HERMITIAN SYMMETRIC SPACES OF TUBE TYPE AND MULTIVARIATE MEIXNER-POLLACZEK POLYNOMIALS

Jacques Faraut & Masato Wakayama

Abstract Harmonic analysis on Hermitian symmetric spaces of tube type is a natural framework for introducing multivariate Meixner-Pollaczek polynomials. Their main properties are established in this setting: orthogonality, generating and determinantal formulae, difference equations. Furthermore, as a by-product, we derive the radial part of the differential equation for the multivariate Laguerre functions and obtain the differential equation for multivariate Laguerre polynomials previously obtained by Baker and Forrester.

2010 Mathematics Subject Classification: *Primary* 32M15, *Secondary* 33C45, 43A90.

Keywords and phrases: *Meixner-Pollaczek polynomial, Laguerre polynomial, symmetric cone, Hermitian symmetric space, Jordan algebra, spherical function.*

Contents

The one variable Meixner-Pollaczek polynomials $P_m^{\alpha}(\lambda;\phi)$ can be defined by the Gaussian hypergeometric representation as

$$P_m^{(\frac{\nu}{2})}\big(\lambda;\phi\big) = \frac{(\nu)_m}{m!} e^{im\theta} {}_2F_1\big(-m, \frac{\nu}{2} + i\lambda; \nu; 1 - e^{-2i\phi}\big).$$

For $\phi = \frac{\pi}{2}$ the Meixner-Pollaczek polynomials $P_m^{(\frac{\nu}{2})}\big(\lambda;\frac{\pi}{2}\big)$ are also obtained as Mellin transforms of Laguerre functions. Their main properties follow from this fact: hypergeometric representation above, orthogonality, generating formula, difference equation, and three terms relation.

These polynomials $P_m^{(\frac{\nu}{2})}\big(\lambda;\frac{\pi}{2}\big)$ have been generalized to the multivariate case. In fact, the multivariable Meixner-Pollaczek (symmetric) polynomials have been essentially considered in the setting of the Fourier analysis on Riemannian symmetric spaces in several papers: [Peetre-Zhang,1992] (Appendix 2: A class of hypergeometric orthogonal polynomials), [Ørsted-Zhang, 1994], section 3.4, [Zhang,2002] and [Davidson-Ólafsson-Zhang,2003]. Also, see [Davidson-Ólafsson,2003] and [Aristidou-Davidson-Ólafsson,2006]. Further, for an arbitrary real value of the multiplicity d, the multivariate Meixner-Pollaczek polynomials are defined in [Sahi-Zhang,2007] in the setting of Heckman-Opdam and Cherednik-Opdam transforms, related to symmetric and non-symmetric Jack polynomials, and generating formulae for them are established. However the case where the parameter ϕ is involved has not been studied so far. Moreover, once we define the multivariate Meixner-Pollaczek polynomials with parameter ϕ, it is also important to clarify a geometric meaning of the parameter. Establishing a natural setting for the study of multivariate Meixner-Pollaczek polynomials with such parameter, one can expect to obtain wider applications such as a study of multi-dimensional Lévi-process, in particular, introducing multi-dimensional Meixner process (see [Schoutens, 2000] for the one dimensional case).

The purpose of this article is to provide a geometric framework for introducing the multivariate Meixner-Pollaczek polynomials (with parameter ϕ) and study their fundamental properties. Our analysis may explain much simpler geometric understanding of several basic properties of the multivariate Meixner-pollaczek polynomials than ever, even in the case $\phi = \frac{\pi}{2}$. For instance, the $\mathfrak{S}_n$-invariant difference operator of which the multivariate Meixner-Pollaczek polynomials are eigenfunctions can be understood by an image of the Euler operator under the composition of three intertwiners: the Cayley transform, the Laplace transform and the spherical Fourier transform.

Let us present in the one variable case the scheme we will develop.

a) The monomials $\phi_m(z) = z^m$ form an orthogonal basis in the weighted Bergman space $\mathcal{H}^2_\nu(D)$ $(\nu > 1)$ of holomorphic functions f on the unit disc $D \subset \mathbb{C}$ with

$$\|f\|^2_\nu := \frac{\nu - 1}{\pi} \int_D |f(w)|^2 (1 - |w|^2)^{\nu-2} m(dw) < \infty.$$

(m denotes the Lebesgue measure on $\mathbb{C}$.) Since

$$\|\phi_m\|^2_\nu = \frac{m!}{(\nu)_m},$$

the reproducing kernel of $\mathcal{H}^2_\nu(D)$ is given by

$$\mathcal{K}_\nu(w, w') = \sum_{m=0}^{\infty} \frac{(\nu)_m}{m!} w^m \bar{w}'^m.$$

It can be written as a generating formula for the functions ϕ_m:

$$\mathcal{G}^{(1)}(\zeta, w) := \sum_{m=0}^{\infty} \frac{(\nu)_m}{m!} \phi_m(\zeta) w^m = (1 - w\zeta)^{-\nu}. \tag{0.1}$$

b) The Cayley transform

$$w \mapsto z = c(w) = \frac{1+w}{1-w}$$

maps the unit disc D onto the right half-plane $T = \{z = x + iy \mid x > 0\}$, and its inverse is given by

$$c^{-1}(z) = \frac{z-1}{z+1}.$$

For a holomorphic function f on D define the function $F = C_\nu f$ on T by

$$F(z) = (C_\nu f)(z) = \Big(\frac{z+1}{2}\Big)^{-\nu} f\Big(\frac{z-1}{z+1}\Big).$$

Then C_ν maps unitarily $\mathcal{H}^2_\nu(D)$ onto the space $\mathcal{H}^2_\nu(T)$ of holomorphic functions F on T such that

$$\|F\|^2_\nu := \frac{\nu - 1}{4\pi} \int_T |F(x+iy)|^2 x^{\nu-2} m(dz) < \infty.$$

The functions $F_m^{(\nu)} = C_\nu \phi_m$ form an orthogonal basis of $\mathcal{H}_\nu^2(T)$. From the generating formula (0.1), by performing the transform C_ν with respect to the variable ζ, one obtains a generating formula for the functions $F_m^{(\nu)}$:

$$\mathcal{G}^{(2)}(z,w) := \sum_{m=0}^{\infty} \frac{(\nu)_m}{m!} F_m^{(\nu)}(z) w^m = \big(\frac{1-w}{2}\big)^{-\nu}\big(z + c(w)\big)^{-\nu}. \tag{0.2}$$

c) Every function F in $\mathcal{H}_\nu^2(T)$ admits a Laplace integral representation:

$$F(z) = (\mathcal{L}_\nu)\psi(z) := \frac{2^\nu}{\Gamma(\nu)} \int_0^\infty e^{-zu} \psi(u) u^{\nu-1} du,$$

with $\psi \in L_\nu^2(0,\infty)$, with the norm

$$\|\psi\|_\nu^2 := \frac{2^\nu}{\Gamma(\nu)} \int_0^\infty |\psi(u)|^2 u^{\nu-1} du,$$

normalized in such a way that $\mathcal{L}_\nu$ is unitary. Define the Laguerre function $\psi_m^{(\nu)}$ as

$$\psi_m^{(\nu)}(u) = e^{-u} L_m^{(\nu-1)}(2u),$$

where $L_m^{(\nu)}$ denotes the classical Laguerre polynomial of degree m. Then

$$(\mathcal{L}_\nu \psi_m^{(\nu)})(z) = \frac{(\nu)_m}{m!} F_m^{(\nu)}(z).$$

Applying the inverse Laplace transform $\mathcal{L}_\nu^{-1}$ to (0.2) one gets the following generating formula for the Laguerre functions:

$$\mathcal{G}^{(3)}(u,w) := \sum_{m=0}^{\infty} \psi_m^{(\nu)}(u) w^m = (1-w)^{-\nu} e^{-uc(w)}. \tag{0.3}$$

d) Finally we perform a modified Mellin transform:

$$\mathcal{M}_\nu \psi(s) := \frac{1}{\Gamma\big(s+\frac{\nu}{2}\big)} \int_0^\infty \psi(u) u^{s+\frac{\nu}{2}-1} du.$$

By the classical Plancherel theorem $\psi \mapsto (\mathcal{M}_\nu\psi)(i\lambda)$ is a unitary isomorphism from $L_\nu^2(0,\infty)$ onto $L^2(\mathbb{R}, M_\nu)$, with

$$M_\nu(d\lambda) = \frac{1}{2\pi} \frac{2^\nu}{\Gamma(\nu)} \big|\Gamma\big(i\lambda + \frac{\nu}{2}\big)\big|^2 d\lambda.$$

The function $q_m^{(\nu)} := \mathcal{M}_\nu \psi_m^\nu$ is a Meixner-Pollaczek polynomial. In fact

$$q_m^{(\nu)}(i\lambda) = \frac{(\nu)_m}{m!} {}_2F_1\big(-m, s+\frac{\nu}{2}; \nu; 2\big) = (-i)^m P_m^{(\frac{\nu}{2})}\big(\lambda; \frac{\pi}{2}\big).$$

Hence the Meixner-Pollaczek polynomials $q_m^{(\nu)}$ form an orthogonal basis of $L^2(\mathbb{R}, M_\nu)$, and

$$\|q_m^{(\nu)}\|_\nu^2 := \int_{-\infty}^{\infty} |q_m^{(\nu)}(i\lambda)|^2 M_\nu(d\lambda) = \frac{(\nu)_m}{m!}.$$

If we apply the tranform $\mathcal{M}_\nu$ to (0.3) with respect to u, we obtain the following generating formula

$$\mathcal{G}_\nu^{(4)}(s,w) := \sum_{m=0}^{\infty} q_m^{(\nu)}(s) w^m = (1-w)^{s-\frac{\nu}{2}}(1+w)^{-s-\frac{\nu}{2}}.$$

(See [Andrews-Askey-Roy,1999], p.348,349, and also [Bump et al.,2000] p.14,15.)

Starting from the Euler equation

$$D_\nu^{(1)}\phi_m := 2w\frac{d}{dw}\phi_m = 2m\psi_m,$$

one obtains a difference equation for the Meixner-Pollaczek polynomial $q_m^{(\nu)}$,

$$D_\nu^{(4)} q_m^{(\nu)}(s) := \big(s+\tfrac{\nu}{2}\big)\big(q_m^{(\nu)}(s+1) - q_m^{(\nu)}(s)\big) - \big(s-\tfrac{\nu}{2}\big)\big(q_m^{(\nu)}(s-1) - q_m^{(\nu)}(s)\big)$$
$$= 2m q_m^{(\nu)}(s),$$

and the three terms relation

$$2s q_m^{(\nu)}(s) = (m+\nu-1) q_{m-1}^{(\nu)}(s) - (m+1) q_{m+1}^{(\nu)}(s).$$

Moreover, by using a Gutzmer formula for the Mellin transform, the orthogonality property extends to the polynomials $P_m^\alpha(\lambda, \phi)$, with $0 < \phi < \pi$.

In the multivariate case we follow the same scheme. Actually, replacing the half-line by a symmetric cone, and the Mellin transform by the spherical Fourier transform, leads to a definition of multivariate Meixner-Pollaczek polynomials together with their properties, analogous to the ones of the one variable Meixner-Pollaczek polynomials.

In Section 1 we recall the basic facts about the spherical Fourier analysis on a symmetric cone. In Section 2 we define the multivariate Meixner-Pollaczek polynomials $Q_{\mathbf{m}}^{(\nu)}(\mathbf{s})$ (the case $\phi = \frac{\pi}{2}$), where $\mathbf{m}$ is a partition, prove that they are orthogonal with respect to a measure M_ν on $\mathbb{R}^n$, and establish a generating formula.

In Section 3, adding a real parameter θ, we introduce the symmetric polynomials $Q_{\mathbf{m}}^{(\nu,\theta)}(\mathbf{s})$ in the variables $\mathbf{s} = (s_1, \ldots, s_n)$ ($Q_{\mathbf{m}}^{(\nu)} = Q_{\mathbf{m}}^{(\nu,0)}$). In the one variable case

$$\begin{aligned} q_m^{(\nu,\theta)}(s) &= e^{im\theta} \tfrac{(\nu)_m}{m!} {}_2F_1\big(-m, s + \tfrac{\nu}{2}; \nu; 2e^{-i\theta}\cos\theta\big) \\ &= (-i)^m P_m^{(\frac{\nu}{2})}\big(-is; \theta + \tfrac{\pi}{2}\big). \end{aligned}$$

The orthogonality property for the polynomials $Q_{\mathbf{m}}^{(\nu,\theta)}(\mathbf{s})$ is obtained by using a Gutzmer formula for the spherical Fourier transform. A generating formula is obtained for these polynomials. In case of the multiplicity $d = 2$, we establish in Section 4 determinantal formulae for multivariate Laguerre and Meixner-Pollaczek polynomials. The last sections are devoted to a difference equation satisfied by the polynomials $Q_{\mathbf{m}}^{(\nu,\theta)}(\mathbf{s})$. Starting from an Euler-type equation involving the parameter θ, this difference equation is obtained in three steps, corresponding to a Cayley transform, an inverse Laplace transform, and a spherical Fourier transform for symmetric cones. The symmetry $\theta \mapsto -\theta$ in the parameter is related to geometric symmetries and to a generalized Tricomi theorem for the Hankel transform on a symmetric cone. As a biproduct we obtain a differential equation for the multivariate Laguerre polynomials, whose radial part is a special case of an equation in [Baker-Forrester,1997]. In the last section we show that multivariate Meixner-Pollaczek polynomials satisfy a Pieri's formula. In the one variable case it reduces to the three terms relation satisfied by the classical Meixner-Pollacek polynomials.

1 Spherical Fourier analysis on a symmetric cone

A reference for this preliminary section is [Faraut-Korányi,1994]. We consider an irreducible symmetric cone Ω in a Euclidean Jordan algebra V. We denote by G the identity component in the group $G(\Omega)$ of linear automorphisms of Ω, and $K \subset G$ is the isotropy subgroup of the unit element $e \in V$.

The Gindikin gamma function Γ_Ω of the cone Ω will be the cornerstone of the analysis we will developp. It is defined, for $\mathbf{s} \in \mathbb{C}^n$, with $\operatorname{Re} s_j > \frac{d}{2}(j-1)$, by

$$\Gamma_\Omega(\mathbf{s}) = \int_\Omega e^{-\operatorname{tr}(u)} \Delta_{\mathbf{s}}(u) \Delta(u)^{-\frac{N}{n}} m(du).$$

The notation tr (u) and $\Delta(u)$ denote the trace and the determinant with respect to the Jordan algebra structure, $\Delta_{\mathbf{s}}$ is the power function, N and n are the dimension and the rank of V, and m is the Euclidean measure associated to the Euclidean structure on V given by $(u|v) = \operatorname{tr}(\mathrm{uv})$. Its evaluation gives

$$\Gamma_\Omega(\mathbf{s}) = (2\pi)^{\frac{N-n}{2}} \prod_{j=1}^{n} \Gamma\big(s_j - \frac{d}{2}(j-1)\big),$$

where d is the multiplicity, related to N and n by the relation $N = n + \frac{d}{2}n(n-1)$.

The spherical function $\varphi_{\mathbf{s}}$, for $\mathbf{s} \in \mathbb{C}^n$, is defined on Ω by

$$\varphi_{\mathbf{s}}(u) = \int_K \Delta_{\mathbf{s}+\rho}(k \cdot u) dk,$$

where $\rho = (\rho_1, \ldots, \rho_n)$, $\rho_j = \frac{d}{4}(2j - n - 1)$, and dk is the normalized Haar measure on the compact group K.

The algebra $\mathbb{D}(\Omega)$ of G-invariant differential operators on Ω is commutative, and the spherical function $\varphi_{\mathbf{s}}$ is an eigenfunction of every $D \in \mathbb{D}(\Omega)$:

$$D\varphi_{\mathbf{s}} = \gamma_D(\mathbf{s})\varphi_{\mathbf{s}}.$$

The function γ_D is a symmetric polynomial function, and the map $D \mapsto \gamma_D$ is an algebra isomorphism from $\mathbb{D}(\Omega)$ onto the algebra $\mathcal{P}(\mathbb{C}^n)^{\mathfrak{S}_n}$ of symmetric polynomial functions, a special case of the Harish-Chandra isomorphism. The symbol σ_D of a partial differential operator D on V is defined by

$$De^{(x|\xi)} = \sigma_D(x, \xi) e^{(x|\xi)} \quad (x, \xi \in V)$$

(D acts on the variable x). If $D \in \mathbb{D}(\Omega)$, then σ_D is a G-invariant polynomial on $V \times V$ in the following sense: for $g \in G$,

$$\sigma_D(g \cdot x, \xi) = \sigma_D(x, g^* \cdot \xi).$$

The map $D \mapsto p(\xi) = \sigma_D(e, \xi)$ is a vector space isomorphism from $\mathbb{D}(\Omega)$ onto the space $\mathcal{P}(V)^K$ of K-invariant polynomials on V.

The spherical Fourier transform $\mathcal{F}\psi$ of a K-invariant function ψ on Ω is given by

$$\mathcal{F}\psi(\mathbf{s}) = \int_\Omega \psi(u)\varphi_{\mathbf{s}}(u)\Delta^{-\frac{N}{n}}(u)m(du).$$

Hence, for $\psi(u) = e^{-\operatorname{tr} u}\Delta^{\frac{\nu}{2}}$ $(\nu > \frac{d}{2}(n-1))$, then

$$\mathcal{F}\psi(\mathbf{s}) = \Gamma_\Omega\big(\mathbf{s} + \frac{\nu}{2} + \rho\big) = (2\pi)^{\frac{N-n}{2}} \prod_{j=1}^n \Gamma\big(s_j + \frac{\nu}{2} - \frac{d}{4}(n-1)\big).$$

For an invariant differential operator $D \in \mathbb{D}(\Omega)$,

$$\mathcal{F}_\nu(D\psi) = \gamma_D(-\mathbf{s})\mathcal{F}_\nu\psi.$$

Recall the spherical Plancherel formula: if the K-invariant function ψ satisfies

$$\int_\Omega |\psi(u)|^2 \Delta(u)^{-\frac{N}{n}} m(du) < \infty,$$

then

$$\int_\Omega |\psi(u)|^2 \Delta(u)^{-\frac{N}{n}} m(du) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} |\mathcal{F}\psi(i\lambda)|^2 \frac{1}{|c(i\lambda)|^2} m(d\lambda),$$

where c is the Harish-Chandra function:

$$c(\mathbf{s}) = c_0 \prod_{j<k} B\big(s_j - s_k, \frac{d}{2}\big).$$

(B is the Euler beta function, the constant c_0 is such that $c(-\rho) = 1$.)

The space $\mathcal{P}(V)$ of polynomials on V decomposes multiplicity free under G as

$$\mathcal{P}(V) = \bigoplus_{\mathbf{m}} \mathcal{P}_{\mathbf{m}},$$

where $\mathcal{P}_{\mathbf{m}}$ is a finite dimensional subspace, irreducible under G. The parameter $\mathbf{m}$ is a partition: $\mathbf{m} = (m_1, \ldots, m_n) \in \mathbb{N}^n$, $m_1 \geq \cdots \geq m_n$. The subspace $\mathcal{P}_{\mathbf{m}}^K$ of K-invariant polynomials in $\mathcal{P}_{\mathbf{m}}$ is one dimensional, generated by the spherical polynomial $\Phi_{\mathbf{m}}$, normalized by the condition $\Phi_{\mathbf{m}}(e) = 1$. The dimension of $\mathcal{P}_{\mathbf{m}}$ will be denoted by $d_{\mathbf{m}}$.

There is a unique invariant differential operator $D^{\mathbf{m}}$ such that

$$D^{\mathbf{m}}\psi(e) = \Big(\Phi_{\mathbf{m}}(\frac{\partial}{\partial u})\psi\Big)(e).$$

We will write $\gamma_{\mathbf{m}} = \gamma_{D^{\mathbf{m}}}$. If a K-invariant function ψ is analytic in a neighborhood of e, it admits a spherical Taylor expansion near e:

$$\psi(e+v) = \sum_{\mathbf{m}} d_{\mathbf{m}} \frac{1}{\left(\frac{N}{n}\right)_{\mathbf{m}}} D^{\mathbf{m}}\psi(e)\Phi_{\mathbf{m}}(v).$$

For $\alpha \in \mathbb{C}$ and a partition $\mathbf{m}$, the generalized Pochhammer symbol $(\alpha)_{\mathbf{m}}$ is defined by

$$(\alpha)_{\mathbf{m}} = \frac{\Gamma_\Omega(\mathbf{m}+\alpha)}{\Gamma_\Omega(\alpha)}.$$

In particular, for $\psi = \varphi_{\mathbf{s}}$, a spherical function,

$$\varphi_{\mathbf{s}}(e+v) = \sum_{\mathbf{m}} d_{\mathbf{m}} \frac{1}{\left(\frac{N}{n}\right)_{\mathbf{m}}} \gamma_{\mathbf{m}}(\mathbf{s})\Phi_{\mathbf{m}}(v).$$

For $\psi = \Phi_{\mathbf{m}}$, we get the spherical binomial formula

$$\Phi_{\mathbf{m}}(e+v) = \sum_{\mathbf{k}\subset\mathbf{m}} \binom{\mathbf{m}}{\mathbf{k}} \Phi_{\mathbf{k}}(v).$$

In fact the generalized binomial coefficient

$$\binom{\mathbf{m}}{\mathbf{k}} = d_{\mathbf{k}} \frac{1}{\left(\frac{N}{n}\right)_{\mathbf{k}}} \gamma_{\mathbf{k}}(\mathbf{m}-\rho)$$

vanishes if $\mathbf{k} \not\subset \mathbf{m}$.

2 Multivariate Meixner-Pollaczek polynomials $Q_{\mathbf{m}}^{(\nu)}$

For $n=1$, we define the Meixner-Pollaczek polynomial $q_m^{(\nu)}$ as follows

$$q_m^{(\nu)}(s) = \frac{(\nu)_m}{m!} {}_2F_1(-m, s+\frac{\nu}{2}; \nu; 2).$$

This definition slightly differs from the classical one $P_m^\alpha(\lambda;\phi)$:

$$q_m^{(\nu)}(i\lambda) = (-i)^m P_m^{\frac{\nu}{2}}\big(\lambda; \frac{\pi}{2}\big).$$

(see for instance [Andrews-Askey-Roy,1999], p.348.) Its expansion can be written

$$q_m^{(\nu)}(s) = \frac{(\nu)_m}{m!} \sum_{k=0}^{m} \frac{[m]_k \big[-s-\frac{\nu}{2}\big]_k}{(\nu)_k} \frac{1}{k!} 2^k.$$

The polynomials $q_m^{(\nu)}(i\lambda)$ are orthogonal with respect to the weight

$$|\Gamma\big(i\lambda + \frac{\nu}{2}\big)|^2 \quad (\nu > 0).$$

Observe that for $n = 1$, $\varphi_s(u) = u^s$, and

$$D^m = u^m \big(\frac{d}{du}\big)^m, \ \gamma_m(s) = [s]_m = s(s-1)\dots(s-m+1).$$

Hence, for higher rank, we see $\gamma_{\mathbf{m}}(\mathbf{s})$ as a multivariate analogue of the Pochhammer symbol $[s]_m$.

We define the multivariate Meixner-Pollaczek polynomial $Q_{\mathbf{m}}^{(\nu)}$ as the following symmetric polynomial in n variables:

$$Q_{\mathbf{m}}^{(\nu)}(\mathbf{s}) = \frac{(\nu)_{\mathbf{m}}}{\big(\frac{N}{n}\big)_{\mathbf{m}}} \sum_{\mathbf{k}\subset\mathbf{m}} d_{\mathbf{k}} \frac{\gamma_{\mathbf{k}}(\mathbf{m}-\rho)\gamma_{\mathbf{k}}\big(-\mathbf{s}-\frac{\nu}{2}\big)}{(\nu)_{\mathbf{k}}} \frac{1}{\big(\frac{N}{n}\big)_{\mathbf{k}}} 2^{|\mathbf{k}|}.$$

For $\nu > \frac{d}{2}(n-1)$ let us denote by $M_\nu(d\lambda)$ the probability measure on $\mathbb{R}^n$ given by

$$M_\nu(d\lambda) = \frac{1}{Z_\nu} \prod_{j=1}^{n} \Big|\Gamma\big(i\lambda_j + \frac{\nu}{2} - \frac{d}{4}(n-1)\big)\Big|^2 \frac{1}{|c(i\lambda)|^2} m(d\lambda),$$

where

$$Z_\nu = \int_{\mathbb{R}^n} \prod_{j=1}^{n} \Big|\Gamma\big(i\lambda_j + \frac{\nu}{2} - \frac{d}{4}(n-1)\big)\Big|^2 \frac{1}{|c(i\lambda)|^2} m(d\lambda).$$

The constant Z_ν can be evaluated by using the spherical Plancherel formula, applied to the function $\psi(u) = e^{-\operatorname{tr} u}\Delta(u)^{\frac{\nu}{2}}$:

$$\int_\Omega e^{-2\operatorname{tr} u} \Delta(u)^{\nu - \frac{N}{n}} m(du)$$
$$= (2\pi)^{N-2n} \int_{\mathbb{R}^n} \prod_{j=1}^n |\Gamma(i\lambda_j + \tfrac{\nu}{2} - \tfrac{d}{4}(n-1)|^2 \tfrac{1}{|c(i\lambda)|^2} m(d\lambda).$$

Therefore

$$Z_\nu = (2\pi)^{2n-N} 2^{-n\nu} \Gamma_\Omega(\nu).$$

Next statement involves the geometry of the Hermitian symmetric space of tube type associated to the symmetric cone Ω. The map $z \mapsto (z-e)(z+e)^{-1}$ maps the tube domain $T_\Omega = \Omega + iV \subset V_{\mathbb{C}}$ onto the bounded Hermitian symmetric domain $\mathcal{D}$. Its inverse is the Cayley transform:

$$c(w) = (e+w)(e-w)^{-1}.$$

Theorem 2.1. *Assume $\nu > \frac{d}{2}(n-1)$.*

(i) *The multivariate Meixner-Pollaczek polynomials $Q_{\mathbf{m}}^{(\nu)}(i\lambda)$ form an orthogonal basis of $L^2(\mathbb{R}^n, M_\nu)^{\mathfrak{S}_n}$. The norm of $Q_{\mathbf{m}}^{(\nu)}$ can be evaluated:*

$$\int_{\mathbb{R}^n} |Q_{\mathbf{m}}^{(\nu)}(i\lambda)|^2 M_\nu(d\lambda) = \frac{1}{d_{\mathbf{m}}} \frac{(\nu)_{\mathbf{m}}}{\left(\frac{N}{n}\right)_{\mathbf{m}}}.$$

(ii) *The polynomials $Q_{\mathbf{m}}^{(\nu)}$ admit the following generating formula: for $\mathbf{s} \in \mathbb{C}^n$, $w \in \mathcal{D}$,*

$$\sum_{\mathbf{m}} d_{\mathbf{m}} Q_{\mathbf{m}}^{(\nu)}(\mathbf{s}) \Phi_{\mathbf{m}}(w) = \Delta(e-w^2)^{-\frac{\nu}{2}} \varphi_{\mathbf{s}}\big(c(w)^{-1}\big).$$

Proof.

a) For $\nu > 2\frac{N}{n} - 1 = 1 + d(n-1)$, $\mathcal{H}_\nu^2(\mathcal{D})$ denotes the weighted Bergman space of holomorphic functions f on $\mathcal{D}$ such that

$$\|f\|_\nu^2 := a_\nu^{(1)} \int_{\mathcal{D}} |f(w)|^2 h(w)^{\nu - 2\frac{N}{n}} m(dw) < \infty.$$

The constant

$$a_\nu^{(1)} = \frac{1}{\pi^n} \frac{\Gamma_\Omega(\nu)}{\Gamma_\Omega\big(\nu - \frac{N}{n}\big)}$$

is such that the function $\Phi_0 \equiv 1$ has norm 1. The spherical polynomials $\Phi_{\mathbf{m}}$ form an orthogonal basis of the space $\mathcal{H}_\nu^2(\mathcal{D})^K$ of K-invariant functions in $\mathcal{H}_\nu^2(\mathcal{D})$, and

$$\|\Phi_{\mathbf{m}}\|_\nu^2 = \frac{1}{d_{\mathbf{m}}} \frac{\left(\frac{N}{n}\right)_{\mathbf{m}}}{(\nu)_{\mathbf{m}}}. \tag{2.1}$$

The reproducing kernel of $\mathcal{H}^2_\nu(\mathcal{D})$ is given by

$$\mathcal{K}_\nu(w,w') = h(w,w')^{-\nu},$$

where $h(w,w')$ is a polynomial holomorphic in w, antiholomorphic in w', and, for w invertible,

$$h(w,w') = \Delta(w)\Delta(w^{-1} - \bar{w}').$$

($\bar{w}'$ is the complex conjugate of w' with respect to the real form V of $V_{\mathbb{C}}$.) By an integration over K one obtains:

$$\mathcal{G}^{(1)}_\nu(\zeta, w) := \sum_{\mathbf{m}} d_{\mathbf{m}} \frac{(\nu)_{\mathbf{m}}}{\left(\frac{N}{n}\right)_{\mathbf{m}}} \Phi_{\mathbf{m}}(\zeta)\Phi_{\mathbf{m}}(w) = \int_K h(w, k\bar{\zeta})^{-\nu} dk. \tag{2.2}$$

b) For a function f holomorphic in $\mathcal{D}$, one defines the function $F = C_\nu f$ on T_Ω by

$$F(z) = \big(C_\nu f\big)(z) = \Delta\Big(\frac{z+e}{2}\Big)^{-\nu} f\big((z-e)(z+e)^{-1}\big).$$

The map C_ν is a unitary isomorphism from $\mathcal{H}^2_\nu(\mathcal{D})$ onto the space $\mathcal{H}^2_\nu(T_\Omega)$ of holomorphic functions on T_Ω such that

$$\|F\|^2_\nu := a^{(2)}_\nu \int_{T_\Omega} |F(z)|^2 \Delta(x)^{\nu - 2\frac{N}{n}} m(dz) < \infty.$$

The constant

$$a^{(2)}_\nu = \frac{1}{(4\pi)^n} \frac{\Gamma_\Omega(\nu)}{\Gamma_\Omega\big(\nu - \frac{N}{n}\big)},$$

is such that the function

$$F^{(\nu)}_0 = C_\nu \Phi_0, \quad F^{(\nu)}_0(z) = \Delta\Big(\frac{z+e}{2}\Big)^{-\nu},$$

has norm 1. The functions $F^{(\nu)}_{\mathbf{m}} = C_\nu \Phi_{\mathbf{m}}$ form an orthogonal basis of the space $\mathcal{H}^2_\nu(T_\Omega)^K$ of K-invariant functions in $\mathcal{H}^2_\nu(T_\Omega)$, and it follows from (2.1) that

$$\|F^{(\nu)}_{\mathbf{m}}\|^2_\nu = \frac{1}{d_{\mathbf{m}}} \frac{\left(\frac{N}{n}\right)_{\mathbf{m}}}{(\nu)_{\mathbf{m}}}. \tag{2.3}$$

Performing in (2.2) the transform C_ν with respect to ζ we get a generating formula for the functions $F_{\mathbf{m}}^{(\nu)}$: for $w \in \mathcal{D}$, $z \in T_\Omega$,

$$\begin{aligned}\mathcal{G}_\nu^{(2)}(z,w) &:= \sum_{\mathbf{m}} d_{\mathbf{m}} \frac{(\nu)_{\mathbf{m}}}{\left(\frac{N}{n}\right)_{\mathbf{m}}} \Phi_{\mathbf{m}}(w) F_{\mathbf{m}}^{(\nu)}(z) \\ &= \Delta\left(\frac{e-w}{2}\right)^{-\nu} \int_K \Delta\big(k \cdot z + c(w)\big)^{-\nu} dk \qquad (2.4)\end{aligned}$$

c) The functions in $\mathcal{H}_\nu^2(T_\Omega)$ admit a Laplace integral representation. The modified Laplace transform $\mathcal{L}_\nu$, given, for a function ψ on Ω, by

$$(\mathcal{L}_\nu \psi(z) = a_\nu^{(3)} \int_\Omega e^{(z|u)} \psi(u) \Delta(u)^{\nu - \frac{N}{n}} m(du),$$

is an isometric isomorphism from the space $L_\nu^2(\Omega)$ of measurable functions ψ on Ω such that

$$\|\psi\|_\nu^2 := a_\nu^{(3)} \int_\Omega |\psi(u)|^2 \Delta(u)^{\nu - \frac{N}{n}} m(du) < \infty,$$

onto $\mathcal{H}_\nu^2(T_\Omega)$. The constant

$$a_\nu^{(3)} = \frac{2^{n\nu}}{\Gamma_\Omega(\nu)}$$

is such that the function $\Psi_0(u) = e^{-\operatorname{tr} u}$ has norm 1, and then $\mathcal{L}_\nu \Psi_0 = F_0$. By the binomial formula

$$\begin{aligned}F_{\mathbf{m}}^{(\nu)}(z) &= \Delta\left(\frac{z+e}{2}\right)^{-\nu} \Phi_{\mathbf{m}}\big((z-e)(z+e)^{-1}\big) \\ &= \Delta\left(\frac{z+e}{2}\right)^{-\nu} \Phi_{\mathbf{m}}\big(e - 2(z+e)^{-1}\big) \\ &= \sum_{\mathbf{k} \subset \mathbf{m}} (-1)^{|\mathbf{k}|} \binom{\mathbf{m}}{\mathbf{k}} \Phi_{\mathbf{k}}\big(2(z+e)^{-1}\big) \Delta\big(2(e+z)^{-1}\big)^{\nu}.\end{aligned}$$

Lemma 2.2.

$$\mathcal{L}_\nu\big(e^{-\operatorname{tr} u} \Phi_{\mathbf{m}}\big)(z) = (\nu)_{\mathbf{m}} \Phi_{\mathbf{m}}\big((z+e)^{-1}\big) \Delta\big(2(e+z)^{-1}\big)^{\nu}.$$

(See Lemma XI.2.3 in [Faraut-Korányi, 1994].)

By Lemma 2.2 the function

$$\Psi_{\mathbf{m}}^{(\nu)} = \frac{(\nu)_{\mathbf{m}}}{\left(\frac{N}{n}\right)_{\mathbf{m}}} \mathcal{L}_{\nu}^{-1}\big(F_{\mathbf{m}}^{(\nu)}\big).$$

is the Laguerre function given by

$$\Psi_{\mathbf{m}}^{(\nu)}(u) = e^{-\operatorname{tr} u} L_{\mathbf{m}}^{(\nu-1)}(2u),$$

where $L_{\mathbf{m}}^{(\nu-1)}$ is the multivariate Laguerre polynomial

$$\begin{aligned} L_{\mathbf{m}}^{(\nu-1)}(x) &= \frac{(\nu)_{\mathbf{m}}}{\left(\frac{N}{n}\right)_{\mathbf{m}}} \sum_{\mathbf{k}\subset\mathbf{m}} \binom{\mathbf{m}}{\mathbf{k}} \frac{1}{(\nu)_{\mathbf{k}}} \Phi_{\mathbf{k}}(-x) \\ &= \frac{(\nu)_{\mathbf{m}}}{\left(\frac{N}{n}\right)_{\mathbf{m}}} \sum_{\mathbf{k}\subset\mathbf{m}} d_{\mathbf{k}} \frac{\gamma_{\mathbf{k}}(\rho-\mathbf{m})}{(\nu)_{\mathbf{k}}} \frac{1}{\left(\frac{N}{n}\right)_{\mathbf{k}}} \Phi_{\mathbf{k}}(-x). \end{aligned}$$

Proposition 2.3. (i) *The multivariate Laguerre functions $\Psi_{\mathbf{m}}^{(\nu)}$ form an orthogonal basis of $L_{\nu}^{2}(\Omega)$, and*

$$\|\Psi_{\mathbf{m}}^{(\nu)}\|_{\nu}^{2} = \frac{1}{d_{\mathbf{m}}} \frac{(\nu)_{\mathbf{m}}}{\left(\frac{N}{n}\right)_{\mathbf{m}}}. \tag{2.5}$$

(ii) *The fonctions $\Psi_{\mathbf{m}}^{(\nu)}$ admit the following generating formula: for $u \in \Omega$, $w \in \mathcal{D}$,*

$$\mathcal{G}_{\nu}^{(3)}(u,w) := \sum_{\mathbf{m}} d_{\mathbf{m}} \Psi_{\mathbf{m}}^{(\nu)}(u) \Phi_{\mathbf{m}}(w) = \Delta(e-w)^{-\nu} \int_{K} e^{-\left(k\cdot u | c(w)\right)} dk. \tag{2.6}$$

The generating formula can also be written

$$\Delta(e-w)^{-\nu} \int_{K} e^{(k\cdot x | w(e-w)^{-1})} dk = \sum_{\mathbf{m}} d_{\mathbf{m}} L_{\mathbf{m}}^{(\nu-1)}(x) \Phi_{\mathbf{m}}(w). \tag{2.6$'$}$$

Formula (2,6’) is proposed as an exercise in [Faraut-Korányi,1994] (Exercise 3, p.347). It is a special case of formula (4.4) in [Baker-Forrester,1997].

Proof. Part (i) follows from the fact that $\mathcal{L}_\nu$ is a unitary isomorphism from $L^2_\nu(\Omega)$ onto $\mathcal{H}^2_\nu(T_\Omega)$, and from (2.3).

The modified Laplace transform of $\mathcal{G}^{(3)}_\nu(u,w)$ with respect to u is equal to $\mathcal{G}^{(2)}_\nu(z,w)$, and one gets (ii) from (2.4). ∎

d) We will evaluate the spherical Fourier transform of the Laguerre functions $\Psi^{(\nu)}_{\mathbf{m}}$. We introduce now the modified spherical Fourier transform $\mathcal{F}_\nu$ as follows: for a function ψ on Ω,

$$(\mathcal{F}_\nu\psi)(\mathbf{s}) = \frac{1}{\Gamma_\Omega\big(\mathbf{s}+\frac{\nu}{2}+\rho\big)}\int_\Omega \psi(u)\varphi_{\mathbf{s}}(u)\Delta(u)^{\frac{\nu}{2}-\frac{N}{n}}m(du).$$

Observe that $\mathcal{F}_\nu\Psi_0 \equiv 1$.

Lemma 2.4. *For* $\operatorname{Re} s_j > \frac{d}{4}(n-1)-\frac{\nu}{2}$,

$$\mathcal{F}_\nu\big(e^{-\operatorname{tr} u}\Phi_{\mathbf{m}}\big)(\mathbf{s}) = (-1)^{|\mathbf{m}|}\gamma_{\mathbf{m}}\big(-\mathbf{s}-\frac{\nu}{2}\big).$$

Proof. Let $\sigma_D(u,\xi)$ be the symbol of $D\in\mathbb{D}(\Omega)$, and $p(\xi)=\sigma_D(e,\xi)$. By the invariance property of σ_D, we have $\sigma_D(u,-e)=p(-u)$, and therefore $De^{-\operatorname{tr} u} = p(-\xi)e^{-\operatorname{tr} u}$. Hence, for $p(\xi)=\Phi_{\mathbf{m}}(\xi)$,

$$\begin{aligned}\mathcal{F}_\nu(e^{-\operatorname{tr} u}\Phi_{\mathbf{m}})(s) &= (-1)^{|\mathbf{m}|}\mathcal{F}_\nu(D^{\mathbf{m}}e^{-\operatorname{tr} u})(s)\\ &= (-1)^{|\mathbf{m}|}\gamma_{\mathbf{m}}\big(-\mathbf{s}-\frac{\nu}{2}\big)\mathcal{F}_\nu(e^{-\operatorname{tr} u})\\ &= (-1)^{|\mathbf{m}|}\gamma_{\mathbf{m}}\big(-\mathbf{s}-\frac{\nu}{2}\big).\end{aligned}$$

∎

From Lemma 2.4 we obtain the evaluation of the spherical Fourier transform of the Laguerre functions: For $\operatorname{Re} s_j > \frac{d}{4}(n-1)-\frac{\nu}{2}$,

$$\mathcal{F}_\nu(\Psi^\nu_{\mathbf{m}})(\mathbf{s}) = Q_{\mathbf{m}}(\mathbf{s}).$$

By the spherical Plancherel formula and part (i) in Proposition 2.3, this proves parts (i) of Theorem 2.1, for $\nu > 1+d(n-1)$:

$$\int_{\mathbb{R}^n}|Q^{(\nu)}_{\mathbf{m}}(i\lambda)|^2 M_\nu(d\lambda) = \frac{1}{d_{\mathbf{m}}}\frac{(\nu)_{\mathbf{m}}}{\left(\frac{N}{n}\right)_{\mathbf{m}}}. \tag{2.7}$$

By analytic continuation it holds for $\nu > \frac{d}{2}(n-1)$.

For proving part (ii) of Theorem 2.1 one performs the spherical Fourier transform to both handsides of part (ii) in Proposition 2.3:

$$\mathcal{G}_\nu^{(4)} := \sum_{\mathbf{m}} d_{\mathbf{m}} Q_{\mathbf{m}}^{(\nu)}(\mathbf{s})\Phi_{\mathbf{m}}(w) = \Delta(e-w^2)^{-\frac{\nu}{2}}\varphi_{\mathbf{s}}\big(c(w)^{-1}\big). \qquad (2.8)$$

This finishes the proof of Theorem 2.1.

We remark that, in [Davidson-Ólafsson-Zang, 2003], a different notation is used for the Meixner-Pollaczek polynomials: their polynomials $p_{\nu,\mathbf{m}}$ (p. 179) are defined through the generating formula above and

$$p_{\nu,\mathbf{m}}(i\mathbf{s}) = d_{\mathbf{m}} Q_{\mathbf{m}}^{(\nu)}(\mathbf{s}).$$

3 Multivariate Meixner-Pollaczek polynomials $Q_{\mathbf{m}}^{(\nu,\theta)}$

The Meixner-Pollaczek polynomials $q_m^{(\nu)}$ we have considered at the beginning of Section 2 correspond to the special value $\phi = \frac{\pi}{2}$ with the classical notation. Using instead $\theta = \phi - \frac{\pi}{2}$, the more general one variable Meixner-Pollaczek polynomials can be written

$$\begin{aligned} q_m^{(\nu,\theta)}(s) &= e^{im\theta}\frac{(\nu)_m}{m!}\,{}_2F_1(-m, s+\frac{\nu}{2};\nu;2e^{-i\theta}\cos\theta) \\ &= e^{im\theta}\frac{(\nu)_m}{m!}\sum_{k=0}^m \frac{[m]_k\big[-s-\frac{\nu}{2}\big]_k}{(\nu)_k}\frac{1}{k!}(2e^{-i\theta}\cos\theta)^k. \end{aligned}$$

In terms of the classical notation $P_m^\alpha(\lambda;\phi)$

$$q_m^{(\nu,\theta)}(i\lambda) = (-i)^m P_m^{\frac{\nu}{2}}\big(\lambda;\theta+\frac{\pi}{2}\big).$$

For $\nu > 0$, $|\theta| < \frac{\pi}{2}$, the polynomials $q_m^{(\nu,\theta)}(i\lambda)$ are orthogonal with respect to the weight

$$e^{2\theta\lambda}\big|\Gamma\big(i\lambda+\frac{\nu}{2}\big)\big|^2.$$

In this section we consider the multivariate Meixner-Pollaczek polynomials $Q_{\mathbf{m}}^{(\nu,\theta)}$ defined by

$$Q_{\mathbf{m}}^{\nu,\theta)}(\mathbf{s}) =$$
$$e^{i|\mathbf{m}|\theta}\frac{(\nu)_{\mathbf{m}}}{\left(\frac{N}{n}\right)_{\mathbf{m}}}\sum_{\mathbf{k}\subset\mathbf{m}} d_{\mathbf{k}}\frac{\gamma_{\mathbf{k}}(\mathbf{m}-\rho)\gamma_{\mathbf{k}}\left(-\mathbf{s}-\frac{\nu}{2}\right)}{(\nu)_{\mathbf{k}}}\frac{1}{\left(\frac{N}{n}\right)_{\mathbf{k}}}(2e^{-i\theta}\cos\theta)^{|\mathbf{k}|}.$$

Theorem 3.1. *Assume* $\nu > \frac{d}{2}(n-1)$, $|\theta| < \frac{\pi}{2}$.

(i) *The multivariate Meixner-Pollaczek polynomials* $Q_{\mathbf{m}}^{(\nu,\theta)}(i\lambda)$ *form an orthogonal basis of* $L^2(\mathbb{R}^n, e^{2\theta(\lambda_1+\cdots+\lambda_n)}M_\nu)^{\mathfrak{S}_n}$. *The norm of* $Q_{\mathbf{m}}^{(\nu,\theta)}$ *can be evaluated:*

$$\int_{\mathbb{R}^n} |Q_{\mathbf{m}}^{(\nu,\theta)}(i\lambda)|^2 e^{2\theta(\lambda_1+\cdots+\lambda_n)}M_\nu(d\lambda) = (\cos\theta)^{-n\nu}\frac{1}{d_{\mathbf{m}}}\frac{(\nu)_{\mathbf{m}}}{\left(\frac{N}{n}\right)_{\mathbf{m}}}.$$

(ii) *The polynomials* $Q_{\mathbf{m}}^{(\nu,\theta)}$ *admit the following generating formula: for* $\mathbf{s}\in\mathbb{C}^n$, $w\in\mathcal{D}$,

$$\sum_{\mathbf{m}} d_{\mathbf{m}} Q_{\mathbf{m}}^{(\nu,\theta)}(\mathbf{s})\Phi_{\mathbf{m}}(w) = \Delta\big((e-e^{i\theta}w)(e+e^{-i\theta}w)\big)^{-\frac{\nu}{2}}\varphi_{\mathbf{s}}\big(c_\theta(w)^{-1}\big),$$

where c_θ *is the modified Cayley transform:*

$$c_\theta(w) = (e+e^{-i\theta}w)(e-e^{i\theta}w)^{-1}.$$

We will prove Theorem 3.1 in several steps.

a) Let us define the Laguerre functions $\Psi_{\mathbf{m}}^{(\nu,\theta)}$:

$$\Psi_{\mathbf{m}}^{(\nu,\theta)}(u) = e^{i|\mathbf{m}|\theta}e^{-\operatorname{tr} u}L_{\mathbf{m}}^{(\nu-1)}(2e^{-i\theta}\cos\theta\, u).$$

For functions ψ on V of the form $\psi(u) = e^{-\operatorname{tr} u}p(u)$, where p is a polynomial, define the inner product

$$(\psi_1|\psi_2)_{(\nu,\theta)} = \frac{2^{n\nu}}{\Gamma_\Omega(\nu)}\int_\Omega \psi_1(e^{i\theta}u)\overline{\psi_2(e^{i\theta}u)}\Delta(u)^{\nu-\frac{N}{n}}m(du).$$

Proposition 3.2. (i) *The Laguerre functions* $\Psi_{\mathbf{m}}^{(\nu,\theta)}$ *are orthogonal with respect to the inner product* $(\cdot|\cdot)_{(\nu,\theta)}$. *Furthermore*

$$\|\Psi_{\mathbf{m}}^{(\nu,\theta)}\|^2_{(\nu,\theta)} = (\cos\theta)^{-n\nu}\frac{1}{d_{\mathbf{m}}}\frac{(\nu)_{\mathbf{m}}}{\left(\frac{N}{n}\right)_{\mathbf{m}}}.$$

(ii) *The Laguerre functions* $\Psi_{\mathbf{m}}^{(\nu,\theta)}$ *satisfy the following generating formula: for* $u \in \Omega$, $w \in \mathcal{D}$,

$$\mathcal{G}^{(3)}_{\nu,\theta}(u,w) := \sum_{\mathbf{m}} d_{\mathbf{m}} \Psi_{\mathbf{m}}^{(\nu,\theta)}(u)\Phi_{\mathbf{m}}(w) = \Delta(e - e^{i\theta}w)^{-\nu} \int_K e^{\left(k\cdot u | c_\theta(w)\right)} dk.$$

Proof. (i) Put $\alpha = e^{i\theta}$, $\beta = 2e^{-i\theta}\cos\theta$. For two polynomials p_1 and p_2 consider the functions

$$\psi_1^{(\theta)}(u) = e^{-\operatorname{tr} u} p_1(\beta u),\ \ \psi_2^{(\theta)}(u) = e^{-\operatorname{tr} u} p_2(\beta u),$$

and their inner product

$$(\psi_1^{(\theta)}|\psi_2^{(\theta)})_{\nu,\theta} = \frac{2^{n\nu}}{\Gamma_\Omega(\nu)} \int_\Omega e^{-\alpha \operatorname{tr} u} p_1(\beta\alpha u)\overline{e^{-\alpha \operatorname{tr} u} p_2(\beta\alpha u)}\Delta(u)^{\nu-\frac{N}{n}} m(du).$$

Observe that $\beta\alpha = 2\cos\theta$, $\alpha + \bar\alpha = 2\cos\theta$. Hence

$$\begin{aligned}
&(\psi_1^{(\theta)}|\psi_2^{(\theta)})_{\nu,\theta}\\
= &\ \frac{2^{n\nu}}{\Gamma_\Omega(\nu)} \int_\Omega e^{-2\cos\theta \operatorname{tr} u} p_1(2\cos\theta u)\overline{p_2(2\cos\theta u)}\Delta(u)^{\nu-\frac{n}{N}} m(du)\\
= &\ \frac{2^{n\nu}}{\Gamma_\Omega(\nu)}(\cos\theta)^{-n\nu} \int_\Omega e^{-2\operatorname{tr} v} p_1(2v)\overline{p_2(2v)}\Delta(v)^{\nu-\frac{N}{n}} m(dv)\\
= &\ (\cos\theta)^{-n\nu}(\psi_1^{(0)}|\psi_2^{(0)}).
\end{aligned}$$

Take

$$p_1(u) = L_{\mathbf{p}}^{(\nu-1)}(u),\ \ p_2(u) = L_{\mathbf{q}}^{(\nu-1)}(u).$$

Then, by part (i) of Proposition 2.3, the statement (i) is proven.

(ii) The sum in the generating formula can be written

$$\sum_{\mathbf{m}} d_{\mathbf{m}} e^{-\operatorname{tr} u} L_{\mathbf{m}}^{(\nu-1)}(2e^{-i\theta}\cos\theta u)\Phi_{\mathbf{m}}(e^{i\theta}w).$$

Hence the generating formula follows from part (ii) in Proposition 2.3. ∎

b) By Lemma 2.4 we obtain the following evaluation of the spherical Fourier transform of the Laguerre functions $\Psi_{\mathbf{m}}^{(\nu,\theta)}$:

$$\mathcal{F}_\nu(\Psi_{\mathbf{m}}^{(\nu,\theta)})(\mathbf{s}) = Q_{\mathbf{m}}^{(\nu,\theta)}(\mathbf{s}).$$

We will need a Gutzmer formula for the spherical Fourier transform on a symmetric cone. Let us first state the following Gutzmer formula for the Mellin transform.

Proposition 3.3. *Let ψ be holomorphic in the following open set in $\mathbb{C}$:*

$$\{\zeta = re^{i\theta} \mid r > 0,\ |\theta| < \theta_0\} \quad \big(0 < \theta_0 < \frac{\pi}{2}\big).$$

The Mellin transform of ψ is defined by

$$\mathcal{M}\psi(s) = \int_0^\infty \psi(r) r^{s-1} dr.$$

Assume that there is a constant $M > 0$ such that, for $|\theta| < \theta_0$,

$$\int_0^\infty |\psi(re^{i\theta})|^2 r^{-1} dr \leq M.$$

Then

$$\int_0^\infty |\psi(re^{i\theta})|^2 r^{-1} dr = \frac{1}{2\pi} \int_{\mathbb{R}} |\mathcal{M}\psi(i\lambda)|^2 e^{2\theta\lambda} d\lambda.$$

Using the decomposition of the symmetric cone Ω as

$$\Omega =]0, \infty[\times \Omega_1,$$

where $\Omega_1 = \{u \in \Omega \mid \Delta(u) = 1\}$, one gets the following Gutzmer formula for Ω:

Proposition 3.4. *Let ψ be a holomorphic function in the tube $T_\Omega = \Omega + iV$. Assume that there are constants $M > 0$ and $0 < \theta_0 < \frac{\pi}{2}$ such that, for $|\theta| < \theta_0$,*

$$\int_\Omega |\psi(e^{i\theta}u)|^2 \Delta(u)^{-\frac{N}{n}} m(du) \leq M.$$

Then, for $|\theta| < \theta_0$,

$$\int_\Omega |\psi(e^{i\theta}u)|^2 \Delta(u)^{-\frac{N}{n}} du$$
$$= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} |\mathcal{F}\psi(i\lambda)|^2 e^{2\theta(\lambda_1 + \cdots + \lambda_n)} \frac{1}{|c(i\lambda)|^2} m(d\lambda).$$

From Proposition 3.2 and 3.4 we obtain parts (i) and (ii) of Theorem 3.1.

A more general Gutzmer formula has been established for the spherical Fourier transform on Riemannian symmetric spaces of non compact type [Faraut,2004].

4 Determinantal formulae

In the case $d=2$, i.e. $V=Herm(n,\mathbb{C})$, $K=U(n)$, there are determinantal formulae for the multivariate Laguerre functions $\Psi_{\mathbf{m}}^{(\nu)}$ and for the multivariate Meixner-Pollaczek polynomials $Q_{\mathbf{m}}^{(\nu,\theta)}$. Consider a Jordan frame $\{c_1,\ldots,c_n\}$ in V, and let $\delta=(n-1,n-2,\ldots,1,0)$.

Theorem 4.1. *Assume $d=2$. The multivariate Laguerre function $\Psi_{\mathbf{m}}^{(\nu)}$ admits the following determinantal formula involving the one variable Laguerre functions $\psi_m^{(\nu)}$: for $u=\sum_{j=1}^n u_i c_i$,*

$$\Psi_{\mathbf{m}}^{(\nu)}(u)=\delta!2^{-\frac{1}{2}n(n-1)}\frac{\det\big(\psi_{m_j+\delta_j}^{(\nu-n+1)}(u_i)\big)_{1\le i,j\le n}}{V(u_1,\ldots,u_n)},$$

where V denote the Vandermonde polynomial:

$$V(u_1,\ldots,u_n)=\prod_{i<j}(u_j-u_i).$$

As a result one obtains the following determinantal formula for the multivariate Laguerre polynomials:

$$\mathbf{L}_{\mathbf{m}}^{\nu}(u)=\delta!\frac{\det\Big(L_{m_j+\delta_j}^{(\nu-n+1)}(u_i)\Big)}{V(u_1,\ldots,u_n)}.$$

Proof. We start from the generating formula for the multivariate Laguerre functions (Proposition 2.3):

$$\begin{aligned}\mathcal{G}_\nu^{(3)}(u,w) &= \sum_{\mathbf{m}} d_{\mathbf{m}}\Phi_{\mathbf{m}}(w)\Psi_{\mathbf{m}}^{(\nu)}(u)\\ &= \Delta(e-w)^{-\nu}\int_K e^{-\big(ku|(e+w)(e-w)^{-1}\big)}dk.\end{aligned}$$

In the case $d=2$, the evaluation of this integral is classical: for $x=\sum_{i=1}^n x_i c_i$, $y=\sum_{j=1}^n y_j c_j$, then

$$\mathcal{I}(x,y)=\int_K e^{(kx|y)}dk=\delta!\frac{\det\big(e^{x_i y_j}\big)}{V(x_1,\ldots,x_n)V(y_1,\ldots,y_n)}.$$

Therefore, for $u = \sum_{i=1}^n u_i c_i$, $w = \sum_{j=1}^n w_j c_j$,

$$\mathcal{G}_\nu^{(3)}(u,w) = \delta! \prod_{j=1}^n (1-w_j)^{-\nu} \frac{\det\Big(e^{-u_i \frac{1+w_j}{1-w_j}}\Big)}{V(u_1,\ldots,u_n)V\big(\frac{1+w_1}{1-w_1},\ldots,\frac{1+w_n}{1-w_n}\big)}.$$

Noticing that

$$\frac{1+w_j}{1-w_j} - \frac{1+w_k}{1-w_k} = 2\frac{w_j - w_k}{(1+w_j)(1+w_k)},$$

we obtain

$$\mathcal{G}_\nu^{(3)}(u,w) = \delta! 2^{-\frac{1}{2}n(n-1)} \frac{\det\Big((1-w_j)^{-(\nu-n+1)} e^{-u_i \frac{1+w_j}{1-w_j}}\Big)}{V(u_1,\ldots,u_n)V(w_1,\ldots,w_n)}.$$

We will expand the above expression in Schur function series by using a formula due to Hua.

Lemma 4.2. *Consider n power series*

$$f_i(w) = \sum_{m=0}^\infty c_m^{(i)} w^m \quad (i = 1,\ldots,n).$$

Then

$$\frac{\det\big(f_i(w_j)\big)}{V(w_1,\ldots,w_n)} = \sum_{\mathbf{m}} a_{\mathbf{m}} s_{\mathbf{m}}(w_1,\ldots,w_n),$$

where $s_{\mathbf{m}}$ is the Schur function associated to the partition $\mathbf{m}$, and

$$a_{\mathbf{m}} = \det\big(c^{(i)}_{m_j+\delta_j}\big).$$

(See [Hua,1963], Theorem 1.2.1, p.22).

Let $\nu' = \nu - n + 1$, and consider the n power series

$$f_i(w) = (1-w)^{-\nu'} e^{-u_i \frac{1+w}{1-w}} = \sum_{m=0}^\infty \psi_m^{(\nu')}(u_i) w^m.$$

Since

$$d_{\mathbf{m}} \Phi_{\mathbf{m}}\Big(\sum_{j=1}^n w_j c_j\Big) = s_{\mathbf{m}}(w_1,\ldots,w_n),$$

we obtain

$$\Psi^{(\nu)}_{\mathbf{m}}(u) = \delta! 2^{-\frac{1}{2}n(n-1)} \frac{\det\big(\psi^{(\nu-n+1)}_{m_j+\delta_j}(u_i)\big)}{V(u_1,\ldots,u_n)}.$$

∎

By using the same method we will obtain a determinantal formula for the multivariate Meixner-Pollaczek polynomials $Q^{(\nu,\theta)}_{\mathbf{m}}$.

Theorem 4.3. *Assume $d=2$. Then*

$$Q^{(\nu,\theta)}_{\mathbf{m}}(\mathbf{s}) = (-2\cos\theta)^{-\frac{1}{2}n(n-1)} \delta! \frac{\det\Big(q^{(\nu-n+1,\theta)}_{m_j+\delta_j}(s_i)\Big)_{1\le i,j\le n}}{V(s_1,\ldots,s_n)},$$

where $q^{(\nu,\theta)}_m$ denotes the one variable Meixner-Pollaczek polynomial.

Proof. We start from the generating formula for the multivariate Meixner-Pollaczek polynomials $Q^{(\nu,\theta)}_{\mathbf{m}}$ (Theorem 3.1, (ii)):

$$\sum_{\mathbf{m}} d_{\mathbf{m}} Q^{(\nu,\theta)}_{\mathbf{m}}(\mathbf{s}) \Phi_{\mathbf{m}}(w) = \Delta\big((e - e^{i\theta}w)(e + e^{-i\theta}w)\big)^{-\frac{\nu}{2}} \varphi_{\mathbf{s}}\big(c_\theta(w)\big)^{-1}\big).$$

For $x = \sum_{i=1}^n x_i c_i$, the spherical function $\varphi_{\mathbf{s}}(x)$ is essentially a Schur function in the variables $x_1,\ldots,x_n$:

$$\varphi_{\mathbf{s}}(x) = \delta! (x_1 x_2 \ldots x_r)^{\frac{1}{2}(n-1)} \frac{\det(x_j^{s_i})}{V(s_1,\ldots,s_n)V(x_1,\ldots,x_n)}.$$

Let us compute now, for $w = \sum_{j=1}^n w_j c_j$,

$$\begin{aligned}
&\Delta\big((e - e^{i\theta}w)(e + e^{-i\theta}w)\big)^{-\frac{\nu}{2}} \varphi_{\mathbf{s}}\big(c_\theta(w)\big)^{-1}\big) \\
&= \delta! \prod_{j=1}^n (1 - 2i\sin\theta w_j - w_j^2)^{-\frac{\nu}{2}} \\
&\prod_{j=1}^n \Big(c_\theta(w_j)\Big)^{\frac{1}{2}(n-1)} \frac{\det\Big(\big(c_\theta(w_j)\big)^{s_i}\Big)}{V(s_1,\ldots,s_n)V\Big(c_\theta(w_1),\ldots,c_\theta(w_n)\Big)}.
\end{aligned}$$

In the same way as for the proof of Theorem 4.1, we obtain

$$\Delta\big((e - e^{i\theta}w)(e + e^{-i\theta})\big)^{-\frac{\nu}{2}}\varphi_{\mathbf{s}}\big(c_\theta(w)\big)^{-1}\big)$$
$$= (-2\cos\theta)^{-\frac{1}{2}n(n-1)}\delta!$$
$$\frac{\det\Big((1 - e^{i\theta}w_j)^{s_i-\frac{\nu}{2}+\frac{1}{2}(n-1)}(1 + e^{-i\theta}w_j)^{-s_i-\frac{\nu}{2}+\frac{1}{2}(n-1)}\Big)}{V(s_1,\dots,s_n)V(w_1,\dots,w_n)}.$$

We apply once more Lemma 4.2 to the n power series

$$f_i(w) = (1 - e^{i\theta}w)^{s_i-\frac{\nu'}{2}}(1 + e^{-i\theta}w)^{-s_i-\frac{\nu'}{2}} = \sum_m^\infty q_m^{(\nu',\theta)}(s_i)w^m$$

with $\nu' = \nu - n + 1$, and obtain finally:

$$Q_{\mathbf{m}}^{(\nu,\theta)}(\mathbf{s}) = (-2\cos\theta)^{-\frac{1}{2}n(n-1)}\delta!\frac{\det\Big(q_{m_j+\delta_j}^{(\nu-n+1,\theta)}(s_i)\Big)}{V(s_1,\dots,s_n)}.$$

□

5 Difference equation for the Meixner-Pollaczek polynomials $Q_{\mathbf{m}}^{(\nu,\theta)}$

We will introduce a difference operator acting on functions in n variables. We recall first the following Pieri's formula for the spherical functions.

Proposition 5.1.

$$(\mathrm{tr}\,\mathrm{x})\varphi_{\mathbf{s}}(\mathrm{x}) = \sum_{\mathrm{j}=1}^{\mathrm{n}} \alpha_{\mathrm{j}}(\mathbf{s})\varphi_{\mathbf{s}+\varepsilon_{\mathrm{j}}}(\mathrm{x}),$$

with

$$\alpha_j(\mathbf{s}) = \prod_{k\neq j}\frac{s_j - s_k + \frac{d}{2}}{s_j - s_k}.$$

($\{\varepsilon_j\}$ denotes the canonical basis of $\mathbb{C}^n$.)

See [Dib, 1990], Proposition 6.1 (with a minor correction), where it is called Kushner's formula. See also [Zhang, 1995], Theorem 1. One observes that

$$\alpha_j(\mathbf{s}) = \frac{c(\mathbf{s})}{c(\mathbf{s}+\varepsilon_j)},$$

in agreement with the asymptotic behaviour of the spherical function $\varphi_{\mathbf{s}}$: for $\mathbf{s} = (s_1, \ldots, s_n) \in \mathbb{C}^n$, with $\operatorname{Re} s_1 > \cdots > \operatorname{Re} s_n$, and $a = \sum_{j=1}^n a_j c_j$ with $a_1 > \cdots > a_n$ ($(c_1, \ldots, c_n)$ is a Jordan frame in V),

$$\varphi_{\mathbf{s}}(\exp ta) \sim c(\mathbf{s}) e^{(\mathbf{s}+\rho|a)t} \quad (t \to \infty).$$

For a partition $\mathbf{m}$, by letting $\mathbf{m} = \mathbf{s} + \rho$, one gets

$$(\mathrm{tr}\, \mathrm{x})\Phi_{\mathbf{m}}(\mathrm{x}) = \sum_{\mathrm{j}=1}^{\mathrm{n}} \mathrm{a_j}(\mathbf{m})\Phi_{\mathbf{m}+\varepsilon_{\mathrm{j}}}(\mathrm{x}),$$

with

$$a_j(\mathbf{m}) = \prod_{k\neq j} \frac{m_j - m_k - \frac{d}{2}(j-k-1)}{m_j - m_k - \frac{d}{2}(j-k)}$$

(in ageement with Lassalle's results [1998], p.320, l.-4).

The difference operator $D_{\nu,\theta}$ is defined by

$$\begin{aligned} &D_{\nu,\theta} f(\mathbf{s}) \\ &= e^{-i\theta} \sum_{j=1}^n \big(s_j + \frac{\nu}{2} - \frac{d}{4}(n-1)\big)\alpha_j(\mathbf{s})\big(f(\mathbf{s}+\varepsilon_j) - f(\mathbf{s})\big) \\ &+ e^{i\theta} \sum_{j=1}^n \big(-s_j + \frac{\nu}{2} - \frac{d}{4}(n-1)\big)\alpha_j(-\mathbf{s})\big(f(\mathbf{s}-\varepsilon_j) - f(\mathbf{s})\big). \end{aligned}$$

Theorem 5.2. *The Meixner-Pollaczek polynomial $Q_{\mathbf{m}}^{(\nu,\theta)}$ is an eigenfunction of the difference operator $D_{\nu,\theta}$:*

$$D_{\nu,\theta} Q_{\mathbf{m}}^{(\nu,\theta)} = 2|\mathbf{m}| \cos\theta \; Q_{\mathbf{m}}^{(\nu,\theta)}.$$

For the proof we will use the scheme we have used in the proof of Theorem 2.1. For $i = 1, 2, 3, 4$, we define the operators $D^{(i)}_{\nu,\theta}$. The operator $D^{(1)}_{\nu,\theta} = D^{(1)}_{\theta}$ is a first order differential operator on the domain $\mathcal{D}$:

$$D^{(1)}_{\theta} f = e^{i\theta}\langle w + e, \nabla f\rangle + e^{-i\theta}\langle w - e, \nabla f\rangle.$$

(For $W_1, w_2 \in V_{\mathbb{C}}$, $\langle w_1, w_2\rangle = \mathrm{tr}\,(\mathrm{w_1 w_2})$.) The operators $D^{(i)}_{\nu,\theta}$, for $i = 2, 3, 4$ are defined by the relations:

$$\begin{aligned} D^{(2)}_{\nu,\theta} C_\nu &= C_\nu D^{(1)}_{\nu,\theta}, \\ \mathcal{L}_\nu D^{(3)}_{\nu,\theta} &= D^{(2)}_{\nu,\theta}\mathcal{L}_\nu, \\ \mathcal{F}_\nu D^{(3)}_{\nu,\theta} &= D^{(4)}_{\nu,\theta}\mathcal{F}_\nu. \end{aligned}$$

The operator $D^{(2)}_{\nu,\theta}$ is a first order differential operator on the tube T_Ω. In Section 7 we will see that $D^{(3)}_{\nu,\theta}$ is a second order differential operator on the cone Ω, and prove that $D^{(4)}_{\nu,\theta}$ is the difference operator $D_{\nu,\theta}$ we have introduced above.

The function

$$\Phi^{(\theta)}_{\mathbf{m}}(w) = \Phi_{\mathbf{m}}(w\cos\theta + ie\sin\theta)$$

is an eigenfunction of the operator $D^{(1)}_{\theta}$:

$$D^{(1)}_{\theta}\Phi^{(\theta)}_{\mathbf{m}} = 2|\mathbf{m}|\cos\theta\ \Phi^{(\theta)}_{\mathbf{m}}.$$

In fact $\Phi_{\mathbf{m}}$ is homogeneous of degree $|\mathbf{m}|$, and satisfies the Euler equation

$$\langle w, \nabla\Phi_{\mathbf{m}}\rangle = |\mathbf{m}|\Phi_{\mathbf{m}}.$$

Hence $F^{(\nu,\theta)}_{\mathbf{m}} = C_\nu\Phi^{(\theta)}_{\mathbf{m}}$ is an eigenfunction of $D^{(2)}_{\nu,\theta}$:

$$D^{(2)}_{\nu,\theta}F^{(\nu,\theta)}_{\mathbf{m}} = 2|\mathbf{m}|\cos\theta\ F^{(\nu,\theta)}_{\mathbf{m}}.$$

Further, since $\mathcal{L}_\nu\Psi^{(\nu,\theta)}_{\mathbf{m}} = \frac{(\nu)_{\mathbf{m}}}{\left(\frac{N}{n}\right)_{\mathbf{m}}}F^{(\nu,\theta)}_{\mathbf{m}}$, we get

$$D^{(3)}_{\nu,\theta}\Psi^{(\nu,\theta)}_{\mathbf{m}} = 2|\mathbf{m}|\cos\theta\ \Psi^{(\nu,\theta)}_{\mathbf{m}}.$$

Finally, since $Q_{\mathbf{m}}^{(\nu,\theta)} = \mathcal{F}_\nu \Psi_{\mathbf{m}}^{(\nu,\theta)}$,

$$D_{\nu,\theta}^{(4)} Q_{\mathbf{m}}^{(\nu,\theta)} = 2|\mathbf{m}| \cos\theta \; Q_{\mathbf{m}}^{(\nu,\theta)}.$$

Hence the proof of Theorem 5.2 amounts to showing that $D_{\nu,\theta}^{(4)} = D_{\nu,\theta}$.

The symmetries $S_\nu^{(i)}$ we will introduce in next Section will be useful for the computation of the operators $D_{\nu,\theta}^{(i)}$.

6 The symmetries $S_\nu^{(i)}$ $(i = 1, 2, 3, 4)$ and the Hankel transform

We start from the symmetry $w \mapsto -w$ of the domain $\mathcal{D}$. Its action on functions is given by

$$S^{(1)} f(w) = f(-w).$$

We carry this symmetry over the tube T_Ω through the Cayley transform and obtain the inversion $z \mapsto z^{-1}$. We define $S_\nu^{(2)}$ such that

$$S_\nu^{(2)} C_\nu = C_\nu S^{(1)}.$$

Hence, for a function F on T_Ω,

$$S_\nu^{(2)} F(z) = \Delta(z)^{-\nu} F(z^{-1}).$$

Further $S_\nu^{(3)}$ is defined by the relation

$$\mathcal{L}_\nu S_\nu^{(3)} = S_\nu^{(2)} \mathcal{L}_\nu.$$

By a generalized theorem of Tricomi (Theorem XV.4.1 in [Faraut-Korányi,1994]), the unitary isomorphism $S_\nu^{(3)}$ of $L_\nu^2(\Omega)$ is the Hankel transform: $S_\nu^{(3)} = U_\nu$,

$$U_\nu \psi(u) = \int_\Omega H_\nu(u, v) \psi(v) \Delta(v)^{\nu - \frac{N}{n}} m(dv).$$

The kernel $H_\nu(u, v)$ has the following invariance property:

$$H_\nu(g \cdot u, v) = H_\nu(u, g^* \cdot v) \quad (g \in G),$$

and

$$H_\nu(u,e) = \frac{1}{\Gamma_\Omega(\nu)} \mathcal{J}_\nu(u),$$

where $\mathcal{J}_\nu$ is a multivariate Bessel function.

Finally we define $S_\nu^{(4)}$ acting on symmetric polynomials in n variables such that

$$S_\nu^{(4)} \mathcal{F}_\nu = \mathcal{F}_\nu S_\nu^{(3)}.$$

Proposition 6.1. *For a symmetric polynomial p,*

$$S_\nu^{(4)} p(\mathbf{s}) = p(-\mathbf{s}).$$

Proof. We will evaluate the spherical Fourier transform $\mathcal{F}_\nu(U_\nu \psi)$. By the invariance property, the kernel $H_\nu(u,v)$ can be written

$$H_\nu(u,v) = h_\nu\big(P(v^{\frac{1}{2}})u\big)\Delta(u)^{-\frac{\nu}{2}}\Delta(v)^{-\frac{\nu}{2}},$$

with $h_\nu(u) = H_\nu(u,e)\Delta(u)^{\frac{\nu}{2}}$, and P is the so-called quadratic representation of the Jordan algebra V. Let us compute first

$$\begin{aligned} &\int_\Omega H_\nu(u,v)\varphi_{\mathbf{s}}(u)\Delta(u)^{\frac{\nu}{2}-\frac{N}{n}} m(du) \\ = \;& \Delta(v)^{-\frac{\nu}{2}} \int_\Omega h_\nu\big(P(v^{\frac{1}{2}})u\big)\varphi_{\mathbf{s}}(u)\Delta(u)^{-\frac{N}{n}} m(du). \end{aligned}$$

By letting $P(v^{\frac{1}{2}})u = u'$, we get

$$\begin{aligned} &\int_\Omega H_\nu(u,v)\varphi_{\mathbf{s}}(u)\Delta(u)^{\frac{\nu}{2}-\frac{N}{n}} m(du) \\ = \;& \Delta(v)^{-\frac{\nu}{2}} \int_\Omega h_\nu(u')\varphi_{\mathbf{s}}\big(P(v^{-\frac{1}{2}})u'\big)\Delta(u)^{-\frac{N}{n}} m(du). \end{aligned}$$

By using K-invariance, and the functional equation of the spherical function $\varphi_{\mathbf{s}}$:

$$\int_K \varphi_{\mathbf{s}}\big(P(v^{-\frac{1}{2}})ku'\big)dk = \varphi_{\mathbf{s}}(v^{-1})\varphi_{\mathbf{s}}(u'),$$

we get

$$\int_\Omega H_\nu(u,v)\varphi_{\mathbf{s}}(u)\Delta(u)^{\frac{\nu}{2}-\frac{N}{n}} m(du) = \varphi_{\mathbf{s}}(v^{-1})\Delta(v)^{-\frac{\nu}{2}}\mathcal{F}(h_\nu)(\mathbf{s}).$$

Recall that $\varphi_{\mathbf{s}}(v^{-1}) = \varphi_{-\mathbf{s}}(v)$. We multiply both sides by $\psi(v)$ and get by integrating with respect to v:

$$\Gamma_\Omega\big(\mathbf{s} + \frac{\nu}{2} + \rho\big)\mathcal{F}_\nu(U_\nu\psi)(\mathbf{s}) = \mathcal{F}h_\nu(\mathbf{s})\Gamma_\Omega\big(-\mathbf{s} + \frac{\nu}{2} + \rho\big)\mathcal{F}_\nu\psi(-\mathbf{s}).$$

Consider the special case $\psi(u) = \Psi_0(u) = e^{-\mathrm{tr}\,\mathrm{u}}$. Since $U_\nu\Psi_0 = \Psi_0$, and $\mathcal{F}_\nu\Psi_0 \equiv 1$, we get

$$\mathcal{F}(h_\nu) = \frac{\Gamma_\Omega\big(\mathbf{s} + \frac{\nu}{2} + \rho\big)}{\Gamma_\Omega\big(-\mathbf{s} + \frac{\nu}{2} + \rho\big)}.$$

Finally

$$\mathcal{F}_\nu(U_\nu\psi)(\mathbf{s}) = \mathcal{F}_\nu\psi(-\mathbf{s}).$$

It follows that $S_\nu^{(4)}p(\mathbf{s}) = p(-\mathbf{s})$. ∎

Corollary 6.2.

$$Q_{\mathbf{m}}^{(\nu,\theta)}(-\mathbf{s}) = (-1)^{|\mathbf{m}|}Q_{\mathbf{m}}^{(\nu,-\theta)}(\mathbf{s}).$$

Proof. This relation follows from

$$S^{(1)}\Phi_{\mathbf{m}}^{(\theta)} = \Phi_{\mathbf{m}}^{(\theta)}(-w) = (-1)^{|\mathbf{m}|}\Phi_{\mathbf{m}}^{(-\theta)}(w),$$

which is easy to check, and Proposition 6.1. ∎

The operator $D_{\nu,\theta}^{(i)}$ $(i = 1, 2, 3, 4)$ can be written

$$D_{\nu,\theta}^{(i)} = e^{i\theta}D_\nu^{(i,+)} + e^{-i\theta}D_\nu^{(i,-)}.$$

For $i = 1$, $D_\nu^{(1,\pm)}$ does not depend on ν, $D_\nu^{(1,\pm)} = D^{(1,\pm)}$:

$$D^{(1,+)}f(w) = \langle w + e, \nabla f(w)\rangle, \quad D^{(1,-)}f(w) = \langle w - e, \nabla f(w)\rangle.$$

Observe that

$$D^{(1,-)} = S^{(1)}D^{(1,+)}S^{(1)}.$$

It follows that, for $i = 2, 3, 4$,

$$D_\nu^{(i,-)} = S_\nu^{(i)}D_\nu^{(i,+)}S_\nu^{(i)}.$$

In next Section we will compute first $D_\nu^{(i,-)}$. The operator $D_\nu^{(i,+)}$ is then obtained by using the above relation. For $i = 3$, we will use the following property of the Hankel transform

Proposition 6.3.

$$U_\nu(\mathrm{tr\,v}\ \psi) = -\Big(\langle \mathrm{u}, \big(\frac{\partial}{\partial \mathrm{u}}\big)^2\rangle + \nu\mathrm{tr}\,\big(\frac{\partial}{\partial \mathrm{u}}\big)\Big)\mathrm{U}_\nu\psi.$$

This is a consequence of Proposition XV.2.3 in [Faraut-Korányi,1994].

7 Proof of Theorem 5.2

a) Recall that $D^{(1,-)}$ is the first order differential operator on the domain $\mathcal{D}$ given by

$$D^{(1,-)} f(w) = \langle w - e, \nabla f(w) \rangle,$$

and $D_\nu^{(2,-)}$ is the first order differential operator on the tube T_Ω such that

$$D_\nu^{(2,-)} C_\nu = C_\nu D^{(1,-)}.$$

Lemma 7.1.

$$D_\nu^{(2,-)} F(z) = -\langle z + e, \nabla F(z) \rangle - n\nu F(z).$$

Proof. Recall that, for a function F on the tube T_Ω,

$$f(w) = (C_\nu^{-1} F)(w) = \Delta(e - w)^{-\nu} F\big(c(w)\big),$$

where c is the Cayley transform

$$c(w) = (e + w)(e - w)^{-1} = 2(e - w)^{-1} - e.$$

Its differential is given by

$$(Dc)_w = 2P\big((e - w)^{-1}\big).$$

We get

$$\nabla f(w) = \nabla\big(\Delta(e - w)^{-\nu}\big) F\big(c(w)\big) + \Delta(e - w)^{-\nu} 2P\big(e - w)^{-1}\big)\Big(\nabla F\big(c(w)\big)\Big).$$

By using

$$\begin{aligned} \nabla\big(\Delta(x)^\alpha\big) &= \alpha \Delta(x)^\alpha x^{-1}, \\ \langle e - w, (e - w)^{-1} \rangle &= n, \\ P\big((e - w)^{-1}\big)(e - w) &= (e - w)^{-1}, \end{aligned}$$

we obtain

$$\begin{aligned} \langle w - e, \nabla f(w) \rangle &= \Delta(e - w)^{-\nu}\Big(-n\nu F\big(c(w)\big) + 2\langle (w - e)^{-1}, \nabla F\big(c(w)\big)\rangle\Big) \\ &= (C_\nu^{-1} G)(z), \end{aligned}$$

with

$$G(z) = -\langle z + e, \nabla F(z)\rangle - n\nu F(z).$$

∎

b) Consider now the differential operator $D_\nu^{(3,-)}$ on the cone Ω such that

$$\mathcal{L}_\nu D_\nu^{(3,-)} = D_\nu^{(2,-)} \mathcal{L}_\nu.$$

Recall that the modified Laplace transform $\mathcal{L}_\nu \psi$ of a function ψ, defined on Ω, is given by

$$F(z) = \mathcal{L}_\nu \psi(z) = \frac{2^{n\nu}}{\Gamma_\Omega(\nu)} \int_\Omega e^{-(z|u)} \psi(u) \Delta(u)^{\nu - \frac{N}{n}} m(du).$$

Lemma 7.2.

$$D_\nu^{(3,-)} \psi(u) = \langle u, \nabla \psi(u)\rangle + \mathrm{tr}\, \mathrm{u}\, \psi(\mathrm{u}).$$

Proof. For $a \in V_{\mathbb{C}}$,

$$\langle a, \nabla F(z)\rangle = \frac{2^{n\nu}}{\Gamma_\Omega(\nu)} \int_\Omega e^{-(z|u)} (-\langle a, u\rangle) \psi(u) \Delta(u)^{\nu - \frac{N}{n}} m(du).$$

Observe that

$$(z|u) e^{-(z|u)} = \langle u, \nabla_u \rangle e^{-(z|u)}.$$

Therefore

$$\langle z, \nabla F(z)\rangle = \frac{2^{n\nu}}{\Gamma_\Omega(\nu)} \int_\Omega (-\langle u, \nabla_u \rangle e^{-(z|u)}) \psi(u) \Delta(u)^{\nu - \frac{N}{n}} m(du).$$

An integration by parts gives

$$= \frac{2^{n\nu}}{\Gamma_\Omega(\nu)} \int_\Omega e^{-(z|u)} (\langle u, \nabla\rangle + n\nu) \psi(u) \Delta^{\nu - \frac{N}{n}} m(du).$$

Finally

$$(D_\nu^{(2,-)} F)(z) = \mathcal{L}_\nu(\langle u, \nabla \psi\rangle + \mathrm{tr}\, \mathrm{u}\, \psi).$$

∎

c) The operator $D_\nu^{(4,-)}$ acting on symmetric functions on $\mathbb{C}^n$ is such that

$$D_\nu^{(4,-)} \mathcal{F}_\nu = \mathcal{F}_\nu D_\nu^{(3,-)}.$$

Recall that the spherical Fourier transform $f = \mathcal{F}_\nu \psi$ of a function ψ, defined on Ω, is given by

$$f(\mathbf{s}) = (\mathcal{F}_\nu \psi)(\mathbf{s}) = \frac{1}{\Gamma_\Omega\big(\mathbf{s} + \frac{\nu}{2} + \rho\big)} \int_\Omega \varphi_{\mathbf{s}}(u)\psi(u)\Delta(u)^{\frac{\nu}{2} - \frac{N}{n}} m(du).$$

Proposition 7.3. *The operator $D_\nu^{(4,-)}$ is the following difference operator: for a function f on $\mathbb{C}^n$,*

$$D_\nu^{(4,-)} f(\mathbf{s}) = \sum_{j=1}^n \big(s_j + \frac{\nu}{2} - \frac{d}{4}(n-1)\alpha_j(\mathbf{s})\big)\big(f(\mathbf{s} + \varepsilon_j) - f(\mathbf{s})\big).$$

Proof. We will compute $\mathcal{F}_\nu(D_\nu^{(3,-)}\psi) = \mathcal{F}_\nu(\langle u, \nabla\psi\rangle + \operatorname{tr} \mathrm{u}\, \psi)$. Consider first

$$\mathcal{F}_\nu(\langle u, \nabla\psi\rangle)(\mathbf{s}) = \frac{1}{\Gamma_\Omega\big(\mathbf{s} + \frac{\nu}{2} + \rho\big)} \int_\Omega \langle u, \nabla\psi(u)\rangle \varphi_{\mathbf{s}+\frac{\nu}{2}}(u)\Delta(u)^{-\frac{N}{n}} m(du).$$

An integration by parts gives, since the function $\varphi_{\mathbf{s}}$ is homogeneous of degree $\sum_{j=1}^n s_j$ (observe that $\sum_{j=1}^n \rho_j = 0$),

$$\begin{aligned}
&= \frac{1}{\Gamma_\Omega\big(\mathbf{s} + \frac{\nu}{2} + \rho\big)} \int_\Omega \psi(u)\big(-\langle u, \nabla_u\rangle \varphi_{\mathbf{s}+\frac{\nu}{2}}(u)\big)\Delta(u)^{-\frac{N}{n}} m(du) \\
&= \frac{1}{\Gamma_\Omega\big(\mathbf{s} + \frac{\nu}{2} + \rho\big)} \int_\Omega \psi(u)\Big(-\sum_{j=1}^n \big(s_j + \frac{\nu}{2}\big)\varphi_{\mathbf{s}}(u)\Delta(u)^{\frac{\nu}{2} - \frac{N}{n}} m(du) \\
&= -\sum_{j=1}^n \big(s_j + \frac{\nu}{2}\big)\mathcal{F}_\nu\psi(\mathbf{s}).
\end{aligned}$$

Recall the Pieri's formula (Proposition 5.1):

$$\operatorname{tr} \mathrm{u}\, \varphi_{\mathbf{s}}(\mathrm{u}) = \sum_{\mathrm{j}=1}^{\mathrm{n}} \alpha_{\mathrm{j}}(\mathbf{s})\varphi_{\mathbf{s}+\varepsilon_{\mathrm{j}}}(\mathrm{u}).$$

Hence

$$
\begin{aligned}
& \mathcal{F}_\nu(\mathrm{tr}\, \mathrm{u}\, \psi)(\mathbf{s}) \\
= \ & \frac{1}{\Gamma_\Omega\big(\mathbf{s}+\frac{\nu}{2}+\rho\big)} \int_\Omega \psi(u)\Big(\sum_{j=1}^n \alpha(\mathbf{s})\varphi_{\mathbf{s}+\varepsilon_j}(u)\Big)\Delta(u)^{\frac{\nu}{2}-\frac{N}{n}} m(du) \\
= \ & \sum_{j=1}^n \frac{\Gamma_\Omega\big(\mathbf{s}+\varepsilon_j+\frac{\nu}{2}+\rho\big)}{\Gamma_\Omega\big(\mathbf{s}+\frac{\nu}{2}+\rho\big)}\alpha_j(\mathbf{s}) \\
& \frac{1}{\Gamma_\Omega\big(\mathbf{s}+\varepsilon_j+\frac{\nu}{2}+\rho\big)} \int_\Omega \psi(u)\varphi_{\mathbf{s}+\varepsilon_j}(u)\Delta^{\frac{\nu}{2}-\frac{N}{n}} m(du) \\
= \ & \sum_{j=1}^n \big(s_j+\frac{\nu}{2}-\frac{d}{4}(n-1)\big)\alpha_j(\mathbf{s})\mathcal{F}_\nu\psi(\mathbf{s}+\varepsilon_j).
\end{aligned}
$$

Finally

$$
\begin{aligned}
& \mathcal{F}_\nu(D_\nu^{(3,-)}\psi)(\mathbf{s}) \\
= \ & \sum_{j=1}^n \big(s_j+\frac{\nu}{2}-\frac{d}{4}(n-1)\big)\alpha_j(\mathbf{s})f(\mathbf{s}+\varepsilon_j) - \sum_{j=1}^n \big(s_j+\frac{\nu}{2}\big)f(\mathbf{s}),
\end{aligned}
$$

with $f=\mathcal{F}_\nu(\psi)$. From $D_\nu^{(3,-)}\Psi_0=0$ and $\mathcal{F}_\nu(\Psi_0)=1$, we get

$$
\sum_{j=1}^n \big(s_j+\frac{\nu}{2}-\frac{d}{4}(n-1)\big)\alpha_j(\mathbf{s}) = \sum_{j=1}^n \big(s_j+\frac{\nu}{2}\big).
$$

Therefore

$$
\begin{aligned}
& \mathcal{F}_\nu(D_\nu^{(3,-)}\psi)(\mathbf{s}) \\
= \ & \sum_{j=1}^n \big(s_j+\frac{\nu}{2}-\frac{d}{4}(n-1)\big)\alpha_j(\mathbf{s})\big(f(\mathbf{s}+\varepsilon_j)-f(\mathbf{s})\big).
\end{aligned}
$$

∎

We finish now the proof of Theorem 5.2. Recall that

$$
D_\nu^{(4,+)} = S_\nu^{(4)} D_\nu^{(4,-)} S_\nu^{(4)}, \quad \text{and} \quad S_\nu^{(4)} f(\mathbf{s}) = f(-\mathbf{s}).
$$

Therefore, by Proposition 7.3,

$$D_\nu^{(4,+)} f(\mathbf{s}) = \sum_{j=1}^n \big(-s_j + \frac{\nu}{2} - \frac{d}{4}(n-1)\big)\alpha_j(-\mathbf{s})\big(f(\mathbf{s}-\varepsilon_j) - f(\mathbf{s})\big).$$

We have establish the formula of Theorem 5.2 since

$$D_{\nu,\theta} = D_{\nu,\theta}^{(4)} = e^{i\theta} D_\nu^{(4,+)} + e^{-i\theta} D_\nu^{(4,-)}.$$

8 Differential equation for the Laguerre polynomials $L_{\mathbf{m}}^{(\nu-1)}$

Theorem 8.1. *The Laguerre polynomial $L = L_{\mathbf{m}}^{(\nu-1)}$ is a solution of the differential equation*

$$\langle x, \Big(\frac{\partial}{\partial x}\Big)^2\rangle L + \langle \nu e - x, \Big(\frac{\partial}{\partial x}\Big)\rangle L + |\mathbf{m}| L = 0.$$

Observe that, for $n = 1$, this is the classical Laguerre differential equation for the ordinary Laguerre polynomial $y = L_m^{(\nu-1)}$:

$$xy'' + (\nu - x)y' + my = 0.$$

An equivalent formula is given in [Davidson-Ólafsson,2003], Theorem 6.1, and in [Aristidou et al.,2007], Theorem 6.3.

Proof. Recall the relation

$$\Psi_{\mathbf{m}}^{(\nu)}(u) = e^{-\operatorname{tr} u} L_{\mathbf{m}}^{(\nu-1)}(2u),$$

and that

$$D_{\nu,0}^{(3)} \Psi_{\mathbf{m}}^{(\nu)} = 2|\mathbf{m}| \Psi_{\mathbf{m}}^{(\nu)}.$$

Furthermore

$$D_{\nu,0}^{(3)} = D_\nu^{(3,+)} + D_\nu^{(3,-)}, \quad D_\nu^{(3,+)} = U_\nu D_\nu^{(3,-)} U_\nu,$$

where $U_\nu = S_\nu^{(3)}$ is the Hankel transform. By Proposition 7.2,

$$D_\nu^{(3,-)}\psi = \langle u, \nabla\psi(u)\rangle + \operatorname{tr} u\, \psi.$$

By using the relation

$$U_\nu(\langle v, \nabla\psi\rangle) = -(\langle u, \nabla\rangle + n\nu)U_\nu\psi,$$

and Proposition 6.3 we obtain

$$D_\nu^{(3,+)} = -\Big(\langle u, \Big(\frac{\partial}{\partial u}\Big)^2\rangle + \nu \mathrm{tr}\Big(\frac{\partial}{\partial \mathrm{u}}\Big) + \langle \mathrm{u}, \Big(\frac{\partial}{\partial \mathrm{u}}\Big)\rangle + \mathrm{n}\nu\Big),$$

and also

$$\begin{aligned} D_{\nu,0}^{(3)} &= D_\nu^{(3,+)} + D_\nu^{(3,-)} \\ &= -\langle u, \Big(\frac{\partial}{\partial u}\Big)^2\rangle - \nu \mathrm{tr}\Big(\frac{\partial}{\partial \mathrm{u}}\Big) + \mathrm{tr}\,\mathrm{u} - \mathrm{n}\nu. \end{aligned}$$

This formula and the relation

$$\Psi_{\mathbf{m}}^{(\nu)}(u) = e^{-\mathrm{tr}\,\mathrm{u}} L_{\mathbf{m}}^{(\nu-1)}(2u),$$

gives Theorem 8.1. ∎

A K-invariant function f on V only depends on the eigenvalues. Define

$$F(x_1, \ldots, x_n) = f(x_1c_1 + \cdots + x_nc_n),$$

where $(c_1, \ldots, c_n)$ is a Jordan frame. Hence F is a symmetric function on $\mathbb{R}^n$.

Corollary 8.2. *The multivariate Laguerre polynomial* $L_{\mathbf{m}}^{(\nu-1)}(x) = L(x_1, \ldots, x_n)$ *is solution of the following equation*

$$\begin{aligned} &\sum_{i=1}^n x_i \frac{\partial^2 L}{\partial x_i^2} + d\sum_{i<j} \frac{1}{x_i - x_j}\Big(x_i\frac{\partial L}{\partial x_i} - x_j\frac{\partial L}{\partial x_j}\Big) \\ &+ \sum_{i=1}^n \Big(\nu - \frac{d}{2}(n-1) - x_i\Big)\frac{\partial L}{\partial x_i} + |\mathbf{m}|L = 0. \end{aligned}$$

This is essentially the differential operator (2.1b) in [Baker-Forrester,1997].

One follows the same lines as in the proof of Proposition VI.4.2 in [Faraut-Korányi,1994].

9 Pieri's formula for the Meixner-Pollaczek polynomials $Q_{\mathbf{m}}^{(\nu,\theta)}$

Theorem 9.1. *The Meixner-Pollaczek polynomials* $Q_{\mathbf{m}}^{\nu,\theta)}$ *satisfy the following Pieri's formula:*

$$\begin{aligned}
&(2|\mathbf{s}|\cos\theta - 2i|2\mathbf{m}+\nu|\sin\theta)Q_{\mathbf{m}}^{(\nu,\theta)}(\mathbf{s}) \\
= &\sum_{j=1}^{n}\Big(m_j+\nu-1-\frac{d}{4}(j-1)\Big)\alpha_j(\mathbf{m}-\varepsilon_j-\rho)d_{\mathbf{m}-\varepsilon_j}Q_{\mathbf{m}-\varepsilon_j}^{(\nu,\theta)}(\mathbf{s}) \\
&-\sum_{j=1}^{n}\Big(m_j+1+\frac{d}{4}(n-j)\Big)\alpha_j(-\mathbf{m}-\varepsilon_j-\rho)d_{\mathbf{m}+\varepsilon_j}Q_{\mathbf{m}}^{(\nu,\theta)}(\mathbf{s}).
\end{aligned}$$

Proof. The generating formula (Theorem 3.1 (ii)), with $\mathbf{s}=\mathbf{m}+\frac{\nu}{2}-\rho$ can be written:

$$\begin{aligned}
&\sum_{\mathbf{k}} d_{\mathbf{k}}Q_{\mathbf{k}}^{(\nu,\theta)}\Big(\mathbf{m}+\frac{\nu}{2}-\rho\Big)\Phi_{\mathbf{k}}(w) \\
= \;&\Delta(e+e^{-i\theta}w)^{-\nu}\Phi_{\mathbf{m}}\Big((e-e^{i\theta}w)(e+e^{-i\theta}w)^{-1}\Big).
\end{aligned}$$

Since

$$\begin{aligned}
&F_{\mathbf{m}}^{(\nu,\theta)}(e^{-i\theta}w) \\
= \;&2^{n\nu}\Delta(e+e^{-i\theta}w)^{-\nu}(-1)^{|\mathbf{m}|}e^{-i|\mathbf{m}|\theta}\Phi_{\mathbf{m}}\Big((e-e^{i\theta}w)(e+e^{-i\theta}w)^{-1}\Big),
\end{aligned}$$

we obtain

$$\sum_{\mathbf{k}} Q_{\mathbf{k}}^{(\nu,\theta)}\Big(\mathbf{m}+\frac{\nu}{2}-\rho\Big)e^{i|\mathbf{k}|\theta}\Phi_{\mathbf{k}}(w) = 2^{-n\nu}(-1)^{|\mathbf{m}|}e^{i|\mathbf{m}|\theta}F_{\mathbf{m}}^{(\nu,\theta)}(w).$$

Recall that the function $F_{\mathbf{m}}^{(\nu,\theta)}$ is an eigenfunction of the differential operator $D_{\nu,\theta}^{(2)}$:

$$D_{\nu,\theta}^{(2)}F_{\mathbf{m}}^{(\nu,\theta)}(w) = 2|\mathbf{m}|\cos\theta F_{\mathbf{m}}^{(\nu,\theta)}(w).$$

It follows that

$$\begin{aligned}
&\sum_{\mathbf{k}} d_{\mathbf{k}}Q_{\mathbf{k}}^{(\nu,\theta)}\Big(\mathbf{m}+\frac{\nu}{2}-\rho\Big)e^{i|\mathbf{k}|\theta}D_{\nu,\theta}^{(2)}\Phi_{\mathbf{k}}(w) \\
= \;&2|\mathbf{m}|\cos\theta\sum_{\mathbf{k}} d_{\mathbf{k}}Q_{\mathbf{k}}^{(\nu,\theta)}\Big(\mathbf{m}+\frac{\nu}{2}-\rho\Big)\Phi_{\mathbf{k}}(w). \qquad (9.1)
\end{aligned}$$

∎

Lemma 9.2. (i)

$$\mathrm{tr}\,\big(\nabla\varphi_{\mathbf{s}}(\mathrm{z})\big) = \sum_{\mathrm{j}=1}^{\mathrm{n}}\big(\mathrm{s_j} + \frac{\mathrm{d}}{4}(\mathrm{n}-1)\big)\alpha_{\mathrm{j}}(-\mathbf{s})\varphi_{\mathbf{s}-\varepsilon_{\mathrm{j}}}(\mathrm{z}).$$

(ii)

$$\begin{aligned} & D^{(2)}_{\nu,\theta}\varphi_{\mathbf{s}}(z) \\ = \ & e^{i\theta}\Big(\sum_{j=1}^{n}\big(s_j - \frac{d}{4}(n-1)+\nu\big)\alpha_j(\mathbf{s})\varphi_{\mathbf{s}+\varepsilon_j}(z) + \big(\sum_{j=1}^{n} s_j\big)\varphi_{\mathbf{s}}(z)\Big) \\ & -e^{-i\theta}\Big(\sum_{j=1}^{n}\big(s_j + \frac{d}{4}(n-1)\big)\alpha_j(-\mathbf{s})\varphi_{\mathbf{s}-\varepsilon_j}(z) + \big(\sum_{j=1}^{n} s_j\big)\varphi_{\mathbf{s}}(z) + n\nu\varphi_{\mathbf{s}}(z)\Big). \end{aligned}$$

((i) is in agreement with Lassalle's results [1998], p.321, first line of (14.1).)

Proof. (i) For $t > 0$ we consider the following Laplace integral:

$$\int_{\Omega} e^{-(x|y)}e^{-t\,\mathrm{tr}\,\mathrm{y}}\varphi_{\mathbf{s}}(y)\Delta(y)^{-\frac{N}{n}}m(dy) = \Gamma_{\Omega}(\mathbf{s}+\rho)\varphi_{-\mathbf{s}}(te+x).$$

Taking the derivatives with respect to t for $t = 0$, one gets:

$$-\int_{\Omega} e^{-(x|y)}\mathrm{tr}\,\mathrm{y}\,\varphi_{\mathbf{s}}(\mathrm{y})\Delta(\mathrm{y})^{-\frac{\mathrm{N}}{\mathrm{n}}}\mathrm{m}(\mathrm{dy}) = \Gamma_{\Omega}(\mathbf{s}+\rho)\,\mathrm{tr}\,\big(\nabla\varphi_{-\mathbf{s}}(\mathrm{x})\big).$$

By using Proposition 5.1:

$$\mathrm{tr}\,\mathrm{y}\,\varphi_{\mathbf{s}}(\mathrm{y}) = \sum_{\mathrm{j}=1}^{\mathrm{n}}\alpha_{\mathrm{j}}(\mathbf{s})\varphi_{\mathbf{s}+\varepsilon_{\mathrm{j}}}(\mathrm{y}),$$

and since

$$\begin{aligned} & \sum_{j=1}^{n}\alpha_j(\mathbf{s})\int_{\Omega} e^{-(x|y)}\varphi_{\mathbf{s}+\varepsilon_j}(y)\Delta(y)^{-\frac{N}{n}}m(dy) \\ = \ & \sum_{j=1}^{n}\alpha_j(\mathbf{s})\Gamma_{\Omega}(\mathbf{s}+\varepsilon_j+\rho)\varphi_{-\mathbf{s}-\varepsilon_j}(x), \end{aligned}$$

one obtains

$$\begin{aligned}\operatorname{tr}\big(\nabla\varphi_{-\mathbf{s}}(\mathrm{x})\big) &= -\sum_{j=1}^{n}\alpha_j(\mathbf{s})\frac{\Gamma_\Omega(\mathbf{s}+\varepsilon_j+\rho)}{\Gamma_\Omega(\mathbf{s}+\rho)}\varphi_{-\mathbf{s}-\varepsilon_j}(x)\\ &= -\sum_{j=1}^{n}\alpha_j(\mathbf{s})\Big(s_j-\frac{d}{4}(n-1)\Big)\varphi_{-\mathbf{s}-\varepsilon_j}(x),\end{aligned}$$

or

$$\operatorname{tr}\big(\nabla\varphi_{\mathbf{s}}(\mathrm{x})\big) = \sum_{\mathrm{j}=1}^{\mathrm{n}}\alpha_{\mathrm{j}}(-\mathbf{s})\Big(\mathrm{s}_{\mathrm{j}}+\frac{\mathrm{d}}{4}(\mathrm{n}-1)\Big)\varphi_{\mathbf{s}-\varepsilon_{\mathrm{j}}}(\mathrm{x}).$$

In fact the explicit formula for Γ_Ω,

$$\Gamma_\Omega(\mathbf{s}+\rho) = (2\pi)^{N-n}\prod_{j=1}^{n}\Gamma\big(s_j-\frac{d}{4}(n-1)\big),$$

gives

$$\frac{\Gamma_\Omega(\mathbf{s}+\varepsilon_j+\rho)}{\Gamma_\Omega(\mathbf{s}+\rho)} = \frac{\Gamma\big(s_j+1-\frac{d}{4}(n-1)\big)}{\Gamma\big(s_j-\frac{d}{4}(n-1)\big)} = s_j-\frac{d}{4}(n-1).$$

(ii) Recall that

$$D_\nu^{(2,-)}F(z) = -\langle z+e,\nabla F(z)\rangle - n\nu F(z).$$

From (i) we obtain

$$D_\nu^{(2,-)}\varphi_{\mathbf{s}}(z) = \sum_{j=1}^{n}\big(s_j+\frac{d}{4}(n-1)\big)\alpha_j(-\mathbf{s})\varphi_{\mathbf{s}-\varepsilon_j}(z) - \big(\sum_{j=1}^{n}s_j+n\nu\big)\varphi_{\mathbf{s}}(z).$$

By using $D_\nu^{(2,+)} = S_\nu^{(2)}D_\nu^{(2,-)}S_\nu^{(2)}$ and $S_\nu^{(2)}\varphi_{\mathbf{s}}(z) = \varphi_{-\mathbf{s}-\nu}(z)$, we get (ii). □

We continue the proof of Theorem 9.1. Let us write (ii) of Lemma 9.2 with $\mathbf{s}=\mathbf{k}-\rho$:

$$\begin{aligned}&D_{\nu,k}^{(2)}\Phi_{\mathbf{k}}(w)\\ = \;& e^{i\theta}\Big(\sum_{j=1}^{n}(k_j+\nu-\frac{d}{2}(j-1)\big)\alpha_j(\mathbf{k}-\rho)\Phi_{\mathbf{k}+\varepsilon_j}(w)+|\mathbf{k}|\Phi_{\mathbf{k}}(w)\Big)\\ & -e^{-i\theta}\Big(\sum_{j=1}^{n}\big(k_j+\frac{d}{2}(n-j)\big)\alpha_j(-\mathbf{k}+\rho)\Phi_{\mathbf{k}-\varepsilon_j}(w)+(|\mathbf{k}|+n\nu)\Phi_{\mathbf{k}}(w)\Big).\end{aligned}$$

(Observe that $\sum_{j=1}^{n} \rho_j = 0$.) Now, equaling the coefficient of $\Phi_{\mathbf{k}}(z)$ in both sides of (9.1), we obtain the formula of Theorem 9.1 for all $\mathbf{s} = \mathbf{m} + \frac{\nu}{2} - \rho$. Since both sides are polynomial functions in $\mathbf{s}$, the equality holds for every $\mathbf{s}$.

Andrews, G. E., R. Askey, and R. Roy (1999). Special functions. *Cambridge.*

Aristidou, M., M. Davidson, and G. Ólafsson (2006). Laguerre functions on symmetric cones and recursion relations in the real case, *J. Comput. Appl. Math.*, **199**, 95–112.

Baker, T. H., P. J. Forrester (1997). The Calogero-Sutherland model and generalized classical polynomials, *Comm. Math. Phys.*, **188**, 175–216.

Bump, D, Choi, K-K., Kurlberg, P. and Vaaler, J. (2000). A local Riemann hypothesis, I, *Math. Z.*, **233**, 1–19.

Davidson, M., G. Ólafsson, and G., Zhang (2003). Laplace and Segal-Bargmann transforms on Hermitian symmetric spaces and orthogonal polynomials, *J. Functional Analysis*, **204**, 157–195.

Davidson, M., G. Ólafsson (2003). Differential recursion relations for Laguerre functions on Hermitian matrices, *Integral transforms and special functions*, **14**, 469–484.

Dib, H. (1990). Fonctions de Bessel sur une algèbre de Jordan, *J. Math. pures et appl.*, **69**, 403–448.

Faraut, J. (2004). Analysis on the crown of a Riemannian symmetric space, *Amer. Math. Soc. Transl.*, **210**, 99–110.

Faraut, J., A. Korányi (1994). Analysis on symmetric cones. *Oxford.*

Hua, L.K. (1963). Harmonic analysis of functions of several variables in the classical domains. *Amer. Math. Soc.*.

Lassalle, M. (1998). Coefficients binomiaux généralisés et polynômes de Macdonald, *J. Funct. Analysis*, **158**, 289–324.

Ørsted, B., G. Zhang (1994). Weyl quantization and tensor products of Fock and Bergman spaces, *Indiana Univ. Math. J.*, **43**, 551–582.

Ørsted, B., G. Zhang (1995). Generalized principal series representations and tube domains, *Duke Math. J.*, **78**, 335–357.

Peetre, J., G. Zhang (1992). Aweighted Plancherel formula III. the case of a hyperbolic matrix domain, *Collect. Math.*, **43**, 273–301.

Sahi, S., G. Zhang (2007). Biorthogonal expansion of non-symmetric Jack functions, *SIGMA*, **3**, 106, 9 pages.

Schoutens, W (2000). Stochastic Processes and Orthogonal Polynomials. *Lecture Notes in Statistics 146. Springer-Verlag, New York.*.

Zhang, G. (1995). Some recurrence formulas for spherical polynomials on tube domains, *Trans. Amer. Math. Soc.*, **347**, 1725–1734.

Zhang, G. (2002). Invariant differential operators on symmetric cones and Hermitian symmetric spaces, *Acta Applicandae Mathematicae*, **73**, 79–94.

Jacques Faraut
Institut de Mathématiques de Jussieu, Université Pierre et Marie Curie
4 place Jussieu, case 247, 75 252 Paris cedex 05, France
faraut@math.jussieu.fr

Masato Wakayama
Institute of Mathematics for Industry, Kyushu University
Motooka, Nishi-ku, Fukuoka 819-0395, Japan
wakayama@imi.kyushu-u.ac.jp