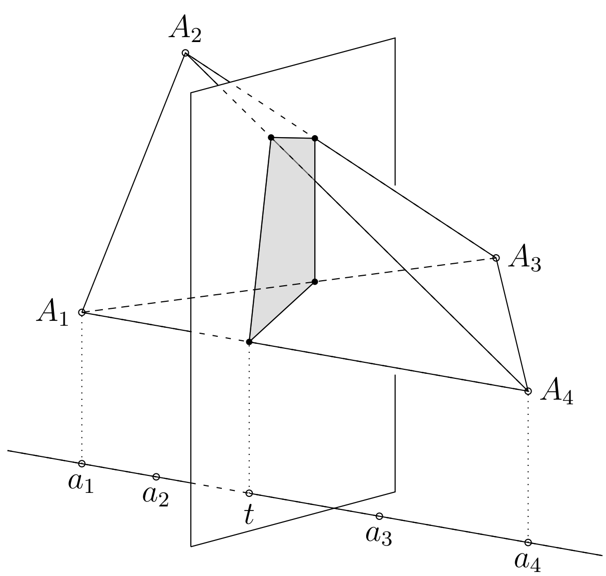


Figure 2. Projection of a simplex.

4 Interlacing property of the eigenvalues

Let $p = p_{n-1}$ be the projection of $\mathcal{H}_n(\mathbb{F})$ onto $\mathcal{H}_{n-1}(\mathbb{F})$ which maps a matrix $X \in \mathcal{H}_n(\mathbb{F})$ to the matrix $Y = p(X) \in \mathcal{H}_{n-1}(\mathbb{F})$ of the $n-1$ first rows and $n-1$ first columns of X.

Theorem 4.1. *The sequence $\mu_1 \leq \cdots \leq \mu_{n-1}$ of the eigenvalues of $Y = p(X)$ interlaces the sequence of the eigenvalues $\lambda_1 \leq \cdots \leq \lambda_n$ of X:*

$$\lambda_1 \leq \mu_1 \leq \lambda_2 \leq \cdots \leq \mu_{n-1} \leq \lambda_n.$$

This interlacing relation will be denoted: $\mu \preceq \lambda$.

Proof. Assume the eigenvalues $\lambda_1, \ldots, \lambda_n$ to be distinct: $\lambda_1 < \cdots < \lambda_n$. Let v_i be a unit eigenvector associated to the eigenvalue λ_i : $Xv_i = \lambda_i v_i$, $\|v_i\| = 1$. Assume also that, for every i, $v_i \notin \mathcal{H}_{n-1}(\mathbb{F})$, i.e. $(v_i|e_n) \neq 0$. We will evaluate in two ways the rationnal function

$$f(z) = \big((zI_n - X)^{-1}e_n|e_n\big) \qquad (z \in \mathbb{C}).$$

On one hand, by the Cramer's formulas,

$$f(z) = \frac{\det^{(n-1)}(zI_{n-1} - Y)}{\det^{(n)}(zI_n - X)} = \frac{\prod_{j=1}^{n-1}(z-\mu_j)}{\prod_{i=1}^{n}(z-\lambda_i)}.$$

The eigenvalues λ_i of X are the poles of f and the eigenvalues μ_j of Y are the zeros of f. On the other hand, by using the spectral decomposition of X,

$$f(z) = \sum_{i=1}^{n} \frac{w_i}{z-\lambda_i}, \qquad \text{with} \quad w_i = |(v_i|e_n)|^2.$$

In fact, for $v \in \mathbb{F}^n$,

$$Xv = \sum_{i=1}^{n} \lambda_i (v|v_i)v_i, \quad (zI_n - X)^{-1}v = \sum_{i=1}^{n} \frac{1}{z-\lambda_i}(v|v_i)v_i.$$

Hence

$$f(z) = \frac{\prod_{j=1}^{n-1}(z-\mu_j)}{\prod_{i=1}^{n}(z-\lambda_i)} = \sum_{i=1}^{n} \frac{w_i}{z-\lambda_i}.$$

The function f is decreasing from $+\infty$ to $-\infty$ on each of the intervals $]\lambda_i, \lambda_{i+1}[$ $(i = 1, \ldots, n-1)$. Therefore each of these intervals contains one and only one zero of f, i.e.

$$\lambda_1 < \mu_1 < \lambda_2 < \cdots < \mu_{n-1} < \lambda_n.$$

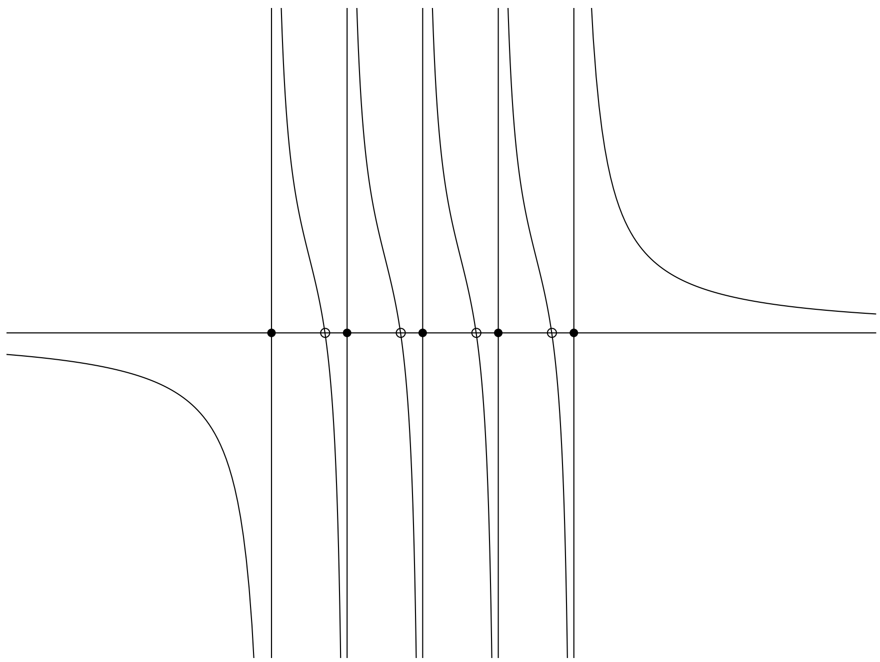


Figure 3. Graph of the rational function $f(z)$.

Note that the residue w_i at the pole λ_i is given by

$$w_i = \frac{\prod_{j=1}^{n-1}(\lambda_i - \mu_j)}{\prod_{j \neq i}(\lambda_i - \lambda_j)}.$$

Note also that

$$w_i > 0, \quad \sum_{i=1}^{n} w_i = 1.$$

To complete the proof one should consider the case of non distinct eigenvalues $\lambda_1, \ldots, \lambda_n$, and the case where some eigenvectors v_i belong to $\mathcal{H}_{n-1}(\mathbb{F})$. $\square$

We can state a more precise theorem:

Theorem 4.2.

$$\Lambda^{(n-1)}\big(p(\mathcal{O}_A)\big) = \{\mu \in (\mathbb{R}^{n-1})_+ \mid \mu \preceq a\}.$$

Proof. By Theorem 4.1,

$$\Lambda^{(n-1)}\big(p(\mathcal{O}_A)\big) \subset \{\mu \in (\mathbb{R}^{n-1})_+ \mid \mu \preceq a\}.$$

Assume the eigenvalues $a_1, \ldots, a_n$ to be distinct. Let $\mu_1, \ldots, \mu_{n-1}$ such that

$$a_1 < \mu_1 < a_2 < \cdots < \mu_{n-1} < a_n.$$

We will show that there exists a matrix $X \in \mathcal{O}_A$ such that $\mu_1, \ldots, \mu_{n-1}$ are the eigenvalues of $Y = p(X)$.

Put

$$w_i = \frac{\prod_{j=1}^{n-1}(a_i - \mu_j)}{\prod_{j\neq i}(a_i - a_j)}.$$

We will show that

$$\frac{\prod_{j=1}^{n-1}(z - \mu_j)}{\prod_{i=1}^{n}(z - a_i)} = \sum_{i=1}^{n} \frac{w_i}{z - a_i},$$

i.e. that, for every $z \in \mathfrak{C}$,

$$\prod_{j=1}^{n-1}(z - \mu_j) = \sum_{i=1}^{n}\Big(w_i \prod_{j\neq i}(z - a_j)\Big).$$

This equality for two polynomials of degree $n-1$ is satisfied for the n numbers $z = a_1, \ldots, z = a_n$, therefore for every z.

Comparing the highest degree terms of both handsides one gets

$$\sum_{i=1}^{n} w_i = 1.$$

Furthermore, from the interlacing property of the sequence $\mu_1, \ldots, \mu_{n-1}$, one deduces that the numbers w_i are > 0.

For each i one fixes $\xi_i \in \mathbb{F}$ such that $|\xi_i|^2 = w_i$. The vector $\xi = (\xi_1, \ldots, \xi_n)$ belongs to the unit sphere of $\mathbb{F}^n$. Therefore there exists $u \in U_n(\mathbb{F})$ such that

$u^* e_n = \xi$. One puts $X = uAu^*$. We saw in the proof of Theorem 4.1 that the eigenvalues $\mu_1^0, \ldots, \mu_{n-1}^0$ of the projection $Y = p(X)$ satisfy

$$\frac{\prod_{j=1}^{n-1}(z - \mu_j^0)}{\prod_{i=1}^{n}(z - a_i)} = \sum_{i=1}^{n} \frac{w_i}{z - a_i},$$

hence $\mu_j^0 = \mu_j$ $(j = 1, \ldots, n-1)$. ∎

5 Baryshnikov's formula

We will determinate the joint distribution of the eigenvalues $\mu_1, \ldots, \mu_{n-1}$ of $Y = p(uAu^*)$, i.e. the image of the Haar measure α under the map

$$U(n) \to (\mathbb{R}^{n-1})_+, \quad u \mapsto \Lambda^{(n-1)}\big(p(uAu^*)\big).$$

We will factorize this map as follows:

$$\begin{array}{rcl} U_n(\mathbb{F}) \xrightarrow{\Phi} & S(\mathbb{F}^n) \xrightarrow{\Psi} \Delta_{n-1} \xrightarrow{\Theta} \{t \in (\mathbb{R}^{n-1})_+ \mid t \preceq a\}, \\ u \mapsto & \xi = ue_n \mapsto w_i = |(ue_n|e_n)|^2 \mapsto (\mu_1, \ldots, \mu_{n-1}). \end{array}$$

Consider the map

$$\Phi : (\mu_1, \ldots, \mu_{n-1}) \mapsto (w_1, \ldots, w_n),$$

defined by

$$w_i = \frac{\prod_{j=1}^{n-1}(a_i - \mu_j)}{\prod_{j \neq i}(a_i - a_j)}.$$

Proposition 5.1. *Let ω be the differential form of degree $n-1$,*

$$\omega = dw_1 \wedge dw_2 \wedge \cdots \wedge dw_{n-1}.$$

Its image under Φ^* is given by

$$\Phi^* \omega = \frac{V_{n-1}(\mu_1; \ldots, \mu_{n-1})}{V_n(a_1, \ldots, a_n)} d\mu_1 \wedge d\mu_2 \wedge \cdots \wedge d\mu_{n-1},$$

where V_n is the Vandermonde polynomial in n variables,

$$V_n(z_1, \ldots, z_n) = \prod_{1 \leq i < j \leq n} (z_j - z_i).$$

Proof. Let us compute the differential of Φ :

$$\frac{\partial w_i}{\partial \mu_j} = -\frac{\prod_{k=1}^{n-1}(a_i - \mu_k)}{\prod_{k\neq i}(a_i - a_k)} \frac{1}{a_i - \mu_j},$$

and its Jacobian determinant:

$$\begin{aligned} &\det\Big(\tfrac{\partial w_i}{\partial \mu_j}\Big)_{1\leq i,j\leq n-1} \\ &= (-1)^{n-1} \textstyle\prod_{i=1}^{n-1} \Big(\tfrac{\prod_{k=1}^{n-1}(a_i-\mu_k)}{\prod_{k\neq i}(a_i-a_k)}\Big) \det\Big(\tfrac{1}{a_i-\mu_j}\Big)_{1\leq i,j\leq n-1}. \end{aligned}$$

We use now the following Cauchy's formula

$$\begin{aligned} &\det\Big(\frac{1}{a_i - \mu_j}\Big)_{1\leq i,j\leq n-1} \\ =\quad & V_{n-1}(a_1, \dots, a_{n-1}) V_{n-1}(\mu_1, \dots, \mu_{n-1}) \prod_{i,l=1}^{n-1} \frac{1}{a_i - \mu_j}. \end{aligned}$$

One gets finally

$$\begin{aligned} &\det\Big(\frac{\partial w_i}{\partial \mu_j}\Big)_{1\leq i,j\leq n} = (-1)^{n-1} \frac{1}{\prod_{i=1}^{n-1}\prod_{k=1,k\neq i}^{n}(a_i - a_k)} \times \\ &\times V_{n-1}(a_1, \dots, a_{n-1}) V_{n-1}(\mu_1, \dots, \mu_{n-1}) \\ &= \pm \frac{V_{n-1}(\mu_1, \dots, \mu_{n-1})}{V_n(a_1, \dots, a_n)}. \end{aligned}$$

∎

Theorem 5.2. *Assume $\mathbb{F} = \mathbb{C}$. The joint distribution of the eigenvalues $\mu_1, \dots, \mu_{n-1}$ is the probability measure $\nu_A^{(n-1)}$ with support*

$$\{\mu \in \mathbb{R}^{n-1} \mid a_1 \leq \mu_1 \leq a_2 \leq \dots \leq \mu_{n-1} \leq a_n\},$$

and density

$$(n-1)! \frac{V_{n-1}(\mu_1, \dots, \mu_{n-1})}{V_n(a_1, \dots, a_n)}.$$

This can be written, for a function f defined on $(\mathbb{R}^{n-1})_+$,

$$\begin{aligned}&\int_{(\mathbb{R}^{n-1})_+} f(t)\nu_A^{(n-1)}(dt)\\ = \ &\frac{(n-1)!}{V_n(a_1,\ldots,a_n)}\int_{a_1}^{a_2} dt_1\int_{a_2}^{a_3} dt_2\ldots\int_{a_{n-1}}^{a_n} dt_{n-1}V_{n-1}(t)f(t).\end{aligned}$$

Proof. The map $\Phi : U(n) \to S(\mathbb{C}^n)$ maps the Haar measure α to the uniform measure σ (Proposition 2.1). The map $\Psi : S(\mathbb{C}^n) \to \Delta_{n-1}$, maps the uniform measure σ to the measure β (Proposition 2.2). The measure β can be defined by the differential form

$$(n-1)!dw_1 \wedge \ldots \wedge dw_{n-1}.$$

By Proposition 5.1, the measure β is transformed into the one given in the statement. ∎

6 The Fourier-Laplace transform of orbital measures

The Fourier-Laplace transform of a bounded measure μ on $\mathcal{H}_n(\mathbb{F})$ is defined by, if $Z \in i\mathcal{H}_n(\mathbb{F})$,

$$\hat{\mu}(Z) = \int_{\mathcal{H}_n(\mathbb{F})} e^{\mathrm{tr}\,(ZX)}\mu(dX).$$

If the support of μ is compact, then this transform is defined for Z in the complexified space $\mathcal{H}_n(\mathbb{F})$: $Sym(n,\mathbb{C})$ if $\mathbb{F}=\mathbb{R}$, $M(n,\mathbb{C})$ if $\mathbb{F}=\mathbb{C}$.

The Fourier-Laplace transform of the orbital measure μ_A is given by

$$\widehat{\mu_A}(Z) = \int_{\mathcal{O}_A} e^{\mathrm{tr}\,(ZX)}\mu_A(dX) = \int_{U_n(\mathbb{F})} e^{\mathrm{tr}\,(ZuAu^*)}\alpha(du).$$

If $Z = \mathrm{diag}(z_1,\ldots,z_n)$, one can write

$$\widehat{\mu_A}(Z) = \mathcal{E}_n(z;a),$$

where $\mathcal{E}_n$ is an analytic function on $\mathbb{C}^n \times \mathbb{C}^n$, biinvariant under the group $\mathfrak{S}_n$ of permutations.

Let μ be a measure on $\mathcal{H}_n(\mathbb{F})$ which is $U_n(\mathbb{F})$-invariant. The integral of a function f defined on $\mathcal{H}_n(\mathbb{F})$ can be written

$$\int_{\mathcal{H}_n(\mathbb{F})} f(X)\mu(dX) = \int_{(\mathbb{R}^n)_+} \Big(\int_{U_n(\mathbb{F})} f(u\mathrm{diag}(t_1,\dots,t_n)u^*)\alpha(du)\Big)\nu(dt),$$

where ν is a measure on $(\mathbb{R}^n)_+$, called the radial part of μ. The Fourier-Laplace transform μ is given by, if $Z = \mathrm{diag}(z_1,\dots,z_n)$,

$$\hat{\mu}(Z) = \int_{(\mathbb{R}^n)_+} \mathcal{E}_n(z;t)\nu(dt).$$

If μ is a probability measure on $\mathcal{H}_n(\mathbb{F})$ which is $U_n(\mathbb{F})$-invariant, its radial part ν is also the joint distribution of the eigenvalues for a random matrix X distributed according to the law μ.

Theorem 6.1. (Harish-Chandra-Itzykson-Zuber) *We assume that* $\mathbb{F} = \mathbb{C}$. *If* $Z = \mathrm{diag}(z_1,\dots,z_n)$,

$$\widehat{\mu_A}(Z) = \delta_n! \frac{1}{V_n(a)V_n(z)} \det(e^{z_i a_j})_{1\le i,j\le n},$$

où

$$\delta_n = (n-1, n-2, \dots, 1, 0), \quad \delta_n! = (n-1)!(n-2)!\dots 2!.$$

In other words

$$\mathcal{E}_n(z,a) = \delta_n! \frac{1}{V_n(a)V_n(z)} \det\big(e^{z_i a_j}\big)_{1\le i,j\le n}.$$

This formula is well defined if the numbers a_i are distinct, and the numbers z_j as well.

Proof.

a) We prove first a recursion formula for $\mathcal{E}_n(z,a)$, valid for $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Let Y denote the projection of X on $\mathcal{H}_{n-1}(\mathbb{F})$. We can write

$$\mathrm{tr}(ZX) = \mathrm{tr}(Z'X) + z_n(\mathrm{tr}(A) - \mathrm{tr}(Y)).$$

In the integral which defines $\mathcal{E}(z,a)$, the integrant

$$e^{\mathrm{tr}(ZX)} = e^{z_n \mathrm{tr}(A)} e^{-z_n \mathrm{tr} Y} e^{\mathrm{tr}(Z'Y)},$$

where $Z' = \mathrm{diag}(z_1, \ldots, z_{n-1})$ only depends on the projection Y. Therefore

$$\mathcal{E}_n(z,a) = e^{z_n \mathrm{tr} A} \int_{\mathcal{H}_{n-1}(\mathbb{F})} e^{-z_n \mathrm{tr} Y} e^{\mathrm{tr}(Z'Y)} \mu_A^{(n-1)}(dY).$$

By using the integral formula (*) we get

$$\mathcal{E}_n(z,a) = e^{z_n \mathrm{tr} A} \int_{(\mathbb{R}^{n-1})_+} \Big(\int_{U_{n-1}(\mathbb{F})} e^{\mathrm{tr}(Z' v T v^*)} \alpha_{n-1}(dv) \Big) \nu_A^{(n-1)}(dt),$$

where $T = \mathrm{diag}(t_1, \ldots, t_{n-1})$. Hence we have established the following recursion formula

$$\mathcal{E}_n(z,a) = e^{z_n \mathrm{tr} A} \int_{(\mathbb{R}^{n-1})_+} \mathcal{E}_{n-1}(z', t) e^{-z_n(t_1 + \cdots + t_{n-1})} \nu_A^{(n-1)}(dt).$$

b) We assume now $\mathbb{F} = \mathbb{C}$, and prove the Harish-Chandra-Itzykson-Zuber formula recursively on n. For $n = 1$ there is nothing to prove. Assume that the formula holds for $n - 1$. Then, by Baryshnikov's formula

$$\mathcal{E}_n(z,a) = \frac{\delta_{n-1}}{V_{n-1}(z') V_n(a)} e^{z_n(a_1 + \cdots + a_n)} \int_{a_1}^{a_2} dt_1 \int_{a_2}^{a_3} dt_2 \ldots \int_{a_{n-1}}^{a_n} \det\Big(e^{(z_j - z_n) t_i} \Big)_{1 \le i,j \le n-1}.$$

Let us compute this integral

$$\begin{aligned} &\int_{a_1}^{a_2} dt_1 \int_{a_2}^{a_3} dt_2 \ldots \int_{a_{n-1}}^{a_n} \det\Big(e^{(z_j - z_n) t_i} \Big)_{1 \le i,j \le n-1} \\ &= \det\Big(\int_{a_i}^{a_{i+1}} e^{z_j - z_n) t_i} dt_i \Big)_{1 \le i,j \le n-1} \\ &= \frac{1}{\prod_{i=1}^{n-1} (z_i - z_n)} \det\Big(e^{z_j - z_n) a_{i+1}} - e^{(z_j - z_n) a_i} \Big)_{1 \le i,j \le n-1}. \end{aligned}$$

It remains to check the identity

$$D := \det\big(e^{z_j a_i} \big)_{1 \le i,j \le n} = e^{z_n(a_1 + \cdots + a_n)} \det\big(e^{a_i(z_i - z_n)} - e^{a_{i+1}(z_j - z_n)} \big)_{1 \le i,j \le n-1}.$$

One writes

$$D = e^{z_n(a_1 + \cdots + a_n)} \begin{vmatrix} e^{a_1(z_1 - z_n)} & e^{a_1(z_2 - z_n)} & \ldots & e^{a_1(z_{n-1} - z_n)} & 1 \\ e^{a_2(z_1 - z_n)} & e^{a_2(z_2 - z_n)} & \ldots & e^{a_2(z_{n-1} - z_n)} & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ e^{a_n(z_1 - z_n)} & e^{a_n(z_2 - z_n)} & \ldots & e^{a_n(z_{n-1} - z_n)} & 1 \end{vmatrix}.$$

One substracts the second row from the first, the third from the second, and so on. One gets finally the identity.

∎

By using this formula we will determine the projection $\mu_A^{(k)}$ on $\mathcal{H}_k(\mathbb{C})$ of the orbital measure μ_A, observing the following: Let μ be a bounded measure on $\mathcal{H}_n(\mathbb{C})$, and $\mu^{(k)}$ the projection of μ on $\mathcal{H}_k(\mathbb{C})$. The Fourier-Laplace transform of $\mu^{(k)}$ is equal to the restriction to $\mathcal{H}_k(\mathbb{C})$ of the Fourier-Laplace transform of μ. But a difficult appears: for $k \leq n-2$, the Harish-Chandra-Itzykson-Zuber formula is not defined for $Z \in \mathcal{H}_k(\mathbb{C})$. We will obtain the Fourier-Laplace transform of $\mu_A^{(k)}$ by going to the limit.

7 Fourier-Laplace transform of the projection of an orbital measure

Consider functions of n variables defined by determinantal formulas of the following type:

$$F(z_1, \ldots, z_n) = \frac{1}{V_n(z)} \det\big(f_i(z_j)\big)_{1 \leq i,j \leq n},$$

where $f_1, \ldots, f_n$ are n analytic functions of one variable defined in a neighborhood of 0, and V_n is the Vandermonde polynomial:

$$V_n(z) = \prod_{1 \leq i < j \leq n} (z_i - z_j).$$

We saw in preceding Section that the Fourier-Laplace transform of an orbital integral is of this type (Theorem 6.1). The formula defines F if the numbers z_j are distinct, and F extends as an analytic function in a neighborhood of 0 in $\mathbb{C}^n$. We will establish an explicit formula for the restriction of F to the subspace $z_n = 0, \ldots, z_{k+1} = 0$.

Theorem 7.1. *For* $0 \leq k \leq n-1$*,*

$$F(z_1, \ldots, z_k, 0, \ldots, 0) = \tfrac{1}{1!2!\ldots(n-k-1)!}$$

$$\frac{1}{V_k(z_1,\dots,z_k)(z_1\dots z_k)^{n-k}} \begin{vmatrix} f_1(z_1) & \dots & f_n(z_1) \\ \vdots & & \vdots \\ f_1(z_k) & \dots & f_n(z_k) \\ f_1^{(n-k-1)}(0) & \dots & f_n^{(n-k-1)}(0) \\ \vdots & & \vdots \\ f_1'(0) & \dots & f_n'(0) \\ f_1(0) & \dots & f_n(0) \end{vmatrix}$$

Proof. We will prove this formula by a backwards recursion on k, starting from $k = n$. Assume that the formula holds for k. We will establish it for $k-1$. Since the value of a determinant does not change if one adds to a row a linear combination of the other rows, we can replace the entries of the k-th row by

$$f_j(z_k) - \big(f_j(0) + z_k f_j'(0) + \frac{1}{2} z_k^2 f_j(0) + \dots + \frac{1}{(n-k-1)!} z_k^{n-k-1} f_j^{(n-k-1)}(0)\big).$$

Observing that

$$\lim_{z_k \to 0} \tfrac{1}{z_k^{n-k}} \Big(f_j(z_k) - \big(f_j(0) + z_k f_j'(0) + \tfrac{1}{2} z_k^2 f_j(0) + \cdots \\ + \tfrac{1}{(n-k-1)!} z_k^{n-k-1} f_j^{(n-k-1)}(0)\big)\Big) = \tfrac{1}{(n-k)!} f_j^{(n-k)}(0),$$

we obtain

$$F(z_1, \dots, z_{k-1}, 0, \dots, 0) = \tfrac{1}{1!2!\dots(n-k)!}$$

$$\frac{1}{V_{k-1}(z_1,\dots,z_{k-1})^{n-k+1}} \begin{vmatrix} f_1(z_1) & \dots & f_n(z_1) \\ \vdots & & \vdots \\ f_1(z_{k-1}) & \dots & f_n(z_{k-1}) \\ f_1^{(n-k)}(0) & \dots & f_n^{(n-k)}(0) \\ \vdots & & \vdots \\ f_1'(0) & \dots & f_n'(0) \\ f_1(0) & \dots & f_n(0) \end{vmatrix}$$

∎

In particular, for $k=1$ we get:

$$F(z_1,0,\dots,0)=\frac{1}{1!2!\dots(n-2)!}\frac{1}{z_1^{n-1}}\begin{vmatrix} f_1(z_1) & \dots & f_n(z_1) \\ f_1^{(n-2)}(0) & \dots & f_n^{(n-2)}(0) \\ \vdots & & \vdots \\ f_1'(0) & \dots & f_n'(0) \\ f_1(0) & \dots & f_n(0) \end{vmatrix}$$

For $k=0$ we get:

$$F(0,\dots,0)=\frac{1}{1!2!\dots(n-1)!}\begin{vmatrix} f_1^{(n-1)}(0) & \dots & f_n^{(n-1)}(0) \\ \vdots & & \vdots \\ f_1'(0) & \dots & f_n'(0) \\ f_1(0) & \dots & f_n(0) \end{vmatrix}$$

If we specialize the formula of Theorem 7.1 to the case:

$$f_j(z_i)=e^{a_j z_i},$$

then, if $Z=\operatorname{diag}(z_1,\dots,z_n)$,

$$\widehat{\mu_A}(Z)=\delta_n!\frac{1}{V_n(a)}F(z_1,\dots,z_n),$$

and we get by restriction the Fourier-Laplace transform of the projection $\mu_A^{(k)}$.

Theorem 7.2. *Assume the numbers a_j to be distinct, and the numbers $z_1,\dots,z_k$ to be distinct and non zero ($0\le k\le n-1$).*

$$\widehat{\mu_A^{(k)}}(z)=(n-k)!\dots(n-1)!$$
$$\frac{1}{V_n(a)V_k(z_1,\dots,z_k)(z_1\dots z_k)^{n-k}}\begin{vmatrix} e^{a_1z_1} & \dots & e^{a_nz_1} \\ \vdots & & \vdots \\ e^{a_1z_k} & \dots & e^{a_nz_k} \\ a_1^{n-k-1} & \dots & a_n^{n-k-1} \\ \vdots & & \vdots \\ a_1 & \dots & a_n \\ 1 & \dots & 1 \end{vmatrix}.$$

In particular, for $k = 1$, $z = \mathrm{diag}(z_1, 0, \ldots, 0)$, we recover the formula of Proposition 3.1.

$$\widehat{\mu_A^{(1)}}(z) = (n-1)! \frac{1}{V_n(a)} \frac{1}{z_1^{n-1}} \begin{vmatrix} e^{a_1 z_1} & \ldots & e^{a_n z_1} \\ a_1^{n-2} & \ldots & a_n^{n-2} \\ \vdots & & \vdots \\ a_1 & \ldots & a_n \\ 1 & \ldots & 1 \end{vmatrix}.$$

8 Olshanski's determinantal formula

Recall the following determinantal formula for the divided differences:

$$f[a_1, \ldots, a_n] = \frac{1}{V_n(a_1, \ldots, a_n)} \begin{vmatrix} 1 & 1 & \ldots & 1 \\ a_1 & a_2 & \ldots & a_n \\ \vdots & \vdots & & \vdots \\ a_1^{n-2} & a_2^{n-2} & \ldots & a_n^{n-2} \\ f(a_1) & f(a_2) & \ldots & f(a_n) \end{vmatrix}$$

This formula can be generalized:

Proposition 8.1. *Let $f_1, \ldots, f_k$ be functions defined on $\mathbb{R}$.*

$$\begin{vmatrix} 1 & 1 & \ldots & 1 \\ a_1 & a_2 & \ldots & a_n \\ \vdots & \vdots & & \vdots \\ a_1^{n-k-1} & a_2^{n-k-1} & \ldots & a_n^{n-k-1} \\ f_1(a_1) & f_1(a_2) & \ldots & f_1(a_n) \\ \vdots & \vdots & & \vdots \\ f_k(a_1) & f_k(a_2) & \ldots & f_k(a_n) \end{vmatrix}$$
$$= \Big(\prod\nolimits_{0<j-i\leq n-k}(a_j - a_i)\Big) \det\big(f_i[a_j, \ldots, a_{j+n-k}]\big)_{1\leq i,j\leq k}.$$

Put $\varphi_k(t) = t^k$. Observe that, for $b_i \in \mathbb{R}$,

$$\varphi_k[b_1, \ldots, b_{k+1}] = 1, \quad \varphi_k[b_1, \ldots, b_\ell] = 0 \text{ for } \ell > k+1.$$

Let D denote the left handside. It can be written

$$D = \begin{vmatrix} \varphi_0(a_1) & \varphi_0(a_2) & \dots & \varphi_0(a_n) \\ \varphi_1(a_1) & \varphi_1(a_2) & \dots & \varphi_1(a_n) \\ \vdots & \vdots & & \vdots \\ \varphi_{n-k-1}(a_1) & \varphi_{n-k-1}(a_2) & \dots & \varphi_{n-k-1}(a_n) \\ f_1(a_1) & f_1(a_2) & \dots & f_1(a_n) \\ \vdots & \vdots & & \vdots \\ f_k(a_1) & f_k(a_2) & \dots & f_k(a_n) \end{vmatrix}$$

One substracts the first column from the second one, the second one from the third one, and so on:

$$C_n \leftarrow C_n - C_{n-1},\ C_{n-1} \leftarrow C_{n-1} - C_{n-2}, \dots, C_2 \leftarrow C_2 - C_1.$$

Then we get

$$D = (a_2 - a_1)(a_3 - a_2) \dots (a_n - a_{n-1}) \times \\ \begin{vmatrix} \varphi_1[a_1, a_2] & \varphi_1[a_2, a_3] & \dots & \varphi_1[a_{n-1}, a_n] \\ \vdots & \vdots & & \vdots \\ \varphi_{n-k-1}[a_1, a_2] & \varphi_{n-k-1}[a_2, a_3] & \dots & \varphi_{n-k-1}[a_{n-1}, a_n] \\ f_1[a_1, a_2] & f_1[a_2, a_3] & \dots & f_1[a_{n-1}, a_n] \\ \vdots & \vdots & & \vdots \\ f_k[a_1, a_2] & f_k[a_2, a_3] & \dots & f_k[a_{n-1}, a_n] \end{vmatrix}$$

Then we repeat the process:

$$D = \big((a_2 - a_1) \dots (a_n - a_{n-1})\big)\big((a_3 - a_1) \dots (a_n - a_{n-2})\big) \times \\ \begin{vmatrix} \varphi_2[a_1, a_2, a_3] & \dots & \varphi_2[a_{n-2}, a_{n-1}, a_n] \\ \vdots & & \vdots \\ \varphi_{n-k-1}[a_1, a_2, a_3] & \dots & \varphi_{n-k-1}[a_{n-2}, a_{n-1}, a_n] \\ f_1[a_1, a_2, a_3] & \dots & f_1[a_{n-2}, a_{n-1}, a_n] \\ \vdots & & \vdots \\ f_k[a_1, a_2, a_3] & \dots & f_k[a_{n-2}, a_{n-1}, a_n] \end{vmatrix}$$

After $n - k$ steps we get the formula of Proposition 8.1. $\square$

By the Hermite-Genocchi formula,

$$\det\big(f_i[a_j,\dots,a_{j+n-k}]\big)_{1\le i,j\le k}$$
$$=\Big(\frac{1}{(n-k)!}\Big)^k \det\Big(\int_{\mathbb{R}} f_i^{(n-k)}(t)M_{n-k+1}(a_j,\dots,a_{j+n-k};t)dt\Big)_{1\le i,j\le k}.$$

We use now the integral Cauchy-Binet formula: Let $u_1,\dots,u_k$ be continuous functions on $\mathbb{R}$, $v_1,\dots,v_k$ continuous functions on $\mathbb{R}$ with compact support, then

$$\int_{\mathbb{R}^k} \det\big(u_j(t_i)\big)_{1\le i,j\le k} \det\big(v_j(t_i)\big)_{1\le i,j\le k} dt_1\dots dt_k$$
$$= k!\det\Big(\int_{\mathbb{R}} u_i(t)v_j(t)dt\Big)_{1\le i,j\le k}.$$

Taking $u_i(t)=f_i^{(n-k)}(t)$ et $v_j(t)=M_{n-k+1}(a_j,\dots,a_{j+n-k};t)$, we get

$$\det\big(f_i[a_j,\dots,a_{j+n-k}]\big)_{1\le i,j\le k}$$
$$=\Big(\frac{1}{(n-k)!}\Big)^k \frac{1}{k!}\int_{\mathbb{R}} \det\big(f_j^{(n-k)}(t_i)\det\big(M_{n-k+1}(a_j,\dots,a_{j+n-k};t)\big)dt_1\dots dt_k.$$

We specialize this formula to the functions $f_i(t)=e^{z_it}$, and obtain, pour $Z=\operatorname{diag}(z_1,\dots,z_k)$,

$$
\begin{aligned}
\widehat{\mu_A^{(k)}}(Z) &= \frac{C(n,k)}{\prod_{j-i\geq n-k+1}(a_j-a_i)} \int_{(\mathfrak{R}^k)_+} \mathcal{E}_k(z_1,\dots,z_k;t) \\
& \det\big(M_{n-k+1}(a_j,\dots,a_{j+n-k};t_i)\big)_{1\leq i,j\leq k} V_k(t)dt_1\dots dt_k,
\end{aligned}
$$

with

$$
C(n,k)=\prod_{i=1}^{k-1}\binom{n-k+i}{i}.
$$

Olshanski's determinantal formula follows:

Theorem 8.2. *The radial part $\nu_A^{(k)}$ of the projection $\mu_A^{(k)}$ on the subspace $\mathcal{H}_k(\mathbb{C})$ of the orbital measure μ_A is given by*

$$
\begin{aligned}
&\nu_A^{(k)}(dt) = \frac{C(n,k)}{\prod_{j-i\geq n-k+1}(a_j-a_i)} \times \\
&\times \det\big(M_{n-k+1}(a_j,\dots,a_{j+n-k};t_i)\big)_{1\leq i,j\leq k} V_k(t)dt_1\dots dt_n.
\end{aligned}
$$

Special cases

a) $k=1$. In this case $C(n,k)=1$, and

$$
\nu_A^{(1)}(dt)=M_n(a_1,\dots,a_n)dt.
$$

b) $k=n-1$. In this case $C(n,k)=(n-1)!$, and

$$
\prod_{j-i\geq 2}(a_j-a_i)=\frac{V_n(a)}{\prod_{j-i=1}(a_j-a_i)}.
$$

Since $a_1<a_2<\dots<a_n$, the determinant

$$
\det\big(M_2(a_j,a_{j+1};t_i)\big)_{1\leq i,j\leq n-1}
$$

vanishes unless $t\preceq a$, and then is equal to

$$
M_2(a_1,a_2;t_1)M_2(a_2,a_3;t_2)\dots M_2(a_{n-1},a_n;t_n).
$$

Since, for $a < b$,

$$M_2(a, b; t) = \frac{1}{b-a} 1_{a \leq t \leq b},$$

one gets

$$\nu_A^{(n-1)}(dt) = \frac{1}{V_n(a)} V_{n-1}(t) 1_{t \preceq a} dt_1 \dots dt_{n-1},$$

and one recovers Baryshnikov's formula.

One can check that Olshanski's formula defines a probability measure by evualating directly the integral

$$Z(a) = \int_{(\mathbb{R}^k)_+} \det\bigl(M_{n-k+1}(a_j, \dots, a_{j+n-k}; t_i) V_k(t) dt_1 \dots dt_k.$$

By the Cauchy-Binet formula,

$$Z(a) = \det\Bigl(\int_{\mathbb{R}} M_{n-k+1}(a_j, \dots, a_{j+n-k}; t) t^{i-1} dt\Bigr).$$

The moments of the Peano measures are known:

$$\int_{\mathbb{R}} M_n(a_1, \dots, a_n; t) t^m dt = \frac{m!(n-1)!}{(m+n-1)!} h_m(a_1, \dots, a_n),$$

where h_m is the complete symmetric function of degree m. Hence

$$Z(a) = \det\Bigl(\frac{(i-1)!(n-k)!}{(n-k+i-1)!} h_{i-1}(a_j, \dots, a_{j+n-k})\Bigr).$$

By using the relation

$$h_m(a_2, \dots, a_n) - h_m(a_1, \dots, a_{n-1}) = (a_1 - a_n) h_{m-1}(a_1, \dots, a_n),$$

which follows from the generating formula:

$$\sum_{m=0}^{\infty} h_m(a_1, \dots, a_n) z^m = \prod_{i=1}^{n} \frac{1}{1 - a_i z},$$

one gets

$$\det\bigl(h_{i-1}(a_j, \dots, a_{j+n-k})\bigr) = \prod_{j-i \geq n-k+1} (a_j - a_i).$$

Finally

$$Z(a) = \frac{1}{C(n,k)} \prod_{j-i \geq n-k+1} (a_j - a_i).$$

9 A branching theorem

Let G be a compact group and π a representation of G on a finite dimensional vector space $\mathcal{V}$. Recall that the character χ_π is the function defined on G by

$$\chi_\pi(g) = \operatorname{tr} \pi(g).$$

It is a central function which can be decomposed in the basis $\{\chi_\lambda\}$ $(\lambda \in \hat{G})$ of the characters of the equivalence classes of irreducible representations of G:

$$\chi_\pi(g) = \sum_{\lambda \in \hat{G}} m_\lambda \chi_\lambda(g).$$

The sum is finite and the coefficients m_λ are integers ≥ 0, called *multiplicities*. This equality is equivalent to the relation

$$\pi = \bigoplus_{\lambda \in \hat{G}} m_\lambda \pi_\lambda.$$

If the numbers m_λ are equal to 0 or 1, one says that the decomposition is *multiplicity free* (see exercice 1).

Consider the restriction $\pi_\lambda\big|_H$ of an irreducible represetnation π_λ of G to a closed subgroup H of G. The character of this restriction is equal to the restriction to H of the character χ_λ of π_λ. In general the representation $\pi_\lambda\big|_H$ is not irreducible, and decomposes in a finite sum of irreducible representations $\pi_\mu^{(H)}$ of H $(\mu \in \hat{H})$:

$$\pi_\lambda\big|_H = \bigoplus_{\mu \in \hat{H}} m(\lambda, \mu) \pi_\mu^{(H)},$$

which involves multiplicities $m(\lambda, \mu)$. Such a relation is called *branching rule*. In order to determine the multiplicities $m(\lambda, \mu)$ one method consists in decomposing the restriction to H of the character χ_λ in the basis of the characters $\chi_\mu^{(H)}$ of the irreducible characters of H : for $h \in H$,

$$\chi_\lambda(h) = \sum_{\mu \in \hat{H}} m(\lambda, \mu) \chi_\mu^{(H)}(h).$$

We will apply this method in the case of $G = U(n)$ and $H = U(n-1)$. The character of the representation $\pi_\lambda^{(n)}$ of the group $U(n)$ with highest weight λ

can be expressed by the Schur function s_λ: if t is a unitary diagonal matrix, $t = \mathrm{diag}(t_1, \ldots, t_n) \in T \simeq (\mathbb{C}^*)^n$, then

$$\chi_\lambda^{(n)}(t) = s_\lambda(t).$$

Recall the definition of the Schur function $s_\lambda(t)$ associated to the signature $\lambda = (\lambda_1, \ldots, \lambda_n)$, $\lambda_1, \ldots, \lambda_n \in \mathbb{Z}$, $\lambda_1 \geq \cdots \geq \lambda_n$. One writes $s_\lambda^{(n)}(t)$ if one wants to specify the numbers of variables, $t = (t_1, \ldots, t_n) \in (\mathbb{C}^*)^n$:

$$s_\lambda^{(n)}(t) = \frac{1}{V_n(t)} \begin{vmatrix} t_1^{\lambda_1+n-1} & t_1^{\lambda_2+n-2} & \ldots & t_1^{\lambda_n} \\ t_2^{\lambda_1+n-1} & t_2^{\lambda_2+n-2} & \ldots & t_2^{\lambda_n} \\ \vdots & \vdots & & \vdots \\ t_n^{\lambda_1+n-1} & t_n^{\lambda_2+n-2} & \ldots & t_n^{\lambda_n} \end{vmatrix},$$

where $V_n(t)$ is the Vandermonde polynomial

$$V_n(t) = \prod_{1 \leq i < j \leq n} (t_i - t_j).$$

For fixed $t_n = 1$, one gets a function of $n-1$ variables $s_\lambda^{(n)}(t_1, \ldots, t_{n-1}, 1)$ which can be written as a linear combination of the Schur functions $s_\mu^{(n-1)}(t_1, \ldots, t_{n-1})$ $(\mu = (\mu_1, \ldots, \mu_{n-1}))$.

Proposition 9.1.

$$s_\lambda^{(n)}(t_1, \ldots, t_{n-1}, 1) = \sum_{\mu \preceq \lambda} s_\mu^{(n-1)}(t_1, \ldots, t_{n-1}),$$

où $\mu \preceq \lambda$ signifie que μ entrelace λ :

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \lambda_3 \geq \cdots \geq \lambda_{n-1} \geq \mu_{n-1} \geq \lambda_n.$$

Proof.

We will use the formula: for $p, q \in \mathbb{Z}$, $p \geq q$,

$$\frac{t^{p+1} - t^q}{t-1} = t^q + t^{q+1} + \cdots + t^p = \sum_{p \geq r \geq q} t^r. \qquad (*)$$

Observe that

$$V_n(t_1, \ldots, t_{n-1}, 1) = V_{n-1}(t_1, \ldots, t_{n-1}) \prod_{i=1}^{n-1} (t_i - 1).$$

in order to evaluate the determinant

$$\begin{vmatrix} t_1^{\lambda_1+n-1} & t_1^{\lambda_2+n-2} & \dots & t_1^{\lambda_n} \\ t_2^{\lambda_1+n-1} & t_2^{\lambda_2+n-2} & \dots & t_2^{\lambda_n} \\ \vdots & \vdots & & \vdots \\ t_{n-1}^{\lambda_1+n-1} & t_{n-1}^{\lambda_2+n-2} & \dots & t_{n-1}^{\lambda_n} \\ 1 & 1 & \dots & 1 \end{vmatrix}$$

one substracts the second column from the first one, then the third one from the second one, and so on. One gets

$$s_\lambda(t_1, \dots, t_{n-1}, 1) = \frac{1}{V_{n-1}(t)} \det(A_{ij})_{1 \le i,j \le n-1},$$

where

$$A_{ij} = \frac{t_i^{\lambda_j+n-j} - t_i^{\lambda_{j+1}+n-j+1}}{t_i - 1}.$$

We apply now formula (*) :

$$A_{ij} = \sum_{\lambda_j \ge \mu \ge \lambda_{j+1}} t_i^{\mu+n-1-j}.$$

The formula of Proposition 9.1 follows. ∎

Let $\pi_\lambda^{(n)}$ denotes the irreducible representation of the unitary group $U(n)$ with highest weight λ.

From Proposition 9.1 one deduces the following branching rule.

Theorem 9.2. (Branching rule) *The restriction of the representation $\pi_\lambda^{(n)}$ to the subgroup $U(n-1)$ decomposes multiplicity free. The irreducible representations $\pi_\mu^{(n-1)}$ of $U(n-1)$ which occur in this decomposition are those for which μ interlaces λ:*

$$\pi_\lambda^{(n)}\big|_{U(n-1)} = \bigoplus_{\mu \preceq \lambda} \pi_\mu^{(n-1)}.$$

The branching rule for the restriction of $\pi_\lambda^{(n)}$ to the subgroup $U(k)$ ($1 \le k \le n-2$) is less simple ;

$$\pi_\lambda^{(n)}\big|_{U(k)} \bigoplus_{\mu \in \widehat{U(k)}} m(\lambda,\mu)\pi_\mu^{(k)},$$

where $m(\lambda, \mu)$ is the number of sequences

$$\nu^{(n-1)} \in \mathbb{Z}^{n-1},\ \nu^{(n-2)} \in \mathbb{Z}^{n-2}, \ldots, \nu^{(k+1)} \in \mathbb{Z}^{k+1},$$

such that

$$\mu \preceq \nu^{(k+1)} \preceq \cdots \preceq \nu^{(n-1)} \preceq \lambda.$$

10 Horn's theorem

In this section we consider the projection q of the space $\mathcal{H}_n(\mathbb{F})$ on the subspace $\mathcal{D}_n \simeq \mathbb{R}^n$ of real diagonal matrices

$$q : \mathcal{H}_n(\mathbb{F}) \to \mathbb{R}^n, \quad X \mapsto (x_1, \ldots, x_n),\ x_i = X_{ii}.$$

For an orbit $\mathcal{O}_A$ of a diagonal matrix $A = \mathrm{diag}(a_1, \ldots, a_n)$ under the action of the unitary group $U_n(\mathbb{F})$, we will see that the projection $q(\mathcal{O}_A)$ is equal to the convex hull of the points $\sigma(a)$, where $\sigma(a)$ is the transform of $a = (a_1, \ldots, a_n)$ under the permutation $\sigma \in \mathfrak{S}_n$: $\sigma(a) = (a_{\sigma(1)}, \ldots, a_{\sigma(n)})$. This the Horn's convexity theorem:

$$q(\mathcal{O}_A) = C(a) := \mathrm{Conv}\big(\{\sigma(a) \mid \sigma \in \mathfrak{S}_n\}\big).$$

The image of the orbital measure μ_A is suppported by $C(a)$, and the density of the projection $N_A = q(\mu_A)$ with respect to the Lebesgue measure of the hyperplane

$$x_1 + \cdots x_n = a_1 + \cdots + a_n$$

is a piecewise polynomial function. This measure has been described by Heckman in a more general setting.

Theorem 10.1. (Horn's convexity Theorem) *For a diagonal matrix* $A = \mathrm{diag}(a_1, \ldots, a_n)$,

$$q(\mathcal{O}_A) = C(a).$$

Proof.

We will sketch the main steps in the proof.

a) *Theorem of Birkhoff*

A real $n \times n$ matrix S is said to be doubly stochastic if, for all i and j, $S_{ij} \geq 0$, and

$$\sum_{k=1}^{n} S_{ik} = 1, \quad \sum_{k=1}^{n} S_{kj} = 1.$$

Example

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

The set of doubly stochastic matrices is convex and compact.

The permutation matrix P_σ associated to the permutation σ is given by

$$(P_\sigma x)_i = x_{\sigma(i)}.$$

The matrix P_σ is doubly stochastic.
Example For $\sigma = (2, 3, 1)$,

$$P_\sigma = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Theorem 10.2. (Birkhoff) *The extremal points of the set of doubly stochastic matrices are the permutation matrices.*

b) We show first that $q(\mathcal{O}_A) \subset C(a)$. Let $X \in \mathcal{O}_A$: $X = uAu^*$ with $u \in U_n(\mathbb{F})$. Then

$$X_{ii} = \sum_{j=1}^{n} |u_{ij}|^2 a_j.$$

The matrix S with $S = |u_{ij}|^2$ is doubly stochastic. By Birkhoff's Theorem it is a convex combination of permutation matrices:

$$S = \sum_{\sigma \in \mathfrak{S}_n} c_\sigma P_\sigma, \quad \text{with } c_\sigma \geq 0, \sum_{\sigma \in \mathfrak{S}_n} c_\sigma = 1.$$

Therefore

$$q(X) = \sum_{\sigma \in \mathfrak{S}_n} c_\sigma P_\sigma a = \sum_{\sigma \in \mathfrak{S}_n} c_\sigma \sigma(a) \in C(a).$$

Hence $q(\mathcal{O}_A) \subset C(a)$.

c) We show now that $C(a) \subset q(\mathcal{O}_A)$. Let $b \in C(a)$. We have to show that there is a matrix $u \in U_n(\mathbb{F})$ such that $q(uAu^*) = b$. Horn shows that, if $b \in C(a)$, then there is an orthogonal matrix u such that $b = Sa$, where S is the doubly stochastic matrix given by $S_{ij} = u_{ij}^2$. Therefore $b = q(uAu^*)$.

$\square$

11 Heckman's measure

Let N_A denote the projection of the orbital measure μ_A on the space $\mathcal{D}_n$ of diagonal matrices: $N_A = q(\mu_A)$. As we observed at the end of Section 6 in an another case, the Fourier-Laplace transform $\widehat{N_A}$ of N_A is the resriction to $\mathcal{D}_n \simeq \mathbb{R}^n$ of the Fourier-Laplace $\widehat{\mu_A}$ of μ_A. By the Harish-Chandra-Itzykson-Zuber formula (Theorem 6.1), for $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$:

$$\widehat{N_A}(z) = \delta_n! \frac{1}{V_n(a)V_n(z)} \det(e^{z_i a_j})_{1 \leq i,j \leq n}.$$

This can be written

$$V_n(z)\widehat{N_A}(z) = \frac{\delta_n!}{V_n(a)} \sum_{\sigma \in \mathfrak{S}_n} \varepsilon(\sigma) \exp\Big(\sum_{i=1}^n z_i a_{\sigma(i)}\Big).$$

This means an equality between two Fourier-Laplace transforms, and implies the following differential equation:

$$V_n\Big(-\frac{\partial}{\partial x}\Big) N_A = \frac{\delta_n!}{V_n(a)} \sum_{\sigma \in \mathfrak{S}_n} \varepsilon(\sigma)\delta_{\sigma(a)},$$

where $\sigma(a) = (a_{\sigma(1)}, \ldots, a_{\sigma(n)})$. For solving this equation we will use an elementary solution of the differential operator $V_n(\frac{\partial}{\partial x})$. Define the distribution E_n:

$$\langle E_n, \varphi \rangle = \int_{\mathbb{R}^{\frac{n(n-1)}{2}}} \varphi\Big(\sum_{i<j} t_{ij}(e_i - e_j)\Big) dt_{ij}.$$

Proposition 11.1. *The distribution E_n is an elementary solution of the differential operator $V_n(\frac{\partial}{\partial x})$:*

$$V_n\Big(\frac{\partial}{\partial x}\Big) E_n = \delta_0.$$

Its support is the following cone

$$\mathrm{supp}(E_n) = \{x \in \mathbb{R}^n \mid x_1 \geq \cdots \geq x_n,\ x_1 + \cdots + x_n = 0\}.$$

The distribution E_n is absolutely continuous with respect to the Lebesgue measure of the hyperplane $x_1 + \cdots + x_n = 0$. The cone $\mathrm{supp}(E_n)$ *decomposes into a finite union of cones, and the restriction of the density to each of these cones is a polynomial homogeneous of degree $\frac{1}{2}(n-1)(n-2)$.*

Proof.

The differential operator $V_n(\frac{\partial}{\partial x})$ is a product of degree one differential operators:

$$V_n\Big(\frac{\partial}{\partial x}\Big) = \prod_{i<j}\Big(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j}\Big).$$

An elementary solution of $\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j}$ is the following Heaviside distribution Y_{ij} defined by

$$\langle Y_{ij}, \varphi\rangle = \int_0^\infty \varphi\big(t(e_i - e_j)\big)dt.$$

Hence the convolution product

$$E_n = \prod_{i<j}^{*} Y_{ij}$$

is an elementary solution of $V_n(\frac{\partial}{\partial x})$. ∎

Define $\check{\varphi}(x) = \varphi(-x)$, and $\check{E}_n$ by $\langle \check{E}_n, \varphi\rangle = \langle E_n, \check{\varphi}\rangle$. Let F and G be distributions on $\mathbb{R}^n$. Assume the support of F to be compact. Let $D = P\big(\frac{\partial}{\partial x}\big)$ be a diffirential operator with constant coefficients. Then

$$DF * G = F * DG = D(F * G).$$

Therefore:

$$\check{E}_n * V_n\Big(-\frac{\partial}{\partial x}\Big)N_A = V_n\Big(\frac{\partial}{\partial x}\Big)\check{E}_n * N_A = N_A.$$

Therefore

$$N_A = \frac{\delta_n}{V_n(a)} \sum_{\sigma\in\mathfrak{S}_n} \varepsilon(\sigma)\check{E}_n * \delta_{\sigma(a)}.$$

The measure N_A is supported by the hyperplane

$$x_1 + \cdots + x_n = a_1 + \cdots + a_n.$$

In fact $\operatorname{supp}(N_A) = q(\mathcal{O}_A) = C(a)$.

Theorem 11.2. *The measure N_A has a density with respect to the Lebesgue measure of the hyperplane*

$$x_1 + \cdots + x_n = a_1 + \cdots + a_n,$$

and the density is piecewise polynomial.

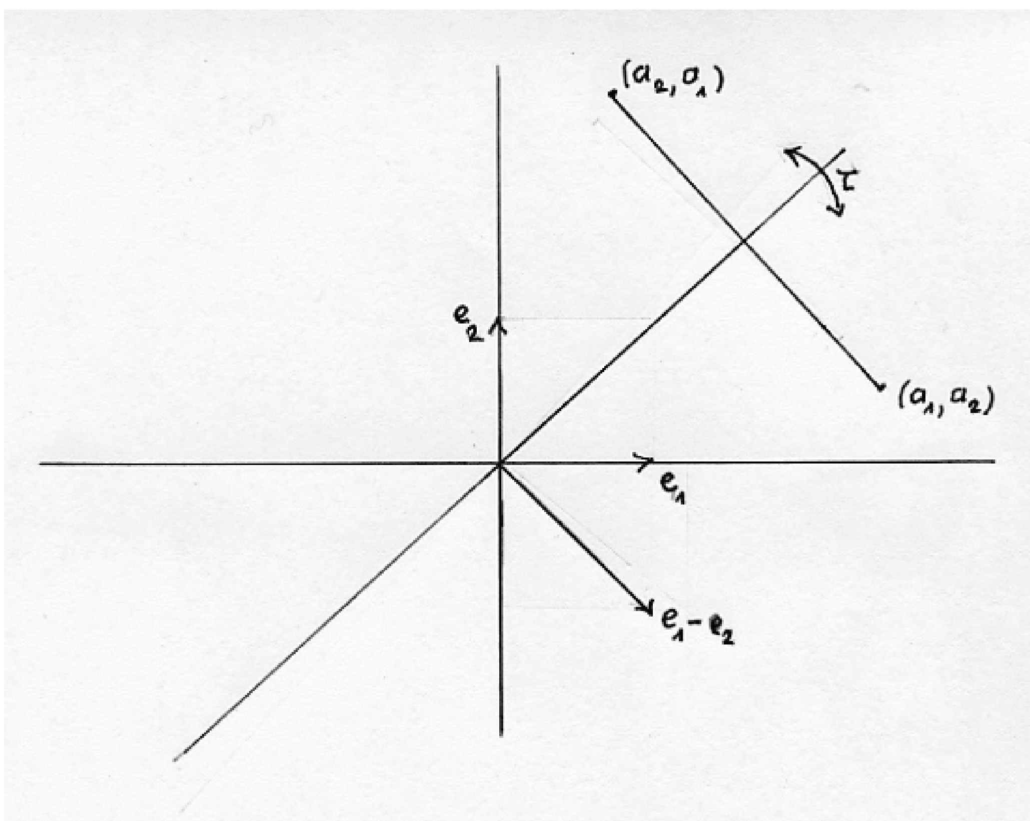
Example, n=2

$$\langle E_2, \varphi\rangle = \int_0^\infty \varphi\big(t(e_1 - e_2)\big)dt.$$

$$\mathfrak{S}_2 = \{Id, \tau\},\ \ \tau : (x_1, x_2) \mapsto (x_2, x_1).$$

$$\langle N_A, \varphi\rangle = \frac{1}{a_1 - a_2}\Big(\int_0^\infty \varphi\big(a - t_1(e_1 - e_2)\big)dt_1 - \int_0^\infty \varphi\big(\tau(a) - t_2(e_1 - e_2)\big)dt_2\Big).$$

$$\langle N_A, \varphi\rangle = \int_0^1 \varphi\big((1-t)a + t\tau(a)\big)dt.$$



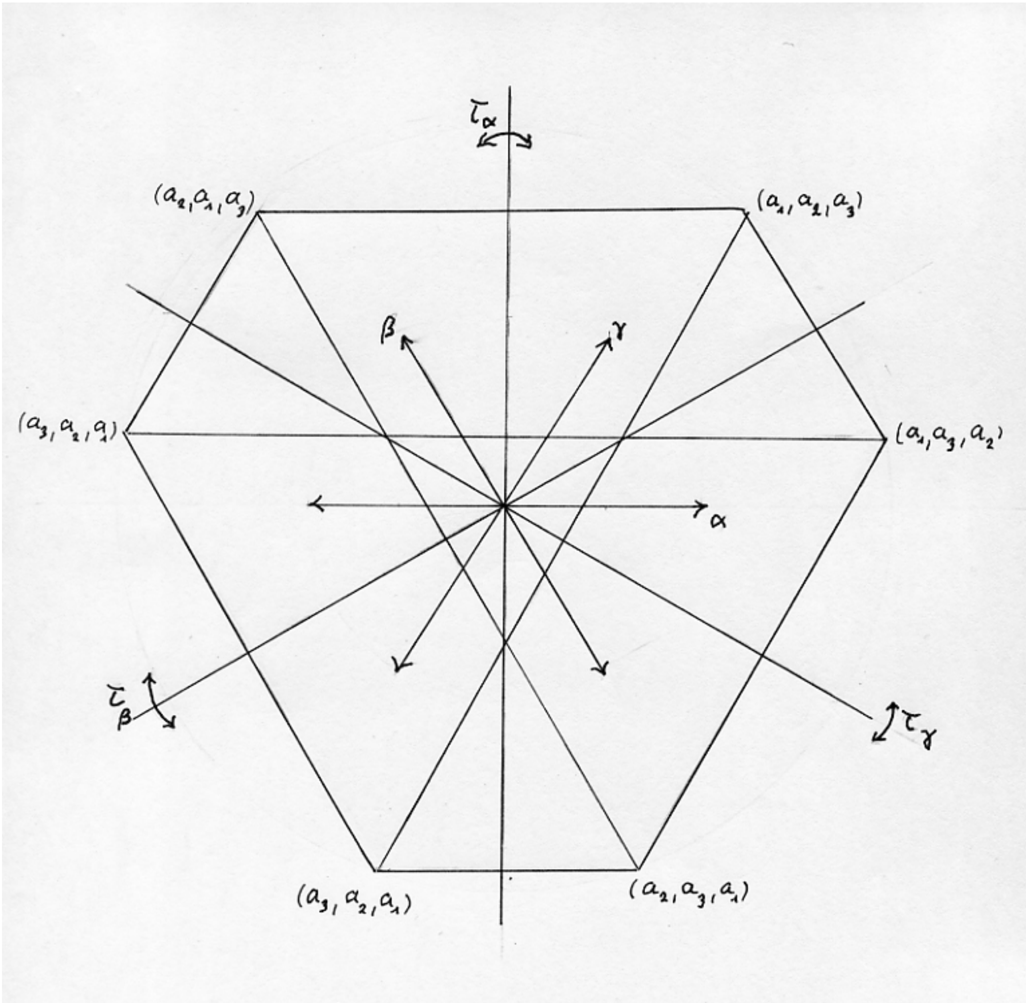
Example, n=3

$\alpha = e_1 - e_2$, $\beta = e_2 - e_3$, $\gamma = e_1 - e_3$, note that $\gamma = \alpha + \beta$.

$$\begin{aligned}
\langle E_3, \varphi \rangle &= \int_{(\mathbb{R}_+)^3} \varphi(u\alpha + v\beta + w\gamma) du dv dw \\
&= \int_{(\mathbb{R}_+)^3} \varphi\big((u+w)\alpha + (v+w)\beta\big) du dv dw. \\
&= \int_{\{0 \le w \le s, 0 \le w \le t\}} f(s\alpha + t\beta) ds dt dw \\
&= \int_{(\mathbb{R}+)^2} \inf(s,t) f(s\alpha + t\beta) ds dt.
\end{aligned}$$

Hence the suppport of E_3 is the angle defined by the rays generated by α and β, with density, if $x = s\alpha + t\beta$, $f(s,t) = \inf(s,t)$.

The support of the measure N_A is the convex hull of the six points $\sigma(a)$ ($\sigma \in \mathfrak{S}_3$). the density of N_A is linear in the three trapezes, and in the three triangles, and constant in the middle triangle.


τ_α
(a_2, a_1, a_3)
(a_1, a_2, a_3)
β
γ
(a_3, a_2, a_1)
(a_1, a_3, a_2)
α
τ_β
τ_γ
(a_3, a_2, a_1)
(a_2, a_3, a_1)

Exercises

1. Let π be a finite dimensional representation of a compact group G on a complex vector space $\mathcal{V}$, and consider its decomposition as direct sum of irreducible representations

$$\pi = \bigoplus_{\lambda \in \hat{G}} m_\lambda \pi_\lambda.$$

a) Show that the commutant in $End(\mathcal{V})$ of $\pi(G)$,

$$\pi(G)' = \{A \in \mathrm{End}(\mathcal{V}) \mid \forall g \in G,\ A\pi(g) = \pi(g)A\},$$

is equal to

$$\pi(G)' = \bigoplus_{\lambda \in \hat{G}} \mathrm{End}(\mathbb{C}^{m_\lambda}).$$

b) Recall that the decomposition is said to be multiplicity free if the multiplicities m_λ are equal to 0 or 1. Show that the commutant $\pi(G)'$ is commutative if and only if the representation π decomposes multiplicity free.

2. The moments $\mathcal{M}_n^{(m)}(a_1, \ldots, a_n)$ of the Peano measure $M_n(a_1, \ldots, a_n; dt)$ are defined by

$$\mathcal{M}_n^{(m)}(a_1, \ldots, a_n) := \int_{\mathfrak{R}} t^m M_n(a_1, \ldots, a_n; dt).$$

Establish the relation

$$\mathcal{M}_n^{(m)}(a_1, \ldots, a_n)t = \frac{m!(n-1)!}{(m+n-1)!} h_m(a_1, \ldots, a_n),$$

where h_m denotes the complete symmetric function.

3. Establish the following relation

$$\begin{aligned} &M_n(a_1, \ldots, a_n; t) \\ &= \frac{n-1}{a_n - a_1} \int_{-\infty}^{t} \big(M_{n-1}(a_1, \ldots, a_{n-1}; s) - M_{n-1}(a_2, \ldots, a_n; s)\big) ds. \end{aligned}$$

4. Originally the Peano measure $M_n(a_1, \ldots, a_n; dt)$ has been introduced in order to express as an integral the error commited when one replace a function by its interpolation polynomial.

a) Let $a_1, \dots, a_n$ be n distinct real numbers, and let f be a function defined on an interval containing the numbers $a_1, \dots, a_n$. Let p be the interpolation polynomial of f with respect to the numbers $a_1, \dots, a_n$: p is the polynomial of degree $\leq n-1$ such that $p(a_i) = f(a_i)$ $(i = 1, \dots, n)$. Show that

$$f(x) = p(x) + (x - a_1)(x - a_2) \dots (x - a_n) f[x, a_1, \dots, a_n].$$

b) Assume the function f to be of class $\mathcal{C}^n$. Show that

$$f(x) = p(x)$$
$$+\tfrac{1}{n!}(x - a_1)(x - a_2) \dots (x - a_n) \int_{\mathbb{R}} f^{(n)}(t) M_{n+1}(x, a_1, \dots, a_n; t) dt.$$

5. Define

$$\tilde{\mathcal{E}}_n(z, x) = \frac{1}{V_n(z)} \det\big(e^{z_i x_j}\big)_{1 \leq i,j \leq n} = \frac{1}{\delta_n!} V_n(x) \mathcal{E}_n(z, x).$$

Show that

$$\tilde{\mathcal{E}}_n(z, x) = \int_{y \preceq x} \tilde{\mathcal{E}}_{n-1}(z', y) e^{(\langle x \rangle - \langle y \rangle) z_n} dy_1 \dots y_{n-1},$$

where $z' = (z_1, \dots, z_{n-1})$, et $\langle x \rangle = x_1 + \dots + x_n$.

6. Let $s_\lambda^{(n)}(t_1, \dots, t_n)$ be the Schur function. Show that its Laurent expansion with respect to t_n is given by

$$s_\lambda^{(n)}(t_1, \dots, t_{n-1}, t_n) = \sum_{\mu \preceq \lambda} s_\mu^{(n-1)}(t_1, \dots, t_{n-1}) t_n^{\langle \mu \rangle - \langle \lambda \rangle},$$

où

$$\langle \lambda \rangle = \lambda_1 + \dots + \lambda_n.$$

One will observe the analogy of this formula with the one of the preceding exercise.

Notes and references

The measure $M_n(a_1, \ldots, a_n; dt)$ has been called Peano measure (or Peano kernel) refering to the following theorem due to Peano : a continuous linear form L on the space $\mathcal{C}^{n+1}([a,b])$ which vanishes on the subspace $\mathcal{P}_n$ of the polynomials of degree $\leq n$ admits the following representation :

$$L(f) = \int_a^b f^{(n+1)}(t)K(t)dt,$$

where

$$K(t) = \frac{1}{n!} L_x\big[(x-t)_+^n\big].$$

(See [Peano,1913], and also [Davis,1963], Section 3.7, [Phillips,2000], Chapter 4.) Observe that a difficulty appear since the function $x \mapsto (x-t)_+^n$ is not of class $\mathcal{C}^{n+1}$. In the present case the linear form L is defined by

$$L(f) = f[a_1, a_2, \ldots, a_n],$$

where $a_1, \ldots, a_n$ are n real numbers.

Theorem 2.3, due to Okounkov, is noticed in [Olshanski-Vershik,1996] : Proposition 8.2, p.172.

Hermite-Genocchi formula (Theorem 3.2) has been established by Hermite et Genocchi independantly in 1878 for which they gave two different proofs.

The oldest reference I know for the intertwining theorem (Theorem 4.1) is a paper by Cauchy : *Sur l'équation à l'aide de laquelle on détermine les inégalités séculaires des planètes* (1829).

The intertwining theorem is also called *Theorem of Rayleigh* refering to the book by Rayleigh *The Theory of Sound* (1877), (new edition by Dover in 1945).

The Harish-Chandra-Itzykson-Zuber formula has been established by Itzykson and Zuber using a method involving the heat equation on the space of Hermitian matrices and a formula for the radial part of the Laplacian [Itzykson-Zuber,1980]. It turns out that the Harish-Chandra-Itzykson-Zuber formula is nothing but a special case of a formula established by Harish-Chandra related to the action of a compact Lie group on its Lie algebra. [Harish-Chandra,1957].

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