ASYMPTOTIC SPHERICAL ANALYSIS

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Abstract. *This is an attempt to present in a unified way results about asymptotics of spherical functions for large dimensions. We consider three cases: multivariate Bessel functions associated to the space of Hermitian matrices, characters of the unitary group, and multivariate Laguerre polynomials associated to the Heisenberg group.*

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Introduction

By asymptotic spherical analysis we mean the study of asymptotics for spherical functions of Gelfand pairs (G, K) as the Lie group G has large dimension. We will consider the general setting of an Olshanski spherical pair, i.e. an inductive limit (G, K) of an increasing sequence of Gelfand pairs $\big(G(n), K(n)\big)$,

$$G = \bigcup_{n=1}^{\infty} G(n), \quad K = \bigcup_{n=1}^{\infty} K(n),$$

and study the asymptotics of a sequence (φ_n) of spherical functions for $\big(G(n), K(n)\big)$, and identify the limit of such a sequence as a spherical function for the Olshanski spherical pair (G, K). We will consider the following three types.

1. Let $V(n)$ be an increasing sequence of real vector spaces, and, for each n, a compact group $K(n)$ acting linearly on $V(n)$. Then $\big(G(n), K(n)\big)$, where $G(n)$ is the generalized motion group $K(n) \ltimes V(n)$, is a Gelfand pair, and the associated spherical functions are generalized Bessel functions. The case of $V(n) = Herm(n, \mathbb{C})$, the space of $n \times n$ Hermitian matrices, and $K(n) = U(n)$, the unitary group, has been considered by Olshanski and Vershik.

2. If $G(n)/K(n)$ is a compact symmetric space, then the corresponding spherical functions can be written in terms of multivariate orthogonal polynomials associated to root systems. The unitary group $U(n)$ is a special case: $G(n) = U(n) \times U(n)$ and $K(n) \simeq U(n)$. Then the

corresponding spherical functions are normalized Schur functions. In this case the asymptotics have been determined by Kerov and Vershik. For the general case of an inductive limit of compact symmetric spaces, asymptotics have been obtained by Okounkov and Olshanski.

3. Let $V(n)$ be an increasing sequence of complex Euclidean vector spaces, and, for each n, $K(n)$ is a compact group acting unitarily on $V(n)$. Then $K(n)$ is a group of automorphisms of the Heisenberg group $H(n) = V(n) \times \mathbb{R}$. If $K(n)$ acts multiplicity free on the space of polynomials on $V(n)$, then $\big(G(n), K(n)\big)$, with $G(n) = K(n) \ltimes H(n)$, is a Gelfand pair. In some cases, the asymptotics of the associated spherical functions have been determined.

In the first case the results are due to Olshanski and Vershik. In the second case they are due to Kerov and Vershik. In fact we will present a method of proof due to Okounkov and Olshanski. The results in the third case are due to the author.

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1. Gelfand pairs and Olshanski spherical pairs

1.1 Spherical functions for a Gelfand pair. Let us first recall what is a spherical function for a Gelfand pair. A pair (G, K), where G is a locally compact group, and K a compact subgroup, is said to be a *Gelfand pair* if the convolution algebra $L^1(K\backslash G/K)$ of K-biinvariant integrable functions on G is commutative. Fix now a Gelfand pair (G, K). A *spherical function* is a continuous function φ on G which is K-biinvariant, with $\varphi(e) = 1$, and satisfies the functional equation

$$\int_K \varphi(xky)\alpha(dk) = \varphi(x)\varphi(y) \quad (x, y \in G),$$

where α is the normalized Haar measure on the compact group K. The characters χ of the commutative Banach algebra $L^1(K\backslash G/K)$ are of the form

$$\chi(f) = \int_G f(x)\varphi(x)m(dx),$$

where φ is a bounded spherical function (m is a Haar measure on the group G, which is unimodular since (G, K) is a Gelfand pair).

Let $\mathcal{P}(K\backslash G/K)$ denote the cone of K-biinvariant continuous functions on G of positive type. To a function $\varphi \in \mathcal{P}(K\backslash G/K)$ one associates by the Gelfand-Naimark-Segal construction a unitary representation $(\pi, \mathcal{H})$ with a cyclic K-invariant vector $u \in \mathcal{H}$ such that

$$\varphi(x) = \big(u|\pi(x)u\big).$$

The triple $(\pi, \mathcal{H}, u)$ is unique up to equivalence. Let $\mathcal{P}_1(K\backslash G/K)$ denote the convex set of the functions $\varphi \in \mathcal{P}(K\backslash G/K)$ with $\varphi(e) = 1$. For a function $\varphi \in \mathcal{P}_1(K\backslash G/K)$, the following properties are equivalent:

(1) φ is spherical,

(2) φ is extremal in $\mathcal{P}_1(K\backslash G/K)$,

(3) the representation $(\pi, \mathcal{H})$ associated to φ via the Gelfand-Naimark-Segal construction is irreducible.

If these properties hold, then $\dim\mathcal{H}^K = 1$, and the representation $(\pi, \mathcal{H})$ is said to be *spherical*, *i.e.* unitary, irreducible with $\dim \mathcal{H}^K = 1$. For the *spherical dual* Ω of the Gelfand pair (G, K), we can give three equivalent definitions:

(1) Ω is the set of spherical functions of positive type,

(2) Ω is the set of extremal points in the convex set $\mathcal{P}_1(K\backslash G/K)$,

(3) Ω is the set of equivalence classes of spherical representations.

On the spherical dual Ω one considers the topology of uniform convergence on compact sets of the corresponding spherical functions.

1.2 Spherical functions for an Olshanski spherical pair. Consider now an increasing sequence of Gelfand pairs $\big(G(n), K(n)\big)$:

$$G(n) \subset G(n+1),\ K(n) \subset K(n+1),\ K(n) = G(n) \cap K(n+1),$$

and define

$$G = \bigcup_{n=1}^{\infty} G(n), \quad K = \bigcup_{n=1}^{\infty} K(n).$$

We say that (G, K) is an *Olshanski spherical pair*. This general setting has been introduced and developped by Olshanski [1990]. A *spherical function* for the Olshanski spherical pair (G, K) is a continuous function φ on G, $\varphi(e) = 1$, which is K-biinvariant and satisfies

$$\lim_{n\to\infty} \int_{K(n)} \varphi(xky)\alpha_n(dk) = \varphi(x)\varphi(y) \quad (x, y \in G),$$

where α_n is the normalized Haar measure on $K(n)$. As in the case of a Gelfand pair, if φ is a spherical function of positive type, there exists a spherical representation $(\pi, \mathcal{H})$ of G (*i.e.* irreducible, unitary, with $\dim \mathcal{H}^K = 1$) such that

$$\varphi(x) = \big(u|\pi(x)u\big),$$

with $u \in \mathcal{H}^K$, $\|u\| = 1$. The previous equivalences (1), (2) and (3) hold for the spherical functions of positive type, and for the spherical dual Ω in case of a Olshanski spherical pair. Furthermore the spherical dual Ω is equipped with a topology.

We will consider the following question. Let Ω_n be the spherical dual for the Gelfand pair $\big(G(n), K(n)\big)$, and let us write a spherical function of positive type for $\big(G(n), K(n)\big)$ as $\varphi_n(\lambda; x)$ ($\lambda \in \Omega_n$, $x \in G(n)$). Further let Ω denote the spherical dual for the Olshanski spherical pair (G, K), and write a spherical function of positive type for (G, K) as $\varphi(\omega; x)$. For which sequences $(\lambda^{(n)})$, with $\lambda^{(n)} \in \Omega_n$, does it exist $\omega \in \Omega$ such that

$$\lim_{n\to\infty} \varphi_n(\lambda^{(n)}; x) = \varphi(\omega; x) \quad (x \in G) \ ?$$

In the cases we will consider there is, for each n, a map $T_n : \Omega_n \to \Omega$ such that, if

$$\lim_{n\to\infty} T_n(\lambda^{(n)}) = \omega$$

for the topology of Ω, then

$$\lim_{n\to\infty} \varphi_n(\lambda^{(n)}; x) = \varphi(\omega; x).$$

It is said that $(\lambda^{(n)})$ is a *Vershik-Kerov sequence.*

2. Generalized motion groups

2.1 Gelfand pair associated to a generalized motion group. Let V be a finite dimensional real Euclidean vector space, and K a closed subgroup of the orthogonal group $O(V)$. Define the generalized motion group $G = K \ltimes V$ with the product

$$(k_1, x_1)(k_2, x_2) = (k_1k_2, x_1 + k_1 \cdot x_2) \quad (k_1, k_2 \in K,\ x_1, x_2 \in V).$$

A K-biinvariant function on G can be seen as a K-invariant function on V, and, as convolution algebras, $L^1(K\backslash G/K) \simeq L^1(V)^K$. Therefore (G, K) is a Gelfand pair, and the functional equation for the spherical functions can be written in this case as

$$\int_K \varphi(x + k \cdot y)\alpha(dk) = \varphi(x)\varphi(y).$$

The spherical functions of positive type are Fourier transforms of K-orbital measures:

$$\varphi(\lambda; x) = \int_K e^{i(k\cdot\lambda|x)}\alpha(dk) \quad (\lambda \in V).$$

They are generalized Bessel functions. Hence the spherical dual Ω for the Gelfand pair (G, K) is the set of K-orbits : $\Omega \simeq K\backslash V/K$.

Example

For $V = \mathbb{R}^n$, $K = O(n)$, the K-orbits are spheres centered at 0. The spherical functions are ordinary Bessel functions of the norm $\|x\|$, and the spherical dual Ω can be identified with $[0, \infty[$.

2.2 The Gelfand pair $\big(U(n) \ltimes Herm(n, \mathbb{C}), U(n)\big)$. We will consider the case where $V(n) = Herm(n, \mathbb{C})$, the space of $n\times n$ Hermitian matrices, with the inner product $(x|y) = \mathrm{tr}(xy)$, and $K(n) = U(n)$, the unitary group, with the usual action on $Herm(n, \mathbb{C})$:

$$k \cdot x = kxk^*.$$

By the classical spectral theorem, every Hermitian matrix is diagonalizable in an orthonormal basis, and the eigenvalues are real. Hence each K-orbit meets a real diagonal matrix. The spherical dual Ω_n can be identified with $\mathbb{R}^n/\mathfrak{S}_n$, where $\mathfrak{S}_n$ is the symmetric group acting on $\mathbb{R}^n$ by permuting the coordinates.

The spherical functions of positive type for the Gelfand pair $\big(G(n), K(n)\big)$ are the following Fourier integrals:

$$\varphi_n(\lambda; x) = \int_{U(n)} e^{i\,\mathrm{tr}(xu\lambda u^*)}\alpha(du),$$

where $\lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ with $\lambda_i \in \mathbb{R}$.

The Itzykson-Zuber integral

$$I(x, y) = \int_{U(n)} e^{\mathrm{tr}(xuyu^*)}\alpha(du)$$

can be evaluated:

$$I(x, y) = \delta! \frac{1}{V(x)V(y)} \det(e^{x_j y_k})_{1\leq j,k\leq n},$$

for $x = \mathrm{diag}(x_1, \ldots, x_n)$, $y = \mathrm{diag}(y_1, \ldots, y_n)$. We use the following notation:

$$V(x) = \prod_{j<k}(x_j - x_k)$$

is the Vandermonde polynomial, $\delta = (n-1, \dots, 1, 0)$, and for a signature $\mathbf{m} = (m_1, \dots, m_n)$, $m_1 \geq m_2 \geq \dots \geq m_n \geq 0$,

$$\mathbf{m}! = m_1! \dots m_n!$$

Hence

$$\varphi_n(\lambda; x) = \delta! \frac{1}{V(\lambda)V(ix)} \det(e^{i\lambda_j x_k})_{1 \leq j,k \leq n}.$$

2.3 The Olshanski spherical pair $\big(U(\infty) \ltimes Herm(\infty, \mathbb{C}), U(\infty)\big)$. The following Olshanski spherical pair (G, K),

$$G = \bigcup_{n=1}^{\infty} G(n), \quad K = \bigcup_{n=1}^{\infty} K(n),$$

is the topic of [Pickrell,1991], and [Olshanski-Vershik,1996]. In this case

$$G = U(\infty) \ltimes Herm(\infty, \mathbb{C}), \quad K = U(\infty),$$

with the notation

$$Herm(\infty, \mathbb{C}) = \bigcup_{n=1}^{\infty} Herm(n, \mathbb{C}), \quad U(\infty) = \bigcup_{n=1}^{\infty} U(n).$$

A spherical function φ for the Olshanski spherical pair (G, K) can be seen as a $U(\infty)$-invariant function on $Herm(\infty, \mathbb{C})$, with $\varphi(0) = 1$, such that

$$\lim_{n \to \infty} \int_{U(n)} \varphi(x + uxu^*) \alpha_n(du) = \varphi(x)\varphi(y),$$

where α_n is the normalized Haar measure of $U(n)$.

A *Pólya function* is a continuous function Φ on $\mathbb{R}$ with $\Phi(0) = 1$, such that the function φ, defined on $Herm(\infty, \mathbb{C})$ by

$$\varphi(x) = \det \Phi(x),$$

is of positive type. The notation $\Phi(x)$ is defined via the functional calculus. In particular, for a diagonal matrix,

$$\varphi\big(\mathrm{diag}(x_1, \dots, x_n, 0, \dots)\big) = \Phi(x_1) \dots \Phi(x_n).$$

Observe that the function φ is K-invariant.

THEOREM 2.1 ([PICKRELL,1991]). — *The Pólya functions are the following ones*

$$\Phi(z) = e^{i\beta z} e^{-\frac{1}{2}\gamma z^2} \prod_{k=1}^{\infty} \frac{e^{-i\alpha_k z}}{1 - i\alpha_k z},$$

with

$$\beta \in \mathbb{R},\ \gamma \geq 0,\ \alpha_k \in \mathbb{R},\ \sum_{k=1}^{\infty} \alpha_k^2 < \infty.$$

Since the spherical functions are of the form $\varphi(x) = \det \Phi(x)$, with a complex valued continuous function Φ defined on $\mathbb{R}$, one obtains:

COROLLARY 2.2 ([PICKRELL, 1991]). — *The spherical functions of positive type for the Olshanski spherical pair*

$$\big(U(\infty) \ltimes Herm(\infty, \mathbb{C}), U(\infty)\big)$$

are precisely the functions

$$\varphi(x) = \det \Phi(x),$$

where Φ is a Pólya function.

Theorem 2.1 is related to results by Schoenberg. A measurable function f on $\mathbb{R}$ is said to be *totally positive* if,

$$\det\big(f(s_i - t_j)\big) \geq 0,$$

for $s_1 < \cdots < s_n$, $t_1 < \cdots < t_n$. A Pólya function, with

$$\gamma + \sum_{k=1}^{\infty} \alpha_k^2 > 0,$$

is the Fourier transform of an integrable totally positive function f on $\mathbb{R}$ with

$$\int_{-\infty}^{\infty} f(t)dt = 1.$$

([Schoenberg, 1951], see also [Faraut,2006].)

The name *Pólya function* comes from the following: Pólya proved that an entire function Ψ with $\Psi(0) = 1$ which is a uniform limit of polynomials with only real zeros is of the form

$$\Psi(z) = e^{-\beta z} e^{-\frac{1}{2}z^2} \prod_{k=1}^{\infty} e^{\alpha_k z}(1 - \alpha_k z),$$

with $\alpha(\alpha_k)$, β, and γ as before ([Pólya,1913]), so that

$$\Phi(\lambda) = \frac{1}{\Psi(i\lambda)}$$

is a Pólya function.

Hence the spherical dual Ω can be identified with the set of triples $\omega = (\alpha, \beta, \gamma)$, with

$$\beta \in \mathbb{R},\ \gamma \geq 0,\ \alpha_k \in \mathbb{R},\ \sum_{k=1}^{\infty} \alpha_k^2 < \infty.$$

(One identifies two triples $\omega = (\alpha, \beta, \gamma)$ and $\omega' = (\alpha', \beta, \gamma)$ if the sets $\{\alpha_k\}$ and $\{\alpha'_k\}$ are the same. We will write

$$\Phi(\omega; z) = e^{i\beta z} e^{-\frac{1}{2}\gamma z^2} \prod_{k=1}^{\infty} \frac{e^{-i\alpha_k z}}{1 - i\alpha_k z} \quad (z \in \mathbb{R}),$$
$$\varphi(\omega; x) = \det \Phi(\omega; x) \quad \big(x \in Herm(\infty, \mathbb{C})\big).$$

For a continuous function f on $\mathbb{R}$ we define the function L_f on Ω by

$$L_f(\omega) = \gamma f(0) + \sum_{k=1}^{\infty} \alpha_k^2 f(\alpha_k),$$

and we consider on Ω the initial topology associated to the functions L_f, and the function $\omega \mapsto \beta$. Then, for z fixed, the function $\omega \mapsto \Phi(\omega; z)$ is continuous on Ω.

2.4 Asymptotics for the spherical functions. We will state in this section the main result of the chapter, and describe in next sections the steps in the proof. For each n let T_n be the following map from the spherical dual $\Omega_n \simeq \mathbb{R}^n$ of the Gelfand pair $\big(G(n), K(n)\big)$ into the spherical dual Ω of the Olshanski spherical pair (G, K):

$$T_n :\ \lambda = (\lambda_1, \ldots, \lambda_n) \mapsto \omega = (\alpha, \beta, \gamma),$$

with

$$\alpha_k = \frac{\lambda_k}{n} \text{ if } 1 \leq k \leq n,\ \alpha_k = 0 \text{ if } k > n,\ \beta = \frac{\lambda_1 + \cdots + \lambda_n}{n},\ \gamma = 0.$$

THEOREM 2.3 (OLSHANSKI-VERSHIK). — *Consider a sequence* $(\lambda^{(n)})$ *with* $\lambda^{(n)} \in \Omega_n$.

(i) *Assume that, for the topology of* Ω,

$$\lim_{n\to\infty} T_n(\lambda^{(n)}) = \omega.$$

Then, for $x \in Herm(\infty, \mathbb{C})$,

$$\lim_{n\to\infty} \varphi_n(\lambda^{(n)}, x) = \det \Phi(\omega; x),$$

uniformly on compact sets.

(ii) *Conversely, assume that*

$$\lim_{n\to\infty} \varphi_n(\lambda^{(n)}; x) = \varphi(x),$$

uniformly on compact sets. Then the sequence $T_n(\lambda^{(n)})$ *converges,*

$$\lim_{n\to\infty} T_n(\lambda^{(n)}) = \omega,$$

and $\varphi(x) = \det \Phi(\omega; x)$.

[Olshanski-Vershik,1996].

Hence we have in this case a description of the Vershik-Kerov sequences $(\lambda^{(n)})$: the sequence $(\lambda^{(n)})$ is a Vershik-Kerov sequence if and only if the sequence $T_n(\lambda^{(n)})$ converges for the topology of Ω.

2.5 Schur expansions. For a signature $\mathbf{m} = (m_1, \ldots, m_n)$, $m_i \in \mathbb{Z}$, $m_1 \geq m_2 \geq \cdots \geq m_n$, one defines the rational function on $(\mathbb{C}^*)^n$

$$A_{\mathbf{m}}(z) = \begin{vmatrix} z_1^{m_1} & z_1^{m_2} & \ldots & z_1^{m_n} \\ z_2^{m_1} & z_2^{m_2} & \ldots & z_2^{m_n} \\ \vdots & \vdots & & \vdots \\ z_n^{m_1} & z_n^{m_2} & \ldots & z_n^{m_n} \end{vmatrix}.$$

In particular, for $\mathbf{m} = \delta = (n-1, \ldots, 1, 0)$, $A_\delta(z)$ is the Vandermonde polynomial

$$A_\delta(z) = V(z) = \prod_{j<k} (z_j - z_k).$$

The *Schur function* $s_{\mathbf{m}}$ is defined by

$$s_{\mathbf{m}}(z) = \frac{A_{\mathbf{m}+\delta}(z)}{V(z)}.$$

This is a symmetric rational function on $(\mathbb{C}^*)^n$. For a positive signature $\mathbf{m} = (m_1, \dots, m_n)$, $m_1 \geq \cdots \geq m_n \geq 0$, $s_{\mathbf{m}}$ is a symmetric polynomial, homogeneous of degree $|\mathbf{m}| = m_1 + \cdots + m_n$.

Proposition 2.4 (Hua's formula). — *Consider n power series*

$$f_i(w) = \sum_{m=0}^{\infty} c_m^{(i)} w^m \quad (w \in \mathbb{C},\ i = 1, \dots, n),$$

which are convergent for $|w| < r$ for some $r > 0$. Define the function F on $\mathbb{C}^n$ by

$$F(z) = F(z_1, \dots, z_n) = \frac{\det\big(f_i(z_j)\big)_{1 \leq i,j \leq n}}{V(z)} \quad (|z_j| < r).$$

Then F admits the following Schur expansion

$$F(z) = \sum_{m_1 \geq \cdots \geq m_n \geq 0} a_{\mathbf{m}} s_{\mathbf{m}}(z),$$

with

$$a_{\mathbf{m}} = \det(c^{(i)}_{m_j+n-j})_{1 \leq i,j \leq n}.$$

([Hua,1963], Chapter II.)

Proof.
In fact

$$\det\big(f_i(z_j)\big)_{1 \leq i,j \leq n} = \sum_{\sigma \in \mathfrak{S}_n} \varepsilon(\sigma) \prod_{i=1}^{n} \Big(\sum_{m=0}^{\infty} c_m^{(i)} z_{\sigma(i)}^m \Big).$$

By permuting the product and the sum we obtain

$$\begin{aligned} &= \sum_{m_1, \dots, m_n = 0}^{\infty} c^{(1)}_{m_1} \dots c^{(n)}_{m_n} \sum_{\sigma \in \mathfrak{S}_n} \varepsilon(\sigma) \prod_{i=1}^{n} z_{\sigma(i)}^{m_i} \\ &= \sum_{m_1, \dots, m_n = 0}^{\infty} c^{(1)}_{m_1} \dots c^{(n)}_{m_n} \det(z_j^{m_i})_{1 \leq i,j \leq n}. \end{aligned}$$

Since $\det(z_j^{m_i}) = 0$ unless the m_i are all distinct, this sum is equal to

$$\begin{aligned} &= \sum_{m_1 > \cdots > m_n \geq 0} \sum_{\tau \in \mathfrak{S}_n} c^{(1)}_{m_{\tau(1)}} \dots c^{(n)}_{m_{\tau(n)}} \det(z_j^{m_{\tau(i)}})_{1 \leq i,j \leq n} \\ &= \sum_{m_1 > \cdots > m_n \geq 0} \sum_{\tau \in \mathfrak{S}_n} \varepsilon(\tau) c^{(1)}_{m_{\tau(1)}} \dots c^{(n)}_{m_{\tau(n)}} \det(z_j^{m_i})_{1 \leq i,j \leq n} \\ &= \sum_{m_1 > \cdots > m_n \geq 0} \det(c^{(i)}_{m_j})_{1 \leq i,j \leq n} \det(z_j^{m_i})_{1 \leq i,j \leq n}. \end{aligned}$$

Finally, with $m_j = k_j + n - j$, we obtain

$$\det\big(f_i(z_j)\big)_{1\leq i,j\leq n} = \sum_{k_1\geq\cdots\geq k_n\geq 0} \det(c^{(i)}_{k_j+n-j})_{1\leq i,j\leq n} A_{\mathbf{k}+\delta}(z),$$

which is the formula of the proposition. ∎

By applying Hua's formula with

$$f_i(w) = e^{x_i w} = \sum_{m=0}^{\infty} \frac{x_i^m}{m!} w^m,$$

one obtains a Schur expansion for the Itzykson-Zuber integral

$$I(x,y) = \int_{U(n)} e^{\mathrm{tr}(xuyu^*)} \alpha(du) \quad \big(x, y \in Herm(n,\mathbb{C})\big).$$

Recall that, for $x = \mathrm{diag}(x_1, \ldots, x_n)$, $y = \mathrm{diag}(y_1, \ldots, y_n)$,

$$I(x,y) = \delta! \frac{1}{V(x)V(y)} \det(e^{x_j y_k})_{1\leq j,k\leq n}.$$

Then, taking

$$c^{(i)}_m = \frac{x_i^m}{m!},$$

one obtains

$$\det(c^{(i)}_{m_j+n-j})_{1\leq i,j\leq n} = \frac{1}{(\mathbf{m}+\delta)!} A_{\mathbf{m}+\delta}(x) = \frac{1}{(\mathbf{m}+\delta)!} V(x) s_{\mathbf{m}}(x),$$

and

$$I(x,y) = \sum_{m_1\geq\cdots\geq m_n\geq 0} \frac{\delta!}{(\mathbf{m}+\delta)!} s_{\mathbf{m}}(x) s_{\mathbf{m}}(y).$$

Therefore:

PROPOSITION 2.5. — *The spherical functions of positive type for the Gelfand pair $\big(G(n), K(n)\big)$ admit the following Schur expansions: for $x = \mathrm{diag}(x_1, \ldots, x_n, 0, \ldots)$,*

$$\varphi_n(\lambda; x) = \sum_{m_1\geq\cdots\geq m_n\geq 0} \frac{\delta!}{(\mathbf{m}+\delta)!} s_{\mathbf{m}}(\lambda) s_{\mathbf{m}}(ix).$$

For writing down the power series expansion of the Pólya function $\Phi(\omega; z)$, one introduces an algebra morphism from the algebra Λ of symmetric functions into the algebra $\mathcal{C}(\Omega)$ of continuous functions on Ω, $f \mapsto \tilde{f}$. Since the Newton power sums p_m,

$$p_m(\xi) = \xi_1^m + \cdots + \xi_n^m + \cdots,$$

generate the algebra Λ, this morphism is determined by the data of the images $\widetilde{p_m}$. By definition this morphism is such that, for $\omega = (\alpha, \beta, \gamma)$,

$$\widetilde{p_1}(\omega) = \beta, \ \widetilde{p_2}(\omega) = \gamma + \sum_{k=1}^{\infty} \alpha_k^2,$$

and, for $m \geq 3$,

$$\widetilde{p_m}(\omega) = \sum_{k=1}^{\infty} \alpha_k^m.$$

THEOREM 2.6. — (i) *The Pólya function $\Phi(\omega, z)$ admits the following power series expansion*

$$\Phi(\omega; z) = \sum_{m=0}^{\infty} \widetilde{h_m}(\omega)(iz)^m.$$

(ii) *For $x = \mathrm{diag}(x_1, \ldots, x_n, 0, \ldots)$,*

$$\det \Phi(\omega, x) = \sum_{m_1 \geq \cdots \geq m_n \geq 0} \widetilde{s_{\mathbf{m}}}(\omega) s_{\mathbf{m}}(ix).$$

Proof. The complete symmetric function h_m is defined by

$$h_m(\xi) = \sum_{|\alpha|=m} \xi^\alpha \quad (\xi^\alpha = \xi_1^{\alpha_1} \ldots \xi_n^{\alpha_n}).$$

For the signature $[m] = (m, 0, \ldots)$, $s_{[m]}(\xi) = h_m(\xi)$. Recall the generating function for the complete symmetric functions h_m:

$$H(\xi; z) = \sum_{m=0}^{\infty} h_m(\xi) z^m = \prod_j \frac{1}{1 - \xi_j z} \quad (z \in \mathbb{C}).$$

Taking the logarithmic derivatives one obtains, for z small enough,

$$\frac{d}{dz} \log H(\xi; z) = -\frac{d}{dz} \sum_j \log(1 - \xi_j z) = \sum_j \frac{\xi_j}{1 - \xi_j z} = \sum_{m=0}^{\infty} p_{m+1}(\xi) z^m,$$

and also

$$\exp\Big(\sum_{m=1}^{\infty} \frac{1}{m} p_m(\xi) z^m\Big) = \sum_{m=0}^{\infty} h_m(\xi) z^m.$$

For a sequence (a_m) $(m \geq 1)$ of complex numbers, let us consider the following expansions

$$\exp\Big(\sum_{m=1}^{\infty} \frac{1}{m} a_m z^m\Big) = 1 + \sum_{m=1}^{\infty} A_m z^m.$$

The coefficient A_m is a polynomial in the coefficients $a_1, \ldots, a_m$:

$$A_m = Q_m(a_1, \ldots, a_m).$$

In particular

$$A_1 = a_1,\ A_2 = \frac{1}{2}(a_1^2 + a_2),$$

and, for all ξ,

$$h_m(\xi) = Q_m\big(p_1(\xi), \ldots, p_m(\xi)\big).$$

Recall the product formula for the Pólya function:

$$\Phi(\omega; z) = e^{i\beta z} e^{-\frac{1}{2}\gamma z^2} \prod_{k=1}^{\infty} \frac{e^{-i\alpha_k z}}{1 - i\alpha_k z}.$$

Let us take its logarithmic derivative:

$$\begin{aligned}\frac{d}{dz} \log \Phi(\omega; z) &= i\beta + i\big(\gamma + p_2(\alpha)\big) iz + i \sum_{m=2}^{\infty} p_{m+1}(\alpha)(iz)^m \\ &= i \sum_{m=0}^{\infty} \widetilde{p_{m+1}}(\omega)(iz)^m.\end{aligned}$$

The Pólya function admits a power expansion near 0:

$$\Phi(\omega; z) = 1 + \sum_{m=1}^{\infty} q_m(\omega)(iz)^m.$$

We have to show that $q_m(\omega) = \widetilde{h_m}(\omega)$. The following identity holds

$$\Phi(\omega; z) = 1 + \sum_{m=1}^{\infty} q_m(\omega)(iz)^m = \exp\Big(\sum_{m=1}^{\infty} \frac{1}{m} \widetilde{p_m}(\omega)(iz)^m\Big).$$

Therefore

$$q_m(\omega) = Q_m\big(\tilde{p}_1(\omega), \ldots, \tilde{p}_m(\omega)\big).$$

Since the map $f \mapsto \tilde{f}$ is an algebra morphism,

$$q_m = Q_m(\widetilde{p_1, \ldots}, p_m) = \widetilde{h_m}. \qquad \square$$

2.6 Proof of Theorem 2.3, part (i). In this proof one applies the following result of classical harmonic analysis:

PROPOSITION 2.7. — *Let ψ_n be a sequence of $\mathcal{C}^\infty$-functions on $\mathbb{R}^d$ of positive type with $\psi_n(0) = 1$, and ψ an analytic function on a neighborhood of 0. Assume that, for every $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d$,*

$$\lim_{n\to\infty} \partial^\alpha \psi_n(0) = \partial^\alpha \psi(0).$$

Then ψ has an analytic extension to $\mathbb{R}^d$, and ψ_n converges to ψ uniformly on compact subsets of $\mathbb{R}^d$.
(Proposition 3.11 in [Faraut,2008].)

Consider a sequence $(\lambda^{(n)})$ with $\lambda^{(n)} \in \Omega_n \simeq \mathbb{R}^n$. Assume that, for the topology of Ω,

$$\lim_{n\to\infty} T_n(\lambda_n^{(n)}) = \omega.$$

We will show that the Taylor coefficients at 0 of the function $\varphi_n(\lambda^{(n)}; x)$ ($x \in Herm(k, \mathbb{C})$) converge to the ones of the function $\det \Phi(\omega; x)$. In fact we will prove

PROPOSITION 2.8. — *Consider a sequence $(\lambda^{(n)})$ with $\lambda^{(n)} \in \Omega_n \simeq \mathbb{R}^n$. Assume that, for the topology of Ω,*

$$\lim_{n\to\infty} T_n(\lambda^{(n)}) = \omega.$$

Then, for every symmetric function $f \in \Lambda$, homogeneous of degree m,

$$\lim_{n\to\infty} \frac{1}{n^m} f(\lambda^{(n)}) = \tilde{f}(\omega).$$

Proof.

It is enough to prove the result in case of $f = p_m$, since the Newton power sums generate the algebra Λ of symmetric functions.

For $m = 1$,

$$p_1(\lambda^{(n)}) = \lambda_1^{(n)} + \cdots + \lambda_n^{(n)}, \quad \widetilde{p_1}(\omega) = \beta.$$

By assumption,

$$\lim_{n\to\infty} \frac{1}{n} p_1(\lambda^{(n)}) = \beta = \widetilde{p_1}(\omega).$$

For $m = 2$,

$$p_2(\lambda^{(n)}) = (\lambda^{(n)})^2 + \cdots + (\lambda_n^{(n)})^2, \quad \widetilde{p_2}(\omega) = \gamma + \sum_{k=1}^{\infty} \alpha_k^2.$$

The assumption means that, for every continuous function φ on $\mathbb{R}$,

$$\lim_{n\to\infty} \sum_{j=1}^{n} \Big(\frac{\lambda_j^{(n)}}{n}\Big)^2 \varphi\Big(\frac{\lambda_j^{(n)}}{n}\Big) = \gamma\varphi(0) + \sum_{k=1}^{\infty} \alpha_k^2 \varphi(\alpha_k).$$

Taking $\varphi \equiv 1$ one obtains

$$\lim_{n\to\infty} \sum_{j=1}^{n} \Big(\frac{\lambda_j^{(n)}}{n}\Big)^2 = \gamma + \sum_{k=1}^{\infty} \alpha_k^2, \quad \text{or} \quad \lim_{n\to\infty} \frac{1}{n^2} p_2(\lambda^{(n)}) = \widetilde{p_2}(\omega).$$

For $m \geq 3$, take $\varphi(s) = s^{n-2}$ (observe that $\varphi(0) = 0$). $\square$

Proof of Theorem 2.3, part (i)

Recall Proposition 2.5: for $x = \mathrm{diag}(x_1, \ldots, x_k, 0, \ldots)$,

$$\varphi_n(\lambda^{(n)}; x) = \sum_{m_1 \geq \cdots \geq m_k \geq 0} \frac{\delta!}{(\mathbf{m}+\delta)!} s_{\mathbf{m}}(\lambda^{(n)}) s_{\mathbf{m}}(ix).$$

For $\mathbf{m} = (m_1, \ldots, m_k)$ fixed,

$$\frac{\delta!}{(\mathbf{m}+\delta)!} \sim \frac{1}{n^{|\mathbf{m}|}} \quad (n \to \infty),$$

where $|\mathbf{m}| = m_1 + \cdots + m_k$. Hence, by Proposition 2.8,

$$\lim_{n\to\infty} \frac{\delta!}{(\mathbf{m}+\delta)!} s_{\mathbf{m}}(\lambda^{(n)}) = \widetilde{s_{\mathbf{m}}}(\omega).$$

We apply Proposition 2.7 with

$$\psi_n(x) = \varphi_n(\lambda^{(n)}; x) = \sum_{m_1 \geq \cdots \geq m_k \geq 0} \frac{\delta!}{(\mathbf{m}+\delta)!} s_{\mathbf{m}}(\lambda^{(n)}) s_{\mathbf{m}}(ix),$$

and

$$\psi(x) = \det \Phi(\omega; x) = \sum_{m_1 \geq \cdots \geq m_k \geq 0} \widetilde{s_{\mathbf{m}}}(\omega) s_{\mathbf{m}}(ix).$$

This finishes the proof of (i).

2.7 Proof of Theorem 2.3, part (ii). We assume that

$$\lim_{n\to\infty} \varphi_n(\lambda^{(n)}; x) = \varphi(x),$$

uniformly on compact sets in $Herm(\infty; \mathbb{C})$. We will show that $(\lambda^{(n)})$ is a Vershik-Kerov sequence, i.e. the sequence $T_n(\lambda^{(n)})$ converges in Ω. The function ψ_n defined on $\mathbb{R}$ by

$$\psi_n(\tau) = \varphi_n(\lambda^{(n)}; x) \text{ with } x = \mathrm{diag}(\tau, 0, \ldots),$$

i.e. the restriction of $x \mapsto \varphi(\lambda^{(n)}; x)$ to $Herm(1, \mathbb{C}) \simeq \mathbb{R}$, being a function of positive type, is the Fourier transform of a probability measure μ_n on $\mathbb{R}$, by Bochner's theorem:

$$\psi_n(\tau) = \int_{\mathbb{R}} e^{it\tau} \mu_n(dt).$$

Furthermore

$$\lim_{n\to\infty} \psi_n(\tau) = \psi(\tau),$$

uniformly on compacts sets in $\mathbb{R}$, where

$$\psi(\tau) = \varphi\big(\mathrm{diag}(\tau, 0, \ldots)\big).$$

The function ψ is the Fourier transform of a probability measure μ, which is the weak limit of the sequence (μ_n) (limit for the tight topology), by Lévy-Cramér's theorem.

From the Schur expansion of $\varphi_n(\lambda; x)$ one obtains the power expansion of ψ_n:

$$\psi_n(\tau) = \sum_{k=0}^{\infty} \frac{(n-1)!}{(k+n-1)!} h_k(\lambda^{(n)}) (i\tau)^k,$$

and therefore the moments of μ_n:

$$\mathfrak{M}_k(\mu_n) = k!\frac{(n-1)!}{(k+n-1)!}h_k(\lambda^{(n)}).$$

LEMMA 2.9. — *Let $\mathcal{M}$ be a set of probability measures on $\mathbb{R}$, relatively compact for the weak topology (tight topology). Assume that, for every $\mu \in \mathcal{M}$,*

$$\mathfrak{M}_4(\mu) = \int_{\mathbb{R}} x^4 \mu(dx) < \infty,$$

and that there is a constant $A > 0$ such that, for every $\mu \in \mathcal{M}$,

$$\mathfrak{M}_4(\mu) \leq A\big(\mathfrak{M}_2(\mu)\big)^2.$$

Then there is a constant $C > 0$ such that, for every $\mu \in \mathcal{M}$,

$$\mathfrak{M}_2(\mu) \leq C.$$

([Okounkov-Olshanski,1998c], Lemma 5.2.)

Proof.
Since $\mathcal{M}$ is relatively compact, for $0 < \varepsilon < \frac{1}{A}$, there is $R > 0$ such that, for every $\mu \in \mathcal{M}$,

$$\mu\big(\{|x| > R\}\big) \leq \varepsilon.$$

By the Schwarz inequality,

$$\Big(\int_{|x|>R} x^2 \mu(dx)\Big)^2 \leq \varepsilon \mathfrak{M}_4(\mu) \leq \varepsilon A\ \big(\mathfrak{M}_2(\mu)\big)^2.$$

Therefore

$$\mathfrak{M}_2(\mu) \leq R^2 + \int_{|x|>R} |x|^2 \mu(dx) \leq R^2 + \sqrt{\varepsilon A}\ \mathfrak{M}_2(\mu),$$

or

$$\mathfrak{M}_2(\mu) \leq \frac{R^2}{1 - \sqrt{\varepsilon A}}.$$

∎

We continue the proof of Theorem 2.3, part (ii). Let us compute the moments of μ_n of order 2 and 4:

$$\mathfrak{M}_2(\mu_n) = 2\,\frac{1}{n(n+1)} h_2(\lambda^{(n)}),$$

$$\mathfrak{M}_4(\mu_n) = 24\,\frac{1}{n(n+1)(n+2)(n+3)} h_4(\lambda^{(n)}).$$

Since there is a constant $A_0 > 0$ such that

$$h_4(x) \leq A_0 h_2(x)^2,$$

there is a constant $A > 0$ such that

$$\mathfrak{M}_4(\mu_n) \leq A\big(\mathfrak{M}_2(\mu_n)\big)^2.$$

By Lemma 2.8 there is a constant $C > 0$ such that

$$\mathfrak{M}_2(\mu_n) \leq C.$$

It follows that there is a constant $R > 0$ such that, if $T_n(\lambda^{(n)}) = \omega_n$, then

$$|\omega_n| \leq R.$$

For $\omega = (\alpha, \beta, \gamma)$, we use the notation

$$|\omega| = \sqrt{\sum_{k=1}^{\infty} \alpha_k^2 + \beta^2 + \gamma}.$$

In fact

$$|\omega_n|^2 = \frac{2}{n^2} h_2(\lambda^{(n)}) = \frac{n(n+1)}{n^2} \mathfrak{M}_2(\mu_n),$$

by the identity

$$2h_2(\xi) = p_1(\xi)^2 + p_2(\xi).$$

Since the set $\Omega_R = \{\omega \in \Omega \mid |\omega| \leq R\}$ is compact for the topology of Ω, it follows that there is a subsequence ω_{n_j} which converges in Ω to ω_0. Then, by the part (i), $\varphi(x) = \varphi(\omega_0; x)$. Hence all converging subsequences have the same limit, and therefore the sequence ω_n itself converges. ∎

3. Infinite dimensional unitary group

3.1 Gelfand pair associated to a compact group. Let U be a compact group, and define $G = U \times U$, $K = \{(u, u) \in G \mid u \in U\}$. The convolution algebra $L^1(K\backslash G/K)$ of K-biinvariant integrable functions F on G can be identified to the convolution algebra $L^1_{\mathrm{central}}(U)$ of central integrable functions f on U. The identification is given by $F(u, v) = f(uv^{-1})$. Hence (G, K) is a Gelfand pair, since the algebra $L^1_{\mathrm{central}}(U)$ is commutative. A spherical function can be seen as a central continuous function on U which satisfies the following functional equation

$$\int_U \varphi(xuyu^{-1})\alpha(du) = \varphi(x)\varphi(y) \quad (x, y \in U).$$

Let $\hat{U}$ be the set of equivalence classes of irreducible representations of U. For each $\lambda \in \hat{U}$, let π_λ be a representation of U in the class λ on a vector space $\mathcal{H}_\lambda$. Let χ_λ denotes its character:

$$\chi_\lambda(x) = \operatorname{tr}\bigl(\pi_\lambda(x)\bigr) \quad (x \in U).$$

Then $d_\lambda = \chi_\lambda(e)$ is the dimension of $\mathcal{H}_\lambda$. The character χ_λ satisfies the following functional equation:

$$\int_U \chi_\lambda(xuyu^{-1})\alpha(du) = \frac{1}{d_\lambda}\chi_\lambda(x)\chi_\lambda(y) \quad (x, y \in U).$$

Hence

$$\varphi(\lambda; x) = \frac{\chi_\lambda(x)}{\chi_\lambda(e)}$$

is a spherical function. One shows that all spherical functions are obtained in that way. Therefore the spherical dual of the Gelfand pair (G, K) is identified to $\hat{U}$.

3.2 The unitary group $U(n)$. We consider the case where U is the unitary group $U(n)$. For $n = 1$,

$$U(1) = \mathbb{T} = \{t \in \mathbb{C} \mid |t| = 1\}.$$

The subgroup of diagonal matrices in $U(n)$ is identified with $\mathbb{T}^n$.

We recall the Weyl's character formula and dimension formula. The unitary dual $\widehat{U(n)}$ is parametrized by signatures:

$$\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n, \quad \lambda_1 \geq \cdots \geq \lambda_n.$$

The corresponding character χ_λ agrees with the Schur function s_λ on $\mathbb{T}^n$:

$$\chi_\lambda\bigl(\operatorname{diag}(t_1, \ldots, t_n)\bigr) = s_\lambda(t) = \frac{\det(t_i^{\lambda_j + n - j})_{1 \leq i,j \leq n}}{V(t)},$$

and the dimension of the representation is given by

$$d_\lambda = s_\lambda(1, \ldots, 1) = \frac{V(\lambda + \delta)}{V(\delta)}.$$

Recall that $\delta = (n-1, \ldots, 1, 0)$, and V is the Vandermonde polynomial.

The spherical dual Ω_n of the Gelfand pair $\big(U(n) \times U(n), U(n)\big)$ is identified to the set of signatures $\lambda = (\lambda_1, \ldots, \lambda_n)$ of length $\leq n$. The corresponding spherical function is the normalized character:

$$\varphi_n(\lambda; u) = \frac{\chi_\lambda(u)}{\chi_\lambda(e)}.$$

Its restriction to the subgroup $\mathbb{T}^n$ of diagonal matrices is given by

$$\varphi_n\big(\lambda; \mathrm{diag}(t_1, \ldots, t_n)\big) = \frac{s_\lambda(t_1, \ldots, t_n)}{s_\lambda(1, \ldots, 1)}.$$

3.3 The infinite dimensional unitary group, Voiculescu functions. We consider now the increasing sequence of Gelfand pairs

$$G(n) = U(n) \times U(n),\ K(n) = \{(u, u) \mid u \in U(n)\} \simeq U(n),$$

and the inductive limit, the Olshanski spherical pair (G, K):

$$G = \bigcup_{n=1}^{\infty} G(n) = U(\infty) \times U(\infty),$$
$$K = \bigcup_{n=1}^{\infty} K(n) = \{(u, u) \mid u \in U(\infty)\} \simeq U(\infty).$$

A spherical function for the pair (G, K) can be seen as a continuous central function φ on $U(\infty)$ such that, for $x, y \in U(\infty)$,

$$\lim_{n \to \infty} \int_{U(n)} \varphi(xuyu^*) \alpha_n(du) = \varphi(x)\varphi(y).$$

Let us first state a basic result by Voiculescu. Consider a power series

$$\Phi(z) = \sum_{m=0}^{\infty} c_m z^m,$$

with

$$c_m \geq 0,\ \sum_{m=0}^{\infty} c_m = 1.$$

The series converges for $|z| \leq 1$, and Φ is a continous function of positive type on

$$U(1) = \{t \in \mathbb{C} \mid |t| = 1\},$$

with $\Phi(1) = 1$. We say that Φ is a *Voiculescu function* if the function φ defined on $U(\infty)$ by $\varphi(u) = \det \Phi(u)$ is of positive type.

THEOREM 3.1. — *The Voiculescu functions are the following ones*

$$\Phi(z) = e^{\gamma(z-1)} \prod_{k=1}^{\infty} \frac{1 + \beta_k(z-1)}{1 - \alpha_k(z-1)},$$

with $\alpha_k \geq 0$, $0 \leq \beta_k \leq 1$, $\gamma \geq 0$,

$$\sum_{k=1}^{\infty} \alpha_k < \infty, \quad \sum_{k=1}^{\infty} \beta_k < \infty.$$

([Voiculescu,1976], Proposition 1.)

THEOREM 3.2. — *The spherical functions of positive type for the Olshanski spherical pair* (G, K) *with* $G = U(\infty) \times U(\infty)$, $K = U(\infty)$ *are precisely the functions*

$$\varphi(u) = \det \Phi_+(u) \det \Phi_-(u^{-1}),$$

where Φ_+ *and* Φ_- *are Voiculescu functions.*
([Voiculescu,1976], [Boyer,1983].)

Let Ω_0 denote the set of parameters $\omega = (\alpha, \beta, \gamma)$ with

$$\alpha = (\alpha_k),\ \alpha_k \geq 0,\ \beta = (\beta_k),\ 0 \leq \beta_k \leq 1,\ \gamma \geq 0,$$
$$\sum_{k=1}^{\infty} \alpha_k < \infty,\ \sum_{k=1}^{\infty} \beta_k < \infty.$$

The Voiculescu function with parameter $\omega = (\alpha, \beta, \gamma)$ will be written $\Phi(\omega; z)$. We will consider on Ω_0 the initial topology with respect to the functions L_f,

$$L_f(\omega) = \gamma f(0) + \sum_{k=1}^{\infty} \alpha_k f(\alpha_k) + \sum_{k=1}^{\infty} \beta_k f(-\beta_k),$$

where f is a continuous function on $\mathbb{R}$. Hence the spherical dual can be identified to the set $\Omega = \Omega_0 \times \Omega_0$ of pairs $\omega = (\omega_+, \omega_-)$.

Let λ be a positive signature, $\lambda = (\lambda_1, \ldots, \lambda_n)$ ($\lambda_1 \geq \cdots \geq \lambda_n \geq 0$). The number λ_i is the number of boxes in the i-th row of the Young diagram of λ. The *conjugate signature* $\lambda' = (\lambda'_1, \lambda'_2, \dots)$ is associated to the transpose diagram. The *Frobenius parameters* $a = (a_1, a_2, \dots)$ and $b = (b_1, b_2, \dots)$ of a positive signature λ are defined by

$$a_i = \lambda_i - i \text{ if } \lambda_i > i,\ a_i = 0 \text{ otherwise,}$$
$$b_j = \lambda'_j - j + 1 \text{ if } \lambda'_j > j - 1,\ b_j = 0 \text{ otherwise.}$$

For instance, if $\lambda = (6, 4, 4, 2, 1)$, then $\lambda' = (5, 4, 3, 1, 1)$ and

$$a = (5, 2, 1, 0, \dots), \quad b = (5, 3, 1, 0, \dots).$$

We define a map from the set Ω_n^+ of positive signature of length $\leq n$ into Ω_0 by:

$$T_n : \lambda \mapsto \omega = (\alpha, \beta, \gamma),$$

with

$$\alpha_k = \frac{a_k}{n},\ \beta_k = \frac{b_k}{n},\ \gamma = 0.$$

THEOREM 3.3. — *Consider a sequence $(\lambda^{(n)})$ of positive signatures with $\lambda^{(n)} \in \Omega_n^+$. Assume that, for the topology of Ω_0,*

$$\lim_{n\to\infty} T_n(\lambda^{(n)}) = \omega.$$

Then, for $x \in U(\infty)$,

$$\lim_{n\to\infty} \varphi_n(\lambda^{(n)}; x) = \det \Phi(\omega; x),$$

uniformly on compact sets.
(See [Okounkov-Olshanski,1998c].)

For the general case, to a signature $\lambda = (\lambda_1, \dots, \lambda_n)$, one associates two positive signatures λ^+, λ^-. If

$$\lambda_1 \geq \cdots \lambda_p \geq 0 \geq \lambda_{p+1} \geq \cdots \geq \lambda_n,$$

then

$$\lambda^+ = (\lambda_1, \dots, \lambda_p), \quad \lambda^- = (-\lambda_n, \dots, -\lambda_{p+1}).$$

The map T_n from Ω_n into $\Omega = \Omega_0 \times \Omega_0$ is given by

$$T_n(\lambda) = \big(T_n(\lambda^+), T_n(\lambda^-)\big).$$

THEOREM 3.4. — *Let* $(\lambda^{(n)})$ *be a sequence of signatures with* $\lambda^{(n)} \in \Omega_n$. *Assume that*

$$\lim_{n\to\infty} T_n(\lambda^{(n)}) = \omega = (\omega^+, \omega^-).$$

then

$$\lim_{n\to\infty} \varphi_n(\lambda^{(n)}; x) = \det \Phi(\omega^+; x) \det \Phi(\omega^-; x^{-1}),$$

uniformly on compact sets.

([Vershik-Kerov,1982], see also [Okounkov-Olshanski,1998c].)

3.4 Power expansions and Schur expansions. In this section we will describe the main ingredients in the proof of Theorems 3.3 and 3.4. The binomial formula for Schur functions generalizes the classical binomial formula,

$$(1+w)^\lambda = \sum_{m=0}^{\infty} \frac{1}{m!}[\lambda]_m w^m,$$

where

$$[a]_m = a(a-1)\ldots(a-m+1).$$

PROPOSITION 3.5 (BINOMIAL FORMULA FOR SCHUR FUNCTIONS).

$$\frac{s_\lambda(1+z_1,\ldots,1+z_n)}{s_\lambda(1,\ldots,1)} = \sum_{m_1\geq\cdots\geq m_n\geq 0} \frac{\delta!}{(\mathbf{m}+\delta)!} s^*_{\mathbf{m}}(\lambda) s_{\mathbf{m}}(z),$$

where $s^*_{\mathbf{m}}$ *is the shifted Schur function*

$$s^*_{\mathbf{m}}(\lambda) = \frac{\det\big([\lambda_i+\delta_i]_{m_j+\delta_j}\big)}{\det\big([\lambda_i+\delta_i]_{\delta_j}\big)}.$$

Proof. This is obtained as an application of Hua's formula (Proposition 2.4) with

$$f_i(w) = (1+w)^{\lambda_i+\delta_i} = \sum_{m=0}^{\infty} \frac{[\lambda_i+\delta_i]_m}{m!} w^m. \qquad \square$$

A function f defined on the set of signatures is said to be *shifted symmetric* if

$$f(\ldots,\lambda_i,\lambda_{i+1},\ldots) = f(\ldots,\lambda_{i+1}-1,\lambda_i+1,\ldots).$$

The algebra of the shifted symmetric functions is denoted by Λ^*. (See [Okounkov-Olshanski,1998a] and [1998b].)

To write down the power series expansion of the Voiculescu functions, one introduces a morphism from the algebra of symmetric functions Λ into the space $\mathcal{C}(\Omega_0)$ of continuous functions on Ω_0, $f \mapsto \tilde{f}$, such that

$$\widetilde{p_1}(\omega) = \sum_{k=1}^{\infty} \alpha_k + \sum_{k=1}^{\infty} \beta_k + \gamma,$$
$$\widetilde{p_m}(\omega) = \sum_{k=1}^{\infty} \alpha_k^m + (-1)^{m-1} \sum_{k=1}^{\infty} \beta_k^m \quad (m \geq 2).$$

Following the same method as in the proof of Theorem 2.6, one establishes the following expansions:

Proposition 3.6.

$$\Phi(\omega; 1+z) = \sum_{m=0}^{\infty} \widetilde{h_m}(\omega) z^m,$$
$$\prod_{j=1}^{n} \Phi(\omega; 1+z_j) = \sum_{m_1 \geq \cdots \geq m_n \geq 0} \widetilde{s_{\mathbf{m}}}(\omega) s_{\mathbf{m}}(z).$$

Finally, there is an analogue of Proposition 2.8:

Proposition 3.7. — *Consider a sequence $(\lambda^{(n)})$ with $\lambda^{(n)} \in \Omega_n^+$. Assume that, for the topology of Ω_0,*

$$\lim_{n\to\infty} T_n(\lambda^{(n)}) = \omega.$$

Then, for every shifted symmetric function $f^* \in \Lambda^*$,

$$\lim_{n\to\infty} f^*(\lambda^{(n)}) = \tilde{f}(\omega),$$

where $m = \deg f^*$, and $\tilde{f} \in \mathcal{C}(\Omega_0)$ is the image of $f \in \Lambda$, which is the homogeneous part of degree m of f^*.

In the outline of the proof of Theorem 3.4 we gave above, we have followed [Okounkov-Olshanski,1998c] where the authors study the asymptotics of the Jack polynomials as the number of variables goes to infinity. The case of the unitary group corresponds to the value $\theta = 2$ of the parameter. For the values $\theta = 1, 2$ and 4, the Jack polynomials are related to the spherical functions of compact symmetric spaces with root system of type A. The asymptotics of the multivariable Jacobi polynomials in case of a root system of type BC is considered in [Okounkov-Olshanski,2006].

4. Spherical analysis on the infinite dimensional Heisenberg group

4.1 Gelfand pair associated to the Heisenberg group. For a finite dimensional complex Euclidean vector space V, we consider the Heisenberg group $H = V \times \mathbb{R}$ with the product

$$(z,t)(z',t') = \big(z+z', t+t' + \mathrm{Im}\,(z'|z)\big).$$

The unitary group $U(V)$ acts on H by automorphisms: $u\cdot(z,t) = (u\cdot z, t)$. For a closed subgroup $K \subset U(V)$, we consider the semi-direct product $G = K \ltimes H$. A K-biinvariant function on G can be seen as a K-invariant function on H, and as convolution algebras, $L^1(K\backslash G/K) \simeq L^1(H)^K$.

THEOREM 4.1. — *(G,K) is a Gelfand pair if and only if K acts multiplicity free on the space $\mathcal{P}(V)$ of polynomials on V.*

[Carcano,1987]

Assume that $\mathcal{P}(V)$ decomposes multiplicity free under the K-action:

$$\mathcal{P}(V) = \bigoplus_{\alpha} \mathcal{P}_\alpha.$$

The subspaces $\mathcal{P}_\alpha$ depending on the parameter α are irreducible for the K-action. A K-invariant function φ on H, with $\varphi(0,0)=1$, will be said to be spherical if

$$\int_K \varphi\big(z + k\cdot z', t+t' + \mathrm{Im}\,(k\cdot z'|z)\big)\alpha(dk) = \varphi(z,t)\varphi(z',t').$$

These Gelfand pairs have been studied by Benson, Jenkins and Ratcliff in a series of papers: [Benson-Jenkins-Ratcliff,1992], [Benson-Ratcliff,1996,1998]. See also [Wolf,2007], Chapter 13.

Spherical functions of positive type and first kind

The *Fock space* $\mathcal{F}_\lambda(V)$ $(\lambda > 0)$ is the space of holomorphic functions ψ on V such that

$$\|\psi\|_\lambda^2 = \Big(\frac{\lambda}{\pi}\Big)^{\dim V} \int_V |\psi(\zeta)|^2 e^{-\lambda\|\zeta\|^2} m(d\zeta) < \infty.$$

The *Bargmann representation* T_λ of the Heisenberg group H on the Fock space is given by

$$\big(T_\lambda(z,t)\psi\big)(\zeta) = e^{\lambda(it-\frac{1}{2}\|z\|^2-(\zeta|z))}\psi(\zeta+z).$$

The group K acts on the Fock space $\mathcal{F}_\lambda(V)$:

$$\big(\pi(k)f\big)(\zeta) = f(k^{-1}\zeta),$$

and the Fock space decomposes multiplicity free under the action of K:

$$\mathcal{F}_\lambda(V) = \widehat{\bigoplus_\alpha}\mathcal{P}_\alpha.$$

If $f \in L^1(H)^K$, then $T_\lambda(f)$ commutes to the K-action. Hence, for every α, the subspace $\mathcal{P}_\alpha$ is an eigenspace of $T_\lambda(f)$ by Schur's lemma. for $\psi \in \mathcal{P}_\alpha$,

$$\big(T_\lambda(f)\psi\big)(\zeta) = \hat{f}(\lambda,\alpha)\psi(\zeta).$$

The character $f \mapsto \hat{f}(\lambda,\alpha)$ of $L^1(H)^K$ is associated to a spherical function $\varphi(\lambda,\alpha;z,t)$:

$$\hat{f}(\lambda,\alpha) = \int_H f(z,t)\varphi(\lambda,\alpha;z,t)m(dz)dt.$$

The spherical functions $\varphi(\lambda,\alpha;z,t)$ are of positive type. They are said to be of first kind.

Spherical functions of positive type and second kind

These spherical functions are related to one dimensional unitary representations of H:

$$\eta_w(z,t) = e^{2i\mathrm{Im}(z|w)} \quad (w \in V).$$

The spherical functions of positive type and second kind are given by

$$\psi(w;z) = \int_K e^{2i\mathrm{Im}(z|k\cdot w)}\alpha(dk),$$

with parameter $w \in V/K$.

The spherical dual decomposes as $\Omega = \Omega^1 \cup \Omega^2$, where Ω^1 is the set of spherical functions of positive type and first kind, and Ω^2 the set of the ones of second kind. See [Benson-Jenkins-Ratcliff,1992], where it is also proved that every bounded spherical function is of positive type.

4.2 The Gelfand pair $\big(K \ltimes (V \times \mathbb{R}), K\big)$ with $V = M(n,\mathbb{C})$, $K = U(n) \times U(n)$. We consider the special case $V = M(n,\mathbb{C})$, with $K = U(n)\times U(n)$ acting on $V = M(n,\mathbb{C})$ by $k\cdot z = uzv^*$ ($k = (u,v) \in K = U(n) \times U(n)$). The space of polynomials $\mathcal{P}(V)$ decomposes multiplicity free as

$$\mathcal{P}(V) = \bigoplus_{\mathbf{m}} \mathcal{P}_{\mathbf{m}}.$$

the parameter $\mathbf{m}$ is a partition of length $\leq n$, $\mathbf{m} = (m_1, \ldots, m_n)$, $m_j \in \mathbb{Z}$, $m_1 \geq \cdots \geq m_n$, and the subspace $\mathcal{P}_{\mathbf{m}}$ is generated by the power function

$$\Delta_{\mathbf{m}}(z) = \Delta_1(z)^{m_1-m_2}\Delta_2(z)^{m_2-m_3}\ldots\Delta_n(z)^{m_n},$$

where $\Delta_1(z), \Delta_2(z), \ldots \Delta_n(z)$ are the principal minors of the matrix z.

The spherical functions of positive type and first kind will be written $\varphi(\lambda, \mathbf{m}; z, t)$, where $\lambda \in \mathbb{R}^*$, and $\mathbf{m}$ is a partition. The first part Ω^1 of the spherical dual can be seen as the set of pairs $(\lambda, \mathbf{m})$.

Theorem 4.2. — *The spherical function $\varphi(\mathbf{m}, \lambda; z, t)$ admits the following expansion:*

$$\varphi(\lambda, \mathbf{m}; z, t) = e^{i\lambda t}e^{-\frac{1}{2}\lambda\|z\|^2}\sum_{\mathbf{k}\subset\mathbf{m}}\Big(\frac{1}{(n)_{\mathbf{k}}}\Big)^2\lambda^{|\mathbf{k}|}s^*_{\mathbf{k}}(\mathbf{m})\chi_{\mathbf{k}}(-zz^*),$$

where $s^*_{\mathbf{k}}$ is a shifted Schur function, and $\chi_{\mathbf{k}}$ is the character of the irreducible representation of $U(n)$ with highest weight $\mathbf{k}$, which extends to the space $V = M(n,\mathbb{C})$.

Recall the usual Pochhammer symbol

$$(\alpha)_k = \alpha(\alpha+1)\ldots(\alpha+k-1) \quad (\alpha \in \mathbb{C}),$$

and the generalized Pochhammer symbol, for a positive signature $\mathbf{k} = (k_1, \ldots, k_n)$,

$$(\alpha)_{\mathbf{k}} = \prod_{i=1}^{n}(\alpha - i + 1)_{k_i}.$$

Recall also that the Schur function is given by:

$$s_{\mathbf{m}}(t_1, \ldots, t_n) = \frac{\det(t_i^{m_j+\delta_j})}{V(t_1, \ldots, t_n)}$$

($\delta = (n-1, \ldots, 1, 0)$), and that

$$\chi_{\mathbf{m}}\big(\mathrm{diag}(t_1, \ldots, t_n)\big) = s_{\mathbf{m}}(t_1, \ldots, t_n).$$

The shifted Schur functions occur in the binomial formula for the Schur functions (see Proposition 3.5):

$$\frac{s_{\mathbf{m}}(1+z_1,\ldots,1+z_n)}{s_{\mathbf{m}}(1,\ldots,1)} = \sum_{\mathbf{k}\subset\mathbf{m}} \frac{\delta!}{(\mathbf{k}+\delta)!} s^*_{\mathbf{k}}(\mathbf{m}) s_{\mathbf{k}}(z).$$

The polynomial

$$L_{\mathbf{m}}(t_1,\ldots,t_n) = \sum_{\mathbf{k}\subset\mathbf{m}} (-1)^{|\mathbf{k}|} \left(\frac{1}{(n)_{\mathbf{k}}}\right)^2 s^*_{\mathbf{k}}(\mathbf{m}) s_{\mathbf{k}}(t)$$

is a multivariate Laguerre polynomial.

The set of orbits V/K can be parametrized by the set of vectors $\rho = (\rho_1,\ldots,\rho_n)$ where $\rho_1 \geq \cdots \geq \rho_n \geq 0$ are the eigenvalues of ww^* for $w \in V = M(n,\mathbb{C})$. The corresponding spherical funcion is given by

$$\psi(\rho; z) = \int_{U(n)\times U(n)} e^{2i\mathrm{Re}\,\mathrm{tr}(uzv^*w)} \beta_n(du)\beta_n(dv),$$

Hence the second part Ω^2 can be seen as the set of the $\rho = (\rho_1,\ldots,\rho_n)$, with $\rho_1 \geq \cdots \geq \rho_n \geq 0$.

It is shown that

$$\psi(\rho; z) = \sum_{\mathbf{k}} \left(\frac{1}{(n)_{\mathbf{k}}}\right)^2 s_{\mathbf{k}}(\rho) \chi_{\mathbf{k}}(-zz^*).$$

The function

$$\psi(\rho_1,\ldots,\rho_n; t_1,\ldots,t_n) = \sum_{\mathbf{k}} (-1)^{|\mathbf{k}|} \left(\frac{1}{(n)_{\mathbf{k}}}\right)^2 s_{\mathbf{k}}(\rho) s_{\mathbf{k}}(t)$$

is a multivariate Besssel function.

As $\lambda \to 0$ and $|\lambda| m_j \to \rho_j$,

$$\lim \varphi(\lambda, \mathbf{m}; z, t) = \psi(\rho; z).$$

In fact the topology of the spherical dual $\Omega = \Omega^1 \cup \Omega^2$ is given as follows: the map $\Omega \to \mathbb{R}^{n+1}$ defined by

$$(\lambda, \mathbf{m}) \mapsto (\lambda, |\lambda| m_1, \ldots, |\lambda| m_n),$$
$$\rho \mapsto (0, \rho_1, \ldots, \rho_n),$$

is a homeomorphism on its image.

4.3 Increasing sequence of Gelfand pairs. We consider the following sequences of groups:

$$\begin{aligned} H(n) =& M(n,\mathbb{C}) \times \mathbb{R}, \\ K(n) =& U(n) \times U(n), \\ G(n) =& K(n) \ltimes H(n). \end{aligned}$$

The spherical dual Ω_n of $\big(G(n), K(n)\big)$ decomposes as:

$$\Omega_n = \Omega_n^1 \cup \Omega_n^2,$$

with

$$\begin{aligned} \Omega_n^1 &= \{(\lambda, \mathbf{m}) \mid \lambda \in \mathbb{R}^*,\ \mathbf{m} \text{ is a partition}, \ell(\mathbf{m}) \leq n\}, \\ \Omega_n^2 &= \{\rho \in \mathbb{R}^n \mid \rho_1 \geq \cdots \geq \rho_n \geq 0\}. \end{aligned}$$

Let us write an expansion valid for spherical functions of both kinds

$$\varphi_n(\sigma; z, t) = e^{i\lambda t} e^{-\frac{1}{2}\lambda \|z\|^2} \sum_{\mathbf{k}} \Big(\frac{1}{(n)_{\mathbf{k}}}\Big)^2 a_{\mathbf{k}}(\sigma) \chi_{\mathbf{k}}(-zz^*).$$

For $\sigma = (\lambda, \mathbf{m}) \in \Omega_n^1$, then

$$a_{\mathbf{k}}(\sigma) = |\lambda|^{|\mathbf{k}|} s_{\mathbf{k}}^*(\mathbf{m}),$$

if $\mathbf{k} \subset \mathbf{m}$, and $a_{\mathbf{k}}(\mathbf{m}) = 0$ otherwise. For $\sigma = \rho \in \Omega_n^2$, then λ is taken to be 0, and

$$a_{\mathbf{k}}(\sigma) = s_{\mathbf{k}}(\rho).$$

Observe that the function a_k is continuous on the spherical dual Ω_n.

We consider the Olshanski spherical pair (G, K) with

$$G = \bigcup_{n=1}^{\infty} G(n), \quad K = \bigcup_{n=1}^{\infty} K(n).$$

The spherical dual Ω, for the Olshanski spherical pair (G, K), is the set of triples $\omega = (\lambda, \alpha, \gamma)$,

$$\lambda \in \mathbb{R},\ \alpha = (\alpha_j),\ \alpha_j \geq 0,\ \sum_{j=1}^{\infty} \alpha_j < \infty,\ \gamma \geq \tfrac{1}{2}\,|\lambda|.$$

One defines a topology on Ω, similarly as in Sections 2 and 3. For a continuous function f on $\mathbb{R}$, we define the functions L_f on Ω by

$$L_f(\omega) = \gamma f(0) + \sum_{k=1}^{\infty} \alpha_k f(\alpha_k).$$

The topology is the initial topology with respect to the functions L_f, and the function $\omega \mapsto \lambda$.

Define the following Pólya type function

$$\Phi(\alpha, \gamma; x) = e^{-\gamma x} \prod_{j=1}^{\infty} \frac{1}{1 + \alpha_j x}.$$

The spherical function of positive type with parameter $\omega = (\lambda, \alpha, \gamma)$, is given by

$$\varphi(\omega; z, t) = e^{i\lambda t} \det \Phi(\alpha, \gamma; zz^*).$$

One defines the map $T_n : \Omega_n \to \Omega,\ \sigma \mapsto \omega = (\lambda, \alpha, \gamma)$ as follows:
If $\sigma = (\lambda, \mathbf{m}) \in \Omega_n^1$, then

$$\alpha_j = \frac{1}{n^2} |\lambda| m_j \ (1 \leq j \leq n),\ \alpha_j = 0 \ (j > n),\ \gamma = 0.$$

If $\sigma = \rho \in \Omega_n^2$, then $\lambda = 0$, and

$$\alpha_j = \frac{1}{n^2} \rho_j \ (1 \leq j \leq n),\ \alpha_j = 0 \ (j > n),\ \gamma = 0.$$

Theorem 4.3. — *Let $(\sigma^{(n)})$ be a sequence with $\sigma^{(n)} \in \Omega_n$. Then*

$$\lim_{n \to \infty} \varphi_n(\sigma^{(n)}; z, t) = \varphi(\omega; z, t)$$

if and only if

$$\lim_{n \to \infty} T_n(\sigma^{(n)}) = \omega$$

for the topology of Ω.

One can find the proof of this theorem in [Faraut,2010a], where we consider the Heisenberg group $H = V \times \mathbb{R}$ with the action of K, for $V = M(n, p; \mathbb{C})$ and $K = U(n) \times U(p)$. In [Faraut,2010b] we consider the cases

$$V = Sym(n, \mathbb{C}), \quad K = U(n),$$
$$V = M(n, \mathbb{C}), \quad K = U(n) \times U(n),$$
$$V = Skew(2n, \mathbb{C}), \quad K \simeq U(2n).$$

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