CIMPA School, Hammamet September 2011

LOGARITHMIC POTENTIAL THEORY, ORTHOGONAL POLYNOMIALS, AND RANDOM MATRICES

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In this course we present the relations which exist between

- The logarithmic potential theory,
- The asymptotic distribution of the zeros of classical orthogonal polynomials,
- The asymptotic distribution of the eigenvalues of random matrices.

Stieltjes considers a finite system of n electric charges moving freely on a line and submitted to an external field. He observes that the minimum of the electrostatistic energy of the system is attained at the points whose coordinates are zeros of certain classical orthogonal polynomials of degree n. This is the electrostatic model for orthogonal polynomials.

In order to study the asymptotic distribution of the zeros of a classical orthogonal polynomial p_n of degree n as $n \to \infty$, one uses a basic result in logarithmic potential theory about the energy and equilibrium measures.

This result is also used in random matrix theory. This is the log gas model of Dyson. In particular, using logarithmic potential theory, one obtains a proof of Wigner Theorem about the convergence of the statistical distribution of the eigenvalues to the semicircle law in case of the Gaussian Orthogonal Ensemble, and the Gaussian Unitary Ensemble.

We will also consider the question: What is the probability for a symmetric matrix to be positive definite ? We will present recent results by Dean and Majumdar whose proof also uses the logarithmic potential theory.

We thank the organizers of the Hammamet CIMPA school for inviting us to present this topic: Piotr Graczyk, Abdelhamid Hassairi, and Gérard Letac. We are very grateful to Marouane Rabaoui who realized the histograms and graphics which illustrate clearly the main results presented in these notes. We also thank the referee for his careful reading of the manuscript.

Chapter I

ELECTROSTATIC MODEL OF STIELTJES FOR ORTHOGONAL POLYNOMIALS

I.1. Orthogonal polynomials. — Let μ be a positive measure on $\mathbb{R}$. We assume that the support of μ is infinite (μ is not a finite linear combination of Dirac measures), and that, for any $k \geq 0$,

$$\int_{\mathbb{R}} |t|^k \mu(dt) < \infty.$$

On the space $\mathcal{P}$ of polynomials in one variable with real coefficients we consider the inner product

$$(p \mid q) = \int_{\mathbb{R}} p(t)q(t)\mu(dt),$$

which makes $\mathcal{P}$ into a preHilbert space. From the system (t^m) $(m \in \mathbb{N})$ the Schmidt orthogonalization produces a sequence (p_m) of orthogonal polynomials: p_m is a polynomial of degree m and

$$\int_{\mathbb{R}} p_m(t)p_n(t)\mu(dt) = 0 \quad \text{if } m \neq n.$$

Hermite polynomials

In this example μ is the Gaussian measure

$$\mu(dt) = e^{-t^2} dt.$$

The Hermite polynomial H_m is defined by

$$H_m(t) = (-1)^m e^{t^2} \left(\frac{d}{dt}\right)^m e^{-t^2} = 2^m t^m + \cdots$$

By m integrations by parts one gets, for any polynomial p,

$$\int_{\mathbb{R}} H_m(t)p(t)e^{-t^2} dt = \int_{\mathbb{R}} p^{(m)}(t)e^{-t^2} dt.$$

Hence, if $\deg p < m$, then $(H_n \mid p) = 0$, and the polynomials H_m are orthogonal with respect to μ. Furthermore, taking $p = H_m$, one gets

$$\|H_m\|^2 = \int_{\mathbb{R}} H_m(t)^2 e^{-t^2} dt = 2^m m! \int_{\mathbb{R}} e^{-t^2} dt = 2^m m! \sqrt{\pi}.$$

Tchebychev polynomials of first kind

The arcsinus law is the probability measure μ defined on $\mathbb{R}$ by

$$\int_{\mathbb{R}} f(t)\mu(dt) = \frac{1}{\pi}\int_{-1}^{1} f(t)\frac{dt}{\sqrt{1-t^2}} = \frac{1}{\pi}\int_0^{\pi} f(\cos\theta)d\theta.$$

The Tchebychev polynomial T_m is defined by

$$T_m(\cos\theta) = \cos m\theta.$$

T_m is a polynomial of degree m,

$$T_m(t) = 2^{m-1}t^m + \cdots$$

The polynomials T_m are orthogonal with respect μ:

$$(T_m \mid T_n) = \frac{1}{\pi}\int_0^{\pi} \cos m\theta \cos n\theta \, d\theta = 0 \quad \text{if } m \neq n,$$

and

$$\|T_m\|^2 = \frac{1}{\pi}\int_{-1}^{1} T_m(t)^2 \frac{dt}{\sqrt{1-t^2}} = \frac{1}{\pi}\int_0^{\pi} \cos^2 m\theta \, d\theta = \begin{cases} 1 \text{ if } m = 0, \\ \frac{1}{2} \text{ if } m \geq 1. \end{cases}$$

Let (p_m) be a sequence of orthogonal polynomials with respect to a measure μ, and let $[a,b]$ be the smallest closed interval which contains the support of μ $(-\infty \leq a < b \leq \infty)$.

Proposition I.1.1. — *The zeros of p_n are real, simple, and belong to $]a,b[$.*

Proof.

Let $x_1, \ldots, x_r$ be the zeros of p_n which are of odd order and belong to $]a,b[$. We will show that $r = n$. We can write

$$p_n(t) = (t - x_1)\ldots(t - x_r)q(t).$$

The polynomial q is of degree $n-r$ and its sign does not change on $]a,b[$. If $r<n$,

$$\int_{\mathbb{R}}(t-x_1)\dots(t-x_r)p_n(t)\mu(dt)=0,$$

or

$$\int_{\mathbb{R}}(t-x_1)^2\dots(t-x_r)^2q(t)\mu(dt)=0,$$

a contradiction. □

The classical orthogonal polynomials are solutions of a second order differential equation. We will use this fact when studying the statistics of the zeros of these polynomials.

Hermite polynomials

The Hermite polynomials H_n are orthogonal with respect to the measure μ given by

$$\int_{\mathbb{R}} f(t)\mu(dt)=\int_{-\infty}^{\infty} f(t)e^{-t^2}dt.$$

The Hermite polynomial H_n is a solution of the differential equation

$$y''-2ty'+2ny=0.$$

Jacobi polynomials

The Jacobi polynomials $P_n^{(\alpha,\beta)}$ are orthogonal with respect to the measure μ given by

$$\int_{\mathbb{R}} f(t)\mu(dt)=\int_{-1}^{1} f(t)(1-t)^\alpha(1+t)^\beta dt.$$

One assumes $\alpha,\beta>-1$. The Jacobi polynomial $P_n^{(\alpha,\beta)}$ is a solution of the differential equation

$$(1-t^2)y''-\big((\alpha+\beta+2)t+\alpha-\beta\big)y'+n(n+\alpha+\beta+1)y=0.$$

Laguerre polynomials

The Laguerre polynomials $L_n^{(\alpha)}$ are orthogonal with respect to the measure μ given by

$$\int_{\mathbb{R}} f(t)\mu(dt)=\int_{0}^{\infty} f(t)e^{-t}t^\alpha dt.$$

One assumes $\alpha > -1$. The Laguerre polynomial $L_n^{(\alpha)}$ is a solution of the differential equation

$$ty'' + (\alpha + 1 - t)y' + ny = 0.$$

I.2 Statistics of the zeros of orthogonal polynomials. — Let (p_n) be a sequence of orthogonal polynomials with respect to a measure μ. Let $x_1^{(n)}, \ldots, x_n^{(n)}$ denote the zeros of p_n. In order to study the asymptotics of the zeros, we consider the probability measure M_n on $\mathbb{R}$ defined by

$$M_n = \frac{1}{n} \sum_{k=1}^{n} \delta_{x_k^{(n)}}.$$

In other words

$$\int_{\mathbb{R}} f(t) M_n(dt) = \frac{1}{n} \sum_{k=1}^{n} f(x_k^{(n)}).$$

Here is the question: *Does the measure M_n converge as $n \to \infty$?* We will present some results about this question.

Zeros of the Tchebychev polynomials T_n

Recall that the Tchebychev polynomial T_n is defined by

$$T_n(\cos\theta) = \cos n\theta.$$

The polynomials T_n are orthogonal with respect to the arcsinus law whose support is $[-1, 1]$. Hence, by Proposition I.1.1, the zeros of T_n belong to $]-1, 1[$. If $x = \cos\theta$ is a zero of T_n, then $\cos n\theta = 0$. Therefore

$$x_k^{(n)} = \cos(2k-1)\frac{\pi}{2n} \quad (k = 1, \ldots, n).$$

Hence

$$\int_{\mathbb{R}} f(t) M_n(dt) = \frac{1}{n} \sum_{k=1}^{n} f\Big(\cos(2k-1)\frac{\pi}{2n}\Big).$$

This sum can be seen as a Riemann sum. If f is continuous on $[-1, 1]$,

$$\begin{aligned}\lim_{n\to\infty} \int_{\mathbb{R}} f(t) M_n(dt) &= \frac{1}{\pi} \lim_{n\to\infty} \sum_{k=1}^{n} f\Big(\cos(2k-1)\frac{\pi}{2n}\Big)\frac{\pi}{n} \\ &= \frac{1}{\pi} \int_0^{\pi} f(\cos\theta)\,d\theta = \frac{1}{\pi} \int_{-1}^{1} f(t) \frac{dt}{\sqrt{1-t^2}}.\end{aligned}$$

In the general case, for instance in case of the Hermite polynomials, the proof is not as simple, because in general there is no explicit formula for the zeros. In Hermite polynomial's case we will see that the measure M_n converges after rescaling. If $x_1^{(n)}, \dots, x_n^{(n)}$ are the zeros of the Hermite polynomial H_n, then, for any bounded continuous function f on $\mathbb{R}$,

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} f\Big(\frac{1}{\sqrt{n}} x_k^{(n)}\Big) = \frac{1}{\pi}\int_{-\sqrt{2}}^{\sqrt{2}} f(t)\sqrt{2-t^2}\,dt.$$

The limit is the so-called semi-circle law of radius $\sqrt{2}$.

I.3 Electrostatistic model of Stieltjes. — Let us consider n particules on the line with positions $x_1, \dots, x_n$ and charges $e_1, \dots, e_n$. We assume that they can freely move and an external field acts on them. The total energy of the system is given by

$$E_n(x_1, \dots, x_n) = \beta \sum_{i<j} \log \frac{1}{|x_i - x_j|} e_i e_j + \sum_{i=1}^{n} e_i Q(x_i),$$

where Q is the potential of the external field. An equilibrium position of the system is a point $x = (x_1, \dots, x_n)$ which minimizes the energy.

We will assume $\beta = 2$, $e_1 = \cdots = e_n = 1$. Then

$$E_n(x_1, \dots, x_n) = 2\sum_{i<j} \log \frac{1}{|x_i - x_j|} + \sum_{i=1}^{n} Q(x_i).$$

Observe that

$$\exp\big(-E(x_1, \dots, x_n)\big) = e^{-\sum_{i=1}^{n} Q(x_i)} \prod_{i<j}(x_i - x_j)^2.$$

We assume that Q is defined and continuous on $]a, b[$, with $-\infty \leq a < b \leq \infty$, goes to $+\infty$ at the boundary points a and b, and, if $]a, b[$ is unbounded,

$$\lim_{|t|\to\infty} \big(Q(t) - \log(t^2+1)\big) = \infty.$$

Hence the energy $E_n(x_1, \dots, x_n)$ is defined for $x_i \in]a, b[$, with values in $]-\infty, \infty]$, bounded from below, and lower semi-continuous. Furthermore the energy goes to infinity at the boundary of the polycube $]a, b[^n$.

Examples of potentials

- $]a, b[=]-\infty, \infty[$, $Q(t) = t^2$. $w(t) = e^{-Q(t)} = e^{-t^2}$ is a Gaussian weight.

- $]a,b[=]-1,1[$,

$$Q(t) = \alpha \log \frac{1}{1-t} + \beta \log \frac{1}{1+t} \quad (\alpha, \beta > 0).$$

$w(t) = e^{-Q(t)} = (1-t)^\alpha (1+t)^\beta$ is a Jacobi weight.

- $]a,b[=]0,\infty[$,

$$Q(t) = t + \alpha \log \frac{1}{t} \quad (\alpha > 0).$$

$w(t) = e^{-Q(t)} = e^{-t} t^\alpha$ is a Laguerre weight.

PROPOSITION I.3.1. — *There is at least one point $x^* = (x_1^*, \ldots, x_n^*)$ in $]a,b[^n$ which minimizes the energy:*

$$E_n(x^*) = \inf_{x \in]a,b[^n} E_n(x).$$

For such a point $x_i^* \neq x_j^*$, if $i \neq j$.

Proof.

This follows from the fact that E_n is lower semi-continuous and goes to $+\infty$ at the boundary of $]a,b[^n$. ∎

The minimum of the energy will be denoted by E_n^*:

$$E_n^* = E_n(x^*) = \inf_{x \in]a,b[^n} E_n(x).$$

If the function Q is of class $\mathcal{C}^1$, the energy is of class $\mathcal{C}^1$ on

$$\{x \in]a,b[^n \mid x_i \neq x_j \text{ if } i \neq j\}.$$

Then the point x^* is a critical point for the energy:

$$\frac{\partial E_n}{\partial x_j}(x^*) = 0 \quad (j = 1, \ldots, n).$$

Let us compute the partial derivatives of the function E_n:

$$\frac{\partial E_n}{\partial x_j} = -2 \sum_{i \neq j} \frac{1}{x_j - x_i} + Q'(x_j).$$

Hence $x = (x_1, \ldots, x_n)$ is a critical point for E_n if

$$Q'(x_j) - 2 \sum_{i \neq j} \frac{1}{x_j - x_i} = 0 \quad (j = 1, \ldots, n).$$

We will use the following lemma whose proof is left to the reader.

LEMMA I.3.2. — *Let f be a function of class $\mathcal{C}^2$ on $]a,b[$, and $t_0 \in]a,b[$. Assume $f(t_0)=0$, $f'(t_0)\neq 0$. Then*

$$\lim_{t\to t_0}\left(\frac{f'(t)}{f(t)}-\frac{1}{t-t_0}\right)=\frac{f''(t_0)}{2f'(t_0)}.$$

To a point $x\in\mathbb{R}^n$ we associate the polynomial

$$p(t)=p_x(t)=(t-x_1)(t-x_2)\dots(t-x_n).$$

Let us compute the logarithmic derivative of p:

$$\frac{p'(t)}{p(t)}=\sum_{i=1}^n\frac{1}{t-x_i},$$

and

$$\frac{p'(t)}{p(t)}-\frac{1}{t-x_j}=\sum_{i\neq j}\frac{1}{t-x_i}.$$

Hence

$$\lim_{t\to x_j}\left(\frac{p'(t)}{p(t)}-\frac{1}{t-x_j}\right)=\sum_{i\neq j}\frac{1}{x_j-x_i},$$

and, by Lemma I.3.2,

$$\lim_{t\to x_j}\left(\frac{p'(t)}{p(t)}-\frac{1}{t-x_j}\right)=\frac{p''(x_j)}{2p'(x_j)}.$$

Therefore

$$\frac{\partial E_n}{\partial x_j}=Q'(x_j)-\frac{p''(x_j)}{p'(x_j)},$$

and $x=(x_1,\dots,x_n)$, with $x_i\neq x_j$ $(i\neq j)$, is a critical point for E_n if

$$p_x''(x_j)-Q'(x_j)p_x'(x_j)=0\quad(j=1,\dots,n).$$

THEOREM I.3.3. — *Assume that*

$$Q'(t)=\frac{A(t)}{B(t)},$$

where A,B are polynomials, $\deg A\leq 1$, $\deg B\leq 2$. *Then $x=(x_1,\dots,x_n)$ is a critical point for E_n if and only if the polynomial*

$$p(t)=p_x(t)=(t-x_1)\dots(t-x_n)$$

is a solution of the differential equation

$$B(t)p''(t) - A(t)p'(t) + Cp(t) = 0.$$

The constant C is determined by looking at the coefficient of t^n.

Proof.
We have seen that $x = (x_1, \dots, x_n)$, with $x_i \neq x_j$ $(i \neq j)$, is a critical point for E_n if and only if

$$p_x''(t) - Q(x_j)p_x'(x_j) = 0 \quad (j = 1, \dots, n)$$

or

$$B(x_j)p_x''(x_j) - A(x_j)p_x'(x_j) = 0 \quad (j = 1, \dots, n).$$

The left handside is a polynomial of degree $\leq n$, vanishing at $x_1, \dots, x_n$. Therefore it is proportional to p_x. ∎

The following corollaries are due to Stieltjes.
a) For $Q(t) = t^2$, $Q'(t) = 2t$, we get the following differential equation

$$y'' - 2ty' + 2ny = 0.$$

The only polynomial solution (up to a constant factor) is the Hermite polynomial H_n.

Corollary I.3.4. — *The minimum of the energy is attained at the $n!$ points whose coordinates are the zeros of the Hermite polynomial H_n.*
b) For

$$Q(t) = \alpha \log \frac{1}{1-t} + \beta \log \frac{1}{1+t}, \quad Q'(t) = \alpha \frac{1}{1-t} - \beta \frac{1}{1+t},$$

we get the following differential equation

$$(1-t^2)y'' - \big((\alpha+\beta)t + \alpha - \beta\big)y' + n(n+\alpha+\beta-1)y = 0.$$

The only polynomial solution (up to a constant factor) is the Jacobi polynomial $P_n^{(\alpha-1,\beta-1)}$.

Corollary I.3.5. — *The minimum of the energy is attained at the $n!$ points whose coordinates are the zeros of the Jacobi polynomial $P_n^{(\alpha-1,\beta-1)}$.*
c) For

$$Q(t) = t + \alpha \log \frac{1}{t}, \; Q'(t) = 1 - \alpha \frac{1}{t},$$

we get the differential equation

$$ty'' + (\alpha - t)y' + ny = 0.$$

The only polynomial solution (up a constant factor) is the Laguerre polynomial $L_n^{(\alpha-1)}$.

Corollary I.3.6. — *The minimum of the energy is attained at the* $n!$ *points whose coordinates are the zeros of the Laguerre polynomial* $L_n^{(\alpha-1)}$.

Recall that the *discriminant* of a polynomial p of degree n with leading coefficient equal to one:

$$p(t) = t^n + a_{n-1}t^{n-1} + \cdots + a_0 \quad (a_n = 1),$$

is the scalar $D(p)$ defined by

$$D(p) = \prod_{i<j}(x_i - x_j)^2,$$

where $x_1, \ldots, x_n$ are the zeros of p. Hence

$$\exp(-E_n^*) = \exp\Big(-\sum_{i=1}^n Q(x_i^{(n)})\Big)D(p_n),$$

where p_n is the polynomial H_n in case (a): $Q(t) = t^2$, or $P_n^{(\alpha-1,\beta-1)}$ in case (b), or $L_n^{(\alpha-1)}$ in case (c), divided by the coefficient of t^n. The discriminant $D(p_n)$ can be evaluated, see [Szegö,1975] p.142. Let us consider the case of the Hermite polynomials, and put

$$D_n = \prod_{i<j}(x_i^{(n)} - x_j^{(n)})^2, \quad D_1 = 1,$$

where $x_1^{(n)}, \ldots, x_n^{(n)}$ are the zeros of H_n.

Proposition I.3.7.

$$D_n = 2^{\frac{n(n-1)}{2}} \prod_{k=2}^n k^k.$$

Proof.

One observes that

$$p_n'(x_i^{(n)}) = \prod_{j\neq i}(x_i^{(n)} - x_j^{(n)}),$$

therefore

$$D_n = (-1)^{\frac{n(n-1)}{2}} \prod_{i=1}^{n} p_n'(x_i^{(n)}).$$

From the relation $H_n'(t) = 2nH_{n-1}(t)$ it follows that $p_n'(t) = np_{n-1}(t)$. Hence

$$D_n = (-1)^{\frac{n(n-1)}{2}} n^n \prod_{i=1}^{n} p_{n-1}(x_i^{(n)}).$$

Define

$$\Delta_n = \prod_{i=1}^{n} p_{n-1}(x_i^{(n)}), \quad \Delta_0 = 1.$$

For $n \geq 1$,

$$\Delta_n = \prod_{i=1}^{n} \prod_{j=1}^{n-1} (x_i^{(n)} - x_j^{(n-1)}) = \prod_{j=1}^{n-1} p_n(x_j^{(n-1)}).$$

We will establlish a relation between Δ_n and Δ_{n-1}. For that we will use the three terms relation

$$H_n(t) = 2tH_{n-1}(t) - 2(n-1)H_{n-2}(t),$$

which gives

$$p_n(t) = tp_{n-1}(t) - \frac{n-1}{2} p_{n-2}(t).$$

It follows that

$$p_n(x_j^{(n-1)}) = -\frac{n-1}{2} p_{n-2}(x_j^{(n-1)}),$$

therefore

$$\Delta_n = \Big(-\frac{n-1}{2}\Big)^{n-1} \prod_{j=1}^{n-1} p_{n-2}(x_j^{(n-1)}) = \Big(-\frac{n-1}{2}\Big)^{n-1} \Delta_{n-1},$$

and

$$D_n = 2^{-\frac{n(n-1)}{2}} \prod_{k=2}^{n} k^k.$$

$\square$

Corollary I.3.8.

$$\exp(-E_n^*) = (2e)^{-\frac{n(n-1)}{2}} \prod_{k=2}^{n} k^k.$$

Proof.

The expansion of the Hermite polynomial is given by

$$H_n(x) = n! \sum_{k=0}^{[\frac{n}{2}]} (-1)^k \frac{(2x)^{n-2k}}{k!(n-2)!}$$
$$2^n \Big(x^n - \frac{n(n-1)}{4} x^{n-2} + \cdots \Big).$$

By using the classical relations between the coefficients and the zeros of a polynomial we obtain

$$\sum_{i=1}^{n} x_i^{(n)} = 0,$$
$$\sum_{i<j} x_i^{(n)} x_j^{(n)} = -\frac{n(n-1)}{4},$$

and

$$\sum_{i=1}^{n} \big(x_i^{(n)}\big)^2 = \Big(\sum_{i=1}^{n} x_i^{(n)}\Big)^2 - 2 \sum_{i<j} x_i^{(n)} x_j^{(n)} = \frac{n(n-1)}{2}.$$

Since

$$\exp(-E_n^*) = \exp\Big(-\sum_{i=1}^{n} (x_i^{(n)})^2\Big) D_n,$$

and we get the formula from Proposition I.3.7. ∎

References

[Stieltjes,1886]; [Szegö,1975], Sections 6.7 and 6.71; [Andrews-Askey-Roy,1999], Section 8.5; [Ismail,2001]; [Chen-Ismail,1997].

Chapter II

LOGARITHMIC POTENTIAL, ENERGY AND EQUILIBRIUM MEASURE

1. Logarithmic potential. — The logarithmic potential of a positive measure μ on $\mathbb{R}$ is the function U^μ defined by

$$U^\mu(x) = \int_{\mathbb{R}} \log \frac{1}{|x-t|} \mu(dt).$$

It is well defined, with values in $]-\infty, \infty]$ if the support of μ is compact, or, more generally, if

$$\int_{\mathbb{R}} \log(1+|t|)\mu(dt) < \infty.$$

Observe that

$$\lim_{|x|\to\infty} \big(U^\mu(x) + \mu(\mathbb{R}) \log|x|\big) = 0.$$

The Cauchy transform G_μ of a bounded measure μ on $\mathbb{R}$ is the function defined on $\mathbb{C} \setminus \operatorname{supp}(\mu)$ by

$$G_\mu(z) = \int_{\mathbb{R}} \frac{1}{z-t} \mu(dt).$$

The Cauchy transform is holomorphic.

Assume that $\operatorname{supp}(\mu) \subset]-\infty, a]$, and

$$\int_{\mathbb{R}} \log(1+|t|)\mu(dt) < \infty.$$

Then the function

$$F(z) = \int_{\mathbb{R}} \log(z-t)\mu(dt)$$

is defined and holomorphic in $\mathbb{C} \setminus]-\infty, a]$. Furthermore $F'(z) = G_\mu(z)$, and

$$\begin{aligned} U^\mu(x) &= -\operatorname{Re} F(x) \quad (x > a), \\ U^\mu(x) &= -\lim_{\varepsilon\to 0} \operatorname{Re} F(x+i\varepsilon) \quad (x \in \mathbb{R}). \end{aligned}$$

In the distribution sense,

$$\frac{d}{dx} U^\mu(x) = -\operatorname{Re} G_\mu(x).$$

We will use some properties of the boundary value distribution of a holomorphic function. Let f be holomorphic in $\mathbb{C} \setminus \mathbb{R}$. It is said to be of moderate growth near $\mathbb{R}$ if, for every compact set $K \subset \mathbb{R}$, there are $\varepsilon > 0$ $N > 0$, and $C > 0$ such that

$$|f(x+iy)| \leq \frac{C}{|y|^N} \quad (x \in K,\ 0 < |y| \leq \varepsilon).$$

Then, the formula, with $\varphi \in \mathcal{D}(\mathbb{R})$,

$$\langle T, \varphi \rangle = \lim_{\varepsilon \to 0, \varepsilon > 0} \int_{\mathbb{R}} \varphi(t)\big(f(t+i\varepsilon) - f(t-i\varepsilon)\big)dt,$$

defines a distribution T on $\mathbb{R}$. It is denoted $T = [f]$, and called the difference of boundary values of f. One shows that the function f extends as a holomorphic function in $\mathbb{C} \setminus \mathrm{supp}([f])$. In particular, if $[f] = 0$, then f extends as a holomorphic function in $\mathbb{C}$.

For $\alpha \in \mathbb{C}$, the distribution Y_α is defined, for $\mathrm{Re}\,\alpha > 0$, by

$$\langle Y_\alpha, \varphi \rangle = \frac{1}{\Gamma(\alpha)} \int_0^\infty \varphi(t) t^{\alpha-1} dt.$$

The distribution Y_α, as a function of α, admits an analytic continuation for $\alpha \in \mathbb{C}$. In particular $Y_0 = \delta_0$, the Dirac measure at 0.

For $\alpha \in \mathbb{C}$, we defines the holomorphic function z^α in $\mathbb{C} \setminus]-\infty, 0]$ as follows: if $z = re^{i\theta}$, with $r > 0$, $-\pi < \theta < \pi$, then

$$z^\alpha = r^\alpha e^{i\alpha\theta}.$$

The function f is of moderate growth near $\mathbb{R}$, and

$$\langle [z^\alpha], \varphi \rangle = -2i\pi \frac{1}{\Gamma(-\alpha)} \langle Y_{\alpha+1}, \check{\varphi} \rangle,$$

where $\check{\varphi}(t) = \varphi(-t)$. In particular

$$[\frac{1}{z}] = -2i\pi\delta_0.$$

Proposition II.1.1. — *Let μ be a bounded positive measure on $\mathbb{R}$.*

(i) *The Cauchy transform G_μ of μ is holomorphic in $\mathbb{C} \setminus \mathrm{supp}(\mu)$, of moderate growth near $\mathbb{R}$, and*

$$[G_\mu] = -2i\pi\mu.$$

(ii) *Assume that the support of* μ *is compact. Let* F *be holomorphic in* $\mathbb{C} \setminus \mathbb{R}$*, of moderate growth near* $\mathbb{R}$*, such that*

$$[F] = -2i\pi\mu.$$

Then F *is holomorphic in* $\mathbb{C} \setminus \operatorname{supp}(\mu)$*. If further*

$$\lim_{|z|\to\infty} F(z) = 0,$$

then $F = G_\mu$*.*

Proof.

(i) follows from

$$[\frac{1}{z}] = -2i\pi\delta_0.$$

(ii) The function $f = G_\mu - F$ satisfies

$$[f] = [G_\mu] - [F] = 0,$$

hence f extends as a holomorphic function in $\mathbb{C}$. From

$$|G_\mu(z)| \leq \mu(\mathbb{R})\frac{1}{|z|},$$

we obtain

$$\lim_{|z|\to\infty} f(z) = 0.$$

By Liouville's theorem, if follows that $f \equiv 0$. ∎

For $a < b$, the function

$$f(z) = \sqrt{(z-a)(z-b)}$$

is first defined as $\sqrt{z-a}\sqrt{z-b}$ in $\mathbb{C}\setminus]-\infty, b]$. For $x > b$, $f(x) = \sqrt{(x-a)(x-b)}$, the usual square root of a positive number. For $x < a$,

$$\lim_{\varepsilon\to 0, \varepsilon>0} f(x \pm i\varepsilon) = e^{\pm i\pi}\sqrt{(a-x)(b-x)} = -\sqrt{(x-a)(x-b)}.$$

Therefore f extends as a holomorphic function on $\mathbb{C} \setminus [a, b]$. Observe that, for $a \leq x \leq b$,

$$\lim_{\varepsilon\to 0, \varepsilon>0} f(x \pm i\varepsilon) = e^{\pm i\frac{\pi}{2}}\sqrt{(x-a)(b-x)} = \pm i\sqrt{(x-a)(b-x)}.$$

It follows that

$$[f] = 2i\sqrt{(t-a)(b-t)}\chi(t),$$

where χ is the indicator function of $[a, b]$. The function f admits, for $|z| > \max(|a|, |b|)$, a Laurent expansion:

$$f(z) = z\sqrt{1-\frac{a}{z}}\sqrt{1-\frac{b}{z}} = z - \frac{a+b}{2} - \frac{(a-b)^2}{8}\frac{1}{z} + \cdots$$

Example 1

Let μ be the arcsinus law:

$$\int_{\mathbb{R}} f(t)\mu(dt) = \frac{1}{\pi}\int_{-1}^{1} f(t)\frac{dt}{\sqrt{1-t^2}}.$$

PROPOSITION II.1.2. — (i) *The Cauchy transform of the arcsinus law μ is defined on $\mathbb{C} \setminus [-1, 1]$.*

$$G_\mu(z) := \frac{1}{\pi}\int_{-1}^{1} \frac{1}{z-t}\frac{dt}{\sqrt{1-t^2}} = \frac{1}{\sqrt{z^2-1}}.$$

(ii) *The logarithmic potential of the arcsinus law μ is given by*

$$\begin{aligned} U^\mu(x) &= \log 2, \quad \text{if } -1 \le x \le 1, \\ &= \log 2 - \log\left||x| + \sqrt{x^2-1}\right|, \quad \text{if } |x| \ge 1. \end{aligned}$$

Proof.
(i) The function

$$F(z) = \frac{1}{\sqrt{z^2-1}}$$

is defined and holomorphic on $\mathbb{C} \setminus [-1, 1]$, and satisfies

$$[F] = -2i\frac{1}{\sqrt{1-t^2}}\chi(t),$$

where χ is the indicator function of $[-1, 1]$. Therefore

$$[F] = -2i\pi\mu.$$

Furthermore

$$\lim_{|z|\to\infty} F(z) = 0.$$

Hence $F = G_\mu$ by Proposition II.1.1.

(ii)

$$\frac{d}{dx}U^\mu(x) = -\mathrm{Re}\, G_\mu(x) = 0 \text{ if } -1 < x < 1,$$
$$= -\frac{1}{\sqrt{1-x^2}} \text{ if } x > 1.$$

It follows that

$$U^\mu(x) = C, \text{ if } -1 \leq x \leq 1,$$
$$= C - \int_1^x \frac{dt}{\sqrt{t^2-1}}, \text{ if } x > 1.$$

Observe that $U^\mu(-x) = U^\mu(x)$. The integral can be computed:

$$\int_1^x \frac{dt}{\sqrt{1-t^2}} = \log(x + \sqrt{x^2-1}).$$

From the relation

$$\lim_{|x|\to\infty} \big(U^\mu(x) + \log|x|\big) = 0,$$

one gets $C = \log 2$. ∎

Example 2

Let μ be the semi-circle law:

$$\int_{\mathbb{R}} f(t)\mu(dt) = \frac{2}{\pi}\int_{-1}^{1} f(t)\sqrt{1-t^2}dt.$$

PROPOSITION II.1.3. — (i) *The Cauchy transform of the semi-circle law is defined on* $\mathbb{C} \setminus [-1,1]$.

$$G_\mu(z) := \frac{2}{\pi}\int_{-1}^{1} \frac{1}{z-t}\sqrt{1-t^2}\, dt = 2(z - \sqrt{z^2-1}).$$

(ii) *The logarithmic potential of the semi-circle law is an even function. It is given by*

$$U^\mu(x) = -x^2 + C, \quad \text{if } -1 \leq x \leq 1,$$
$$= -x^2 + C + 2\int_1^x \sqrt{t^2-1}\, dt, \quad \text{if } x > 1,$$

with $C = \log 2 + \frac{1}{2}$.

Proof.

(i) The function $f(z) = \sqrt{z^2 - 1}$ defined on $\mathbb{C} \setminus [-1, 1]$ satisfies

$$[f] = 2i\sqrt{1 - t^2}\chi(t) = i\pi\mu,$$

where χ is the indicator function of $[-1, 1]$, and

$$\lim_{|z|\to\infty} \big(f(z) - z\big) = 0.$$

By Proposition II.1.1 It follows that

$$G_\mu(z) = 2(z - \sqrt{z^2 - 1}).$$

(ii)

$$\begin{aligned} \frac{d}{dx} U^\mu(x) = -\mathrm{Re}\, G_\mu(x) &= -2x \text{ if } -1 \le x \le 1, \\ &= -2x + 2\sqrt{x^2 - 1} \text{ if } x > 1. \end{aligned}$$

Therefore

$$\begin{aligned} U^\mu(x) &= -x^2 + C \text{ if } -1 \le x \le 1, \\ &= -x^2 + C + 2\int_1^x \sqrt{t^2 - 1}dt \text{ if } x > 1. \end{aligned}$$

Lemma II.1.4.

$$\int_1^x \sqrt{t^2 - 1}\, dt = \frac{1}{2}x^2 - \frac{1}{2}\log x - \frac{1}{4} - \frac{1}{2}\log 2 + o(1).$$

Proof. The integral can be computed

$$\int_1^x \sqrt{t^2 - 1}dt = \frac{1}{2}x\sqrt{x^2 - 1} - \frac{1}{2}\log(x + \sqrt{x^2 - 1}).$$ ∎

From the relation

$$\lim_{x\to\infty} \big(U^\mu(x) + \log x\big) = 0,$$

it follows, by Lemma II.1.4, that $C = \frac{1}{2} + \log 2$. ∎

Example 3

For $c \geq 1$, consider the Marchenko-Pastur law

$$\int_{\mathbb{R}} f(t)\mu(dt) = \frac{1}{2\pi}\int_a^b f(t)\sqrt{(t-a)(b-t)}\frac{dt}{t},$$

where $a = (\sqrt{c}-1)^2$, $b = (\sqrt{c}+1)^2$.

PROPOSITION II.1.5. — (i) *The Cauchy transform of the measure μ is defined in $\mathbb{C} \setminus [a,b]$.*

$$\begin{aligned} G_\mu(z) :&= \frac{1}{2\pi}\int_a^b \frac{1}{z-t}\sqrt{(t-a)(b-t)}\frac{dt}{t} \\ &= \frac{z-(c-1)-\sqrt{(z-a)(z-b)}}{2z}. \end{aligned}$$

(ii) *The logarithmic potential of the measure μ is given on $[0,\infty[$ by*

$$\begin{aligned} &U^\mu(x) \\ &= -\frac{1}{2}\Big(x+(c-1)\log\frac{1}{x}\Big) + C + \frac{1}{2}\int_x^a \sqrt{(a-t)(b-t)}\frac{dt}{t} \quad \text{if } 0 < x \leq a, \\ &= -\frac{1}{2}\Big(x+(c-1)\log\frac{1}{x}\Big) + C \quad \text{if } a \leq x \leq b, \\ &= -\frac{1}{2}\Big(x+(c-1)\log\frac{1}{x}\Big) + C + \frac{1}{2}\int_b^x \sqrt{(t-a)(t-b)}\frac{dt}{t} \quad \text{if } x \geq b. \end{aligned}$$

with $C = \frac{1}{2}(c+1-c\log c)$.

Proof.

(i) The function

$$f(z) = \frac{\sqrt{(z-a)(z-b)}}{z}$$

is holomorphic in $\mathbb{C} \setminus ([a,b] \cup \{0\})$, with a simple pole at $z=0$ and residu $-\sqrt{ab}$. Therefore

$$\begin{aligned} [f] &= 2i\frac{\sqrt{(t-a)(b-t)}}{t}\chi(t) + 2i\pi\sqrt{ab}\delta_0 \\ &= i\pi\mu + 2i\pi(c-1)\delta_0, \end{aligned}$$

where χ is the indicator function of $[a,b]$. Furthermore

$$\lim_{|z|\to\infty}\big(f(z)-1\big) = 0.$$

By Proposition II.1.1, it follows that

$$G_\mu(z) = \frac{z-(c-1)-\sqrt{(z-a)(z-b)}}{2z}.$$

(ii) The proof is as in Examples 1 and 2, except the computation of the constant C which is not as simple. ∎

Example 4

For $-1<a<b<1$ consider the probability measure μ defined by

$$\int_{\mathbb{R}} f(t)\mu(dt) = \frac{1}{A}\int_a^b f(t)\frac{\sqrt{(t-a)(b-t)}}{1-t^2}dt,$$

with

$$A = \int_a^b \frac{\sqrt{(t-a)(b-t)}}{1-t^2}dt.$$

Define α, β by

$$\sqrt{(1-a)(1-b)} = \frac{2\alpha}{1+\alpha+\beta}, \quad \sqrt{(1+a)(1+b)} = \frac{2\beta}{1+\alpha+\beta}.$$

Proposition II.1.6. — (i)

$$A = \frac{\pi}{1+\alpha+\beta}.$$

(ii) *The Cauchy transform of μ is given, for $z\in\mathbb{C}\setminus[a,b]$, by*

$$\begin{aligned}G_\mu(z) &= \frac{1+\alpha+\beta}{\pi}\int_a^b \frac{1}{z-t}\frac{\sqrt{(t-a)(b-t)}}{1-t^2}dt\\ &= (1+\alpha+\beta)\frac{\sqrt{(z-a)(z-b)}}{z^2-1} - \frac{\alpha}{z-1} - \frac{\beta}{z+1}.\end{aligned}$$

(iii) *The logarithmic potential of μ is given, for $x\in]-1,1[$, by*

$$\begin{aligned}U^\mu(x) &= \alpha\log(1-x)+\beta\log(1+x)+C\\ &\quad+(1+\alpha+\beta)\int_x^a \frac{\sqrt{(a-t)(b-t)}}{1-t^2}dt, \quad \text{if } -1<x\le a,\\ &= \alpha\log(1-x)+\beta(1+x)+C, \quad \text{if } a\le x\le b,\\ &= \alpha\log(1-x)+\beta\log(1+x)+C\\ &\quad+(1+\alpha+\beta)\int_b^x \frac{\sqrt{(t-a)(t-b)}}{1-t^2}dt, \quad \text{if } b\le x<1.\end{aligned}$$

Proof.
Define, for $z \in \mathbb{C} \setminus ([a,b] \cup \{-1,1\})$,

$$f(z) = \frac{\sqrt{(z-a)(z-b)}}{z^2-1}.$$

The function f is holomorphic with poles in ± 1, and

$$\begin{aligned} \mathrm{Res}(f,1) &= \frac{1}{2}\sqrt{(1-a)(1-b)} = \frac{\alpha}{1+\alpha+\beta}, \\ \mathrm{Res}(f,-1) &= \frac{1}{2}\sqrt{(1+a)(1+b)} = \frac{\beta}{1+\alpha+\beta}. \end{aligned}$$

Hence the function

$$f_1(z) = f(z) - \frac{\alpha}{1+\alpha+\beta}\frac{1}{z-1} - \frac{\beta}{1+\alpha+\beta}\frac{1}{z+1}$$

is holomorphic in $\mathbb{C} \setminus [a,b]$, and

$$\begin{aligned} &[f_1] = -2iA\mu, \\ &f_1(z) \sim \frac{1}{1+\alpha+\beta}\frac{1}{z} \quad (|z| \to \infty). \end{aligned}$$

Therefore

$$G_\mu(z) = \frac{\pi}{A} f_1(z), \quad A = \frac{\pi}{1+\alpha+\beta}.$$

This proves (i). Part (ii) is proved as previously. ∎

2. Energy, equilibrium measure. — Let us first recall some basic facts about the tight topology. Let $\mathfrak{M}^1(\Sigma)$ be the set of probability measures on the closed set $\Sigma \subset \mathbb{R}$. We consider the tight topology. For this topology a sequence (μ_n) converges to a measure μ if, for every continuous bounded function f on Σ,

$$\lim_{n\to\infty} \int_\Sigma f(x)\mu_n(dx) = \int_\Sigma f(x)\mu(dx).$$

This topology is metrizable. If Σ is bounded, then $\mathfrak{M}^1(\Sigma)$ is compact.

Prokhorov Criterium *A subset $M \subset \mathfrak{M}^1(\Sigma)$ is relatively compact if and only if, for every $\varepsilon > 0$, there is a compact $K \subset \Sigma$ such that, for every $\mu \in M$,*

$$\mu(\Sigma \setminus K) \leq \varepsilon.$$

This criterium has the following useful consequence.

Let $M \subset \mathfrak{M}^1(\Sigma)$. Assume that there is a function h on Σ with

$$\lim_{|x|\to\infty} h(x) = \infty,$$

and a constant $C > 0$ such that, for every $\mu \in M$,

$$\int_\Sigma h(x)\mu(dx) \leq C.$$

Then M is relatively compact.

We return now to the logarithmic potential theory.

Let Σ be a closed interval ($\Sigma = \mathbb{R}$, $[a, \infty[$, $]-\infty, b]$ or $[a, b]$), and Q a function defined on Σ with values on $]-\infty, \infty]$, continuous on $\mathrm{int}(\Sigma)$. If Σ is unbounded, it is assumed that

$$\lim_{|x|\to\infty} \Big(Q(x) - \log(x^2 + 1)\Big) = \infty.$$

Some simple examples

- $\Sigma = \mathbb{R}$, $Q(x) = x^2$.
- $\Sigma = [-1, 1]$, $Q(x) = 0$,
- $\Sigma = [0, \infty[$, $Q(x) = x$.

If μ is a probability measure supported by Σ, the energy $E(\mu)$ of μ is defined by

$$\begin{aligned} E(\mu) &= \int_{\Sigma\times\Sigma} \log\frac{1}{|x-y|}\mu(dx)\mu(dy) + \int_\Sigma Q(x)\mu(dx) \\ &= \int_\Sigma U^\mu(x)\mu(dx) + \int_\Sigma Q(x)\mu(dx). \end{aligned}$$

Put

$$k(x, y) = \log\frac{1}{|x-y|} + \frac{1}{2}Q(x) + \frac{1}{2}Q(y).$$

From the inequality

$$|x - y| \leq \sqrt{x^2+1}\sqrt{y^2+1},$$

it follows that

$$k(x, y) \geq \big(Q(x) - \log(x^2 + 1)\big) + \big(Q(y) - \log(y^2 + 1)\big) \geq m,$$

with

$$m = \inf_{x \in \Sigma} \big(Q(x) - \log(x^2 + 1) \big).$$

Since we can write

$$E(\mu) = \int_{\Sigma \times \Sigma} k(x, y) \mu(dx) \mu(dy),$$

we obtain

$$m \leq E(\mu) \leq \infty.$$

PROPOSITION II.2.1. — *If (μ_n) is a sequence of probability measures supported by Σ which converges to a measure μ for the tight topology, then*

$$E(\mu) \leq \liminf_{n \to \infty} E(\mu_n).$$

This means that the map

$$\mathfrak{M}^1(\Sigma) \to]-\infty, \infty]$$

is lower semi-continuous.

Proof.

The cut kernel

$$k^\ell(x, y) = \inf\big(k(x, y), \ell\big)$$

at the level $\ell > 0$ is continuous and bounded, and $k^\ell(x, y) \leq k(x, y)$. For each n

$$\int_{\Sigma^2} k^\ell(x, y) \mu_n(dx) \mu_n(dy) \leq E(\mu_n).$$

Since $\mu_n \otimes \mu_n \to \mu \otimes \mu$,

$$\lim_{n \to \infty} \int_{\Sigma^2} k^\ell(x, y) \mu_n(dx) \mu_n(dy) = \int_{\Sigma^2} k^\ell(x, y) \mu(dx) \mu(dy).$$

Then take the $\liminf$:

$$\int_{\Sigma^2} k^\ell(x, y) \mu(dx) \mu(dy) \leq \liminf_{n \to \infty} E(\mu_n),$$

and, as $\ell \to \infty$, by the monotone convergence theorem,

$$E(\mu) = \int_{\Sigma^2} k(x, y) \mu(dx) \mu(dx) \leq \liminf_{n \to \infty} E(\mu_n). \qquad \square$$

Proposition II.2.2. — *Let μ be a signed measure on $\mathbb{R}$ with compact support and zero integral. Then*

$$\int_{\mathbb{R}^2} \log \frac{1}{|x-y|} \mu(dx)\mu(dy) = \int_0^\infty \frac{|\widehat{\mu}(t)|^2}{t} dt,$$

where $\widehat{\mu}$ is the Fourier transform of μ:

$$\widehat{\mu}(t) = \int_{\mathbb{R}} e^{itx} \mu(dx).$$

Proof.

(a) For $\varepsilon > 0$, define

$$F_\varepsilon(x) = \int_0^\infty e^{-\varepsilon t} \frac{1 - \cos tx}{t} dt.$$

Since

$$F_\epsilon'(x) = \int_0^\infty e^{-\varepsilon t} \sin tx dt = \frac{x}{\varepsilon^2 + x^2},$$

we obtain

$$F_\varepsilon(x) = \frac{1}{2} \log(\varepsilon^2 + x^2) + C.$$

Observing that $F_\varepsilon(0) = 0$, we get

$$C = -\frac{1}{2} \log \varepsilon^2 = -\log \varepsilon.$$

Finally

$$F_\varepsilon(x) = \log \sqrt{\varepsilon^2 + x^2} - \log \varepsilon.$$

(b) Let μ be a Radon measure on $\mathbb{R}$ with compact support and zero integral. Then

$$\hat{\mu}(0) = 0, \quad \hat{\mu}(-t) = \overline{\hat{\mu}(t)},$$

and

$$\int_{\mathbb{R}^2} \log\Big(\big((x-y)^2 + \varepsilon^2\big)^{\frac{1}{2}} \mu(dx)\mu(dy) = -\int_0^\infty e^{-\varepsilon t} \frac{|\hat{\mu}(t)|^2}{t} dt.$$

Decomposing μ as the difference of two positive measures: $\mu = \mu^+ - \mu^-$, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^2} \Big(\big((x-y)^2 + \varepsilon^2\big)^{-\frac{1}{2}}\Big) \big(\mu^+(dx)\mu^+(dy) + \mu^-(dx)\mu^-(dy)\big) \\
&= \int_{\mathbb{R}^2} \log\Big(\big((x-y)^2 + \varepsilon^2\big)^{-\frac{1}{2}}\Big) \big(\mu^+(dx)\mu^-(dy) + \mu^-(dx)\mu^+(dy)\big) \\
&+ \int_0^\infty e^{-\varepsilon t} \frac{|\hat{\mu}(t)|^2}{t} dt.
\end{aligned}$$

We apply to each integral the monotone convergence theorem as $\varepsilon \to 0$. Observe that

$$\log\Big(\big((x-y)^2+\varepsilon^2\big)^{-\frac{1}{2}}\Big) \nearrow \log\frac{1}{|x-y|},$$

and is bounded from below on the support of μ by

$$A = \log\frac{1}{R}, \quad \text{with } R = \sup_{x,y\in \mathrm{supp}(\mu)} \sqrt{(x-y)^2+1}.$$

Finally

$$\begin{aligned}
&\int_{\mathbb{R}^2} \log\frac{1}{|x-y|}\big(\mu^+(dx)\mu^+(dy)+\mu^-(dx)\mu^-(dy)\big)\\
&=\int_{\mathbb{R}^2} \log\frac{1}{|x-y|}\big(\mu^+(dx)\mu^-(dy)+\mu^-(dx)\mu^+(dy)\big)\\
&+\int_0^\infty \frac{|\hat{\mu}(t)|^2}{t}dt.
\end{aligned}$$

□

We have seen that $E(\mu)$ is bounded from below : $E(\mu)\geq m$. We define

$$E^* = \inf\{E(\mu) \mid \mu\in\mathfrak{M}^1(\Sigma)\}.$$

Then $E^*\geq m$. If $\mu(dx)=f(x)dx$, where f is a continuous function with compact support $\subset \mathrm{int}(\Sigma)$, the potential U^μ is a continuous function, and $E(\mu)<\infty$. Therefore

$$m\leq E^*<\infty.$$

Theorem II.2.3. — *There is a unique measure $\mu^*\in\mathcal{M}^1(\Sigma)$ such that*

$$E(\mu^*)=E^*.$$

The support of μ^* is compact.

This measure μ^* is called the *equilibrium measure.*

Proof.

a) *Existence*

By Proposition II.2.1, for $C>E^*$, the set

$$M_C=\{\mu\in\mathfrak{M}^1(\Sigma) \mid E(\mu)\leq C\}$$

is closed. If Σ is bounded, then $\mathfrak{M}^1(\Sigma)$ is compact, hence M_C is compact. If Σ is unbounded we will use Prokhorov Criterium for proving that M_C is compact. We have seen that

$$k(x,y)\geq \frac{1}{2}h(x)+\frac{1}{2}h(y).$$

Therefore

$$\int_\Sigma h(x)\mu(dx) \le E(\mu),$$

and, if $\mu \in M_C$,

$$\int_\Sigma h(x)\mu(dx) \le C.$$

The function $\mu \mapsto E(\mu)$ is lower semi-continuous on the compact M_C, therefore attains its infimum: there exists $\mu \in M_C$ such that $E(\mu) = E^*$.

b) Let $\mu \in \mathfrak{M}^1(\Sigma)$ such that $E(\mu) = E^*$. We will see that the support of μ is compact. Let A be a Borel set, and χ its characteristic function. Put

$$\mu_t = \frac{1+t\chi}{1+t\mu(A)}\mu.$$

For $-1 < t < 1$, $\mu_t \in \mathfrak{M}^1(\Sigma)$, and

$$E(\mu_t) = \int_{\Sigma^2} k(x,y)\frac{\big(1+t\chi(x)\big)\big(1+t\chi(y)\big)}{\big(1+t\mu(A)\big)^2}\mu(dx)\mu(dy).$$

Since $\mu_0 = \mu$, the minimum of $E(\mu_t)$ is attained at $t = 0$, hence

$$\frac{d}{dt}\Big|_{t=0} E(\mu_t) = 0.$$

Let us compute this derivative

$$\begin{aligned}\frac{d}{dt}\Big|_{t=0} E(\mu_t) = &\int_{\Sigma^2} k(x,y)\big(\chi(x)+\chi(y)\big)\mu(dx)\mu(dy)\\ &- 2\mu(A)\int_{\Sigma^2} k(x,y)\mu(dx)\mu(dy).\end{aligned}$$

By using the inequality

$$k(x,y) \ge \frac{1}{2}h(x) + \frac{1}{2}h(y),$$

we obtain

$$2\mu(A)E(\mu) \ge \int_A h(x)\mu(dx) + \mu(A)\int_\Sigma h(x)\mu(dx),$$

or

$$\int_A \Big(h(x) + \int_{\mathbb{R}} h(x)\mu(dx) - 2E(\mu)\Big)\mu(dy) \le 0.$$

Since $\lim_{|x|\to\infty} h(x) = \infty$, there exists $\alpha > 0$ such that, if $|y| > \alpha$,

$$h(y) + \int_{\mathbb{R}} h(x)\mu(dx) - 2E(\mu) > 0.$$

Take $A = \mathbb{R} \setminus [-\alpha, \alpha]$, then $\mu(A) = 0$. Hence the support of μ is contained in $[-\alpha, \alpha]$, therefore compact.

c) *Uniqueness*

We will see that the function $\mu \mapsto E(\mu)$, defined on the set $\mathfrak{M}_c^1(\Sigma)$ of probability measures supported by Σ with compact support, is strictly convex. In fact, for μ_0, $\mu_1 \in \mathfrak{M}_c^1(\Sigma)$, ($\mu_0 \neq \mu_1$), put

$$\mu_t = (1-t)\mu_0 + t\mu_1.$$

Then

$$E(\mu_t) = at^2 + bt + c,$$

with

$$\begin{aligned} a &= \int_{\Sigma^2} \log \frac{1}{|x-y|} \nu(dx)\nu(dy) \quad (\nu = \mu_1 - \mu_2), \\ b &= \int_{\Sigma} \big(U^{\mu}(x) + Q(x)\big)\nu(dx), \\ c &= I(\mu_0). \end{aligned}$$

By Proposition II.2.2, a is > 0, therefore the function $t \mapsto E(\mu_t)$ is strictly convex: for $0 < t < 1$,

$$E(\mu_t) < (1-t)E(\mu_0) + tE(\mu_1).$$

This implies uniqueness. ∎

It can be useful to observe the action of a linear transformation:

Lemma II.2.4. — *Let the transformation $h(s) = as + b$ map Σ onto Σ'. If Q is defined on Σ', then $Q \circ h$ is defined on Σ. If μ is a probability measure μ on Σ, then $\nu = h(\mu)$ is the probability measure on Σ' defined by*

$$\int_{\Sigma'} f(t)\nu(dt) = \int_{\Sigma} f\big(h(s)\big)\mu(ds).$$

Then

$$E_{(\Sigma', Q)}\big(h(\mu)\big) = E_{(\Sigma, Q\circ h)}(\mu) - \log|a|.$$

3. Determination of the equilibrium measure via a variational problem. — The following statement, which is not the best possible, will be useful for the examples we have in mind.

Proposition II.3.1. — *Let $\mu \in \mathfrak{M}^1(\Sigma)$ with compact support. Assume that the potentiel U^μ of μ is continuous and that there is a constant C such that*

(i) $U^\mu(x) + \frac{1}{2}Q(x) \geq C$ on Σ,

(ii) $U^\mu(x)+\frac{1}{2}Q(x) = C$ on $\mathrm{supp}(\mu)$. Then μ is the equilibrium measure: $\mu = \mu^*$.

The constant C is called the *(modified) Robin constant*. Observe that

$$E^* = C + \frac{1}{2}\int_\Sigma Q(x)\mu^*(dx).$$

Proof.

For two measures μ and ν we can write

$$E(\mu+\nu) = E(\mu)+2\int_\Sigma \Big(U^\mu(x)+\frac{1}{2}Q(x)\Big)\nu(dx)+\int_{\Sigma^2} \log\frac{1}{|x-y|}\nu(dx)\nu(dy).$$

Writing $\mu^* = \mu + (\mu^* - \mu)$, we obtain

$$\begin{aligned} E(\mu^*) = E(\mu) &+ 2\int_\Sigma \Big(U^\mu(x) + \frac{1}{2}Q(x)\Big)(\mu^* - \mu)(dx) \\ &+ \int_{\Sigma^2} \log\frac{1}{|x-y|}(\mu^* - \mu)(dx)(\mu^* - \mu)(dy). \end{aligned}$$

By the hypothesis,

$$\int_\Sigma \Big(U^\mu(x) + \frac{1}{2}Q(x)\Big)\mu^*(dx) \geq C,$$
$$\int_\Sigma \Big(U^\mu(x) + \frac{1}{2}Q(x)\Big)\mu(dx) = C.$$

By Proposition II.2.2,

$$\int_{\Sigma^2} \log\frac{1}{|x-y|}(\mu^* - \mu)(dx)(\mu^* - \mu)(dx) \geq 0.$$

Therefore $E(\mu^*) \geq E(\mu)$, which implies that $\mu = \mu^*$. ∎

Examples

The weight corresponding to the potential Q is defined as

$$w(x) = e^{-Q(x)}.$$

1) *Legendre weight*

$$\Sigma = [-1, 1], \ Q(x) = 0, \ w(x) = 1.$$

Let μ be the arcsinus law. We saw in Section 1 that $U^\mu(x) = \log 2$ on $[-1, 1]$. Therefore μ is the equilibrium measure, $\mu = \mu^*$, and $E^* = C = \log 2$.

2) *Gaussian weight*

$$\Sigma = \mathbb{R}, \ Q(x) = x^2, \ w(x) = e^{-x^2}.$$

The equilibrium measure is the semi-circular law μ of radius $\sqrt{2}$:

$$\int_{\mathbb{R}} f(t)\mu(dt) = \frac{1}{\pi}\int_{-\sqrt{2}}^{\sqrt{2}} f(t)\sqrt{2 - t^2}dt.$$

In fact

$$U^\mu(x) + \frac{1}{2}x^2 \geq C \quad (x \in \mathbb{R}),$$
$$U^\mu(x) + \frac{1}{2}x^2 = C \quad (x \in [-\sqrt{2}, \sqrt{2}]),$$

with $C = \frac{1}{2} + \frac{1}{2}\log 2$, and

$$E^* = C + \frac{1}{2}\int_{\mathbb{R}} Q(t)\mu(dt) = \frac{3}{4} + \frac{1}{2}\log 2.$$

3) *Laguerre weight*

$$\Sigma = [0, \infty[, \ Q(x) = x + \alpha \log\frac{1}{x}, \ w(x) = e^{-x}x^\alpha,$$

with $\alpha \geq 0$. The equilibrium measure μ is the Marchenko-Pastur law with $c = 1 + \alpha$:

$$\int_{\mathbb{R}} f(x)\mu(dx) = \frac{1}{2\pi}\int_0^\infty f(x)\sqrt{(x-a)(b-x)}\frac{dx}{x},$$

with

$$a = (\sqrt{\alpha + 1} - 1)^2, \ b = (\sqrt{\alpha + 1} + 1)^2.$$

4) *Jacobi weight*

$$\Sigma = [-1, 1],$$
$$Q(x) = p \log \frac{1}{1-x} + q \log \frac{1}{1-x}, .$$
$$w(x) = (1-x)^p (1+x)^q.$$

The equilibrium measure is given by

$$\int_{\mathbb{R}} f(x) \mu^*(dx) = \frac{1}{\pi}(1+\alpha+\beta) \int_a^b f(x) \frac{\sqrt{(x-a)(b-x)}}{1-x^2} dx,$$

with $\alpha = \frac{p}{2}$, $\beta = \frac{q}{2}$. Recall that a and b are determined by

$$\sqrt{(1-a)(1-b)} = \frac{2\alpha}{1+\alpha+\beta}, \quad \sqrt{(1+a)(1+b)} = \frac{2\beta}{1+\alpha+\beta}.$$

5) *Freud weight*

$$\Sigma = \mathbb{R},\ Q(x) = c|x|^\alpha,\ w(x) = e^{-c|x|^\alpha},$$

with $c > 0, \alpha > 0$. The equilibrium measure is a Ullman distribution. There is a choice of $c = c_\alpha$ such that

$$\int_{\mathbb{R}} f(x) \mu^*(dx) = \int_{-1}^{1} f(x) h_\alpha(x) dx,$$

with

$$h_\alpha(x) = \frac{\alpha}{\pi} \int_{|x|}^{1} \frac{t^{\alpha-1}}{\sqrt{t^2 - x^2}} dt.$$

For $\alpha = 2$, this is the semi-circle law:

$$h_2(x) = \frac{2}{\pi} \sqrt{1-x^2}.$$

See [Saff-Totik,1997], Chapter IV, Theorem 5.1.

We have seen how to check that a probabillity measure μ is the equilibrium measure.

Data : Σ a closed interval, and Q a continuous function on the interior of Σ such that

$$\lim_{|t|\to\infty} \big(Q(t) - \log(1+t^2)\big) = \infty.$$

Steps

(1) Determine the Cauchy transform of μ:

$$G_\mu(z) = \int_\Sigma \frac{1}{z-t}\mu(dt).$$

In several examples we have used the residue theorem.

(2) Determine the logarithmic potential of μ:

$$U_\mu(x) = \int_\Sigma \log\frac{1}{|x-t|}\mu(dt),$$

by using the relation

$$\frac{d}{dx}U^\mu(x) = -\mathrm{Re}\, G_\mu(x).$$

Hence the logarithmic potential is determined up to a constant.

(3) Check that there is a constant C such that

$$U^\mu(x) + \frac{1}{2}Q(x) \geq C \quad \text{for } x \in \Sigma,$$
$$U^\mu(x) + \frac{1}{2}Q(x) = C \quad \text{for } x \in \mathrm{supp}(\mu).$$

If it holds, then we know that μ is the equilibrium measure: $\mu = \mu^*$.

(4) Determine the constant C by using

$$\lim_{|x|\to\infty}\big(U^\mu(x) + \log x\big) = 0.$$

Then the equilibrium energy is

$$E^* = E(\mu) = C + \frac{1}{2}\int_\Sigma Q(x)\mu(dx).$$

We end this section with the Pastur Formula.

Proposition II.3.2. — *Let $\Sigma = \mathbb{R}$, and Q a polynomial of even degree $2k$ $(k \geq 1)$, convex. Then the equilibrium μ^* is given by*

$$\int_\mathbb{R} f(x)\mu^*(dx) = \frac{1}{\pi}\int_a^b f(x)q(x)\sqrt{(x-a)(b-x)}dx,$$

where q is the polynomial of degree $2k-2$ given by

$$q(x)=\frac{1}{2\pi}\int_a^b \frac{Q'(x)-Q'(t)}{x-t}\frac{dt}{\sqrt{(t-a)(b-t)}}.$$

The numbers a and b are determined by the conditons

$$\int_a^b \frac{Q'(t)}{\sqrt{(t-a)(b-t)}}dt=0, \quad \int_a^b \frac{tQ'(t)}{\sqrt{(t-a)(b-t)}}dt=2\pi.$$

As a special case we obtain Example 2 of this section: $Q(x)=x^2$. Then

$$\frac{Q'(z)-Q'(t)}{z-t}=2.$$

Hence

$$q(z)=\frac{1}{\pi}\int_a^b \frac{dt}{\sqrt{(t-a)(b-t)}}=1.$$

The numbers a and b are determined by

$$\int_a^b \frac{2t}{\sqrt{t-a)(b-t)}}dt=0, \quad \int_a^b \frac{2t^2}{\sqrt{(t-a)(b-t)}}dt=2\pi.$$

The first equation gives $a+b=0$, and the second $a^2=b^2=2$. Therefore μ^* is the semi-circle law of radius $\sqrt{2}$.

Proof.

Consider a measure μ of the form

$$\mu(dt)=u(t)dt,$$

where u is a continuous function with support $[a,b]$, and let G be its Cauchy transform: for $z\in\mathbb{C}\setminus[a,b]$,

$$G(z):=\int_a^b \frac{u(t)}{z-t}dt.$$

By Proposition II.3.1, if μ is the equilibrium measure, then, for $a\le x\le b$,

$$\mathrm{Re}\ G(x)=-\frac{1}{2}Q'(x).$$

Define

$$\tilde{G}(z) = \frac{G(z)}{\sqrt{(z-a)(z-b)}}.$$

The function $\tilde{G}$ is holomorphic for $z \in \mathbb{C} \setminus [a, b]$, and

$$[\tilde{G}] = -i \frac{Q'(t)}{\sqrt{(t-a)(b-t)}} \chi(t),$$

where χ is the indicator function of $[a, b]$. By using Liouville's theorem as in the proof of Proposition II.1.1, we obtain

$$\tilde{G}(z) = \frac{1}{2\pi} \int_a^b \frac{1}{z-t} \frac{Q'(t)}{\sqrt{(t-a)(b-t)}} dt.$$

This can be written

$$\tilde{G}(z) = \frac{1}{2\pi} \int_a^b \frac{Q'(t) - Q'(z)}{z-t} \frac{dt}{\sqrt{(t-a)(b-t)}} dt$$
$$+ Q'(z) \frac{1}{2\pi} \int_a^b \frac{1}{z-t} \frac{dt}{\sqrt{(t-a)(b-t)}}.$$

By Proposition II.1.2,

$$\frac{1}{\pi} \int_a^b \frac{1}{z-t} \frac{dt}{\sqrt{(t-a)(b-t)}} = \frac{1}{\sqrt{(z-a)(z-b)}}.$$

Hence

$$\tilde{G}(z) = -q(z)\sqrt{(z-a)(z-b)} + \frac{Q'(z)}{2\sqrt{(z-a)((z-b)}},$$

or

$$G(z) = -q(z)\sqrt{(z-a)(z-b)} + \frac{1}{2} Q'(z).$$

This implies, by the relation $[G] = -2i\pi\mu$ (Proposition II.1.1), that

$$u(t) = q(t)\sqrt{(t-a)(b-t)}\chi(t).$$

Let us consider the Laurent expansion of $\tilde{G}(z)$:

$$\tilde{G}(z) = \frac{a_0}{z} + \frac{a_1}{z^2} + \cdots$$

Then

$$\begin{aligned}
G(z) &= \tilde{G}(z)\sqrt{(z-a)(z-b)} \\
&= \Big(\frac{a_0}{z} + \frac{a_1}{z^2} + \cdots\Big)\Big(z - \frac{a+b}{2} - \frac{(a-b)^2}{8}\frac{1}{z} + \cdots\Big) \\
&= a_0 + \Big(a_1 - a_0\frac{a+b}{2}\Big)\frac{1}{z} + \cdots
\end{aligned}$$

Since $\lim_{z\to\infty} zG(z) = 1$, we get

$$\begin{aligned}
a_0 &= \frac{1}{2\pi}\int_a^b \frac{Q'(t)}{\sqrt{(t-a)(b-t)}}dt = 0, \\
a_1 &= \frac{1}{2\pi}\int_a^b \frac{tQ't)}{\sqrt{(t-a)(b-t)}}dt = 1.
\end{aligned}$$

By the way we used previously we get

$$\begin{aligned}
\frac{d}{dx}U^\mu(x) + \frac{1}{2}Q'(x) &= -q(x)\sqrt{(a-x)(b-x)}, \text{ if } x < a, \\
&= 0, \text{ if } a \le x \le b, \\
&= q(x)\sqrt{(x-a)(x-b)}, \text{ if } x > b.
\end{aligned}$$

Therefore there is a constant C such that

$$\begin{aligned}
U^\mu(x) + \frac{1}{2}Q'(x) &= C, \text{ if } a \le x \le b, \\
&\ge C \text{ everywhere.}
\end{aligned}$$

By Proposition II.3.1 this shows that μ is actually the equilibrium measure.

$\square$

References

[Saff-Totik,1997]; [Dyson,1962]; [Deift,1998], Chapter 6; [Anderson-Guionnet-Zeitouni,2010], Chapter 2; [Pastur-Shcherbina,2010], Part 1.

A possible reference for the classical potential theory: [Helms,2009].

Chapter III

STATISTICS OF THE ZEROS OF CLASSICAL ORTHOGONAL POLYNOMIALS

1. Statistic of the zeros of Jacobi polynomials. — We have considered in Chapter I the energy

$$E_n(x_1,\ldots,x_n)=\sum_{i\neq j}\log\frac{1}{|x_i-x_j|}+\sum_{i=1}^{n}Q(x_i),$$

on $\Sigma=[-1,1]$, with

$$Q(t)=(\alpha+1)\log\frac{1}{1-t}+(\beta+1)\log\frac{1}{1+t}.$$

The function Q corresponds to the Jacobi weight:

$$e^{-Q(t)}=(1+t)^{\alpha+1}(1-t)^{\beta+1}.$$

We saw that the coordinates of the point $x^*=(x_1^*,\ldots,x_n^*)$ which minimizes the energy are the zeros of the Jacobi polynomial $P_n^{(\alpha,\beta)}$ (Corollary I.3.5). The minimum E_n^* of the energy,

$$E_n^*=E_n(x_1^*,\ldots,x_n^*)=\inf_{x\in\Sigma^n}E_n(x_1,\ldots,x_n)$$

is such that

$$\exp(-E_n^*)=\exp\Bigl(-\sum_{i=1}^{n}Q(x_i^*)\Bigr)D(P_n^{(\alpha,\beta)}),$$

where $D(P)$ denotes the discriminant of the polynomial P.

Let $x_1^{(n)},\ldots,x_n^{(n)}$ denote the zeros of $P_n^{(\alpha,\beta)}$ and define the probability measure on $[-1,1]$

$$M_n=\frac{1}{n}\sum_{i=1}^{n}\delta_{x_i^{(n)}}.$$

We have considered in Chapter II the energy of a probability measure $\mu\in\mathfrak{M}^1(\Sigma)$:

$$E(\mu)=\int_{\Sigma^2}\log\frac{1}{|s-t|}\mu(ds)\mu(dt).$$

The equilibrium measure is the arcsinus law, and the minimum of the energy

$$E^* = \inf_{\mu \in \mathfrak{M}^1(\Sigma)} E(\mu)$$

is equal to $\log 2$.

Theorem III.1.1. — *The measure M_n converges to the arcsinus law for the tight topolygy. This means that, for a continuous function f on $[-1, 1]$,*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} f(x_i^{(n)}) = \frac{1}{\pi} \int_{-1}^{1} f(t) \frac{dt}{\sqrt{1-t^2}}.$$

Furthermore

$$\lim_{n\to\infty} \frac{1}{n^2} E_n^* = E^* \quad (= \log 2).$$

Proof.
We saw in Chapter II, Section 3, that the equilibrium measure μ^*, which realizes the minimum E^* of the energy

$$E(\mu) = \int_{[-1,1]^2} \log \frac{1}{|s-t|} \mu(ds)\mu(dt),$$

is the arcsinus law. Define

$$w_n = \frac{1}{n(n-1)} \inf_{x \in [-1,1]^n} E_n(x).$$

For $\mu \in \mathcal{M}^1([-1, 1])$,

$$\begin{aligned} &\int_{[-1,1]^n} E_n(x_1, \ldots, x_n)\mu(dx_1)\ldots\mu(dx_n) \\ &= n(n-1)E(\mu) + n \int_{[-1,1]} Q(t)\mu(dt), \end{aligned}$$

and, for $\mu = \mu^*$, we get

$$w_n \le E^* + \frac{1}{n-1} \int_{[-1,1]} Q(t)\mu^*(dt),$$

and

$$\limsup_{n\to\infty} w_n \le E^*.$$

Consider, for $\ell > 0$, the cut kernel

$$k^\ell(s,t) = \inf\Big(\log \frac{1}{|s-t|}, \ell\Big),$$

and the corresponding energy of a measure

$$E^\ell(\mu) = \int_{[-1,1]^2} k^\ell(s,t)\mu(ds)\mu(dt).$$

Then

$$\begin{aligned} E^\ell(M_n) &= \frac{1}{n^2}\sum_{i,j=1}^{n} k^\ell(x_i^{(n)}, x_j^{(n)}) \\ &\le \frac{1}{n^2}\sum_{i\neq j}\log\frac{1}{|x_i^{(n)} - x_j^{(n)}|} + \frac{\ell}{n} \\ &\le \frac{1}{n^2}\big(E_n(x^{(n)}) - \sum_{i=1}^{n} Q(x_i^{(n)})\big) + \frac{\ell}{n} \\ &\le \frac{n-1}{n}w_n - \frac{\gamma}{n} + \frac{\ell}{n}, \end{aligned}$$

where

$$\gamma = \inf_{-1<t<1} Q(t) \ge -(\alpha+\beta+2)\log 2.$$

Since $\mathcal{M}^1([-1,1])$ is compact for the tight topology, there is a converging subsequence:

$$\lim_{j\to\infty} M_{n_j} = \mu_0.$$

We obtain

$$E^\ell(\mu_0) \le \liminf_{j\to\infty} w_{n_j},$$

and, as $\ell \to \infty$, by the monotone convergence theorem,

$$E(\mu_0) \le \liminf_{j\to\infty} w_{n_j}.$$

Thefore

$$E^* \le E(\mu_0) \le \liminf_{j\to\infty} w_{n_j} \le \limsup_{j\to\infty} w_{n_j} \le E^*.$$

Hence $E(\mu_0) = E^*$. This implies that $\mu_0 = \mu^*$. We have proved that μ^* is the only possible limit for a subsequence of the sequence (M_n). It follows that the sequence (M_n) itself converges:

$$\lim_{n\to\infty} M_n = \mu^*.$$

Furthermore

$$\lim_{n\to\infty} w_n = E^*. \qquad \square$$

2. Statistics of the zeros of Hermite polynomials. — Now we consider the case $\Sigma = \mathbb{R}$, $Q(t) = t^2$ then the energy is given by

$$E_n(x) = \sum_{i \neq j} \log \frac{1}{|x_i - x_j|} + \sum_{i=1}^{n} Q(x_j).$$

By Corollary I.3.4 the minimum of the energy is attained at the $n!$ points whose coordinates are the zeros $x_1^{(n)}, \ldots, x_n^{(n)}$ of the Hermite polynomial H_n. Define the probability measure M_n on $\mathbb{R}$ by

$$M_n = \frac{1}{n} \sum_{i=1}^{n} \delta_{x_i^{(n)}}.$$

Consider first the two first moments of M_n. The expansion of the Hermite polynomial is given by

$$\begin{aligned} H_n(x) &= n! \sum_{k=0}^{[\frac{n}{2}]} (-1)^k \frac{(2x)^{n-2k}}{k!(n-2)!} \\ &= 2^n \Big(x^n - \frac{n(n-1)}{4} x^{n-2} + \cdots \Big). \end{aligned}$$

By using the classical relations between the coefficients and the zeros of a polynomial we obtain

$$\begin{aligned} \sum_{i=1}^{n} x_i^{(n)} &= 0, \\ \sum_{i<j} x_i^{(n)} xj^{(n)} &= -\frac{n(n-1)}{4}, \end{aligned}$$

and

$$\sum_{i=1}^{n} \big(x_i^{(n)}\big)^2 = \Big(\sum_{i=1}^{n} x_i^{(n)} \Big)^2 - 2 \sum_{i<j} x_i^{(n)} x_j^{(n)} = \frac{n(n-1)}{2}.$$

Therefore

$$\begin{aligned} m_1 &= \int_{\mathbb{R}} t M_n(dt) = \frac{1}{n} \sum_{i=1}^{n} x_i^{(n)} = 0, \\ m_2 &= \int_{\mathbb{R}} t^2 M_n(dt) = \frac{1}{n} \sum_{i=1}^{n} \big(x_i^{(n)}\big)^2 = \frac{n-1}{2}. \end{aligned}$$

This suggests that M_n does not converge and that a rescaling is necessary for getting a convergence. Put

$$\widetilde{M_n} = \frac{1}{n}\sum_{i=1}^{n} \delta_{\alpha_i^{(n)}}, \quad \alpha_i^{(n)} = \frac{1}{\sqrt{n}} x_i^{(n)}.$$

Similarly we rescale the energy E_n: define

$$\begin{aligned}\widetilde{E_n}(x) &= \sum_{i\neq j} \log \frac{1}{|x_i - x_j|} + \sum_{i=1}^{n} Q(\sqrt{n}x_i) \\ &= \sum_{i\neq j} \log \frac{1}{|x_i - x_j|} + n\sum_{i=1}^{n} Q(x_i).\end{aligned}$$

The minimum $\tilde{E}_n^*$ of $\tilde{E}_n(x)$ is attained at the $n!$ points whose coordinates are the numbers $\alpha_i^{(n)}$.

For a measure $\mu \in \mathfrak{M}^1(\mathbb{R})$, we consider the energy

$$E(\mu) = \int_{\mathbb{R}^2} \log \frac{1}{|s-t|}\mu(ds)\mu(dt) + \int_{\mathbb{R}} Q(t)\mu(dt).$$

We saw in Chapter II, Section 3 that the equilibrium measure μ^* which realizes the minimum E^* of the energy is the semi-circle law of radius $\sqrt{2}$, and

$$E^* = \inf_{\mu\in\mathfrak{M}^1(\mathbb{R})} E(\mu) = \frac{3}{4} + \frac{1}{2}\log 2.$$

THEOREM III.2.1. — *The measure $\widetilde{M_n}$ converges to the semi-circle law with radius $\sqrt{2}$ for the tight topology. This means that, for every bounded continuous function f on $\mathbb{R}$,*

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} f\Big(\frac{1}{\sqrt{n}} x_i^{(n)}\Big) = \frac{1}{\pi}\int_{-\sqrt{2}}^{\sqrt{2}} f(t)\sqrt{2-t^2}dt.$$

Furthermore

$$\lim_{n\to\infty} \frac{1}{n^2}\widetilde{E_n^*} = E^* = \frac{3}{4} + \frac{1}{2}\log 2.$$

Proof.

As in Section 1 define

$$w_n = \frac{1}{n(n-1)} \inf_{x\in\mathbb{R}^n} \widetilde{E_n}(x).$$

For $\mu \in \mathfrak{M}^1(\mathbb{R})$,

$$\int_{\mathbb{R}^n} \widetilde{E_n}(x_1, \dots, x_n)\mu(dx_1)\dots\mu(dx_n) = n(n-1)E(\mu) + n\int_{\mathbb{R}} Q(t)\mu(dt).$$

For $\mu = \mu^*$, we get

$$w_n \leq E^* + \frac{1}{n-1}\int_{\mathbb{R}} Q(t)\mu^*(dt),$$

and

$$\limsup_{n\to\infty} w_n \leq E^*.$$

Recall the notation

$$k(s,t) = \log\frac{1}{|s-t|} + \frac{1}{2}Q(s) + \frac{1}{2}Q(t),$$

and that

$$k(s,t) \geq \frac{1}{2}h(s) + \frac{1}{2}h(t),$$

with $h(t) = Q(t) - \log(1+t^2)$. Observe that

$$\sum_{i\neq j} k(x_i, x_j) = \sum_{i\neq j}\log\frac{1}{|x_i - x_j|} + (n-1)\sum_{i=1}^{n} Q(x_i).$$

Hence, since $Q(t) \geq 0$,

$$\begin{aligned}\widetilde{E_n}(x) &= \sum_{i\neq j} k(x_i,x_j) + \sum_{i=1}^{n} Q(x_i)\\ &\geq \frac{1}{2}\sum_{i\neq j}\big(h(x_i) + h(x_j)\big) = (n-1)\sum_{i=1}^{n} h(x_i).\end{aligned}$$

It follows that

$$\widetilde{E_n}\big(\alpha_1^{(n)}, \dots, \alpha_n^{(n)}\big) \geq n(n-1)\int_{\mathbb{R}} h(t)\widetilde{M_n}(dt),$$

Hence

$$\int_{\mathbb{R}} h(t)\widetilde{M_n}(dt) \leq w_n \leq E^* + \frac{1}{n-1}\int_{\mathbb{R}} Q(t)\mu^*(dt).$$

Since the right handside is bounded, by Prokhorov criterium this proves that the sequence $(\widetilde{M_n})$ is relatively compact for the tight topology. Therefore there is a converging subsequence:

$$\lim_{j\to\infty} \widetilde{M_{n_j}} = \mu_0.$$

Consider, for $\ell > 0$, the cut kernel

$$k^\ell(s,t) = \inf\bigl(k(s,t), \ell\bigr),$$

and define

$$E^\ell(\mu) = \int_{\mathbb{R}^2} k^\ell(s,t)\mu(ds)\mu(dt).$$

Observing that $Q(t) \geq 0$, we get

$$E^\ell(\widetilde{M_n}) \leq \frac{n-1}{n} w_n + \frac{\ell}{n}.$$

As $j \to \infty$ we get

$$E^\ell(\mu_0) \leq \liminf_{j\to\infty} w_{n_j},$$

and, by the monotone convergence theorem,

$$\lim_{\ell\to\infty} E^\ell(\mu_0) = E(\mu_0).$$

Therefore

$$E^* \leq E(\mu_0) \leq \liminf_{j\to\infty} w_{n_j} \leq \limsup_{j\to\infty} w_{n_j} \leq E^*.$$

The proof finishes as the one of Theorem III.1.1. $\square$

Remark

One establishes easily that

$$\tilde{E}_n^* = E_n^* + \frac{n(n-1)}{2}\log n.$$

Hence, by Theorem III.2.1,

$$\frac{1}{n^2}E_n^* = -\frac{1}{2}\log n + \Bigl(\frac{3}{4} + \frac{1}{2}\log 2\Bigr) + o(1).$$

This can also be obtained directly from the result we saw in Chapter I (Corollary I.3.8):

$$E_n^* = \frac{n(n-1)}{2}(\log 2 + 1) - \sum_{k=2}^{n} k \log k.$$

Lemma III.2.2.

$$\lim_{n\to\infty}\Big(\sum_{k=1}^{n} k\log k - \frac{1}{2}\log n\Big) = -\frac{1}{4}.$$

Proof.
In fact, by an elementary property of the Riemann integral of a continuous function,

$$\int_0^1 x \log x dx = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \frac{k}{n}\log\frac{k}{n},$$

and

$$\int_0^1 x \log x dx = -\frac{1}{4}. \qquad \square$$

It follows that

$$\begin{aligned}\frac{1}{n^2}E_n^* + \frac{1}{2}\log n &= \frac{1}{2} + \frac{1}{2}\log 2 + \frac{1}{2}\log n - \sum_{k=2}^{n} k\log k + o(1)\\ &= \frac{1}{2} + \frac{1}{2}\log 2 + \frac{1}{4} + o(1) = \frac{3}{4} + \frac{1}{2}\log 2 + o(1).\end{aligned}$$

3. Statistics of the zeros of Laguerre polynomials. — Let $x_1^{(n)}, \dots, x_n^{(n)}$ be the zeros of the Laguerre polynomial $L_n^{(\alpha-1)}$ ($\alpha > 0$). Define on $\mathbb{R}_+^n$ the energy

$$E_n(x_1, \cdots, x_n) = \sum_{i\neq j} \log\frac{1}{|x_i - x_j|} + \sum_{i=1}^{n} Q(x_i),$$

with

$$Q(t) = t + \alpha\log\frac{1}{t}.$$

The minimum of the energy E_n is attained at the $n!$ points whose coordinates are the zeros $x_1^{(n)}, \dots, x_n^{(n)}$ of the Laguerre polynomial $L_n^{(\alpha-1)}$. Define the probability measure

$$M_n = \frac{1}{n} \sum_{i=1}^{n} \delta_{x_i^{(n)}}.$$

As we did for the Hermite polynomials, we compute the two first moments of the measure M_n:

$$m_1 = \int_{\mathbb{R}+} t M_n(dt) \sim n, \quad m_2 = \int_{\mathbb{R}_+} t^2 M(dt) \sim cn^2.$$

This suggests that we should rescale M_n: define

$$\widetilde{M_n} = \frac{1}{n} \sum_{i=1}^{n} \delta_{\alpha_i^{(n)}}, \quad \text{with } \alpha_i^{(n)} = \frac{1}{n} x_i^{(n)},$$

and

$$\widetilde{E_n}(x_1, \dots, x_n) = \sum_{i \neq j} \log \frac{1}{|x_i - x_j|} + \sum_{i=1}^{n} Q(nx_i).$$

The minimum $\widetilde{E_n}^*$ of $\widetilde{E_n}$ is attained to the $n!$ points whose coordinates are the numbers $\alpha_i^{(n)}$. As in previous sections define

$$w_n = \frac{1}{n(n-1)} \widetilde{E_n}^*.$$

THEOREM III.3.1. — *The measure $\widetilde{M_n}$ converges to the Marchenko-Pastur law with $c = 1$ for the tight topology. This means that, for every bounded continuous function f on $\mathbb{R}_+$,*

$$\lim_{n\to\infty} \sum_{i=1}^{n} f\left(\frac{1}{n} x_i^{(n)}\right) = \frac{1}{2\pi} \int_0^4 f(t) \sqrt{\frac{4-t}{t}} dt.$$

Proof. We define the energy of a probability measure $\mu \in \mathcal{M}^1(\mathbb{R}_+)$ as

$$E_0(\mu) = \int_{\mathbb{R}_+^2} \log \frac{1}{|s-t|} \mu(ds)\mu(dt) + \int_{\mathbb{R}_+} Q_0(t)\mu(dt),$$

with $Q_0(t) = t$. We saw in Chapter II that the equilibrium measure is the Marchenko-Pastur, with $c = 1$.

For $\mu \in \mathcal{M}^1(\mathbb{R}_+)$,

$$\begin{aligned}&\int_{\mathbb{R}_+^n} \widetilde{E_n}(x_1, \dots, x_n)\mu(dx_1)\dots\mu(dx_n)\\&= n(n-1)\int_{\mathbb{R}_+^2} \log\frac{1}{|s-t|}\mu(ds)\mu(dt) + n\int_{\mathbb{R}_+} Q(nt)\mu(dt).\end{aligned}$$

Since

$$Q(nt) = nt + \alpha\log\frac{1}{t} - \alpha\log n,$$

we get

$$\begin{aligned}&\int_{\mathbb{R}_+^n} \widetilde{E_n}(x_1, \dots, x_n)\mu(dx_1)\dots\mu(dx_n)\\&= n(n-1)E_0(\mu) + n\int_{\mathbb{R}_+} Q(t)\mu(dt) - \alpha n\log n.\end{aligned}$$

It follows that

$$w_n \le E_0^* + \frac{1}{n-1}\int_{\mathbb{R}_+} Q(t)\mu(dt), \tag{3.1}$$

and $\limsup_{n\to\infty} w_n \le E_0^*$. With the notation

$$\begin{aligned}k_0(s,t) &= \log\frac{1}{|s-t|} + \frac{1}{2}Q_0(s) + \frac{1}{2}Q_0(t),\\h_0(t) &= Q_0(t) - \log(1+t^2),\end{aligned}$$

we get

$$\begin{aligned}\widetilde{E_n}(x_1, \dots, x_n) &= \sum_{i\neq j} k_0(x_i, x_j) + \sum_{i=1}^n Q(x_i) - \alpha n\log n\\&\ge \frac{1}{2}\sum_{i\neq j}\big(h(x_i) + h(x_j)\big) - \alpha n\log n.\end{aligned}$$

Therefore

$$\widetilde{E_n}(\alpha_1^{(n)}, \dots, \alpha_n^{(n)}) \ge n(n-1)\int_{\mathbb{R}_+} h_0(t)\widetilde{M_n}(dt) - \alpha n\log n,$$

and

$$\int_{\mathbb{R}_+} h_0(t)\widetilde{M_n}(dt) \le w_n + \alpha\frac{\log n}{n-1} \le E_0^* + \frac{1}{n-1}\int_{\mathbb{R}_+} Q(t)\mu^*(dt) + \alpha\frac{\log n}{n-1},$$

by using (3.1). Since the right hand side is bounded, by Prokhorov Criterium this proves that the sequence $(\widetilde{M_n})$ is relatively compact. The proof finishes as the ones of Theorems III.1.1 and III.2.1. $\square$

Remark

The Laguerre polynomials $L_n^{-\frac{1}{2}}$ and $L_n^{\frac{1}{2}}$ are related to Hermite polynomials:

$$H_{2n}(x) = (-1)^n 2^{2n} n!\ L_n^{-\frac{1}{2}}(x^2),$$

$$H_{2n+1}(x) = (-1)^n 2^{2n+1} n!\ x L_n^{\frac{1}{2}}(x^2).$$

Hence, if $x_i^{(2n)}$ is a zero of H_{2n}, then $y_i^{(n)} = \big(x_i^{(2n)}\big)^2$ is a zero of $L_n^{-\frac{1}{2}}$, and if $x_i^{(2n+1)}$ is a zero of H_{2n+1}, then $y_i^{(n)} = \big(x_i^{(2n)}\big)^2$ is a zero of $L_n^{\frac{1}{2}}$. This is reflected by the following facts:

- The scaling is in $\sqrt{n}$ in the case of the asymptotics of the zeros of Hermite polynomials, and is in n in case of the asymptotics of the zeros of Laguerre polynomials.
- The image of the semi-circle law with radius $\sqrt{2}$ by the map $x \mapsto y = 2x^2$ is the Marchenko-Pastur with parameter $c = 1$.

4. Weighted transfinite diameter, Fekete points. — As in Chapter II, Σ is a closed interval, and Q is a function defined on Σ with values on $]-\infty, \infty]$, continuous on $int(\Sigma)$. If Σ is unbounded, it is assumed that

$$\lim_{|x|\to\infty} \big(Q(x) - \log(x^2+1)\big) = \infty.$$

Consider, for $x = (x_1, \dots, x_n) \in \Sigma^n$ the *weighted geometric mean*

$$\mathfrak{M}_n^Q(x) = \Big(\prod_{i<j} (x_i - x_j)^2 e^{-Q(x_i)} e^{-Q(x_j)}\Big)^{\frac{1}{n(n-1)}},$$

and its supremum

$$\delta_n^Q = \sup_{x\in\Sigma^n} \mathfrak{M}_n^Q(x).$$

We will show that the limit

$$\delta^Q = \lim_{n\to\infty} \delta_n^Q$$

exists and that

$$\delta^Q = \exp(-E^*).$$

It is called the *weighted transfinite diameter* of Σ (the transfinite diameter if $Q = 0$).

With the notation

$$K_n(x) = \sum_{i \neq j} \log \frac{1}{|x_i - x_j]} + (n-1) \sum_{i=1}^{n} Q(x_i),$$

we can write

$$\mathfrak{M}_n^Q(x) = \exp\bigl(-\frac{1}{n(n-1)} K_n(x)\bigr).$$

Recall the notation: for $t, s \in \Sigma$,

$$k(s,t) = \log \frac{1}{|s-t|} + \frac{1}{2}Q(s) + \frac{1}{2}Q(t).$$

and

$$k(s,t) \geq m.$$

The function K_n can be written

$$K_n(x) = \sum_{i \neq j} k(x_i, x_j).$$

Hence

$$K_n(x) \geq n(n-1)m.$$

If μ is a probability measure supported by Σ, then

$$\begin{aligned}\int_{\Sigma^n} K_n(x)\mu(dx_1)\dots\mu(dx_n) &= \sum_{i \neq j} \int_{\Sigma^2} k(x_i, x_j)\mu(dx_i)\mu(dx_j) \\ &= n(n-1)E(\mu).\end{aligned}$$

In particular, for $\mu = \mu^*$, the equilibrium measure,

$$\int_{\Sigma^n} K_n(x)\mu^*(dx_1)\dots\mu^*(dx_n) = n(n-1)E^*.$$

Define

$$\kappa_n = \frac{1}{n(n-1)} \inf_{x \in \Sigma^n} K_n(x).$$

We have seen that $m \leq \kappa_n \leq E^*$. The statement

$$\lim_{n \to \infty} \delta_n^Q = \exp(-E^*)$$

is equivalent to the following

$$\lim_{n \to \infty} \kappa_n = E^*.$$

An n-Fekete point is a point $x = (x_1, \ldots, x_n) \in \Sigma^n$ such that

$$K_n(x) = n(n-1)\kappa_n.$$

For each n consider a n-Fekete point $x^{(n)}$ and the probability measure λ_n on $\mathbb{R}$

$$\lambda_n = \frac{1}{n} \sum_{i=1}^{n} \delta_{x_i^{(n)}}.$$

THEOREM III.4.1.
(i) $\lim_{n\to\infty} \kappa_n = E^*$,
(ii) *The measure λ_n converges to the equilibrium measure μ^* for the tight topology.*

The proof is similar to the one of Theorem III.2.1.

References

[Ullman,1972]; [Erdös-Freud,1974]; [Fekete,1923]; [Mahskar-Saff,1984]; [Deift,1998], Chapter 6; [Simeonov,2003]; [Deano-Huybrechs-Kuijlaars,2010].

Chapter IV

STATISTICAL EIGENVALUE DISTRIBUTION FOR RANDOM MATRICES

1. Statistical eigenvalue distribution. — Let $H_n = Herm(n, \mathbb{C})$ be the space of $n \times n$ Hermitian matrices with coefficients in $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$. On H_n one considers the probability

$$\mathbb{P}_n(dx) = \frac{1}{C_n} \exp\bigl(-\gamma \operatorname{tr}(x^2)\bigr) m_n(dx),$$

with $\gamma > 0$, and m_n is the Euclidean measure on H_n associated to the inner product $(x|y) = \operatorname{tr}(xy)$. C_n is a normalization constant

$$C_n = \int_{H_n} \exp\bigl(-\gamma \operatorname{tr}(x^2)\bigr) m_n(dx) = \Bigl(\sqrt{\frac{\pi}{\gamma}}\Bigr)^N,$$

where

$$N = \dim_{\mathbb{R}} H_n = n + \frac{\beta}{2} n(n-1), \quad \beta = \dim_{\mathbb{R}} \mathbb{F} = 1,\ 2,\ \text{or } 4.$$

The probability $\mathbb{P}_n$ is invariant under the action of the unitary group $U_n = U(n, \mathbb{F})$ of unitary matrices with entries in $\mathbb{F}$.

If $\mathbb{F} = \mathbb{R}$, then $U(n, \mathbb{F}) = O(n)$ the orthogonal group, and the probability space $(H_n, \mathbb{P}_n)$ is called the *Gaussian Orthogonal Ensemble* (GOE).

If $\mathbb{F} = \mathbb{C}$, then $U(n, \mathbb{C}) = U(n)$ the usual unitary group, and $(H_n, \mathbb{P}_n)$ is called the *Gaussian Unitary Ensemble* (GUE).

If $\mathbb{F} = \mathbb{H}$, then $U(n, \mathbb{H}) \simeq Sp(n)$, the compact symplectic group, and $(H_n, \mathbb{P}_n)$ is called the *Gaussian Symplectic Ensemble.*

One is interested in the distribution of the eigenvalues of a random matrix $x \in H_n$ for large n.

The *empirical eigenvalue distribution* of a matrix $x \in H_n$ is the probability measure $\mu_n^{(x)}$ on $\mathbb{R}$ defined by

$$\mu_n^{(x)} = \frac{1}{n} \sum_{k=1}^{n} \delta_{\lambda_k^{(x)}},$$

where $\lambda_1^{(x)}, \ldots, \lambda_n^{(x)}$ are the eigenvalues of x. Observe that, for $B \subset \mathbb{R}$,

$$\mu_n^{(x)}(B) = \frac{1}{n}\#\{\text{eigenvalues of } x \text{ in } B\}.$$

The *statistical eigenvalue distribution* is the probability measure on $\mathbb{R}$ defined by

$$\mu_n(B) = \mathbb{E}_n\big(\mu_n^{(x)}(B)\big).$$

We can write, if χ_B denotes the characteristic function of the set B,

$$\mu_n^{(x)}(B) = \frac{1}{n}\Big(\sum_{k=1}^{n} \chi_B(\lambda_k^{(x)})\Big) = \frac{1}{n}\operatorname{tr}\big(\chi_B(x)\big),$$

with the notation of the functional calculus. Hence

$$\mu_n(B) = \frac{1}{n}\int_{H_n} \operatorname{tr}\chi_B(x)\mathbb{P}_n(dx),$$

and, if f is a bounded measurable function on $\mathbb{R}$,

$$\int_{\mathbb{R}} f(t)\mu_n(dt) = \frac{1}{n}\int_{H_n} \operatorname{tr}\big(f(x)\big)\mathbb{P}_n(dx).$$

The first moment of the statistical eigenvalue distribution vanishes:

$$m_1(\mu_n) = \frac{1}{n}\int_{H_n} \operatorname{tr}(x)\mathbb{P}_n(dx) = 0.$$

Let us compute its second moment:

$$m_2(\mu_n) = \frac{1}{n}\int_{H_n} \operatorname{tr}(x^2)\mathbb{P}_n(dx).$$

We use the formula

$$C_n(\gamma) = \int_{H_n} \exp\big(-\gamma\operatorname{tr}(x^2)\big)\mathbb{P}_n(dx) = \Big(\sqrt{\frac{\pi}{\gamma}}\Big)^N,$$

where $N = \dim_{\mathbb{R}} H_n$. Hence

$$m_2(\mu_n) = -\frac{1}{n}\frac{d}{d\gamma}\log C_n(\gamma)\Big|_{\gamma=1} = \frac{N}{n} = 1 + \beta\frac{n-1}{2}.$$

This suggests that μ_n does not converge, and that a rescaling is necessary. In fact:

THEOREM IV.1.1 (WIGNER). — *After rescaling, the statistical eigenvalue distribution* μ_n *converges to the semi-circle law: for every bounded continuous function* f *on* $\mathbb{R}$,

$$\lim_{n\to\infty} \int_{\mathbb{R}} f\Big(\frac{t}{\sqrt{n}}\Big)\mu_n(dt) = \frac{2}{\pi r^2}\int_{-r}^{r} f(u)\sqrt{r^2-u^2}\,du,$$

with $r = \sqrt{\frac{\beta}{\gamma}}$.

2. Weyl integration formula. — By the classical spectral theorem, every matrix $x \in H_n$ can be diagonalized in an orthonormal basis, and the eigenvalues are real. In other words the map

$$U_n \times D_n \to H_n, \quad (u,a) \mapsto uau^*,$$

is surjective, where D_n is the space of real diagonal matrices, $D_n \simeq \mathbb{R}^n$.

PROPOSITION IV.2.1. — *If the function* f *is integrable on* H_n*, then*

$$\int_{H_n} f(x)m_n(dx) = c_n \int_{D_n}\int_{U_n} f(uau^*)\alpha_n(du)|\Delta(a)|^\beta da_1 \dots da_n,$$

where $a = (a_1, \dots, a_n)$, Δ *is the Vandermonde determinant,*

$$\Delta(a) = \prod_{j<k}(a_k - a_j),$$

α_n *is the normalized Haar measure of the compact group* U_n, c_n *is a constant and*

$$\beta = \dim_{\mathbb{R}} \mathbb{F} = 1, 2, \text{ or } 4.$$

If the function f is U_n-invariant:

$$f(uxu^*) = f(x) \quad (u \in U_n),$$

then $f(x)$ only depends on the eigenvalues $\lambda_1, \dots, \lambda_n$ of x:

$$f(x) = F(\lambda_1, \dots, \lambda_n),$$

where F is a symmetric function on $\mathbb{R}^n$. In this case the Weyl formula simplifies

$$\int_{H_n} f(x)m_n(dx) = c_n \int_{\mathbb{R}^n} F(a_1, \dots, a_n)|\Delta(a)|^\beta da_1 \dots da_n.$$

3. The density of the statistical eigenvalue distribution. — Let $\Omega_n \subset H_n$ be a U_n-invariant open set, of the form

$$\Omega_n = \{x = uau^* \in H_n \mid u \in U_n,\ a_i \in \omega\},$$

where $\omega = \operatorname{int}(\Sigma)$ is the interior of a closed interval Σ. Hence Ω_n is the set of matrices $x \in H_n$ whose eigenvalues belong to ω. Let Q be a positive continuous function on ω, such that, for every $m \in \mathbb{N}$,

$$\int_\omega e^{-Q(t)}|t|^m dt < \infty.$$

We consider on Ω_n the probability measure given by

$$\mathbb{P}_n(dx) = \frac{1}{C_n} \exp\big(-\operatorname{tr} Q(x)\big) m_n(dx),$$

where $Q(x)$ is defined via the functional calculus.

Examples

a) The main example will be $\Sigma = \mathbb{R}$, and $Q(t) = \gamma t^2$. Then $\Omega_n = H_n$ and $(H_n, \mathbb{P}_n)$ is the Gaussian Orthogonal Ensemble (resp. Gaussian Unitary Ensemble, Gaussian Symplectic Ensemble).

$$\exp\big(-\operatorname{tr} Q(x)\big) = \exp\big(-\gamma \operatorname{tr}(x^2)\big).$$

b) For $\Sigma = [0, \infty[$, Ω_n, the cone of positive definite Hermitian matrices, and

$$Q(t) = t + \alpha \log \frac{1}{t},$$

we get the Wishart Ensemble, or Laguerre Ensemble (Wishart Orthognal Ensemble, Wishart Unitary Ensemble, Wishart Symplectic Ensemble), with

$$\exp\big(-\operatorname{tr} Q(x)\big) = e^{-\operatorname{tr} x}(\det x)^\alpha.$$

c) If $\Sigma = [-1, 1]$, then Ω_n is a matrix interval:

$$\Omega_n = \{x \in H_n \mid \|x\|_{\mathrm{op}} < 1\} = \{x \in H_n \mid -I < x < I\}.$$

For

$$Q(t) = p \log \frac{1}{1-t} + q \log \frac{1}{1+t},$$

we get the Jacobi Ensemble,

$$\exp\bigl(-\operatorname{tr} Q(x)\bigr) = \bigl(\det(I-x)\bigr)^p \bigl(\det(I+x)\bigr)^q.$$

The probability measure $\mathbb{P}_n$ is U_n-invariant. By the Weyl integration formula, if f is U_n-invariant,

$$f(x) = F(\lambda_1, \ldots, \lambda_n),$$

we get

$$\int_{H_n} f(x)\mathbb{P}_n(dx) = \int_{\omega^n} F(\lambda_1, \ldots, \lambda_n) q_n(\lambda_1, \ldots, \lambda_n) d\lambda_1 \ldots d\lambda_n,$$

with

$$q_n(\lambda_1, \ldots, \lambda_n) = \frac{1}{Z_n} e^{-\sum_{i=1}^n Q(\lambda_n)} |\Delta(\lambda)|^\beta,$$

and

$$Z_n = \int_{\omega^n} e^{-\sum_{i=1}^n Q(\lambda_n)} |\Delta(\lambda)|^\beta d\lambda_1 \ldots d\lambda_n.$$

In particular, if

$$f(x) = \frac{1}{n} \operatorname{tr} \varphi(x) = \frac{1}{n} \sum_{i=1}^n \varphi(\lambda_i),$$

where φ is a measurable function on ω, we get

$$\begin{aligned}
\frac{1}{n} \int_{\Omega_n} \operatorname{tr} \varphi(x) \mathbb{P}_n(dx) &= \frac{1}{n} \sum_{i=1}^n \int_{\omega^n} \varphi(\lambda_i) q_n(\lambda) d\lambda_1 \ldots d\lambda_n \\
&= \int_{\omega^n} \varphi(\lambda_1) q_n(\lambda_1, \ldots, \lambda_n) d\lambda_1 \ldots d\lambda_n \\
&= \int_\omega \varphi(t) w_n(t) dt,
\end{aligned}$$

with

$$w_n(t) = \int_{\omega^{n-1}} q_n(t, \lambda_2, \ldots, \lambda_n) d\lambda_2 \ldots d\lambda_n.$$

This means that the statistical eigenvalue distribution μ_n has density w_n,

$$\mu_n(dt) = w_n(t) dt.$$

a) In the first example Z_n is the Mehta integral

$$Z_n = \int_{\mathbb{R}^n} e^{-\gamma\|x\|^2} |\Delta(x)|^\beta dx_1 \dots dx_n.$$

b) In the second example Z_n is the Siegel integral

$$Z_n = \int_{[0,\infty[^n} e^{-(x_1+\cdots+x_n)} (x_1 \dots x_n)^\alpha |\Delta(x)|^\beta dx_1 \dots dx_n.$$

c) In the third example Z_n is the Selberg integral

$$Z_n = \int_{[-1,1]^n} \prod_{i=1}^n (1-x_i)^p (1+x_i)^q |\Delta(x)|^\beta dx_1 \dots dx_n.$$

4. Asymptotic of the integral $\mathcal{Z}_n$. — Consider the integral

$$\mathcal{Z}_n = \int_{\Sigma^n} \exp\Bigl(-n \sum_{i=1}^n Q(x_i)\Bigr) |\Delta(x)|^\beta dx_1 \dots dx_n.$$

The integrant can be written

$$\exp -n^2 \Bigl(\frac{\beta}{2} \sum_{i \neq j} \log \frac{1}{|x_i - x_j|} \frac{1}{n^2} + \sum_{i=1}^n Q(x_i) \frac{1}{n}\Bigr).$$

Define the energy of a probability measure μ on Σ:

$$E(\mu) = \frac{\beta}{2} \int_{\Sigma^2} \log \frac{1}{|s-t|} \mu(ds)\mu(dt) + \int_\Sigma Q(t)\mu(dt).$$

Heuristically

$$\mathcal{Z}_n = \int_{\Sigma^n} \exp\Bigl(-n^2 E(\mu_n^{(x)})\Bigr) dx_1 \dots dx_n,$$

with

$$\mu_n^{(x)} = \frac{1}{n} \sum_{i=1}^n \delta_{x_i}.$$

Of course this is not correct since $E(\mu_n^{(x)}) = \infty$.

The method of proof is inspired from the Laplace method. Let us present this method in a simple case. Consider the integrals:

$$Z(\lambda) = \int_U e^{-\lambda\varphi(x)} a(x) m(dx),$$
$$I(\lambda; f) = \frac{1}{Z(\lambda)} \int_U f(x) e^{-\lambda\varphi(x)} a(x) m(dx).$$

U is an open set in $\mathbb{R}^n$, φ is a continuous function on U such that

$$\lim_{\|x\|\to\infty} \varphi(x) = \infty,$$

and φ attains its infimum in only one point x_0. The function a is a positive, continuous, and integrable. The function f is continuous and bounded.

Proposition IV.4.1.

(*i*) $$\lim_{\lambda\to\infty} \frac{1}{\lambda} \log Z(\lambda) = -\varphi(x_0).$$

(*ii*) $$\lim_{\lambda\to\infty} I(\lambda; f) = f(x_0).$$

Proof.

a) For $\lambda > 0$,

$$e^{-\lambda\varphi(x)} \leq e^{-\lambda\varphi(x_0)},$$

hence

$$Z(\lambda) \leq e^{-\lambda\varphi(x_0)} \int_U a(x) m(dx),$$

and

$$\frac{1}{\lambda} \log Z(\lambda) \leq -\varphi(x_0) + \frac{1}{\lambda} \log \int_U a(x) m(dx).$$

Therefore

$$\limsup_{\lambda\to\infty} \frac{1}{\lambda} \log Z(\lambda) \leq -\varphi(x_0).$$

b) Let $\alpha > \varphi(x_0)$. Then

$$\mathcal{V} = \{x \in U \mid \varphi(x) < \alpha\}$$

is a non-empty open set, and

$$Z(\lambda) \geq \mathrm{vol}(\mathcal{V}) e^{-\alpha\lambda},$$
$$\frac{1}{\lambda} \log Z(\lambda) \geq -\alpha + \frac{1}{\lambda} \log \mathrm{vol}(\mathcal{V}).$$

Therefore

$$\liminf_{\lambda\to\infty} \frac{1}{\lambda} \log Z(\lambda) \geq -\alpha.$$

This holds for every $\alpha > \varphi(x_0)$. Hence

$$\liminf_{\lambda\to\infty} \frac{1}{\lambda} \log Z(\lambda) \geq -\varphi(x_0).$$

c) Let $\mathcal{W}$ be a neighborhood of x_0 and

$$\beta = \inf_{x\in U\setminus\mathcal{W}} \varphi(x).$$

By hypothesis, $\beta > \varphi(x_0)$. Choose α such that $\beta > \alpha > \varphi(x_0)$. By b) there is a constant C_α such that

$$Z(\lambda) \geq C_\alpha e^{-\alpha\lambda}.$$

Therefore

$$\begin{aligned}&\frac{1}{Z(\lambda)} \int_{U\setminus\mathcal{W}} e^{-\lambda\varphi(x)} a(x)m(dx)\\ &\leq \frac{1}{C_\alpha} e^{\alpha\lambda} e^{-\beta\lambda} \int_{U\setminus\mathcal{W}} a(x)m(dx) = Ce^{-(\beta-\alpha)\lambda}.\end{aligned}$$

The rest of the proof of (ii) is standard. ∎

We will follow the lines of the previous proof.

Proposition IV.4.2.

$$\lim_{n\to\infty} \frac{1}{n^2} \log \mathcal{Z}_n = -E^*.$$

Proof.

a) Define, for $x \in \mathbb{R}^n$,

$$K_n(x) = \frac{\beta}{2} \sum_{i\neq j} \log \frac{1}{|x_i - x_j|} + (n-1) \sum_{i=1}^{n} Q(x_i),$$

and

$$\kappa_n = \frac{1}{n(n-1)} \inf_{x\in\mathbb{R}^n} K_n(x).$$

We saw that

$$\lim_{n\to\infty} \kappa_n = E^*.$$

Since $K_n(x) \geq n(n-1)\kappa_n$,

$$\mathcal{Z}_n \leq e^{-n(n-1)\kappa_n}\Big(\int_\Sigma e^{-Q(t)}dt\Big)^n,$$

and

$$\frac{1}{n^2}\log \mathcal{Z}_n \leq -\frac{n-1}{n}\kappa_n + \frac{1}{n}\log\int_\Sigma e^{-Q(t)}dt.$$

Therefore

$$\limsup_{n\to\infty}\frac{1}{n^2}\log \mathcal{Z}_n \leq -E^*.$$

b) For a probability measure μ on Σ,

$$\int_{\Sigma^n} K_n(x)\mu(dx_1)\dots\mu_n(dx_n) = n(n-1)E(\mu).$$

Assume that $\mu(dt) = u(t)dt$, where u is a continuous function supported by $\overline{U} \subset \Sigma$, with $u(t) > 0$ on the open set U. We can write

$$\mathcal{Z}_n = \int_{\Sigma^n}\exp\Big(-\big(K_n(x) + \sum_{i=1}^n Q(x_i) + \sum_{i=1}^n \log u(x_i)\big)\Big)\prod_{i=1}^n u(x_i)dx_i.$$

We will apply the following Jensen's inequality: Let (X,ν) be a probability space, and Φ a convex function on $\mathbb{R}$. Then, for a real measurable function f on X,

$$\Phi\big(\int_X f(x)\nu(dx)\big) \leq \int_X \Phi\big(f(x)\big)\nu(dx).$$

We take $X = \Sigma^n$, $\nu(dx) = \prod_{i=1}^n u(x_i)dx_i$, $\Phi(x) = \exp(x)$, and

$$f(x) = -\big(K_n(x) + \sum_{i=1}^n Q(x_i) + \sum_{i=1}^n \log u(x_i)\big).$$

We get

$$\mathcal{Z}_n \geq \exp\big(-n(n-1)E(\mu)\big)\Big(\int_\Sigma -Q(t)u(t)\Big)^n\Big(\int_\Sigma -\log u(t)dt\Big)^n.$$

It follows that

$$\lim_{n\to\infty}\frac{1}{n^2}\log \mathcal{Z}_n \geq -E(\mu).$$

If the equilibrium measure is of this form, this finishes the proof. If not one has to prove that, for every $\varepsilon > 0$, there is a probability measure μ of that form such that $E(\mu) \leq E(\mu^*) + \varepsilon$. $\square$

Let us consider the Mehta integral: for $\beta > 0$,

$$I_n(\beta) = \int_{\mathbb{R}^n} \exp\Big(-\frac{1}{2}\sum_{i=1}^n x_i^2\Big) \prod_{i<j} |x_i - x_j|^\beta dx_1 \ldots dx_n.$$

After rescaling we get

$$Z_n(\beta) = (\sqrt{2n})^{\left(n+\frac{\beta}{2}n(n-1)\right)} I_n(\beta).$$

Define, for $\mu \in \mathfrak{M}^1(\mathbb{R})$,

$$\begin{aligned} E_\beta(\mu) :&= \frac{\beta}{2}\int_{\mathbb{R}^2} \log\frac{1}{|s-t|}\mu(ds)\mu(dt) + \int_{\mathbb{R}} t^2 \mu(dt) \\ &= \frac{\beta}{2}\Big(\int_{\mathbb{R}^2} \log\frac{1}{|s-t|}\mu(ds)\mu(dt) + \int_{\mathbb{R}} \big(\sqrt{\frac{2}{\beta}}t\big)\mu(dt)\Big). \end{aligned}$$

In section II.3 (2) Gaussian weight), we saw, for $\beta = 2$, that

$$E_2^* = \frac{3}{4} + \frac{1}{2}\log 2,$$

and, by Lemma II.2.4,

$$E_\beta^* = \frac{\beta}{2}\Big(E_2^* + \log\sqrt{\frac{2}{\beta}}\Big) = \frac{3\beta}{8} + \frac{\beta}{4}\log\frac{4}{\beta}.$$

By Proposition IV.4.2,

$$\lim_{n\to\infty} \frac{1}{n^2}\log \mathcal{Z}_n(\beta) = -E_\beta^* = -\Big(\frac{3\beta}{8} + \frac{\beta}{4}\log\frac{4}{\beta}\Big).$$

Since

$$\log I_n(\beta) = \frac{1}{2}\big(n + \frac{\beta}{2}n(n-1)\big)\log 2n + \log \mathcal{Z}_n(\beta),$$

we get

$$\frac{1}{n^2}\log I_n(\beta) = \frac{\beta}{4}\log n - \Big(\frac{3\beta}{8} + \frac{\beta}{4}\log\frac{2}{\beta}\Big) + o(1).$$

This result can be obtained from the explicit evaluation of the Mehta integral:

$$I_n(\beta) = (2\pi)^{\frac{n}{2}} \prod_{j=1}^n \frac{\Gamma\big(j\frac{\beta}{2}+1\big)}{\Gamma\big(\frac{\beta}{2}+1\big)}.$$

Hence

$$\log I_n(\beta) = \sum_{j=1}^{n} \log \Gamma\big(j\frac{\beta}{2}+1\big) + n \log \sqrt{2\pi} + n \log \Gamma\big(\frac{\beta}{2}+1\big).$$

By the Stirling formula,

$$\log \Gamma(x+1) = x \log x - x + O(\log x),$$

we get

$$\sum_{j=1}^{n} \gamma\big(j\frac{\beta}{2}+1\big) = \frac{\beta}{2}\Big(\sum_{j=1}^{n} j \log j + \frac{n(n-1)}{2} \log \frac{\beta}{2}\Big) - \frac{\beta}{2}\frac{n(n-1)}{2} + O(n \log n).$$

We saw that (Lemma III.2.2)

$$\frac{1}{n^2}\sum_{j=1}^{n} j \log j = \frac{1}{2}\log n - \frac{1}{4} + o(1).$$

We obtain finally, in agrreement with the previous result,

$$\frac{1}{n^2}\log I_n(\beta) = \frac{\beta}{4}\log n - \Big(\frac{3\beta}{8} + \frac{\beta}{4}\log\frac{2}{\beta}\Big) + o(1).$$

5. Generalized Wigner Theorem. — Define the rescaled statistical eigenvalue distribution μ_n:

$$\int_{\Sigma} f(t)\mu_n(dt) = \mathcal{E}_n\Big(\frac{1}{n}\sum_{i=1}^{n} f(x_i)\Big),$$

where $\mathcal{E}_n$ is the expectation related to the probability measure $\mathcal{P}_n$ on Σ^n defined by

$$\mathcal{P}_n(dx) = \frac{1}{\mathcal{Z}_n}\exp\Big(-n\sum_{i=1}^{n} Q(x_i)\Big)|\Delta(x)|^{\beta} dx_1 \dots dx_n.$$

Theorem IV.5.1. — *The measure μ_n converges to the equilibrium measure μ^* for the tight topology:*

$$\lim_{n\to\infty} \mu_n = \mu^*.$$

It means that, for every bounded continuous function f,

$$\lim_{n\to\infty}\int_\Sigma f(t)\mu_n(dt)=\int_\Sigma f(t)\mu^*(dt).$$

The probability $\mathcal{P}_n$ concentrates in a neighborhood of the points where the function

$$K_n(x)=\sum_{i\neq j}\log\frac{1}{|x_i-x_j|}+(n-1)\sum_{i=1}^n Q(x_i)$$

attains its infimum:

LEMMA IV.5.2. — *For $\eta>0$ define*

$$A_{\eta,n}=\{x\in\Sigma^n \mid K_n(x)\le (E^*+\eta)n^2\}.$$

The set $A_{\eta,n}$ is compact and

$$\lim_{n\to\infty}\mathcal{P}_n(A_{\eta,n})=1.$$

Proof.
From the definition of $A_{\eta,n}$ it follows that

$$\mathcal{P}_n(\Sigma^n\setminus A_{\eta,n})\le\frac{1}{\mathcal{Z}_n}e^{-(E^*+\eta)n^2}\Big(\int_\Sigma e^{-Q(t)}dt\Big)^n.$$

By Proposition IV.4.1, for every $\varepsilon>0$ there is N such that, if $n\ge N$,

$$\frac{1}{\mathcal{Z}_n}\le e^{(E^*+\varepsilon)n^2}.$$

Choose $\varepsilon<\eta$. ∎

Proof of Theorem IV.5.1 Let f be a bounded continuous function on Σ, and F_n the function defined on Σ^n by

$$F_n(x)=\frac{1}{n}\sum_{i=1}^n f(x_i).$$

a) Fix $\eta>0$. The set $A_{\eta,n}$ is compact, hence the continuous function F_n attains its supremum on $A_{\eta,n}$ at a point

$$x^{(\eta,n)}=(x_1^{(\eta,n)},\ldots,x_n^{(\eta,n)})\in A_{\eta,n}.$$

We obtain

$$\begin{aligned}\int_\Sigma f(t)\mu_n(dt) &\le F(x^{(\eta,n)})\mathcal{P}_n(A_{\eta,n}) + \|f\|_\infty \mathcal{P}_n(\Sigma^n \setminus A_{\eta,n}) \\ &\le F(x^{(\eta,n)}) + \|f\|_\infty \big(1 - \mathcal{P}_n(A_{\eta,n})\big).\end{aligned}$$

To the point $x^{(\eta,n)}$ we associate the following probability measure on Σ:

$$\nu_\eta^{(n)} = \frac{1}{n}\sum_{i=1}^n \delta_{x_i^{(\eta,n)}}.$$

The previous inequality can be written

$$\int_\Sigma f(t)\mu_n(dt) \le \int_\Sigma f(t)\nu_\eta^{(n)}(dt) + \|f\|_\infty \big(1 - \mathcal{P}_n(A_{\eta,n})\big).$$

The truncated energy E^ℓ of the measure $\nu_\eta^{(n)}$ satisfies:

$$E^\ell(\nu_\eta^{(n)}) \le \frac{\ell}{n} + (E^* + \eta).$$

From the inequality

$$K_n(x) \ge (n-1)\sum_{i=1}^n h(x_i),$$

it follows that

$$\int_\Sigma h(t)\nu_\eta^{(n)}(dt) \le \frac{n}{n-1}(E^* + \eta).$$

This implies that the sequence $\nu_\eta^{(n)}$ is relatively compact for the tight topology. There is a sequence n_j going to ∞ such that the subsequence $\nu_\eta^{(n_j)}$ converges:

$$\lim_{j\to\infty} \nu_\eta^{(n_j)} = \nu_\eta.$$

We may also assume that

$$\lim_{j\to\infty} \int_\Sigma f(t)\mu_{n_j}(dt) = \limsup_{n\to\infty} \int_\Sigma f(t)\mu_n(dt).$$

The limit measure ν_η satisfies

$$E^\ell(\nu_\eta) \le E^* + \eta,$$

and, as $\ell \to \infty$,

$$E(\nu_\eta) \leq E^* + \eta.$$

Furthermore

$$\limsup_{n\to\infty} \int_\Sigma f(t)\mu_n(dt) \leq \int_\Sigma f(t)\nu_\eta(dt).$$

b) The inequality

$$E(\nu_\eta) \leq E^* + \eta$$

implies that the measure ν_η converges to the equilibrium measure μ^* as $\eta \to 0$. Therefore

$$\limsup_{n\to\infty} \int_\Sigma f(t)\mu_n(dt) \leq \int_\Sigma f(t)\mu^*(dt).$$

c) Applying the previous result to $-f$ instead of f, one gets

$$\liminf_{n\to\infty} \int_\Sigma f(t)\mu_n(dt) \geq \int_\Sigma f(t)\mu^*(dt),$$

and finally

$$\lim_{n\to\infty} \int_\Sigma f(t)\mu_n(dt) = \int_\Sigma f(t)\mu^*(dt).$$ □

In the special case $\Sigma = \mathbb{R}$, and $Q(t) = t^2$, we obtain Wigner's theorem (Theorem IV.1.1). In fact in this case the equilibrium measure μ^* is the semi-circle law of radius $r = \sqrt{\beta}$:

Corollary IV.5.2. — *If $\Sigma = \mathbb{R}$, and $Q(t) = t^2$, the measure μ_n converges to the semi-circle law of radisu $r = \sqrt{\beta}$: for every bounded continuous function f on $\mathbb{R}$,*

$$\lim_{n\to\infty} \int_{\mathbb{R}} f(t)\mu_n(dt) = \frac{2}{\pi\beta} \int_{-\sqrt{\beta}}^{\sqrt{\beta}} f(u)\sqrt{\beta - u^2}\,du.$$

References

[Mehta,1991]; [Deift,1998]; [Anderson-Guionnet-Zeitouni,2010]; [Pastur-Shcherbina,2010]; [Haagerup-Thorbjørnsen,2003]; [Faraut,2012].

Chapter V

THE WISHART UNITARY ENSEMBLE

1. The Wishart unitary ensemble. — Let Ω_n be the cone of positive definite $n \times n$ Hermitian matrices in the real vector space $H_n = Herm(n, \mathbb{C})$. For $p > n-1$, the Wishart law W_n^p is the probability measure on Ω_n defined by

$$\int_{\Omega_n} f(x) W_n^p(dx) = \frac{1}{\Gamma_n(p)} \int_{\Omega_n} f(x) e^{-\operatorname{tr} x} (\det x)^{p-n} m_n(dx),$$

for a bounded measurable function f. The function Γ_n is the gamma function of the cone Ω_n:

$$\Gamma_n(p) = \int_{\Omega_n} e^{-\operatorname{tr} x} (\det x)^{p-n} m_n(dx).$$

The probability space (Ω_n, W_n^p) is called the *Wishart unitary ensemble*. In fact the Wishart law W_n^p is invariant under the action of the unitary group $U(n)$ given by the transformations

$$x \mapsto uxu^* \quad \big(u \in U(n)\big).$$

The gamma function Γ_n can be computed:

$$\Gamma_n(p) = (2\pi)^{\frac{n(n-1)}{2}} \prod_{j=1}^{n} \Gamma(p-j+1).$$

The probability space (Ω_n, W_n^p) can be obtained from the general construction we introduced in Chapter IV:

$$\Sigma = [0, \infty[, \quad Q(t) = t + (p-n) \log \frac{1}{t}, w(t) = e^{-Q(t)} = e^{-t} t^{p-n}.$$

By the Weyl integration formula,

$$Z_n := \int_{]0,\infty[^n} e^{-(a_1 + \cdots + a_n)} (a_1 \ldots a_n)^{p-n} \Delta(a)^2 da_1 \ldots da_n = A_n \Gamma_n(p).$$

The Laplace transform of the Wishart law has a simple expression:

Proposition V.1.1. — *For* $\zeta = \xi + i\eta \in H_n + iH_n \simeq M_n(\mathbb{C})$ *with* $\xi + I \in \Omega_n$,

$$\mathcal{L}W_n^p(\zeta) := \int_{\Omega_n} e^{-\operatorname{tr}(\zeta x)} W_n^p(dx) = \det(I+\zeta)^{-p}.$$

Proof.
One starts from the formula

$$\int_{\Omega_n} e^{-\operatorname{tr}(x)} (\det x)^{p-n} m_n(dx) = \Gamma_n(p),$$

and changes the variable: one puts $x = gx'g^*$ with $g \in GL(n, \mathbb{C})$. Then

$$m_n(dx) = |\det g|^{2n} m_n(dx'),$$

and

$$\begin{aligned}\int_{\Omega_n} & e^{-\operatorname{tr}(x)} (\det x)^{p-n} m_n(dx) \\ &= |\det g|^{2p} \int_{\Omega_n} e^{-\operatorname{tr}(gx'g^*)} (\det x')^{p-n} m_n(dx').\end{aligned}$$

Therefore, for $y = g^*g$,

$$\int_{\Omega_n} e^{-\operatorname{tr}(x'y)} (\det x')^{p-n} m_n(dx') = \Gamma_n(p)(\det y)^{-p}.$$

Since, for $y \in \Omega_n$, there exists $g \in GL(n, \mathbb{C})$ such that $y = g^*g$, the proposition is proven for $\operatorname{Im} \zeta = \eta = 0$. The two functions $\zeta \mapsto \mathcal{L}W_n^p(\zeta)$ and $\zeta \mapsto (I+\zeta)^{-p}$ are holomorphic in the open set

$$\{\zeta = \xi + i\eta \mid \xi + I \in \Omega_n\} = (\Omega_n - I) + iH_n,$$

and agree for $\zeta = \xi \in \Omega_n - I$, hence agree on $(\Omega_n - I) + iH_n$. ∎

On the space $M(n, p; \mathbb{C})$ of $n \times p$ complex matrices let us denote by $\mathbb{G}$ the Gaussian probability measure

$$\mathbb{G}(d\xi) = \frac{1}{\pi^{np}} e^{-\operatorname{tr}(\xi\xi^*)} m(d\xi).$$

We consider the quadratic map

$$q : M(n, p; \mathbb{C}) \to \overline{\Omega_n}, \quad \xi \mapsto \xi\xi^*.$$

Proposition V.1.2. — *If $p \geq n$, then the image by the map q of the Gaussian probability $\mathbb{G}$ is the Wishart law W_n^p.*

This means that, for a function f on $\overline{\Omega_n}$ which is integrable with respect to W_n^p,

$$\int_{\Omega_n} f(x) W_n^p(dx) = \int_{M(n,p;\mathbb{C})} f(\xi\xi^*)\mathbb{G}(d\xi).$$

Proof.

The measure $\mu = q(\mathbb{G})$ is the measure on $\overline{\Omega_n}$ such that, for a function f on $\overline{\Omega_n}$, measurable and bounded,

$$\int_{\overline{\Omega_n}} f(x)\mu(dx) = \int_{M(n,p;\mathbb{C})} f\big(q(\xi)\big)\mathbb{G}(d\xi).$$

Let us compute the Laplace transform of the image $\mu = q(\mathbb{G})$. By taking

$$f(x) = e^{-\operatorname{tr}(x\zeta)},$$

with $\zeta = \xi + i\eta \in H_n + iH_n$, $\xi + I \in \Omega_n$, we obtain

$$\begin{aligned}\mathcal{L}\mu(\zeta) &= \frac{1}{\pi^{np}} \int_{M(n,p;\mathbb{C})} e^{-\operatorname{tr}(\zeta\xi\xi^*)} e^{-\operatorname{tr}(\xi\xi^*)} m(d\xi) \\ &= \frac{1}{\pi^{np}} \int_{M(n,p;\mathbb{C})} e^{-\operatorname{tr}\big((I+\zeta)\xi\xi^*)\big)} m(d\xi) \\ &= \det(I+\zeta)^p.\end{aligned}$$

By Proposition V.1.1 and the injectivity of the Laplace transform, this proves the proposition.

∎

If $p < n$, then the image of $\mathbb{G}$ is a well defined probability measure supported on the boundary $\partial\Omega_n$ of the cone Ω_n. It is singular with respect to the Euclidean measure m_n. We will also denote it by W_n^p. In fact it can be obtained by analytic continuation from W_n^p, $p > n-1$, with respect to p. Therefore we obtain a familily of probability measures W_n^p for p in the so called *Wallach set*

$$\{0, 1, \ldots, n-1\} \cup]n-1, \infty[.$$

2. The statistical eigenvalue distribution. — The statiscal eigenvalue distribution μ_n^p is defined by

$$\int_{[0,\infty[} f(t)\mu_n^p(dt) = \frac{1}{n}\int_{\Omega_n} \operatorname{tr}\big(f(x)\big) W_n^p(dx).$$

For $p > n-1$, this measure is absolutely continuous with respect to the Lebesque measure:

$$\mu_n^p(dt) = w_n^p(t)dt.$$

In fact, define

$$q_n^p(a_1, \ldots, a_n) = \frac{1}{Z_n^p} e^{-(a_1+\cdots+a_n)} \prod_{j=1}^n a_j^{p-n} \Delta(a)^2,$$

where

$$Z_n^p = \int_{[0,\infty[^n} e^{-(a_1+\cdots+a_n)} \prod_{j=1}^n a_j^{p-n} \Delta(a)^2 da_1 \ldots da_n.$$

Then

$$w_n^p(t) = \int_{[0,\infty[^{n-1}} q_n^p(t, a_2, \ldots, a_n) da_2 \ldots da_n.$$

As for the classical Wigner theorem, we will rescale the statiscal eigenvalue distribution in order to get the convergence. Define $\tilde{\mu}_n$ by:

$$\int_{[0,\infty[} f(t)\tilde{\mu}_n^p(dt) = \int_{[0,\infty[} f\left(\frac{t}{n}\right)\mu_n^p(dt).$$

We obtain

$$\int_{[0,\infty[} f(t)\tilde{\mu}_n^p(dt)$$
$$= \frac{1}{Z_n^p} \int_{[0,\infty[^n} \frac{1}{n}\Big(\sum_{i=1}^n f(x_i)\Big) \exp\Big(-n\sum_{i=1}^n Q(x_i)\Big)\Delta(x)^2 dx_1 \ldots dx_n,$$

with

$$Q(t) = t + \Big(\frac{p}{n} - 1\Big) \log \frac{1}{t}.$$

To determine the limit of μ_n^p we will use the results established in Chapter II about logarithmic potential theory.

3. The Marchenko-Pastur law. — For $c > 1$ define

$$Q(t) = t + (c-1)\log\frac{1}{t},$$

on $\Sigma = [0, \infty[$, and consider the energy of a probability measure on Σ:

$$E(\mu) = \int_{\Sigma^2} \log \frac{1}{|s-t|} \mu(dt)\mu(ds) + \int_\Sigma Q(t)\mu(dt).$$

We saw that the infimum E^* of the energy is attained at a unique measure μ^*, called the equilibrium measure. We saw also that, if μ is a measure with the following property: there is a constant C such that

$$\begin{aligned} U^\mu(x) + \frac{1}{2}Q(x) &\geq C \quad (x \in \Sigma), \\ U^\mu(x) + \frac{1}{2}Q(x) &= C \quad (x \in \mathrm{supp}(\mu), \end{aligned}$$

then $\mu = \mu^*$.

For $c \geq 1$, the Marchenko-Pastur distribution μ_c is defined by

$$\int_\Sigma f(t)\mu_c(dt) = \frac{1}{2\pi}\int_a^b f(t)\sqrt{(t-a)(t-b)}\frac{dt}{t},$$

where

$$a = (\sqrt{c}-1)^2, \; b = (\sqrt{c}+1)^2.$$

By Proposition II.1.5, the Marchenko-Pastur μ_c is the equilibrium measure.

4. Convergence to the Marchenk-Pastur distribution. — We assume that p depends on n: $p = p(n)$.

THEOREM V.4.1. — *Assume that*

$$\lim_{n\to\infty} \frac{p(n)}{n} = c \geq 1.$$

Then, for a bounded continuous function f on Σ,

$$\lim_{n\to\infty} \int_\Sigma f\Big(\frac{t}{n}\Big)\mu_n^p(dt) = \int_\Sigma f(t)\mu_c(dt).$$

This is a consequence of the results we saw in Chapter IV.

In the case of $p < n$, the matrix $x = \xi\xi^*$ has $(n-p)$ zero eigenvalues, therefore

$$\mathrm{tr}\, f(x) = f(\lambda_1) + \cdots + f(\lambda_p) + (n-p)f(0).$$

It follows that the statistical eigenvalue distribution μ_n^p has an atom in 0:

$$\int_\Sigma f(t)\mu_n^p(dt) = \Big(1 - \frac{p}{n}\Big)f(0) + \int_0^\infty f(t)w_n^p(t)dt.$$

For $0 < c < 1$, the Marchenko-Pastur distribution μ_c has an atom at 0 as well:

$$\int_\Sigma f(t)\mu_c(dt) = (1-c)f(0) + \frac{1}{2\pi}\int_a^b f(t)\sqrt{(t-a)(t-b)}\frac{dt}{t}.$$

The theorem still holds with $c > 0$ instead of $c \geq 1$.

References

[Anderson-Guionnet-Zeitouni,2010]; [Pastur-Shcherbina,2010], Chapter 7; [Haagerup-Thorbjørsen,2003]; [Faraut,2012].

Chapter VI

WHAT IS THE PROBABILITY FOR A SYMMETRIC OR A HERMITIAN MATRIX TO BE POSITIVE DEFINITE ?

The result we will present is part of the paper

D. S. Dean, S. N. Majumdar
Extreme value statistics of eigenvalues of Gaussian random matrices (2008)

1. The probability for a matrix to be positive definite. — Let $(H_n, \mathbb{P}_n)$ be the Gaussian Orthogonal Ensemble, the Gaussian Unitary Ensemble, or the Gaussian Symplectic Ensemble.

As before $\Omega_n \subset H_n$ denotes the cone of positive definite Hermitian matrices. The question is: What can be said about the numbers $p_n = \mathbb{P}_n(\Omega_n)$, the probability for a matrix $x \in H_n$, to be positive definite ?

It is the probability that all the eigenvalues are > 0. If the eigenvalues were independent, it would be equal to $\frac{1}{2^n}$. But it is not the case.

The simplest case: $n = 2$, $H_n = Sym(2, \mathbb{R})$. We use the coordinates

$$x = \begin{pmatrix} x_1 + x_2 & x_3 \\ x_3 & x_1 - x_2 \end{pmatrix}$$

Then $\mathrm{tr}(x^2) = 2(x_1^2 + x_2^2 + x_3^2)$, and Ω_2 is the circular cone

$$\Omega_2 = \{x = (x_1, x_2, x_2) \mid x_1^2 - x_2^2 - x_3^2 > 0,\ x_1 > 0\}.$$

The number p_2 is the area of the intersection of Ω_2 with the unit sphere $S(\mathbb{R}^3)$, the area being normalized so that the area of $S(\mathbb{R}^3)$ is equal to one. This means that p_2 is the normalized solid angle of the cone Ω_2. One computes easily

$$p_2 = \frac{2 - \sqrt{2}}{4}.$$

Then $p_2 \simeq 0.14$, much less that $\frac{1}{4}$. On the opposite the probabibility for a matrix to have signature $(1, 1)$ is $\frac{\sqrt{2}}{2}$, much more that 0.5. The interpretation of p_n as a solid angle holds in any dimension: p_n equals the area of the intersection of Ω_n with the unit sphere $S(H_n)$, the area being normalized so that the area of $S(H_n)$ is equal to one.

In his thesis, Kuriki has determined the probability, for a real symmetric matrix of order ≤ 5, that the eigenvalues are $\geq \ell$. For $\ell = 0$, one gets the probabilities p_n ([Kuriki,1992], p.35):

$$p_3 = \frac{\pi - 2\sqrt{2}}{4\pi},\ p_4 = \frac{(4-\sqrt{2})\pi - 8}{16\pi},\ p_5 = \frac{3\pi - 8 - \sqrt{2}}{24\pi}.$$

In the paper of Dean and Majumdar it is proven:

THEOREM VI.1.1.

$$\lim_{n\to\infty} \frac{1}{n^2} \log p_n = -\beta \frac{\log 3}{4}.$$

Hence p_n converges to zero very rapidly: p_n is like

$$e^{-cn^2}, \quad c = \beta \frac{\log 3}{4}.$$

We consider the Gaussian probability

$$\mathbb{P}_n(dx) = \frac{1}{Z_n} \exp\Big(-\frac{\beta}{2}\operatorname{tr}(x^2)\Big) m_n(dx),$$

where

$$C_n = \int_{H_n} \exp\Big(-\frac{\beta}{2}\operatorname{tr}(x^2)\Big) m_n(dx).$$

Hence

$$p_n = \mathbb{P}_n(\Omega_n) = \frac{1}{C_n}\int_{\Omega_n} \exp\Big(-\frac{\beta}{2}\operatorname{tr}(x^2)\Big) m_n(dx).$$

By using the Weyl integration formula it can also be written, since the ratio is not affected by the rescaling,

$$p_n = \frac{\mathcal{Z}_n^+}{\mathcal{Z}_n},$$

with

$$\mathcal{Z}_n = \int_{\mathbb{R}^n} e^{-n\frac{\beta}{2}(a_1^2+\cdots+a_n^2)} |\Delta(a)|^\beta da_1 \dots da_n.$$

$$\mathcal{Z}_n^+ = \int_{[0,\infty[^n} e^{-n\frac{\beta}{2}(a_1^2+\cdots+a_n^2)} |\Delta(a)|^\beta da_1 \dots da_n.$$

Consider, for $\Sigma = \mathbb{R}$, and $Q(t) = t^2$,

$$E(\mu) = \int_{\mathbb{R}^2} \log\frac{1}{|s-t|}\mu(ds)\mu(dt) + \int_{\mathbb{R}} Q(t)\mu(dt),$$

and denote by E^* the equilibrium energy. In Chapter IV we consider a slightly different definition:

$$\tilde{E}(\mu) = \frac{\beta}{2} \int_{\mathbb{R}^2} \log \frac{1}{|s-t|} \mu(ds)\mu(dt) + \int_{\mathbb{R}} Q(t)\mu(dt).$$

By Lemma II.2.4, the modified equilibrium energy $\tilde{E}^*$ is related to E^* by

$$\tilde{E}^* = \frac{\beta}{2}\Big(E^* - \frac{1}{2}\log\frac{\beta}{2}\Big).$$

Therefore, by Proposition IV.4.1

$$\lim_{n\to\infty} \frac{1}{n^2} \log \mathcal{Z}_n = -\frac{\beta}{2}\big(E^* - \frac{1}{2}\log\frac{\beta}{2}\big).$$

Similarly consider, for $\Sigma = [0, \infty[$, $Q(t) = t^2$, the energy

$$E_+(\mu) = \int_{\mathbb{R}_+^2} \log \frac{1}{|s-t|} \mu(ds)\mu(dt) + \int_{\mathbb{R}_+} Q(t)\mu(dt).$$

and denote by E_+^* the equilibrium energy. Then

$$\lim_{n\to\infty} \frac{1}{n^2} \log \mathcal{Z}_n^+ = -\frac{\beta}{2}\big(E_+^* - \frac{1}{2}\log\frac{\beta}{2}\big).$$

Therefore

$$\lim_{n\to\infty} \frac{1}{n^2} \log p_n = -\frac{\beta}{2}(E_+^* - E^*).$$

We have seen in Chapter II that

$$E^* = \frac{3}{4} + \frac{1}{2}\log 2.$$

We will see in the next section that

$$E_+^* = \frac{3}{4} + \frac{1}{2}\log 2 + \frac{1}{2}\log 3.$$

Since

$$p_n = \frac{\mathcal{Z}_n}{\mathcal{Z}_n^+},$$

this proves Theorem VI.1.1. ∎

2. The Dean-Majumdar distribution. — Recall that, for $\Sigma = \mathbb{R}$, and $Q(t) = t^2$, the energy of a probability measure μ is

$$E(\mu) = \int_{\mathbb{R}^2} \log \frac{1}{|s-t|} \mu(ds)\mu(dt) + \int_{\mathbb{R}} Q(t)\mu(dt).$$

The equilibrium measure is the semi-circle law of radius $\sqrt{2}$:

$$\int_{\mathbb{R}} f(t)\mu^*(dt) = \frac{1}{\pi} \int_{-\sqrt{2}}^{\sqrt{2}} f(t)\sqrt{2-t^2}dt,$$

and the equilibrium energy is

$$E^* = \frac{3}{4} + \frac{1}{2}\log 2.$$

Now we replace $\mathbb{R}$ by $\Sigma = [0, \infty[$, and keep $Q(t) = t^2$. Define the energy of a probability measure μ on $[0, \infty[$ as

$$E_+(\mu) = \int_{\mathbb{R}_+^2} \log \frac{1}{|s-t|} \mu(ds)\mu(dt) + \int_{\mathbb{R}_+} Q(t)\mu(dt).$$

We will see that the equilibrium measure is then the probability measure μ defined by

$$\int_{\mathbb{R}} f(t)\mu(dt) = \frac{1}{\pi} \int_0^b f(t)\big(t + \frac{b}{2}\big)\sqrt{\frac{b}{t} - 1}\, dt,$$

with $b = \frac{2}{3}\sqrt{6}$. This measure is considered in the paper [Dean-Majumdar,2008].

THEOREM VI.2.1. — *The Cauchy transform of μ is given, for $z \in \mathbb{C} \setminus [0, b]$, by*

$$G_\mu(z) = z - \big(z + \frac{b}{2}\big)\sqrt{1 - \frac{b}{z}},$$

and its logarithmic potential by

$$\begin{aligned} U^\mu(x) &= -\frac{1}{2}x^2 + C, \quad \text{if } 0 \le x \le b, \\ &= -\frac{1}{2}x^2 + C + \int_b^x \big(t + \frac{b}{2}\big)\sqrt{1 - \frac{b}{t}}dt, \quad \text{if } x \ge b. \end{aligned}$$

with

$$C = \frac{1}{2} + \frac{1}{2}\log 2 + \frac{1}{2}\log 3.$$

Proof.
The function

$$f(z)=\big(z+\frac{b}{2}\big)\sqrt{1-\frac{b}{z}}$$

is holomorphic in $\mathbb{C}\setminus[0,b]$, and the difference of its boundary values equals

$$[f]=2i\big(t+\frac{b}{2}\big)\sqrt{\frac{b}{t}-1}\chi(t),$$

where χ is the indicator function of $[0,b]$. Furthermore the Laurent expansion of f is

$$f(z)=z-\frac{1}{z}-\frac{b}{3}\frac{1}{z^2}-\frac{1}{2}\frac{1}{z^3}+\cdots$$

Therefore μ is a probability measure whose Cauchy transform is

$$G_\mu(z)=z-\big(z+\frac{b}{2}\big)\sqrt{1-\frac{b}{z}}.$$

Furthermore the first two moments of μ are

$$m_1(\mu)=\frac{b}{3},\ m_2(\mu)=\frac{1}{2}.$$

Hence

$$\begin{aligned}\frac{d}{dx}U^\mu(x)=-\mathrm{Re}\,G_\mu(x)=-x\quad&\text{if } 0\le x\le b,\\ =-x+\big(x+\frac{b}{2}\big)\sqrt{1-\frac{b}{t}}\quad&\text{if } x\ge b,\end{aligned}$$

and

$$\begin{aligned}U^\mu(x)&=-\frac{1}{2}x^2+C,\quad \text{if } 0\le x\le b,\\ &=-\frac{1}{2}x^2+C+\int_b^x\big(t+\frac{b}{2}\big)\sqrt{1-\frac{b}{t}}dt,\quad \text{if } x\ge b.\end{aligned}$$

The constant C will be computed by using

$$\lim_{x\to\infty}U^\mu(x)+\log x=0.$$

Let us compute the integral

$$F(x)=\int_b^x\big(t+\frac{b}{2}\big)\sqrt{1-\frac{b}{t}}dt=F_1(x)+F_2(x),$$

with
$$F_1(x)=\int_b^x \sqrt{t(t-b)}dt,\ F_2(x)=\frac{b}{2}\int_b^x \sqrt{1-\frac{b}{t}}dt.$$
For the first integral we change the variable: $t=\frac{b}{2}(1+u)$ and get
$$F_1(x)=\frac{b^2}{4}\int_1^{\frac{2}{b}x-1}\sqrt{u^2-1}du=\frac{2}{3}\Phi\big(\frac{2}{b}x-1\big),$$
with
$$\Phi(v)=\int_1^v \sqrt{u^2-1}du.$$
For the second integral we put $t=bu^2$ and get
$$F_2(x)=b^2\int_1^{\sqrt{\frac{x}{b}}}\sqrt{u^2-1}du=\frac{8}{3}\Phi\big(\sqrt{\frac{x}{b}}\big).$$
By Lemma II.1.4
$$\Phi(v)=\frac{1}{2}v^2-\frac{1}{2}\log v-\frac{1}{4}-\frac{1}{2}\log 2+o(1).$$
Hence
$$\begin{aligned}F_1(x)&=\frac{2}{3}\Big(\frac{1}{2}\big(\frac{2}{b}x-1\big)^2-\frac{1}{2}\log\big(\frac{2}{b}x-1\big)-\frac{1}{4}-\frac{1}{2}\log 2\Big)+o(1)\\&=\frac{1}{2}x^2-\frac{b}{2}x-\frac{1}{3}\log x+\frac{1}{6}-\frac{1}{6}\log 2-\frac{1}{6}\log 3+o(1),\end{aligned}$$
and
$$\begin{aligned}F_2(x)&=\frac{8}{3}\Big(\frac{1}{2}\big(\sqrt{\frac{x}{b}}\big)^2-\frac{1}{2}\log\sqrt{\frac{x}{b}}-\frac{1}{4}-\frac{1}{2}\log 2\Big)+o(1)\\&=\frac{b}{2}x-\frac{2}{3}\log x-\frac{2}{3}-\frac{1}{3}\log 2-\frac{1}{3}\log 3+o(1).\end{aligned}$$
Finally
$$F(x)=F_1(x)+F_2(x)=\frac{1}{2}x^2-\log x-\big(\frac{1}{2}+\frac{1}{2}\log 2+\frac{1}{2}\log 3\big)+o(1).$$
It follows that
$$C=\frac{1}{2}+\frac{1}{2}\log 2+\frac{1}{2}\log 3.$$
∎

Corollary VI.2.2. — *For $\Sigma=[0,\infty[$, $Q(t)=t^2$, the equilibrium measure is the Dean-Majumdar distribution μ, and the equilibrium energy is*
$$E_+^*=\frac{3}{4}+\frac{1}{2}\log 2+\frac{1}{2}\log 3.$$

Proof.

By Theorem VI.2.1,

$$\begin{aligned} U^\mu(x) + \frac{1}{2}Q(x) &\geq C \quad \text{on } [0,\infty[, \\ &= C \quad \text{on } \operatorname{supp}(\mu). \end{aligned}$$

By Proposition II.3.1, this implies that μ is the equilibrium measure. Furthermore

$$E_+^* = C + \frac{1}{2}\int_\Sigma Q(t)\mu(dt) = C + \frac{1}{2}m_2(\mu).$$

We have seen that $m_2(\mu) = \frac{1}{2}$. Hence

$$E_+^* = C + \frac{1}{4} = \frac{3}{4} + \frac{1}{2}\log 2 + \frac{1}{2}\log 3. \qquad \square$$

Consider on Ω_n the Gaussian probability measure

$$\mathbb{P}_n(dx) = \frac{1}{p_n}\exp(-\operatorname{tr} x^2)m(dx),$$

and let μ_n be the statistical distribution of the eigenvalues of a matrix in Ω_n. By the generalized Wigner Theorem (Theorem IV.5.1), the statistical distribution μ_n of the eigenvalues of a matrix x in Ω_n converges, after scaling, to the Dean-Majumdar distribution.

Corollary VI.2.3. — *For a bounded continuous function f on $[0,\infty[$,*

$$\lim_{n\to\infty}\int_{\mathbb{R}_+} f\Big(\frac{t}{\sqrt{n}}\Big)\mu_n(dt) = \int_{\mathbb{R}_+} f\Big(\sqrt{\frac{\beta}{2}}u\Big)\mu(du),$$

where μ is the Dean-Majumdar distribution.

If $x \in H_n$ has p positive eigenvalues and q negative eigenvalues, one says that x has signature (p,q), or index (p,q). It is natural to study the distribution of the random variable $p(x)$, and its asymptotic as $n \to \infty$. See

The index distribution of Gaussian random matrices (2009)
S. N. Majumdar, C. Nadal, University of Paris-Sud (Orsay)
A. Scardicchio, P. Vivo, Abdus Salam International Centre for Theoretical Physics (Trieste)

Once more this question is solved by using Logarithmic Potential Theory. It amounts to solving:

For $\ell \in \mathbb{R}$, determine the probability measure μ which minimazes the energy

$$E(\mu) = \int_{\mathbb{R}^2} \log \frac{1}{|s-t|} \mu(ds)\mu(dt) + \int_{\mathbb{R}} t^2 \mu(dt),$$

among the probability measures μ for which

$$\mu([0, \infty[) \leq \ell.$$

This is solved by the Lagrange multipliers method.

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Histograms of the zeros of classical orthogonal polynomials. Let P be a polynomial of degree n with n real zeros $x_1, \dots, x_n$. Fix $h > 0$ and define

$$k_j = \#\{x_i \mid jh \leq x_i < (j+1)h\}.$$

To obtain the histogram of the zeros x_i with step h, one draws the colloction of the rectangles

$$R_j = \{(x, y) \mid jh \leq x < (j+1)h,\ 0 \leq y \leq k_j\}.$$

For getting the histogram of the zeros of the Legendre polynomial, the Hermite polynomial L_n, and the Laguerre polynomial, one has to compute the zeros $x_i^{(n)}$. For that one has used the fact that they are eigenvalues of a tridiagonal symmetric $n \times n$ matrix. Let (P_n) be a sequence of orthogonal polynomials, and consider the normalized polynomials

$$p_n(x) = \frac{1}{\|P_n\|} P_n(x).$$

The polynomials p_n satisfy a three terms recursion relation of the form

$$x p_n(x) = \alpha_n p_{n+1}(x) + \beta_n p_n(x) + \alpha_{n-1} p_{n-1}(x).$$

Consider the infinite dimensional tridiagonal symmetric matrix

$$T = \begin{pmatrix} \beta_0 & \alpha_0 & & & & \\ \alpha_0 & \beta_1 & \alpha_1 & & & \\ & \alpha_1 & \beta_2 & \alpha_2 & & \\ & & \alpha_2 & \beta_3 & \alpha_3 & \\ & & & \ddots & \ddots & \ddots \end{pmatrix},$$

and let T_n denote the $n \times n$ matrix of the first lines and n first rows of T. Then the eigenvalues of T_n are the zeros of p_n.

a) The Legendre polynomials P_n (special case of the Jacobi polynomials $P_n = P_n^{(0,0)}$) satisfy the relation

$$x P_n = \frac{n}{2n+1} P_{n-1} + \frac{n+1}{2n+1} P_{n+1},$$

and

$$\|P_n\|^2 = \int_{-1}^{1} P_n(x)^2 dx = \frac{2}{2n+1}.$$

Therefore the normalized polynomials $p_n = \frac{1}{\|P_n\|} P_n$ satisfy the relation

$$xp_n = \frac{n}{\sqrt{(2n+1)(2n-1)}} p_{n-1}(x) + \frac{n+1}{\sqrt{(2n+3)(2n+1)}} p_{n+1}(x).$$

In this case

$$\alpha_n = \frac{n+1}{\sqrt{(2n+1)(2n+3)}}, \ \beta_n = 0.$$

b) The Hermite polynomials H_n satisfy the relation

$$xH_n(x) = nH_{n-1}(x) + \frac{1}{2} H_{n+1}(x),$$

and

$$\|H_n\|^2 = \int_{-\infty}^{\infty} H_n(x)^2 e^{-x^2} dx = 2^n n! \sqrt{\pi}.$$

The normalized polynomials $p_n = \frac{1}{\|H_n\|} H_n$ satisfy

$$xp_n(x) = \sqrt{\frac{n}{2}} p_{n-1}(x) + \sqrt{\frac{n+1}{2}} p_{n+1}(x).$$

In this case

$$\alpha_n = \sqrt{\frac{n+1}{2}}, \ \beta_n = 0.$$

c) The Laguerre polynomials L_n ($L_n = L_n^{(0)}$) satisfy the relation

$$xL_n(x) = -nL_{n-1}(x) + (2n+1)L_n(x) - (n+1)L_{n+1}(x),$$

and they are normalized

$$\|L_n\|^2 = \int_0^{\infty} L_n(x)^2 e^{-x} dx = 1.$$

Therefore, in this case,

$$\alpha_n = -(n+1), \ \beta_n = 2n+1.$$

Histograms and graphics

The histograms and graphics have been realized by Marouane Rabaoui.

1. Histogram of the zeros of the Legendre polynomial P_n, for $n = 2000$.
2. Graphic of the density of the arcsinus law.
3. Histogram of the zeros of the Hermite polynomial H_n, for $n = 2000$.
4. Graphic of the density of the semicircle law.
5. Histogram of the zeros of the Laguerre polynomial L_n, for $n = 2000$.
6. Graphic of the density of the Marchenko-Pastur law, for $c = 1$.
7. Histogram of the eigenvalues of a random $n \times n$ symmetric matrix, for $n = 4000$, relatively to a Gaussian probability.