Infinite Dimensional Spherical Analysis

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Introduction

In these notes we present some new developments in asymptotic harmonic analysis due to Vershik, Kerov, Okunkov, and Olshanski. The purpose of this analysis is to study the asymptotics of functions related to harmonic analysis on groups or homogeneous spaces as the dimension goes to infinity. The subject is not new. In 1938 Schoenberg considered the case of infinite dimensional Euclidean spaces. For instance Schoenberg determined the continuous functions Φ on $[0, \infty[$ such that, for every n, the function $\Phi\big(\sqrt{x_1^2 + \cdots + x_n^2}\big)$ is of positive type on $\mathbb{R}^n$. He considered also a similar problem on spheres and Krein on hyperbolic spaces. Olshanski developped a general theory of the inductive limit (G, K) of an increasing sequence $\big(\big(G(n), K(n)\big)$ of Gelfand pairs, introducing the notion of spherical function for such a pair ([Olshanski, 1990]). Several results about specific infinite dimensional pairs can be stated in this general framework. A natural queston arises: is it possible to obtain the spherical functions for the inductive limit (G, K) as limits of spherical functions for the Gelfand pair $\big(G(n), K(n)\big)$? As far I know there is presently no general answer. In these notes we will present results by Olshanski, Vershik, Kerov, and Okunkov for certain sequences of Riemannian symmetric spaces as the rank goes to infinity.

In the first chapter we present some basic results about Olshanski spherical pairs, from the paper [Olshanski, 1990]. We review in Chapter 2 basic properties of Schur functions and Schur expansions, then introduce the shifted Schur functions, with some of their properties, from the paper [Okunkov-Olshanski,1998a]. In Chapter 3 we consider the space of infinite dimensional Hermitian matrices with the action of the infinite dimensional unitary group, and describe the asymptotics of the orbital integrals. This is mainly a review of the main results in [Olshanski-Vershik,1996]. Chapter 4 deals with asymptotics of the characters of the unitary group $U(n)$. The main result is from [Vershik-Kerov,1982], but we present the method of proof from [Okunkov-Olshanski,1998b]. We end by a short review of some results about inductive limits of compact symmetric spaces from the recent paper [Okunkov-Olshanski,2006]. Further references can be found in [Faraut,2006].

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Contents

Chapter 1

Olshanski spherical pairs

An Olshanski spherical pair is the inductive limit of an increasing sequence of Gelfand pairs. In this first chapter we present the definition and the main properties of the spherical functions of an Olshanski spherical pair.

1.1 Gelfand pairs

Let us first recall the definition and the basic properties of Gelfand pairs. Let G be a locally compact group, and K a compact subgroup. The space $L^1(K\backslash G/K)$ of K-biinvariant integrable functions on G is a convolution algebra. The pair (G,K) is said to be a *Gelfand pair* if the algebra $L^1(K\backslash G/K)$ is commutative. We consider in the sequel of this section a Gelfand pair (G,K).

A continuous function $\varphi \not\equiv 0$ on G is said to be *spherical* if, for $x,y \in G$,

$$\int_K \varphi(xky)dk = \varphi(x)\varphi(y), \qquad \text{for any } x,y \in G, \tag{1.1}$$

where dk is the normalized Haar measure on K. A spherical function φ is K-biinvariant, and $\varphi(e)=1$.

Let $\mathcal{P}(K\backslash G/K)$ be the convex cone of continuous K-biinvariant functions on G of positive type. Recall that a function φ on G is said to be of *positive type* if

$$\sum_{i,j=1}^{N} \varphi(g_i^{-1}g_j)c_i\overline{c_j} \geq 0.$$

for any $g_1,\dots g_N \in G$ and $c_1,\dots,c_N \in \mathbb{C}$. We will denote by $\mathcal{P}_1(K\backslash G/K)$ the convex set of functions φ in $\mathcal{P}(K\backslash G/K)$ with $\varphi(e)=1$.

Theorem 1.1 *Let (G, K) be a Gelfand pair and $\varphi \in \mathcal{P}_1(K\backslash G/K)$. Then the following properties are equivalent:*

(i) φ is spherical.

(ii) φ is an extremal point in the convex set $\mathcal{P}_1(K\backslash G/K)$, i.e., if φ is expressed as $\varphi = \alpha\varphi_1 + (1-\alpha)\varphi_2$ with some $\varphi_1, \varphi_2 \in \mathcal{P}_1(K\backslash G/K)$ and $0 < \alpha < 1$, then $\varphi = \varphi_1 = \varphi_2$.

(iii) The representation π^φ associated to φ by the Gelfand-Naimark-Segal construction is irreducible.

We will recall in Section 1.3 the Gelfand-Naimark-Segal construction.

Let Ω be the set of spherical functions of positive type. From Theorem 1.1, Ω is the set of extremal points of $\mathcal{P}_1(K\backslash G/K)$:

$$\Omega = \mathrm{Ext}(\mathcal{P}_1(K\backslash G/K)).$$

We will see that, for an irreducible unitary representation $(\pi, \mathcal{H})$,

$$\mathrm{dim}\mathcal{H}^{\mathrm{K}} = 0 \text{ or } 1,$$

where $\mathcal{H}^K$ denotes the subspace of K-invariant vectors. If $\mathrm{dim}\mathcal{H}^{\mathrm{K}} = 1$, we will say that the representation $(\pi, \mathcal{H})$ is *spherical.* Hence the set Ω can also be seen as the set of equivalence classes of spherical representations $(\pi, \mathcal{H})$ of G . For that reason we will call Ω the *spherical dual* of the pair (G, K).

One considers on the set Ω of spherical functions of positive type the topology of uniform convergence on compact subsets of G. Then it can be shown that Ω is locally compact.

Theorem 1.2 (Bochner-Godement) *For each $\varphi \in \mathcal{P}(K\backslash G/K)$, there exists a unique positive bounded measure μ on Ω such that*

$$\varphi(g) = \int_\Omega \omega(g)\mu(d\omega).$$

(See, for instance, [Faraut,1982].)

1.2 Olshanski spherical pairs

Let $(G(n), K(n))_{n\geq 1}$ be a increasing sequence of Gelfand pairs: $G(n)$ is a closed subgroup of $G(n+1)$, $K(n)$ of $K(n+1)$, and $K(n) = G(n) \cap K(n+1)$. Define

$$G = \bigcup_{n=1}^{\infty} G(n), \qquad K = \bigcup_{n=1}^{\infty} K(n).$$

We consider on G the inductive limit topology. Then K is a closed subgroup of G. But in general G is not locally compact, and K is not compact. We say that the pair (G,K) is an *Olshanski spherical pair.*

Let us give a simple example of such a sequence $(G(n),K(n))$.

Example

Let $K(n)=O(n)$ be the orthogonal group and let $G(n)=O(n)\ltimes\mathbb{R}^n$ the motion group. The product in $G(n)$ is given by:

$$(g_1,\xi_1)\cdot(g_2,\xi_2)=(g_1g_2,\xi_1+g_1\xi_2),$$

$(g_1,g_2\in O(n),\ \xi_1,\xi_2\in\mathbb{R}^n)$. Then

$$K=O(\infty)=\bigcup_{n=1}^{\infty}O(n),$$

the *infinite dimensional orthogonal group.* An element $k=(k_{ij})_{i,j\geq 1}$ in $O(\infty)$ satisfies $k_{ij}=\delta_{ij}$ for i and j large enough. Define $\mathbb{R}^{(\infty)}$ by

$$\mathbb{R}^{(\infty)}=\bigcup_{n=1}^{\infty}\mathbb{R}^n.$$

A vector $\xi\in\mathbb{R}^{(\infty)}$ is a sequence $\xi=(\xi_1,\xi_2,\dots)$ of real numbers with $\xi_i=0$ for i large enough.

The group $O(\infty)$ naturally acts on $\mathbb{R}^{(\infty)}$, and $G=O(\infty)\ltimes\mathbb{R}^{(\infty)}$.

Let (G,K) be an Olshanski spherical pair, inductive limit of an increasing sequence of Gelfand pairs $\big(G(n),K(n)\big)$. A K-biinvariant continuous function φ on G is said to be *spherical* if, for $x,y\in G$,

$$\lim_{n\to\infty}\int_{K(n)}\varphi(xky)\alpha_n(dk)=\varphi(x)\varphi(y), \tag{1.2}$$

where α_n is the normalized Haar measure on the compact group $K(n)$.

We will see that Theorem 1.1 holds for an Olshanski spherical pair.

1.3 Gelfand-Naimark-Segal construction

Let G be a topological group, and K a closed subgroup. Consider a unitary representation $(\pi,\mathcal{H})$ of G. A vector $u\in\mathcal{H}$ is said to be *cyclic* if the subspace of $\mathcal{H}$ generated by the vectors $\pi(g)u$ for $g\in G$ is dense in $\mathcal{H}$.

Proposition 1.3 (Gelfand-Naimark-Segal construction) *For $\varphi \in \mathcal{P}_1(K\backslash G/K)$, there exists a unitary representation $(\pi^\varphi, \mathcal{H}^\varphi)$ of G, and a K-invariant cyclic unit vector u such that*

$$\varphi(g) = (u|\pi(g)u).$$

Furthermore, the triple $(\pi^\varphi, \mathcal{H}^\varphi, u)$ is unique, up to isomorphism.

Two triples $(\pi_1, \mathcal{H}_1, u_1)$ and $(\pi_2, \mathcal{H}_2, u_2)$ are said to be isomorphic if there is a isometric isomorphism $A : \mathcal{H}_1 \to \mathcal{H}_2$ such that, for $g \in G$,

$$A\pi_1(g) = \pi_2(g)A,$$

and $Au_1 = u_2$.
Proof
Let $\mathcal{H}_0^\varphi$ be the space of functions on G of the form

$$f(x) = \sum_{i=1}^{N} c_i \varphi(g_i^{-1}x),$$

with $g_1, \ldots, g_N \in G$, $c_1, \ldots, c_N \in \mathbb{C}$. Clearly, $\mathcal{H}_0^\varphi$ is the subspace of $\mathcal{C}(G/K)$, the space of right K-invariant continuous functions on G.

The norm of such a function f is defined by

$$\|f\|^2 = \sum_{i,j=1}^{N} \varphi(g_i^{-1}g_j)c_i\overline{c_j}.$$

Since φ is of positive type, this number is ≥ 0. Writing

$$f = \mu * \varphi, \text{ with } \mu = \sum_{i=1}^{N} c_i \delta_{g_i},$$

we obtain

$$\|f\|^2 = \int_G (\mu * \varphi)(x)\overline{\mu(dx)}.$$

By the Schwarz inequality, with $\nu = \sum_{i=1}^N d_i \delta_{g_i}$,

$$\left|\int_G f(x)\nu(dx)\right|^2 = \left|\sum_{i,j=1}^{N} \varphi(g_i^{-1}g_j)c_i\overline{d_j}\right|^2 \tag{1.3}$$

$$\leq \sum_{i,j=1}^{N} \varphi(g_i^{-1}g_j)c_i\overline{c_j} \cdot \sum_{i,j=1}^{N} \varphi(g_i^{-1}g_j)d_i\overline{d_j},$$

Therefore, if $\|f\|^2 = 0$, then $f \equiv 0$. Observe that in general the representation

$$f(x) = \sum_{i=1}^{N} c_i \varphi(g_i^{-1} x)$$

of a function f is not unique. The above inequality shows that $\|f\|^2$ only depends on f, and not on the chosen representation. Hence $\|f\|$ is indeed a norm on $\mathcal{H}_0^\varphi$, and $\mathcal{H}_0^\varphi$ is a preHilbert space, with the inner product, for

$$f(x) = \sum_{i=1}^{N} c_i \varphi(g_i^{-1} x), \; f'(x) = \sum_{j=1}^{N'} c'_j \varphi\big((g'_j)^{-1} x\big),$$

defined by

$$(f|f') = \sum_{i=1}^{N} \sum_{j=1}^{N'} \varphi(g_i^{-1} g'_j) c_i \overline{c'_j}.$$

Define the representation π^φ of G on $\mathcal{H}_0^\varphi$ by the left action

$$(\pi^\varphi(g)f)(x) = f(g^{-1}x), \qquad f \in \mathcal{H}_0^\varphi, \; g, x \in G.$$

Then π^φ is unitary. In fact, if

$$f(x) = \sum_{i=1}^{N} c_i \varphi(g_i^{-1} x),$$

then

$$(\pi^\varphi(g)f)(x) = \sum_{i=1}^{N} c_i \varphi((g g_i)^{-1} x),$$

and hence

$$\begin{aligned} \|\pi^\varphi(g)f\|^2 &= \sum_{i,j=1}^{N} \varphi((g g_i)^{-1}(g g_j)) c_i \overline{c_j} \\ &= \sum_{i,j=1}^{N} \varphi(g_i^{-1} g_j) c_i \overline{c_j} = \|f\|^2. \end{aligned}$$

Let $\mathcal{H}^\varphi$ be the Hilbert completion of $\mathcal{H}_0^\varphi$. Since, for $f \in \mathcal{H}^\varphi$, $|f(x)| \leq \|f\|$ (by letting $\nu = \delta_x$ in (1.3)), the Hilbert space $\mathcal{H}^\varphi$ can be realized as a subspace of $\mathcal{C}_{\mathrm{b}}(G/K)$, the space of bounded functions in $\mathcal{C}(G/K)$. By definition of $\mathcal{H}_0^\varphi$, the vector $\varphi \in \mathcal{H}^\varphi$ is cyclic, K-invariant, and satisfies

$$\varphi(g) = (\varphi|\pi^\varphi(g)\varphi).$$

Let $(\pi^\varphi, \mathcal{H}^\varphi, u^\varphi)$ be the triple obtained via the Gelfand-Naimark-Segal construction, and $(\pi, \mathcal{H}, u)$ be a triple with a K-invariant and cyclic unit vector u in $\mathcal{H}$ such that

$$\varphi(g) = (u|\pi(g)u).$$

Let us define the map $A : \mathcal{H}^\varphi \to \mathcal{H}$, by

$$Af = \sum_{i=1}^{N} c_i \pi(g_i)u,$$

if

$$f(x) = \sum_{i=1}^{N} c_i \varphi(g_i^{-1}x).$$

Then

$$\begin{aligned}\|Af\|_{\mathcal{H}}^2 &= \sum_{i,j=1}^{N} c_i \overline{c_j} \big(\pi(g_i)u|\pi(g_j)u\big) \\ &= \sum_{i,j=1}^{N} c_i \overline{c_j} \varphi(g_i^{-1}g_j) = \|f\|^2,\end{aligned}$$

Since u is cyclic, the range of A: $A(\mathcal{H}^\varphi)$ is dense in $\mathcal{H}$. It follows that A extends as an isometric isomorphism from $\mathcal{H}^\varphi$ onto $\mathcal{H}$. Furthermore

$$A\pi^\varphi(g) = \pi(g)A, \;\; Au^\varphi = u.$$

$\square$

1.4 Extremal functions in $\mathcal{P}_1(K\backslash G/K)$ and irreducibility

As in the previous section, G is a topological group and K a closed subgroup.

Proposition 1.4 *For $\varphi \in \mathcal{P}_1(K\backslash G/K)$, let $(\pi, \mathcal{H})$ be the unitary representation obtained by the Gelfand-Naimark-Segal construction (Proposition 1.3). The following properties are equivalent.*

(i) φ is extremal in the convex set $\mathcal{P}_1(K\backslash G/K)$.

(ii) The unitary representation $(\pi^\varphi, \mathcal{H}^\varphi)$ is irreducible.

Proof

(i) $\Rightarrow$ (ii). Assume φ extremal and let $u \in \mathcal{H}$ be a K-invariant cyclic unit vector. Suppose that $\mathcal{H}$ decomposes as the sum $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ of two orthogonal closed invariant subspaces. The vector u decomposes as $u = u_1' + u_2'$, $(u_i' \in \mathcal{H}_i)$. Put $\alpha = \|u_1'\|^2$. Then $0 \leq \alpha \leq 1$ since

$$1 = \|u\|^2 = \|u_1'\|^2 + \|u_2'\|^2.$$

If either $\alpha = 0$ or $\alpha = 1$, then we have either $u_1' = u$ or $u_2' = u$. Since u is cyclic, either $\mathcal{H} = \mathcal{H}_1$ or $\mathcal{H} = \mathcal{H}_2$, which means that $\mathcal{H}$ is irreducible.

Assume now that $0 < \alpha < 1$, and put

$$u_1 = \frac{u_1'}{\sqrt{\alpha}}, \qquad u_2 = \frac{u_2'}{\sqrt{1-\alpha}},$$
$$\varphi_1(g) = (u_1|\pi(g)u_1), \qquad \varphi_2(g) = (u_2|\pi^{\varphi}(g)u_2).$$

Then $\varphi = \alpha\varphi_1 + (1-\alpha)\varphi_2$. Since φ is extremal, $\varphi = \varphi_1 = \varphi_2$. Observing that

$$\big(u_i|\pi(g)u_i\big) = \big(u_i|\pi(g)u\big) \quad (i = 1, 2),$$

we get

$$\big(u_1|\pi(g)u\big) = \big(u_2|\pi(g)u\big),$$

and, since u is cyclic, $u_1 = u_2$: a contradiction. We have proven that π is irreducible.

(ii) $\Rightarrow$ (i). Assume π irreducible and that φ is expressed as $\varphi = \alpha\varphi_1 + (1-\alpha)\varphi_2$ for some $\varphi_1, \varphi_2 \in \mathcal{P}_1(K\backslash G/K)$, and $0 < \alpha < 1$. For $f \in \mathcal{H}_0$, expressed as

$$f(x) = \sum_{i=1}^{N} c_i \varphi(g_i^{-1}x),$$

put

$$H(f) = \alpha \sum_{i,j=1}^{N} \varphi_1(g_j^{-1}g_i)c_i\overline{c_j}.$$

This defines an invariant Hermitian form on $\mathcal{H}_0$. Furthermore, since

$$\alpha \sum_{i,j=1}^{N} \varphi_1(g_i^{-1}g_j)c_i\overline{c_j} \leq \sum_{i,j=1}^{N} \varphi(g_i^{-1}g_j)c_i\overline{c_j},$$

we get

$$0 \leq H(f) \leq \|f\|^2,$$

hence H extends as a continuous invariant Hermitian form on $\mathcal{H}$. This form can be written $H(f) = (Af|f)$, where A is a selfadjoint operator on $\mathcal{H}$, $0 \leq A \leq I$, which commutes with the representation π: $A\pi(g) = \pi(g)A$. By Schur's Lemma, $A = \lambda I$, with $0 \leq \lambda \leq 1$. It follows that $\alpha\varphi_1 = \lambda\varphi$. Since $\varphi(e) = \varphi_1(e) = 1$, we get $\lambda = \alpha$, and $\varphi_1 = \varphi$. This means that φ is extremal.

$\square$

1.5 Spherical functions and irreducibility

Let G be a topological group, and $(K(n))_{n\geq 1}$ an increasing sequence of compact subgroups of G. Put $K = \bigcup_{n=1}^{\infty} K(n)$. For a unitary representation $(\pi, \mathcal{H})$ of G, the orthogonal projection P_n onto the space $\mathcal{H}^{K(n)}$ of $K(n)$-invariant vectors is given by

$$P_n v = \int_{K(n)} \pi(k) v \alpha_n(dk) \quad (v \in \mathcal{H}),$$

where α_n is the normalized Haar measure of $K(n)$. The sequence of the subspaces $\mathcal{H}^{K(n)}$ is decreasing, and the projections P_n strongly converge to the projection P onto

$$\mathcal{H}^K = \bigcap_{n=1}^{\infty} \mathcal{H}^{K(n)}.$$

It follows that, if $\mathcal{Y} \subset \mathcal{H}$ is an invariant closed subspace, then $P(\mathcal{Y}) \subset \mathcal{Y}$.

Proposition 1.5 *Let $(\pi, \mathcal{H})$ be a unitary representation of G with a K-invariant cyclic vector $u \in \mathcal{H}$. If $\dim \mathcal{H}^K = 1$, then π is irreducible.*

Proof

Let $\mathcal{Y}$ be a closed G-invariant subspace of $\mathcal{H}$. We will show that either $\mathcal{Y} = \{0\}$ or $\mathcal{Y} = \mathcal{H}$. If $P(\mathcal{Y}) = \{0\}$, then $\mathcal{Y}$ is orthogonal to $u \in \mathcal{H}^K$. Since u is cyclic, it follows that $\mathcal{Y} = \{0\}$. If $P(\mathcal{Y}) \neq \{0\}$, then $\mathcal{H}^K \subset \mathcal{Y}$, and $\mathcal{Y} = \mathcal{H}$ since $u \in \mathcal{H}^K$ is cyclic. Thus, we have proven that the representation π is irreducible. $\square$

We assume now that (G, K) is an Olshanski spherical pair, inductive limit of an increasing sequence $\big(G(n), K(n)\big)$ of Gelfand pairs.

Proposition 1.6 *Let (G, K) be an Olshanski spherical pair. For any irreducible unitary representation $(\pi, \mathcal{H})$ of G,*

$$\dim \mathcal{H}^K \leq 1.$$

Proof

Assume $\mathcal{H}^K \neq \{0\}$. Since $\big(G(n), K(n)\big)$ is a Gelfand pair, the convolution algebra $L^1\big(K(n)\backslash G(n)/K(n)\big)$ is commutative, and the algebra $M^b\big(K(n)\backslash G(n)/K(n)\big)$ of K-biinvariant bounded measures is commutative as well. In particular, for $x, y \in G(n)$,

$$\alpha_n * \delta_x * \alpha_n * \delta_y * \alpha_n = \alpha_n * \delta_y * \alpha_n * \delta_x * \alpha_n,$$

and, since $P_n = \pi(\alpha_n)$,

$$P_n\pi(x)P_n\pi(y)P_n = P_n\pi(y)P_n\pi(x)P_n.$$

Observing that $P_{n+1} = P_nP_{n+1} = P_{n+1}P_n$, we obtain, for $m, m' \geq 0$,

$$P_{n+m}\pi(x)P\pi(y)P_{n+m'} = P_{n+m}\pi(y)P\pi(x)P_{n+m'}.$$

As $m, m' \to \infty$, and then $n \to \infty$, we obtain

$$P\pi(x)P\pi(y)P = P\pi(y)P\pi(x)P,$$

since P_n strongly converges to P.

Let $\mathcal{A}$ be the closed algebra (for the operator norm) generated by the operators $P\pi(x)P$, for $x \in G$. As proven above, the algebra $\mathcal{A}$ is commutative. The space $\mathcal{H}^K$ is invariant under $\mathcal{A}$. Since an irreducible representation of a commutative Banach algebra is one dimensional, it is sufficient to prove that $\mathcal{H}^K$ is irreducible under $\mathcal{A}$.

Assume that $\mathcal{H}^K = \mathcal{H}_1 \oplus \mathcal{H}_2$, where $\mathcal{H}_1$ and $\mathcal{H}_2$ are two $\mathcal{A}$-invariant orthogonal subspaces of $\mathcal{H}^K$. Let $u_1 \in \mathcal{H}_1$ ($u_1 \neq 0$). For any $u_2 \in \mathcal{H}_2$ and $x \in G$, $\big(P\pi(x)Pu_1|u_2\big) = 0$. Since $Pu_1 = u_1$, $Pu_2 = u_2$, this means that $(\pi(x)u_1|u_2) = 0$. We use now the fact that the representation π is irreducible, and hence that any non zero vector is cyclic, in particular u_1 is cyclic. This implies $u_2 = 0$, and $\mathcal{H}_2 = \{0\}$. ∎

Proposition 1.7 *Let (G, K) be an Olshanski spherical pair. For $\varphi \in \mathcal{P}_1(K\backslash G/K)$, the following properties are equivalent:*

(i) φ is spherical.

(ii) The representation $(\pi, \mathcal{H})$ associated to φ by the Gelfand-Naimark-Segal construction is irreducible.

Proof

Recall that $\varphi(g) = \big(u|\pi(g)u\big)$, where u is a cyclic unit vector in $\mathcal{H}^K$.

(ii) $\Rightarrow$ (i). Assume the representation $(\pi, \mathcal{H})$ irreducible. By Proposition 1.6 we know that $\dim \mathcal{H}^K = 1$. Therefore the orthogonal projection P onto $\mathcal{H}^K$ can be written

$$Pv = (v|u)u.$$

For $y \in G$, and any $v \in \mathcal{H}$,

$$\big(v|P\pi(y)Pu\big) = \big(Pv|\pi(y)u\big) = (v|u)\big(u|\pi(y)u\big) = \varphi(y)(v|u).$$

Therefore $P\pi(y)Pu = \varphi(y)u$. Hence, for $x \in G$,

$$P\pi(x)P\pi(y)Pu = \varphi(y)P\pi(x)Pu = \varphi(x)\varphi(y)u,$$

and

$$\big(u|\pi(x)P\pi(y)u\big) = \varphi(x)\varphi(y).$$

Since the projections P_n strongly converge to P, we get

$$\begin{aligned}\varphi(x)\varphi(y) &= \lim_{n\to\infty} \big(u|\pi(x)P_n\pi(y)u\big)\\ &= \lim_{n\to\infty} \int_{K(n)} (u|\pi(xky)u)\alpha_n(dk)\\ &= \lim_{n\to\infty} \int_{K(n)} \varphi(xky)\alpha_n(dk),\end{aligned}$$

which means that φ is spherical.

(i) $\Rightarrow$ (ii). Assume φ spherical. We will show that, for $g \in G$, $P\pi(g)u = \varphi(g)u$. If this holds, then the subspace $\mathcal{H}^K$ is one dimensional: $\mathcal{H}^K = \mathbb{C}u$. Therefore, by Proposition 1.5, the representation π is irreducible. By assumption, for $x, y \in G$,

$$\begin{aligned}\varphi(x)\varphi(y) &= \lim_{n\to\infty} \int_{K(n)} \varphi(xky)\alpha_n(dk)\\ &= \lim_{n\to\infty} \big(u|\pi(x)P_n\pi(y)u\big) = \big(u|\pi(x)P\pi(y)u\big).\end{aligned}$$

This can be written as

$$\big(\pi(x^{-1})u|P\pi(y)u\big) = \varphi(y)\big(\pi(x^{-1})u|u\big).$$

Since u is cyclic, we obtain $P\pi(y)u = \varphi(y)u$. $\square$

Let us mention that the Bochner-Godement theorem (Theorem 1.2) has been recently extended to Olshanski spherical pairs ([Rabaoui,2008]).

1.6 Examples

We end this chapter by giving two simple examples of Olshanski spherical pairs.

Example 1

We come back to the example of Section 1.2. Let $G(n) = O(n) \ltimes \mathbb{R}^n$ and $K(n) = O(n)$. Then $G = O(\infty) \ltimes \mathbb{R}^{(\infty)}$ and $K = O(\infty)$. For an element $x = (g, \xi) \in G$, we denote by $\|x\|$ the radius of ξ: $\|x\| = \|\xi\| = \sqrt{\xi_1^2 + \xi_2^2 + \cdots}$ for $\xi = (\xi_1, \xi_2, \ldots, 0, 0, \ldots) \in \mathbb{R}^{(\infty)}$. Let φ be a K-biinvariant function of G. Then, for any $g_1, g_2, g \in O(\infty)$ and $\xi \in \mathbb{R}^{(\infty)}$,

$$\varphi(g, \xi) = \varphi((g_1, 0) \cdot (g, \xi) \cdot (g_2, 0)) = \varphi(g_1 g g_2, g_1 \xi).$$

Therefore $\varphi(g, \xi)$ only depends on the radius $\|\xi\|$, i.e., there exists a function Φ on $\mathbb{R}_{\geq 0}$ such that

$$\varphi(x) = \Phi(\|x\|^2).$$

Assume φ spherical:

$$\lim_{n\to\infty} \int_{K(n)} \varphi(xky)\alpha_n(dk) = \varphi(x)\varphi(y), \qquad x, y \in G. \tag{1.4}$$

By classical harmonic analysis,

$$\int_{K(n)} \varphi(xky)\alpha_n(dk) = c_n \int_0^\pi \Phi(a^2 + b^2 + 2abcos\theta) \sin^{n-1}\theta d\theta.$$

where $a = \|x\|$ and $b = \|y\|$. Note that the constant c_n is given by

$$c_n = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{n}{2}\right)}.$$

One shows easily that, if f is a continuous function on $[0, \pi]$,

$$\lim_{n\to\infty} c_n \int_0^\pi f(\theta) \sin^{n-1}\theta d\theta = f\left(\frac{\pi}{2}\right).$$

Therefore we obtain the following functional equation

$$\Phi(a^2 + b^2) = \Phi(a^2)\Phi(b^2).$$

This equation implies that $\Phi(a) = e^{-\lambda a}$ for some $\lambda \in \mathbb{C}$. Furthermore φ is of positive type if and only if $\lambda \geq 0$. Hence the spherical functions φ for the Olshanski spherical pair (G, K) are given by

$$\varphi(x) = e^{-\lambda\|x\|^2}, \qquad \lambda \geq 0.$$

Thus, the spherical dual Ω can be identifed with $[0,\infty[$. Observe that a spherical function, which is essentially a function on $\mathbb{R}^{(\infty)}$, extends as a continuous function on

$$\ell^2(\mathbb{N}) = \left\{ (\xi_1, \xi_2, \dots) \;\middle|\; \xi_k \in \mathbb{R},\ \sum_{k\geq 1} \xi_k^2 < \infty \right\}.$$

Example 2

Let $G(n) = O(n+1)$ and $K(n) = O(n)$. Here $K(n)$ is seen as a subgroup of $G(n)$ as follows:

$$O(n) \ni u \mapsto \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} \in O(n+1).$$

Let $\{e_0, e_1, \dots, e_{n+1}\}$ be the canonical basis of $\mathbb{R}^{n+1}$. A K-biinvariant continuous function φ on G can be written as

$$\varphi(g) = \Phi((ge_0|e_0)),$$

where Φ is a continuous function on $[-1,1]$. We get

$$\int_{K(n)} \varphi(xky)\alpha_n(dk) = c_n \int_0^{\pi} \Phi(\cos a \cos b + \sin a \sin b \cos\theta) \sin^{n-1}\theta d\theta$$

where $\cos a = (xe_0|e_0)$ and $\cos b = (ye_0|e_0)$ (c_n is the same constant as in Example 1). If φ is spherical, then

$$\Phi(\cos a \cos b) = \Phi(\cos a)\Phi(\cos b).$$

Finally, the spherical functions φ in $\mathcal{P}_1(K\backslash G/K)$ are the following:

$$\varphi(g) = (ge_0|e_0)^m, \qquad (m \in \mathbb{N}).$$

Thus the spherical dual Ω can be identified with $\mathbb{N}$.

Chapter 2

Schur functions

In order to study the spherical functions and their asymptotics we will need expansions involving Schur functions and shifted Schur functions.

2.1 Schur functions and Schur expansions

For a *signature* $\lambda = (\lambda_1, \dots, \lambda_n)$, i.e.$\lambda_i \in \mathbb{Z}^n$ and $\lambda_1 \geq \cdots \geq \lambda_n$, we define the rational function $A_\lambda(z) = A_\lambda(z_1, \dots, z_n)$ on $(\mathbb{C}^*)^n$ by

$$A_\lambda(z) = \begin{vmatrix} z_1^{\lambda_1} & z_1^{\lambda_2} & \dots & z_1^{\lambda_n} \\ z_2^{\lambda_1} & z_2^{\lambda_2} & \dots & z_2^{\lambda_n} \\ \vdots & \vdots & \ddots & \vdots \\ z_n^{\lambda_1} & z_n^{\lambda_2} & \dots & z_n^{\lambda_n} \end{vmatrix}.$$

In particular, for $\lambda = \delta := (n-1, \dots, 1, 0)$, $A_\delta(z)$ is the Vandermonde polynomial

$$A_\delta(z) = V(z) := \prod_{1 \leq j < k \leq n} (z_j - z_k).$$

The *Schur function* s_λ is defined by

$$s_\lambda(z) = \frac{A_{\lambda+\delta}(z)}{V(z)}.$$

This is a symmetric rational function defined on $(\mathbb{C}^*)^n$, and the Schur functions s_λ where $\lambda = (\lambda_1, \dots, \lambda_n)$ run over all signatures of length $\leq n$, constitute a basis of the space of symmetric Laurent polynomials in n variables. For a *partition* $\mathbf{m} = (m_1, \dots, m_n)$: $m_1 \geq \ldots \geq m_n \geq 0$, $s_{\mathbf{m}}$ is a symmetric polynomial.

Let us mention two special cases:

a) Let $\mathbf{m} = (1^k) := (\underbrace{1, \dots, 1}_{k}, 0, \dots, 0)$, where $0 \leq k \leq n$. Then

$$s_{(1^k)}(z) = e_k(z),$$

the k-th elementary symmetric function

$$e_k(z) := \sum_{1 \leq j_1 < \cdots < j_k \leq n} z_{j_1} \dots z_{j_k}. \tag{2.1}$$

To see it observe that the generating function of the elementary symmetric functions e_k is given by

$$E(z; t) := \sum_{k=0}^{n} e_k(z) t^k = \prod_{j=1}^{n} (1 + t z_j).$$

Let us consider the Vandermonde polynomial in $n+1$ variables $V(t, z_1, \dots, z_n)$. It can be written as

$$\begin{aligned} V(t, z_1, \dots, z_n) &= \prod_{i=1}^{n} (t - z_i) \prod_{i<j} (z_i - z_j) \\ &= V(z_1, \dots, z_n) \sum_{k=0}^{n} (-1)^k e_k(z) t^{n-k}, \end{aligned}$$

and also as a determinant:

$$V(t, z_1, \dots, z_n) = \begin{vmatrix} t^n & t^{n-1} & \dots & t & 1 \\ z_1^n & z_1^{n-1} & \dots & z_1 & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ z_n^n & z_n^{n-1} & \dots & z_n & 1 \end{vmatrix}$$

Let us expand this determinant with respect to the first row:

$$V(t, z_1, \dots, z_n) = \sum_{k=0}^{n} (-1)^k t^{n-k} A_{1^k + \delta}(z).$$

Therefore

$$A_{1^k + \delta}(z) = V(z_1, \dots, z_n) e_k(z),$$

or

$$s_{(1^k)}(z) = e_k(z).$$

b) If $\mathbf{m} = (m) := (m, 0, \ldots, 0)$, with $m \geq 0$, then

$$s_{(m)}(z) = h_m(z),$$

the m-th complete symmetric function,

$$h_m(z) := \sum_{|\alpha|=m} z^\alpha = \sum_{\substack{\alpha_1,\ldots,\alpha_n \geq 0 \\ \alpha_1+\cdots+\alpha_n=m}} z_1^{\alpha_1} \cdots z_n^{\alpha_n}. \tag{2.2}$$

To see it, let us show that the generating function of the complete symmetric functions is given, for $|t|$ small enough, by

$$H(z,t) := \sum_{m=0}^{\infty} h_m(z) t^m = \prod_{j=1}^{n} \frac{1}{1 - tz_j}.$$

In fact

$$\begin{aligned}\sum_{\alpha \in \mathbb{N}^n} (tz)^\alpha &= \sum_{m=0}^{\infty} \Big(\sum_{|\alpha|=m} z^\alpha \Big) t^m = \sum_{m=0}^{\infty} h_m(z) t^m \\ &= \prod_{j=1}^{n} \sum_{\alpha_j=0}^{\infty} (tz_j)^{\alpha_j} = \prod_{j=1}^{n} \frac{1}{1 - tz_j}.\end{aligned}$$

Let us now compute the sum of the following power series, for $|t|$ small enough,

$$\sum_{m=0}^{\infty} t^m A_{(m)+\delta}(z) = \begin{vmatrix} \sum_{m=0}^{\infty} t^m z_1^{m+n-1} & z_1^{n-2} & \ldots & z_1 & 1 \\ \sum_{m=0}^{\infty} t^m z_2^{m+n-1} & z_2^{n-2} & \ldots & z_2 & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ \sum_{m=0}^{\infty} t^m z_n^{m+n-1} & z_n^{n-2} & \ldots & z_n & 1 \end{vmatrix} = \begin{vmatrix} \frac{z_1^{n-1}}{1-tz_1} & z_1^{n-2} & \ldots & z_1 & 1 \\ \frac{z_2^{n-1}}{1-tz_2} & z_2^{n-2} & \ldots & z_2 & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ \frac{z_n^{n-1}}{1-tz_n} & z_n^{n-2} & \ldots & z_n & 1 \end{vmatrix}.$$

Since

$$z_i^k + t\frac{z_i^{k+1}}{1 - tz_i} = \frac{t_i^k}{1 - zt_i},$$

this determinant is equal to

$$\begin{vmatrix} \frac{z_1^{n-1}}{1-tz_1} & \frac{z_1^{n-2}}{1-tz_1} & \ldots & \frac{z_1}{1-tz_1} & \frac{1}{1-tz_1} \\ \frac{z_2^{n-1}}{1-tz_2} & \frac{z_2^{n-2}}{1-tz_2} & \ldots & \frac{z_2}{1-tz_2} & \frac{1}{1-tz_2} \\ \vdots & \vdots & & \vdots & \vdots \\ \frac{z_z^{n-1}}{1-tz_n} & \frac{z_n^{n-2}}{1-tz_n} & \ldots & \frac{z_n}{1-tz_n} & \frac{1}{1-tz_n} \end{vmatrix} = V(z) \prod_{j=1}^{n} \frac{1}{1 - tz_j} = V(z) \sum_{m=0}^{\infty} t^m h_m(z).$$

Therefore

$$A_{(m)+\delta}(z) = V(z)h_m(z),$$

or

$$s_{(m)}(z) = h_m(z).$$

Proposition 2.1 (Hua's formula) *Consider n power series*

$$f_i(w) = \sum_{m=0}^{\infty} c_m^{(i)} w^m, \qquad w \in \mathbb{C},\ i = 1, \ldots, n$$

which are convergent for $|w| < r$ for some $r > 0$. Define the function F on $\mathbb{C}^n$ by

$$F(z) = F(z_1, \ldots, z_n) = \frac{\det(f_i(z_j))_{1\le i,j\le n}}{V(z)}, \qquad |z_j| < r.$$

Then F admits the following Schur expansion

$$F(z) = \sum_{\substack{\mathbf{m}=(m_1,\ldots,m_n)\\ m_1\ge\cdots\ge m_n\ge 0}} a_{\mathbf{m}} s_{\mathbf{m}}(z)$$

with

$$a_{\mathbf{m}} = \det(c^{(i)}_{m_j+n-j})_{1\le i,j\le n}.$$

([Hua,1963], Chapter II.)

Proof

In fact,

$$\det(f_i(z_j))_{1\le i,j\le n} = \sum_{\sigma\in\mathfrak{S}_n} \epsilon(\sigma) \prod_{i=1}^{n} \Big(\sum_{m=0}^{\infty} c_m^{(i)} z_{\sigma(i)}^m \Big).$$

If we permute the product $\prod_i$ and the sum $\sum_m$, we obtain

$$= \sum_{m_1,\ldots,m_n=0}^{\infty} c_{m_1}^{(1)} \cdots c_{m_n}^{(n)} \sum_{\sigma\in\mathfrak{S}_n} \epsilon(\sigma) \prod_{i=1}^{n} z_{\sigma(i)}^{m_i} = \sum_{m_1,\ldots,m_n=0}^{\infty} c_{m_1}^{(1)} \cdots c_{m_n}^{(n)} \det(z_j^{m_i})_{1\le i,j\le n}.$$

Since $\det(z_j^{m_i}) = 0$ unless the m_i are all distinct, this sum is equal to

$$\begin{aligned}
&= \sum_{m_1>\cdots>m_n\ge 0} \sum_{\tau\in\mathfrak{S}_n} c^{(1)}_{m_{\tau(1)}} \cdots c^{(n)}_{m_{\tau(n)}} \det(z_j^{m_{\tau(i)}})_{1\le i,j\le n} \\
&= \sum_{m_1>\cdots>m_n\ge 0} \sum_{\tau\in\mathfrak{S}_n} \epsilon(\tau) c^{(1)}_{m_{\tau(1)}} \cdots c^{(n)}_{m_{\tau(n)}} \det(z_j^{m_i})_{1\le i,j\le n} \\
&= \sum_{m_1>\cdots>m_n\ge 0} \det(c^{(i)}_{m_j})_{1\le i,j\le n} \det(z_j^{m_i})_{1\le i,j\le n}.
\end{aligned}$$

Finally, with $m_j = k_j + n - j$, we obtain

$$\det(f_i(z_j))_{1\le i,j\le n} = \sum_{k_1\ge\cdots\ge k_n\ge 0} \det(c^{(i)}_{k_j+n-j})_{1\le i,j\le n} A_{(k_1,k_2,\ldots,k_n)+\delta}(z),$$

which is the claim of the proposition. □

Looking at the value at $z = 0$, since

$$c^{(i)}_m = \frac{1}{m!} f_i^{(m)}(0) = \frac{1}{m!}\frac{d^m}{dw^m} f_i(w)|_{w=0},$$

we obtain

$$\begin{aligned}\lim_{z_1,\ldots,z_n\to 0} \frac{\det(f_i(z_j))_{1\le i,j\le n}}{V(z)} =& F(\mathbf{0}) = a_{\mathbf{0}} = \det(c^{(i)}_{n-j})_{1\le i,j\le n} \\ =& \frac{1}{\delta!}\det(f_i^{(n-j)}(0))_{1\le i,j\le n}.\end{aligned} \tag{2.3}$$

For a partition $\mathbf{m} = (m_1, \ldots, m_n)$ we have used the notation

$$\mathbf{m}! = m_1! \cdots m_n!$$

We present some applications of Hua's formula.

Proposition 2.2 *For a signature* $\lambda = (\lambda_1, \ldots, \lambda_n)$,

$$s_\lambda(1, \ldots, 1) = \frac{V(\lambda+\delta)}{V(\delta)}.$$

Note that $V(\delta) = \delta!$. Observe that this formula is nothing but the Weyl dimension formula in case of the unitary group $U(n)$. In fact the value $s_\lambda(1, \ldots, 1)$ is the dimension d_λ of the irreducible representation of $U(n)$ associated with the highest weight λ.

Proof

Let $f_i(w) = (1 + w)^{\alpha_i}$ with $\alpha_i = \lambda_i + n - i$. We see that

$$F(\mathbf{0}) = \lim_{z_1,\ldots,z_n\to 0} \frac{\det(f_i(z_j))_{1\le i,j\le n}}{V(z)} = s_\lambda(1, \ldots, 1).$$

Since

$$f_i^{(n-j)}(0) = \alpha_i(\alpha_i - 1)\cdots(\alpha_i - n + j - 1),$$

it follows by (2.3) that

$$\begin{aligned}s_\lambda(1, \ldots, 1) = F(\mathbf{0}) =& \frac{1}{\delta!}\det(\alpha_i(\alpha_i - 1)\cdots(\alpha_i - n + j - 1))_{1\le i,j\le n} \\ =& \frac{1}{\delta!}\det(\alpha_i^{n-j})_{1\le i,j\le n} = \frac{V(\lambda+\delta)}{\delta!}.\end{aligned}$$

□

Proposition 2.3 *For $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$,*

$$\frac{\det(e^{x_i y_j})_{1 \le i,j \le n}}{V(x)V(y)} = \sum_{m_1 \ge \cdots \ge m_n \ge 0} \frac{1}{(\mathbf{m}+\delta)!} s_{\mathbf{m}}(x) s_{\mathbf{m}}(y).$$

Proof

Apply Hua's formula (Proposition 2.1) with $f_i(w) = e^{x_i w} = \sum_{m=0}^{\infty} \frac{x_i^m}{m!} w^m$. Then

$$\det(c^{(i)}_{m_j+n-j})_{1 \le i,j \le n} = \frac{1}{(\mathbf{m}+\delta)!} A_{\mathbf{m}+\delta}(x) = \frac{1}{(\mathbf{m}+\delta)!} V(x) s_{\mathbf{m}}(x),$$

Proposition 2.3 immediately follows. ∎

Proposition 2.4

$$e^{z_1 + \cdots + z_n} = \delta! \sum_{m_1 \ge \cdots \ge m_n \ge 0} \frac{1}{(\mathbf{m}+\delta)!} d_{\mathbf{m}} s_{\mathbf{m}}(z),$$

where $d_{\mathbf{m}} = s_{\mathbf{m}}(1, \ldots, 1)$.

∎

As a special case of (2.3) we obtain

$$\lim_{y_1, \cdots, y_n \to 1} \frac{\det(e^{x_i y_j})_{1 \le i,j \le n}}{V(y)} = \frac{1}{\delta!} \det\Big(\frac{d^{n-j}}{dw^{n-j}} e^{x_i w}\Big|_{w=1}\Big)_{1 \le i,j \le n} \tag{2.4}$$

$$= \frac{1}{\delta!} \det(x_i^{n-j} e^{x_i})_{1 \le i,j \le n} = \frac{1}{\delta!} e^{x_1 + \cdots + x_n} V(x). \tag{2.5}$$

Therefore, letting $y_j \to 1,\ 1 \le j \le n$, in Proposition 2.3, we get the claim. ∎

Proposition 2.5 (Cauchy identity)

$$\prod_{i,j=1}^{n} \frac{1}{1 - x_i y_j} = \sum_{m_1 \ge \cdots \ge m_n \ge 0} s_{\mathbf{m}}(x) s_{\mathbf{m}}(y).$$

Proof

Take

$$f_i(w) = \frac{1}{1 - x_i w} = \sum_{m=0}^{\infty} x_i^m w^m$$

in Hua's formula (Proposition 2.1). Then we obtain the identity

$$\det\Big(\frac{1}{1-x_i y_j}\Big)_{1\le i,j\le n} = \sum_{m_1\ge\cdots\ge m_n\ge 0} s_{\mathbf{m}}(x)s_{\mathbf{m}}(y).$$

The claim follows from the well-known evaluation for Cauchy's determinant

$$\det\Big(\frac{1}{1-x_i y_j}\Big)_{1\le i,j\le n} = V(x)V(y)\prod_{i,j=1}^{n}\frac{1}{1-x_i y_j}.$$

∎

Proposition 2.6 (Voiculescu's formula) *Consider a Laurent series*

$$f(w)=\sum_{m=-\infty}^{\infty} c_m w^m$$

Then

$$\prod_{j=1}^{n} f(z_j) = \sum_{\substack{\mathbf{m}\in\mathbb{Z}^n\\ m_1\ge\cdots\ge m_n}} a_{\mathbf{m}} s_{\mathbf{m}}(z)$$

with $a_{\mathbf{m}}=\det(c_{m_i-i+j})_{1\le i,j\le n}$.

([Voiculescu,1976], Lemme 2.)

Proof

Let us expand the product,

$$V(z)f(z_1)\cdots f(z_n)=\sum_{\sigma\in\mathfrak{S}_n}\sum_{p_1,\ldots,p_n\in\mathbb{Z}}\epsilon(\sigma)c_{p_1}\cdots c_{p_n} t_1^{p_1+\delta_{\sigma(1)}}\cdots t_1^{p_n+\delta_{\sigma(n)}},$$

where $\delta_j = n-j$. The number $a_{\mathbf{m}}$ is the coefficient of the monomial $z_1^{m_1+\delta_1}\cdots z_n^{m_n+\delta_n}$ in this sum. It comes from the terms for which $p_i+\delta_{\sigma(i)} = m_i+\delta_i$, or $p_i = m_i - i + \sigma(i)$. Therefore we obtain

$$a_{\mathbf{m}} = \sum_{\sigma\in\mathfrak{S}_n}\epsilon(\sigma)\prod_{i=1}^{n} c_{m_i-i+\sigma(i)} = \det(c_{m_i-i+j})_{1\le i,j\le n}.$$

∎

Proposition 2.7 (Jacobi-Trudi identity) *For a partition* $\mathbf{m}$ *(*$m_1\ge\cdots\ge m_n\ge 0$*),*

$$s_{\mathbf{m}}(x)=\det(h_{m_i-i+j}(x))_{1\le i,j\le n},$$

where h_k *is the complete symmetric function (see* (2.2)*).*

Proof

Take

$$f(w) = \prod_{i=1}^{n} \frac{1}{1 - x_i w} = \sum_{m=0}^{\infty} h_m(x) w^m$$

in Proposition 2.6. Then

$$\prod_{j=1}^{n} f(z_j) = \prod_{i,j=1}^{n} \frac{1}{1 - x_i z_j} = \sum_{m_1 \geq \cdots \geq m_n \geq 0} \det(h_{m_i - i + j}(x))_{1 \leq i,j \leq n} s_{\mathbf{m}}(z). \quad (2.6)$$

On the other hand, it follows by the Cauchy identity (Proposition 2.5) that

$$\prod_{i,j=1}^{n} \frac{1}{1 - x_i z_j} = \sum_{m_1 \geq \cdots \geq m_n \geq 0} s_{\mathbf{m}}(x) s_{\mathbf{m}}(z).$$

Comparing with (2.6), we obtain the Jacobi-Trudi identity. ∎

2.2 Binomial formula for Schur functions, and shifted Schur functions

The results in this section are from [Okunkov-Olshanski,1998a]. For a partition $\mathbf{m}$ ($m_1 \geq \cdots \geq m_n \geq 0$), the *shifted Schur function* $s^*_{\mathbf{m}}$ is defined, for a signature $\lambda = (\lambda_1, \ldots, \lambda_n)$, by

$$s^*_{\mathbf{m}}(\lambda) = s^*_{\mathbf{m}}(\lambda_1, \ldots, \lambda_n) = \frac{\det([\lambda_i + \delta_i]_{m_j + \delta_j})_{1 \leq i,j \leq n}}{\det([\lambda_i + \delta_i]_{\delta_j})_{1 \leq i,j \leq n}},$$

where $\delta_j = n - j$ and $[a]_m = a(a-1)\cdots(a-m+1)$.

Theorem 2.8 (Binomial formula for Schur functions) *For a signature $\lambda = (\lambda_1, \ldots, \lambda_n)$,*

$$\frac{s_\lambda(1+z_1, \ldots, 1+z_n)}{s_\lambda(1, \ldots, 1)} = \sum_{m_1 \geq \cdots \geq m_n \geq 0} \frac{\delta!}{(\mathbf{m}+\delta)!} s^*_{\mathbf{m}}(\lambda) s_{\mathbf{m}}(z). \quad (2.7)$$

For $n = 1$ it reduces to the classical binomial formula ($\lambda \in \mathbb{C}$):

$$(1+w)^\lambda = \sum_{m=0}^{\infty} \frac{1}{m!} [\lambda]_m w^m.$$

Proof

We apply Hua's formula (Proposition 2.1) with

$$f_i(w) = (1+w)^{\lambda_i+\delta_i} = \sum_{m=0}^{\infty} \frac{[\lambda_i+\delta_i]_m}{m!} w^m, \qquad 1 \le i \le n.$$

Then we have

$$\frac{A_{\lambda+\delta}(1+z_1,\dots,1+z_n)}{V(z)} = \sum_{m_1\ge\cdots\ge m_n\ge 0} \frac{1}{(\mathbf{m}+\delta)!} \det([\lambda_i+\delta_i]_{m_j+\delta_j})_{1\le i,j\le n} s_{\mathbf{m}}(z).$$

Finally, by Proposition 2.2,

$$s_\lambda(1,\dots,1) = \frac{V(\lambda+\delta)}{V(\delta)} = \frac{\det([\lambda_i+\delta_i]_{\delta_j})_{1\le i,j\le n}}{\delta!}.$$

∎

Let us state some properties of the shifted Schur functions.

Stability: for a partition $\mathbf{m} = (m_1,\dots,m_n)$,

$$s^*_{\mathbf{m}}(\lambda_1,\dots,\lambda_n,\lambda_{n+1})\big|_{\lambda_{n+1}=0} = s^*_{\mathbf{m}}(\lambda_1,\dots,\lambda_n).$$

Therefore we can regard $s^*_{\mathbf{m}}$ as a function on $\mathbb{R}^{(\infty)}$.

Shifted symmetry: an ordinary Schur function is symmetric, i.e., $s_{\mathbf{m}}(\dots,\lambda_i,\lambda_{i+1},\dots) = s_{\mathbf{m}}(\dots,\lambda_{i+1},\lambda_i,\dots)$, while a shifted Schur function is *shifted symmetric*, i.e.,

$$s^*_{\mathbf{m}}(\dots,\lambda_i,\lambda_{i+1},\dots) = s^*_{\mathbf{m}}(\dots,\lambda_{i+1}-1,\lambda_i+1,\dots).$$

Let us mention two important special cases.

a) The *shifted elementary symmetric function* e^*_k:

$$e^*_k(\lambda) := \sum_{1\le j_1<\cdots<j_k} (\lambda_{j_1}-k+1)(\lambda_{j_2}-k+2)\cdots\lambda_{j_k}.$$

We will see in next section that $e^*_k(\lambda) = s^*_{(1^k)}(\lambda)$.

b) The *shifted complete symmetric function* h^*_m:

$$h^*_m(\lambda) := \sum_{1\le j_1\le\cdots\le j_m} (\lambda_{j_1}-m+1)(\lambda_{j_2}-m+2)\cdots\lambda_{j_m}.$$

We will prove in next section that $h^*_m(\lambda) = s^*_{(m)}(\lambda)$. By Theorem 2.8 we obtain the following power expansion:

$$\frac{s_\lambda(1+w,1,\dots,1)}{s_\lambda(1,\dots,1)} = \sum_{m=0}^{\infty} \frac{(n-1)!}{(m+n-1)!} h^*_m(\lambda) w^m.$$

Ordinary Schur functions $s_{\mathbf{m}}$ are homogeneous polynomials but shifted Schur functions $s^*_{\mathbf{m}}$ are not homogeneous. Note that

$$s^*_{\mathbf{m}}(\lambda) = s_{\mathbf{m}}(\lambda) + (\text{terms of degree} < |\mathbf{m}|),$$

where $|\mathbf{m}| = m_1 + m_2 + \cdots$.

2.3 Generating function for shifted elementary symmetric functions

Theorem 2.9 (i) *For a signature $\lambda = (\lambda_1, \dots, \lambda_n)$,*

$$s^*_{(1^k)}(\lambda) = e^*_k(\lambda).$$

(ii) *Define the generating function F^* for the shifted elementary symmetric functions e^*_k by*

$$F^*(\lambda; t) = \sum_{k=0}^{n} (-1)^k e^*_k(\lambda)[t]_{n-k}.$$

Then

$$F^*(\lambda; t) = \prod_{j=1}^{n} (t - \lambda_j - n + j).$$

Our definition slightly differs from the definition given by Okunkov and Olshanski. They define

$$E^*(\lambda; u) = \sum_{k=0}^{n} \frac{e^*_k(\lambda)}{[u]_k},$$

and then

$$E^*_k(\lambda, u) = \prod_{j=1}^{n} \frac{u - j + 1 + \lambda_j}{u - j + 1}.$$

In fact the two statements are equivalent: for $t = n - 1 - u$, by using the relation

$$\frac{[u]_n}{[u]_k} = (-1)^k [t]_{n-k},$$

one sees that

$$F^*(\lambda; t) = (-1)^n [u]_n E^*(\lambda; u).$$

Proof

a) We will first prove (ii) by recursion on n. We will use

$$[t]_k = t[t-1]_{k-1},$$

and the relation

$$e_k^*(\lambda_1, \ldots, \lambda_n) = e_k^*(\lambda_1, \ldots, \lambda_{n-1}) + e_{k-1}^*(\lambda_1 + 1, \ldots, \lambda_{n-1} + 1)\lambda_n.$$

Observe that

$$\begin{aligned}(t - \lambda_1 - n + 1)(t - \lambda_2 - n + 2) \ldots (t - \lambda_{n-1} - 1) &= F^*(\lambda_1, \ldots, \lambda_{n-1}; t - 1) \\ &= F^*(\lambda_1 + 1, \ldots, \lambda_{n-1} + 1; t)\end{aligned}$$

For $n = 1$ it is trivial. Assume that (ii) holds for $n - 1$. Then

$$\begin{aligned}
&F^*(\lambda_1, \ldots, \lambda_n; t) = \\
&(t - \lambda_1 - n + 1)(t - \lambda_2 - n + 2) \ldots (t - \lambda_{n-1} - 1) \times (t - \lambda_n) \\
&= tF^*(\lambda_1, \ldots, \lambda_{n-1}; t - 1) - \lambda_n F^*(\lambda_1 + 1, \ldots, \lambda_{n-1} + 1; t) \\
&= \sum_{k=0}^{n-1} (-1)^k e_k^*(\lambda_1, \ldots, \lambda_{n-1} t[t-1]_{n-k-1} - \sum_{k=0}^{n-1} (-1)^k e_k^*(\lambda_1 + 1, \ldots, \lambda_{n-1} + 1)\lambda_n [t]_{n-k-1} \\
&= \sum_{k=0}^{n-1} (-1)^k e_k^*(\lambda_1, \ldots, \lambda_{n-1})[t]_{n-k} + \sum_{k=1}^{n} (-1)^k e_{k-1}^*(\lambda_1 + 1, \ldots, \lambda_{n-1} + 1)\lambda_n [t]_{n-k} \\
&= [t]_n + \sum_{k=1}^{n-1} \Big(e_k^*(\lambda_1, \ldots, \lambda_{n-1}) + e_{k-1}^*(\lambda_1 + 1, \ldots, \lambda_{n-1} + 1)\lambda_n \Big) [t]_{n-k} \\
&+ (-1)^n e_{n-1}^*(\lambda_1 + 1, \ldots, \lambda_{n-1} + 1)\lambda_n \\
&= \sum_{k=0}^{n} (-1)^k e_k^*(\lambda_1, \ldots, \lambda_n)[t]_{n-k}.
\end{aligned}$$

b) Consider the following $(n+1)\times(n+1)$ determinant:

$$D^*(\lambda;t) = \begin{vmatrix} [t]_n & [t]_{n-1} & \dots & [t]_1 & 1 \\ [\lambda_1+n-1]_n & [\lambda_1+n-1]_{n-1} & \dots & [\lambda_1+n-1]_1 & 1 \\ [\lambda_2+n-2]_n & [\lambda_2+n-2]_{n-1} & \dots & [\lambda_2+n-2]_1 & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ [\lambda_n]_n & [\lambda_n]_{n-1} & \dots & [\lambda_n]_1 & 1 \end{vmatrix}$$

By using the fact that the value of a determinant does not change when one adds to a column a combination of the other ones, this determinant reduces to a Vandermonde determinant, and one obtains

$$\begin{aligned} D^*(\lambda;t) &= V(\lambda+\delta)\prod_{j=1}^{n}(t-\lambda_j-n+j) \\ &= V(\lambda+\delta)F^*(\lambda;t). \end{aligned}$$

Therefore, by a),

$$D^*(\lambda;t) = V(\lambda+\delta)\sum_{k=0}^{n}(-1)^k e_k^*(\lambda)[t]_{n-k}.$$

On the other hand, by expanding this determinant with respect to the first row one obtains

$$D^*(\lambda,t) = V(\lambda+\delta)\sum_{k=0}^{n}(-1)^k s^*_{(1^k)}(\lambda_1,\dots,\lambda_n)[t]_{n-k}.$$

This proves (i):

$$s_{(1^k)}(\lambda) = e_k^*(\lambda).$$

2.4 Generating function for shifted complete symmetric functions as a factorial expansion

A *factorial expansion* is an expansion of the following form

$$c_0 + \frac{c_1}{z} + \frac{c_2}{z(z+1)} + \dots + \frac{c_m}{z(z+1)\dots(z+m-1)} + \cdots$$

(See [Nörlund,1914].)

Consider the following Mellin transform

$$f(z) = \int_0^1 \varphi(t) t^{z-1} dt.$$

Assume that the function φ is analytic for $|t-1| < 1$, and consider its power expansion at $t = 1$:

$$\varphi(t) = \sum_{m=0}^{\infty} a_m (t-1)^m.$$

Assume that

$$\sum_{m=0}^{\infty} |a_m| \int_0^1 (1-t)^m t^{\sigma-1} dt < \infty.$$

Then, for $\operatorname{Re} z \geq \sigma$,

$$\begin{aligned} f(z) &= \sum_{m=0}^{\infty} (-1)^m a_m \int_0^1 (1-t)^m t^{z-1} dt \\ &= \sum_{m=0}^{\infty} (-1)^m a_m \frac{m!}{z(z+1)\dots(z+m)}. \end{aligned}$$

This is a factorial expansion with

$$c_0 = 0,\ c_{m+1} = (-1)^m m! a_m.$$

In the special case $\varphi(t) = t^{-\alpha}$ $(\alpha \in \mathbb{R})$,

$$\varphi(t) = (t-1+1)^{-\alpha} = \sum_{m=0}^{\infty} \frac{\alpha(\alpha)\dots(\alpha+m-1)}{m!} (1-t)^m,$$

one obtains the *Stirling series* (Stirling, 1730)

$$\frac{1}{z-\alpha} = \frac{1}{z} + \frac{\alpha}{z(z+1)} + \cdots + \frac{\alpha(\alpha+1)\dots(\alpha+m-1)}{z(z+1)\dots(z+m)} + \cdots$$

which converges for $\operatorname{Re} z > \max(\alpha, 0)$. By putting $\alpha = -\lambda$, $z = -(u+1)$, one obtains the *modified Stirling series*

$$\frac{u+1}{u-\lambda+1} = \sum_{m=0}^{\infty} \frac{[\lambda]_m}{[u]_m},$$

which converges for $\operatorname{Re} u < \min(\lambda - 1, 0)$. Recall that

$$[a]_m = a(a-1)\dots(a-m+1).$$

Theorem 2.10 (i) *For a signature* $\lambda = (\lambda_1, \ldots, \lambda_n)$,

$$s^*_{(m)}(\lambda) = h^*_m(\lambda).$$

(ii) *Define the generating function for the shifted complete symmetric functions* h^*_m *by*

$$H^*(\lambda; u) = \sum_{m=0}^{\infty} h^*_m(\lambda) \frac{1}{[u]_m}.$$

Then $H^*(\lambda, u)$ *is the following rational function:*

$$H^*(\lambda; u) = \prod_{j=1}^{n} \frac{u+j}{u+j-\lambda_j}.$$

Notice that, for $n = 1$, (i) means that

$$h^*_m(\lambda) = [\lambda]_m,$$

and (ii) is nothing but the evaluation of the modified Stirling series.

Proof
We will first show that

$$\sum_{m=0}^{\infty} s^*_{(m)}(\lambda) \frac{1}{[u]_m} = \prod_{j=1}^{n} \frac{u+j}{u+j-\lambda_j},$$

and then that

$$s^*_{(m)}(\lambda) = h^*_m(\lambda).$$

a) Define

$$\mathcal{H}^*(\lambda; u) = \sum_{m=0}^{\infty} s^*_{(m)}(\lambda) \frac{1}{[u]_m}.$$

Recall that

$$s^*_{(m)}(\lambda) = \frac{1}{V(\lambda+\delta)} \begin{vmatrix} [\lambda_1+n-1]_{m+n-1} & [\lambda_1+n-1]_{n-2} & \ldots & 1 \\ [\lambda_2+n-1]_{m+n-1} & [\lambda_2+n-1]_{n-2} & \ldots & 1 \\ \vdots & \vdots & & \vdots \\ [\lambda_n]_{m+n-1} & [\lambda_n]_{n-2} & \ldots & 1 \end{vmatrix}$$

By using the identity

$$[a]_{m+n-1} = [a]_{n-1}[a-n+1]_m,$$

and the modified Stirling series we obtain

$$\mathcal{H}^*(\lambda;u) = \frac{1}{V(\lambda+\delta)} \begin{vmatrix} [\lambda_1+n-1]_{n-1}\frac{u+1}{u+1-\lambda_1} & [\lambda_1+n-1]_{n-2} & \dots & 1 \\ [\lambda_2+n-2]_{n-1}\frac{u+1}{u+2-\lambda_2} & [\lambda_2+n-2]_{n-2} & \dots & 1 \\ \vdots & \vdots & & \vdots \\ [\lambda_n]_{n-1}\frac{u+1}{u+1-\lambda_n} & [\lambda_n]_{n-2} & \dots & 1 \end{vmatrix}$$

Hence $u+1$ factors out. Then we add the first column of the determinant to the second one and the i-th row becomes

$$[\lambda_i+n-i]_{n-1}\frac{1}{u+i-\lambda_i} \quad [\lambda_i+n-i]_{n-2}\frac{u+2}{u+i-\lambda_i} \quad [\lambda_i+n-i]_{n-3} \quad \dots \quad 1$$

Now $u+2$ factors out, and so on. One obtains finally

$$\begin{aligned} V(\lambda+\delta)\mathcal{H}^*(\lambda;u) &= (u+1)(u+2)\dots(u+n) \\ &\begin{vmatrix} [\lambda_1+n-1]_{n-1}\frac{1}{u+1-\lambda_1} & [\lambda_1+n-1]_{n-2}\frac{1}{u+1-\lambda_1} & \dots & \frac{1}{u+1-\lambda_1} \\ [\lambda_2+n-2]_{n-1}\frac{1}{u+2-\lambda_2} & [\lambda_2+n-2]_{n-2}\frac{1}{u+2-\lambda_2} & \dots & \frac{1}{u+2-\lambda_2} \\ \vdots & \vdots & & \vdots \\ [\lambda_n]_{n-1}\frac{1}{u+n-\lambda_n} & [\lambda_n]_{n-2}\frac{1}{u+n-\lambda_n} & \dots & \frac{1}{u+n-\lambda_n} \end{vmatrix} \\ &= \frac{(u+1)(u+2)\dots(u+n)}{(u+1-\lambda_1)(u+2-\lambda_2)\dots(u+n-\lambda_n)} V(\lambda+\delta). \end{aligned}$$

b) We prove now that

$$s^*_{(m)}(\lambda) = h^*_m(\lambda),$$

by recursion with respect to n. For $n=1$, this means that

$$h^*_m(\lambda) = [\lambda]_m.$$

Assume that the formula holds for $m-1$. By a), and by using the Stirling series, for a signature $\lambda = (\lambda_1, \dots, \lambda_n)$,

$$\mathcal{H}^*(\lambda,u) = \prod_{i=1}^n \frac{u+i}{u+i-\lambda_i} = \prod_{i=1}^{n-1} \frac{u+i}{u+i-\lambda_i} \sum_{q=0}^\infty \frac{[\lambda_n]_q}{[u+n-1]_q}.$$

Then, by observing that

$$\frac{(u+1)(u+2)\dots(u+n-1)}{[u+n-1]_q} = \frac{(u-q+1)(u-q+2)\dots(u-q+n-1)}{[u]_q},$$

we get

$$\mathcal{H}^*(u,\lambda) = \prod_{i=1}^{n-1} \frac{u-q+i}{u+i-\lambda_i} \sum_{q=0}^{\infty} \frac{[\lambda_n]_q}{[u]_q}$$
$$= \prod_{i=1}^{n-1} \frac{(u-q)+i}{(u-q)+i-(\lambda-q)} \sum_{q=0}^{\infty} \frac{[\lambda_n]_q}{[u]_q}.$$

We use now the recursion hypothesis:

$$\mathcal{H}^*(\lambda;u) = \sum_{p=0}^{\infty} \frac{h_p^*(\lambda_1-q,\ldots,\lambda_{n-1}-q)}{[u-q]_p} \sum_{q=0}^{\infty} \frac{[\lambda_n]_q}{[u]_q}$$
$$= \sum_{m=0}^{\infty} \Big(\sum_{p+q=m} \frac{h_p^*(\lambda_1-q,\ldots,\lambda_{n-1}-q)[\lambda_n]_q}{[u-q]_p[u]_q} \Big).$$

By using the identity

$$[u-q]_p[u]_q = [u]_{p+q},$$

we get finally

$$\mathcal{H}^*(\lambda;u) = \sum_{m=0}^{\infty} \Big(\sum_{p+q=m} h_p^*(\lambda_1-q,\ldots,\lambda_{n-1}-q)[\lambda_n]_q \Big) \frac{1}{[u]_m}.$$

Observing that

$$\sum_{p+q=m} h_p^*(\lambda_1-q,\ldots,\lambda_{n-1}-q)[\lambda_n]_q = h_m^*(\lambda),$$

we get

$$\mathcal{H}^*(\lambda;u) = \sum_{m=0}^{\infty} h_m^*(\lambda) \frac{1}{[u]_m}.$$

We use now the fact that, if two factorial expansions are equal, their coefficients are equal (See [Nörlund,1914]). Therefore

$$s_{(m)}^*(\lambda) = h_m^*(\lambda),$$

and

$$\mathcal{H}^*(\lambda;u) = H^*(\lambda;u).$$

∎

Consider the logarithmic derivative of $H^*(\lambda; u)$:

$$\frac{d}{du} \log H^*(\lambda; u) = \sum_{j=1}^{n} \Big(\frac{1}{u+j} - \frac{1}{u+j-\lambda_j} \Big).$$

It can be written as a power series in $\frac{1}{u}$:

$$\frac{d}{du} \log H^*(\lambda, u) = -\sum_{m=0}^{\infty} q_m^*(\lambda) \frac{1}{u^{m+1}},$$

with

$$q_m^*(\lambda) = \sum_{j=1}^{n} \big((\lambda_j - j)^m - (-j)^m \big).$$

This provides a new family of shifted symmetric functions. The functions q_m^* can be seen as shifted analogues of the Newton power series. They will occur in chapter 4 in the proof of Theorem 4.6.

Chapter 3

Infinite dimensional Hermitian matrices

In this chapter we consider the Olshanski spherical pair

$$\big(U(\infty) \ltimes Herm(\infty, \mathbb{C}), U(\infty)\big).$$

We will present results from [Olshanski-Vershik,1996].

3.1 Gelfand pair associated to a motion group

Let V be a finite dimensional real vector space $V \cong \mathbb{R}^n$, and K a compact subgroup of $GL(V)$. Define the generalized motion group $G = K \ltimes V$ with the product

$$(k_1, a_1) \cdot (k_2, a_2) = (k_1 k_2, a_1 + k_1 a_2), \qquad k_1, k_2 \in K,\ a_1, a_2 \in V. \tag{3.1}$$

If φ is a K-biinvariant function on G, then, since

$$\varphi((k, a)) = \varphi((k_1, 0)\cdot(k, a)\cdot(k_2, 0)) = \varphi((k_1 k k_2, k_1 a)), \qquad k_1, k_2, k \in K,\ a \in V,$$

the function φ only depends on a:

$$\varphi(g) = \tilde{\varphi}(a), \qquad g = (k, a) \in K \ltimes V, \tag{3.2}$$

where $\tilde{\varphi}$ is a K-invariant function on V. The correspondence $\varphi \leftrightarrow \widetilde{\varphi}$ identifies the convolution algebras $L^1(K\backslash G/K)$ and $L^1(K\backslash V)$. Since the convolution algebra $L^1(K\backslash V)$ is commutative, (G, K) is a Gelfand pair.

A spherical function φ for the Gelfand pair (G, K) satisfies the functional equation

$$\int_K \varphi((k_1, a_1) \cdot (k, 0) \cdot (k_2, a_2))dk = \varphi((k_1, a_1))\varphi((k_2, a_2)) \tag{3.3}$$

for any $k_1, k_2 \in K$, $a_1, a_2 \in V$, and dk is the normalized Haar measure on K. It follows by (3.1) and (3.2) that the left hand side equals

$$\int_K \varphi((k_1 k k_2, a_1 + k_1 k a_2))dk = \int_K \tilde{\varphi}(a_1 + k_1 k a_2)dk.$$

so that the functional equation (3.3) can be rewritten as

$$\int_K \tilde{\varphi}(a_1 + k_1 k a_2)dk = \tilde{\varphi}(a_1)\tilde{\varphi}(a_2).$$

Replacing a_1 with $k_1 a_1$, we finally obtain the equation

$$\int_K \tilde{\varphi}(a_1 + k a_2)dk = \tilde{\varphi}(a_1)\tilde{\varphi}(a_2), \qquad a_1, a_2 \in V. \tag{3.4}$$

Conversely, if $\tilde{\varphi}$ is a K-invariant continuous function on V which satisfies the functional equation (3.4), and if φ is the function on G given by (3.2), then φ is a spherical function for the Gelfand pair (G, K). For that reason we make the following definition: a K-invariant continuous function φ on V is said to be *spherical* if

$$\int_K \varphi(x + k \cdot y)\alpha(dk) = \varphi(x)\varphi(y).$$

Let us consider on V a K-invariant inner product. The spherical functions of positive type are Fourier transform of orbital measures on V:

$$\varphi(\lambda; x) = \int_K e^{-i(k\cdot\lambda, x)}\alpha(dk).$$

Let us check that the function $\varphi(x) = \varphi(\lambda; x)$ is spherical:

$$\begin{aligned} \int_K \varphi(x + k \cdot y)\alpha(dk) &= \int_K \Big(\int_K e^{-i(k_1\cdot\lambda, x + k\cdot y)}\alpha(dk_1) \Big) \alpha(dk) \\ &= \int_K e^{-(k_1\cdot\lambda, x)} \Big(\int_K e^{-i(k^* k_1 \cdot\lambda, y)}\alpha(dk) \Big) \alpha(dk_1) \\ &= \varphi(x)\varphi(y), \end{aligned} \tag{3.5}$$

by the invariance of the Haar measure α.

Hence the spherical dual Ω can be identified with the set of K-orbits in V: $\Omega \simeq K\backslash V$.

3.2 The Gelfand pair $(U(n) \ltimes \mathrm{Herm}(n,\mathbb{C}), U(n))$

In the present chapter, we consider the case where $V = \mathrm{Herm}(n,\mathbb{C})$, the space of $n \times n$ Hermitian matrices, and $K = U(n)$. Here K acts on V by

$$k \cdot x = kxk^* \quad \big(x \in Herm(n,\mathbb{C}), k \in U(n)\big)$$

A spherical function is a continuous solution of the following functional equation:

$$\int_{U(n)} \varphi(x + kyk^*)dk = \varphi(x)\varphi(y), \qquad x, y \in \mathrm{Herm}(n,\mathbb{C}).$$

We consider on $Herm(n,\mathbb{C})$ the following inner product $(x|y) = \mathrm{tr}(xy)$. By the spectral theorem, every Hermitian matrix is diagonalizable by unitary matrices and its eigenvalues are real, therefore

$$\Omega \simeq U(n)\backslash Herm(n,\mathbb{C}) \simeq \mathbb{R}^n/\mathfrak{S}_n.$$

A $U(n)$-invariant function $\varphi(x)$ on $Herm(n,\mathbb{C})$ only depends on the eigenvalues $\lambda_1, \dots, \lambda_n$ of $x \in V$, i.e., φ can be seen as a symmetric function of n real variables $\lambda_1, \dots, \lambda_n$.

Consider the orbital integral

$$I(x,y) = \int_{U(n)} e^{\mathrm{tr}(xuyu^*)}\alpha(du), \qquad x, y \in \mathrm{Herm}(n,\mathbb{C}),$$

where α is the normalized Haar measure on $U(n)$.

Theorem 3.1 *Let x and y be diagonal matrices $x = \mathrm{diag}(x_1, \dots, x_n)$ and $y = \mathrm{diag}(y_1, \dots, y_n)$. Then we have*

$$\begin{aligned} I(x,y) =& \delta! \frac{1}{V(x)V(y)} \det(e^{x_i y_j}) \\ =& \sum_{m_1 \geq \cdots \geq m_n \geq 0} \frac{\delta!}{(\mathbf{m}+\delta)!} s_{\mathbf{m}}(x_1, \dots, x_n) s_{\mathbf{m}}(y_1, \dots, y_n). \end{aligned}$$

Proof

Let $\chi_{\mathbf{m}}$ be the character of the irreducible representation of $U(n)$ with highest weight $\mathbf{m} = (m_1, \dots, m_n)$. By the Weyl character formula, it can be expressed as a Schur function: if $z_1, \dots, z_n$ are the eigenvalues of the matrix $u \in U(n)$, then, $\chi_{\mathbf{m}}(u) = s_{\mathbf{m}}(z_1, \dots, z_n)$. If $\mathbf{m}$ is a partition: $m_1 \geq \ldots \geq m_n \geq 0$, then the character $\chi_{\mathbf{m}}$ extends as a polynomial function on the vector space $M(n,\mathbb{C})$ of $n \times n$ complex matrices.

By Proposition 2.4,

$$I(x,y) = \sum_{m_1 \geq \cdots \geq m_n \geq 0} \frac{\delta!}{(\mathbf{m}+\delta)!} d_{\mathbf{m}} \int_{U(n)} \chi_{\mathbf{m}}(xuyu^*)\alpha(du).$$

Using the functional equation of the characters :

$$\int_{U(n)} \chi_{\mathbf{m}}(xuyu^*)\alpha(du) = \frac{1}{d_{\mathbf{m}}} \chi_{\mathbf{m}}(x)\chi_{\mathbf{m}}(y),$$

(see Proposition 4.1 below) we get

$$I(x,y) = \sum_{m_1 \geq \cdots \geq m_n \geq 0} \frac{\delta!}{(\mathbf{m}+\delta)!} \chi_{\mathbf{m}}(x)\chi_{\mathbf{m}}(y).$$

Furthermore, by Proposition 2.3, this equals

$$\frac{\delta!}{V(x)V(y)} \det(e^{x_i y_j}).$$

$\square$

3.3 Multiplicative property of spherical functions

Let $G(n) = U(n) \ltimes \mathrm{Herm}(n,\mathbb{C})$ and $K(n) = U(n)$. We regard $\mathrm{Herm}(n,\mathbb{C})$ as the subspace of $\mathrm{Herm}(n+1,\mathbb{C})$ by

$$\mathrm{Herm}(n,\mathbb{C}) \ni x \mapsto \left(\begin{array}{c|c} x & 0 \\ \hline 0 & 0 \end{array}\right) \in \mathrm{Herm}(n+1,\mathbb{C}).$$

Define the Olshanski spherical pair (G,K) as the inductive limit of the Gelfand pairs $\big(G(n),K(n)\big)$:

$$G = \bigcup_{n=1}^{\infty} G(n) = U(\infty) \ltimes \mathrm{Herm}(\infty,\mathbb{C}), \quad K = \bigcup_{n=1}^{\infty} K(n) = U(\infty).$$

For $m \leq n$, define the subgroup $K_m(n)$ of $K(n)$ by

$$K_m(n) = \left\{ \left(\begin{array}{c|c} I_m & 0 \\ \hline 0 & v \end{array}\right) \middle| v \in U(n-m) \right\} \simeq K(n-m)$$

and the subgroup K_m of K by $K_m = \bigcup_{n=m}^{\infty} K_m(n)$.

Proposition 3.2 (Weyl's integral formula) *Let f be a continuous $K_m(n)$-biinvariant bounded function on $K(n)$. If $2m \leq n$, then*

$$\int_{K(n)} f(k)\alpha_n(dk) = \int_{[0,\frac{\pi}{2}]^m} \int_{K(m)\times K(m)} f(h_1 a(\theta) h_2)\alpha_m(dh_1)\alpha_m(dh_2) D_{m,n}(\theta) d\theta_1 \cdots d\theta_m,$$

where

$$a(\theta) = \begin{pmatrix} \cos\theta_1 & & & -\sin\theta_1 & & & \\ & \ddots & & & \ddots & & 0 \\ & & \cos\theta_m & & & -\sin\theta_m & \\ \sin\theta_1 & & & \cos\theta_1 & & & \\ & \ddots & & & \ddots & & 0 \\ & & \sin\theta_m & & & \cos\theta_m & \\ & 0 & & & 0 & & I_{n-2m} \end{pmatrix}$$

and

$$D_{m,n}(\theta) = a_{m,n} \left| \prod_{1\leq i<j\leq m} \sin^2(\theta_i+\theta_j)\sin^2(\theta_i-\theta_j) \cdot \prod_{i=1}^m (\sin 2\theta_i)(\sin\theta_i)^{2(n-2m)} \right|.$$

The constant $a_{m,n}$ is such that $\int_{[0,\frac{\pi}{2}]^m} D_{m,n}(\theta)d\theta_1 \cdots d\theta_m = 1$. □

Using Proposition 3.2 we obtain:

Proposition 3.3 *Let f be a K_m-biinvariant continuous bounded function on K. Then*

$$\lim_{n\to\infty} \int_{K(n)} f(k)\alpha_n(dk) = \int_{K(m)\times K(m)} f(h_1 w_m h_2)\alpha_m(dh_1)\alpha_m(dh_2)$$

with

$$w_m = a\Big(\frac{\pi}{2}, \ldots, \frac{\pi}{2}\Big) = \begin{pmatrix} 0 & -I_m & 0 \\ I_m & 0 & 0 \\ 0 & 0 & I_{n-2m} \end{pmatrix}.$$

Proof

It follows from the following

Lemma *Let X be a compact space, and μ a positive measure such that every nonempty open set has a positive measure. Let $\delta \geq 0$ be a continuous function on X which attains its maximum at only one point x_0. Define*

$$\frac{1}{a_n} = \int_X \delta(x)^n \mu(dx),$$

and, for a continuous function f on X,

$$L_n(f) = a_n \int_X f(x)\delta(x)^n \mu(dx).$$

Then

$$\lim_{n\to\infty} L_n(f) = f(x_0).$$

Corollary 3.4 *Let φ be a K-invariant continuous bounded function on* $\mathrm{Herm}(\infty, \mathbb{C})$*. If $x = \mathrm{diag}(x_1, \ldots, x_m, 0, 0, \ldots)$ and $y = \mathrm{diag}(y_1, \ldots, y_m, 0, 0, \ldots)$, then*

$$\lim_{n\to\infty} \int_{K(n)} \varphi(x + kyk^*)\alpha_n(dk) = \varphi(\mathrm{diag}(x_1, \ldots, x_m, y_1, \ldots, y_m, 0, 0, \ldots)).$$

Proof

Since the function $K \ni k \mapsto \varphi(x + kyk^*)$ is K_m-biinvariant, we can apply Proposition 3.3:

$$\lim_{n\to\infty} \int_{K(n)} \varphi(x+kyk^*)\alpha_n(dk) = \int_{K(m)\times K(m)} \varphi(x+h_1 w_m h_2 y h_2^* w_m^{-1} h_1^*)\alpha_m(dh_1)\alpha_m(dh_2).$$

One sees that

$$x + h_1 w_m h_2 y h_2^* w_m^{-1} h_1^* = \begin{pmatrix} x & 0 & 0 \\ 0 & h_2 y h_2^* & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence we have the desired statement. ∎

Theorem 3.5 (Multiplicative property) *Let φ be a continuous bounded function on* $\mathrm{Herm}(\infty, \mathbb{C})$ *with $\varphi(0) = 1$, which is $U(\infty)$-invariant. Then φ is spherical if and only if there exists a continuous function Φ on $\mathbb{R}$ with $\Phi(0) = 1$ such that*

$$\varphi(\mathrm{diag}(x_1, x_2, \ldots, x_n, 0, 0, \ldots)) = \Phi(x_1)\Phi(x_2)\cdots\Phi(x_n). \tag{3.6}$$

This can be written, by using functional calculus,

$$\varphi(x) = \det \Phi(x).$$

Proof

Assume φ spherical. Then, for $x = \mathrm{diag}(x_1, \ldots, x_m, 0, 0, \ldots)$, $y = \mathrm{diag}(y_1, \ldots, y_m, 0, 0, \ldots)$,

$$\lim_{n\to\infty} \int_{U(n)} \varphi(x + kyk^*)\alpha_n(dk) = \varphi(x)\varphi(y).$$

By Corollary 3.4

$$\varphi(x)\varphi(y) = \varphi(\mathrm{diag}(x_1, \ldots, x_m, y_1, \ldots, y_m, 0, 0, \ldots)).$$

Hence $\varphi(\mathrm{diag}(x_1, \ldots, x_m, 0, 0, \ldots)) = \Phi(x_1) \cdots \Phi(x_m)$, where Φ is the restriction of φ to $\mathbb{R} = \mathrm{Herm}(1, \mathbb{C})$.

Conversely, if φ has the property (3.6), by Corollary 3.4,

$$\lim_{n\to\infty} \int_{U(n)} \varphi(x + kyk^*)\alpha_n(dk) = \varphi(x)\varphi(y).$$

Therefore φ is spherical. $\square$

3.4 Pólya functions

We will describe the functions Φ occuring in Theorem 3.5 when the spherical function is of positive type. We say that a continuous function Φ on $\mathbb{R}$, with $\Phi(0) = 1$, is a *Pólya function* if the function φ defined on $Herm(\infty, \mathbb{C})$ by

$$\varphi(x) = \det \Phi(x)$$

is of positive type. It amounts to saying that, for every n, the function φ defined on $Herm(n, \mathbb{C})$ by

$$\varphi(x) = \det \Phi(x)$$

is of positive type. This formula means that φ is K-invariant, and

$$\varphi\big(\mathrm{diag}(x_1, \ldots, x_n, 0, \ldots)\big) = \Phi(x_1) \ldots \Phi(x_n).$$

Theorem 3.6 *The Pólya functions are the following ones:*

$$\Phi(z) = e^{-i\beta z} e^{-\frac{\gamma}{2} z^2} \prod_{k=1}^{\infty} \frac{e^{i\alpha_k z}}{1 + i\alpha_k z} \quad (z \in \mathbb{R}),$$

where $\beta \in \mathbb{R}$, $\gamma \geq 0$, $\alpha_k \in \mathbb{R}$ *with*

$$\sum_{k=1}^{\infty} \alpha_k^2 < \infty.$$

The name *Pólya* refers to the following result by Pólya:

Let Ψ be an entire function with $\Psi(0) = 1$. Then Ψ is the uniform limit on compact sets in $\mathbb{C}$ of polynomials with only real zeros if and only if Ψ has the following form

$$\Psi(s) = e^{-\beta s} e^{-\frac{\gamma}{2}s^2} \prod_{k=1}^{\infty} e^{\alpha_k s}(1 - \alpha_k),$$

where $\beta \in \mathbb{R}$, $\gamma \geq 0$, $\alpha_k \in \mathbb{R}$ with

$$\sum_{k=1}^{\infty} \alpha_k^2 < \infty.$$

Such an entire function Ψ is said to belong to the *Pólya-Laguerre class.*

First let us check that the function given by the formula of Theorem 3.6 is a Pólya function.

If $\Phi(z) = e^{-\beta z}$, with $\beta \in \mathbb{R}$, then

$$\varphi(x) = \det \Phi(x) = e^{-i\beta \operatorname{tr}(x)}$$

is of positive type.

If $\Phi(z) = e^{-\frac{\gamma}{2}z^2}$, with $\gamma > 0$, then

$$\varphi(x) = \det \Phi(x) = e^{-\frac{\gamma}{2}\operatorname{tr}(x^2)}$$

is a Gaussian function, which is of positive type.

Consider the following Wishart distribution, image by the map

$$\xi \mapsto y = \alpha \xi \xi^*, \ \mathbb{C}^n \to Herm(n, \mathbb{C}),$$

of the Gaussian measure

$$\pi^{-n} e^{-\|\xi\|^2} m(d\xi)$$

on $\mathbb{C}^n$ (m is the Euclidean measure). Its Fourier transform is

$$\begin{aligned}
\widehat{W}(x) = \pi^{-n} \int_{Herm(n,\mathbb{C}} e^{-i(x|y)} W(dy) &= \pi^{-n} \int_{\mathbb{C}^n} e^{-i\alpha(x|\xi\xi^*)} e^{-\|\xi\|^2} m(d\xi) \\
&= \pi^{-n} \int_{\mathbb{C}^n} e^{-(I+i\alpha x)\xi|\xi)} m(d\xi) \\
&= \det(I + i\alpha x)^{-1}.
\end{aligned}$$

Therefore, if $\Phi(z) = (1 + i\alpha z)^{-1}$, then

$$\det \Phi(x) = \det(I + i\alpha x)^{-1},$$

and Φ is a Pólya function.

Since products and limits of functions of positive type are of positive type, it follows that the function given in the formula of Theorem 3.6 is a Pólya function.

Pólya functions have been considered by Schoenberg in connection with totally positive functions. Recall that a function f on $\mathbb{R}$ is said to be *totally positive* if

$$\det\bigl(f(s_i - t_j)\bigr)_{1 \leq i,j \leq n} \geq 0,$$

for all reals numbers $s_< \leq s_n$, $t_1, \ldots, t_n$.

Theorem 3.7 (Schoenberg,1951) *Pólya functions, for which*

$$\gamma + \sum_{k=1}^{\infty} \alpha_k^2 > 0,$$

(i.e. not of the form $e^{-i\beta z}$,) are Fourier transforms of totally positive integrable functions f with

$$\int_{\mathbb{R}} f(t)dt = 1.$$

From Theorem 3.5 and 3.6 one obtains:

Theorem 3.8 (Pickrell,1991) *The spherical functions of positive type for the Olshanski spherical pair $\bigl(U(\infty) \ltimes Herm(\infty, \mathbb{C}), U(\infty)\bigr)$ are given by the following functions φ on $H(erm(\infty, \mathbb{C})$:*

$$\varphi(x) = \det \Phi(x),$$

where Φ is a Pólya function. This means that

$$\varphi\bigl(\mathrm{diag}(x_1, \ldots, x_n, 0, \ldots)\bigr) = \Phi(x_1) \ldots \Phi(x_n).$$

The spherical unitary dual Ω can be identified with the set of triples $\omega = (\alpha, \beta, \gamma)$ where $\beta \in \mathbb{R}$, $\gamma \geq 0$, $\alpha_k \in \mathbb{R}$ with

$$\sum_{k=1}^{\infty} \alpha_k^2 < \infty.$$

More precisely, to an element ω in Ω corresponds the set $\{\alpha_k\}$. One should consider an element ω up to permutation of the numbers α_k. The Pólya function corresponding to ω will be written $\Phi(\omega; z)$:

$$\Phi(\omega; z) = e^{-i\beta z} e^{-\frac{\gamma}{2} z^2} \prod_{k=1}^{\infty} \frac{e^{i\alpha_k z}}{1 + i\alpha_k z}.$$

For a continuous function f on $\mathbb{R}$ we define the function L_f on Ω by

$$L_f(\omega) = \gamma f(0) + \sum_{k=1}^{\infty} \alpha_k^2 f(\alpha_k).$$

We consider on Ω the initial topology associated to the functions L_f, and the function $\omega \mapsto \beta$. For z fixed, the function $\omega \mapsto \Phi(\omega, z)$ is continuous on Ω for that topology. This can be seen by looking at the logarithmic derivative

$$\frac{d}{dz}\Phi(\omega; z) = -i\beta - \big(\gamma + p_2(\alpha)\big) z - i \sum_{m=2}^{\infty} p_{m+1}(\alpha)(-iz)^n,$$

where, for $m \geq 2$,

$$p_m(\alpha) = \sum_{k=1}^{\infty} \alpha_k^m.$$

Observe that

$$\begin{aligned} &\gamma + p_2(\alpha) = L_f(\omega), \text{ with } f(s) = 1, \\ &p_m(\alpha) = L_f(\omega), \text{ with } f(s) = s^{m-2} \quad (m \geq 3). \end{aligned}$$

3.5 Convergence of probability measures and functions of positive type

Let μ be a positive measure on $\mathbb{R}^d$. Define

$$\mathcal{D}(\mu) = \{x \in \mathbb{R}^d \mid \int_{\mathbb{R}^d} e^{(x|\xi)} \mu(d\xi) < \infty\}.$$

If $\mathcal{D}(\mu) \neq \emptyset$, the Fourier-Laplace of the measure μ is defined for z in the tube $\mathbb{R}^d + i\mathcal{D}$ by

$$\mathcal{F}\mu(z) = \int_{\mathbb{R}^d} e^{-i(z|\xi)} \mu(d\xi).$$

Theorem 3.9 *Assume that the interior Ω of $\mathcal{D}(\mu)$ is not empty. Then the Fourier-Laplace transform $\mathcal{F}\mu$ of μ is holomorphic in the tube $\mathbb{R}^d + i\Omega$.*

Proof

a) Assume first that the open ball $B(0, R) \subset \Omega$: for every y with $\|y\| < R$,

$$\int_{\mathbb{R}^d} e^{(y|\xi)} \mu(d\xi) < \infty.$$

We will show that, for every r with $0 < r < R$,

$$\int_{\mathbb{R}^d} e^{r\|\xi\|} \mu d\xi) < \infty.$$

For $\|y\| > r$, observe that the set

$$\{\xi \in \mathbb{R}^d \mid (y|\xi) \geq r\|\xi\|\}$$

is a conical neighbourhood of the semi-axis $\mathbb{R}_+ y$. One can find a finite number of points $y_1, \ldots, y_N$, with $\|y_i\| < R$, and decompose $\mathbb{R}^d$ in a disjoint union of conical sets $E_1, \ldots, E_N$ such that, for $\xi \in E_i$,

$$(y_i|\xi) \geq r\|\xi\|.$$

Therefore

$$\int_{E_i} e^{r\|\xi\|} \mu(d\xi) \leq \int_{E_i} e^{(y_i|\xi)} \mu(d\xi) < \infty,$$

and

$$\int_{\mathbb{R}^d} e^{(r\|\xi\|} \mu(d\xi) = \sum_{i=1}^{N} \int_{E_i} e^{r\|\xi\|} \mu(d\xi) < \infty.$$

It follows that the Fourier-Laplace transform of μ is holomorphic on the tube $\mathbb{R}^d + iB(0, R)$.

b) Assume now that $B(y_0, R) \subset \Omega$. Applying the result of a) to the measure $e^{(y_0|\xi)} \mu(d\xi)$ proves that the Fourier-Laplace of the measure μ is holomorphic on the tube $\mathbb{R}^d + iB(y_0, R)$. □

Proposition 3.10 *Let ψ be an analytic function in a neighborhood of 0 in $\mathbb{R}^d$, with $\psi(0) = 1$. Assume that there is a probability measure μ on $\mathbb{R}^d$ with finite moments such that, for every $\alpha = (\alpha_1, \ldots, \alpha_d)$,*

$$\int_{\mathbb{R}^d} (-i\xi)^\alpha \mu(d\xi) = \partial_\alpha \psi(0).$$

Then ψ extends as an analytic function on $\mathbb{R}^d$ which is the Fourier transform of the measure μ: $\psi = \hat{\mu}$. Furthermore, if such a probability measure exists, it is unique.

Proof

Assume that the function ψ is holomorphic in the ball $B(0,R)\subset\mathbb{C}^d$. For $a\neq 0$ fixed in $\mathbb{R}^d$ with $\|a\|<R$, consider the function f_a in one variable defined by

$$f_a(w)=\psi(aw).$$

The function f_a is holomorphic in the disc $D(0,\rho)$ with radius $\rho=\frac{R}{\|a\|}>1$. The derivatives of f_a at 0 are given by

$$\begin{aligned} f_a^{(m)}(0) &= \left(\left(a_1\frac{\partial}{\partial x_1}+\cdots+a_d\frac{\partial}{\partial x_n}\right)^m\psi\right)(0)\\ &= \int_{\mathbb{R}^d}(a|-i\xi)^m\mu(d\xi). \end{aligned}$$

Let us evaluate the function f_a at i and $-i$:

$$\begin{aligned} \frac{1}{2}\big(f_a(i)+f_a(-i)\big) &= \sum_{k=0}^{\infty}\frac{1}{(2k)!}f_a^{(2k)}(0)(-1)^k\\ &= \sum_{k=0}^{\infty}\frac{1}{(2k)!}\int_{\mathbb{R}^d}(a|\xi)^{2k}\mu(d\xi)\\ &= \int_{\mathbb{R}^d}\cosh(a|\xi)\mu(d\xi). \end{aligned}$$

It follows that, for every a with $\|a\|<R$,

$$\int_{\mathbb{R}^d}e^{(a|\xi)}\mu(d\xi)<\infty.$$

As we saw in the proof of Proposition 3.9, the Fourier-Laplace transform $\mathcal{F}\mu$ of the measure μ is holomorphic in the tube $\mathbb{R}^d+iB(0,R)$. Looking at the derivatives at 0 one checks that $\psi=\mathcal{F}\mu$ in a neighbourhood of 0. Uniqueness of the measure μ follows from uniqueness of the analytic continuation. $\square$

Proposition 3.11 *Let ψ_n be a sequence of $\mathcal{C}^\infty$-functions on $\mathbb{R}^d$ of positive type with $\psi_n(0)=1$, and ψ an analytic function on a neighborhood of 0. Assume that, for every $\alpha=(\alpha_1,\ldots,\alpha_d)$,*

$$\lim_{n\to\infty}\partial^\alpha\psi_n(0)=\partial^\alpha\psi(0).$$

Then ψ has an analytic extension to $\mathbb{R}^d$, and ψ_n converges to ψ uniformly on compacts subsets of $\mathbb{R}^d$.

Proof

The function ψ_n is the Fourier transform of a probability measure μ_n:

$$\psi_n(x) = \int_{\mathbb{R}^d} e^{-i(x|\xi)} \mu_n(d\xi).$$

Since

$$\lim_{n\to\infty} \Delta\psi_n(0) = \Delta\psi(0),$$

where Δ is the Laplace operator of $\mathbb{R}^d$, it follows that there is a constant $C > 0$ such that

$$\int_{\mathbb{R}^d} \|\xi\|^2 \mu_n(d\xi) \leq C.$$

Therefore the set $\{\mu_n\}$ is relatively compact for the weak topology (tight topology), and there exists a subsequence μ_{n_k} which converges to a probability measure μ for the weak topology. Furthermore, for every $N > 0$, since

$$\lim_{n\to\infty} (I + \Delta)^N \psi_n(0) = (I + \Delta)^N \psi(0),$$

there is a contant $C_N > 0$ with

$$\int_{\mathbb{R}^d} (1 + \|\xi\|^2)^N \mu_n(d\xi) \leq C_N.$$

Consider the function f defined by

$$f(\xi) = (1 + \|\xi\|^2)^N \Big(1 - \frac{\|\xi\|^2}{R^2}\Big),$$

if $\|\xi\| < R$, and $f(\xi) = 0$ for $\|\xi\| > R$. The function f is continuous with compact support. Then

$$\lim_{k\to\infty} \int_{\mathbb{R}^d} f(\xi)\mu_{n_k}(d\xi) = \int_{\mathbb{R}^d} f(\xi)\mu(d\xi).$$

It follows that, for every R

$$\int_{\mathbb{R}^d} (1 + \|\xi\|^2)^N \Big(1 - \frac{\|\xi\|^2}{R^2}\Big)\mu(d\xi) \leq C_N,$$

and, by the Lebesgue monotone convergence theorem, as $R \to \infty$, that

$$\int_{\mathbb{R}^d} (1 + \|\xi\|^2)^N \mu(d\xi) \leq C_N.$$

We will show that, if p is a polynomial, then

$$\lim_{k\to\infty}\int_{\mathbb{R}^d} p(\xi)\mu_{n_k}(d\xi) = \int_{\mathbb{R}^d} p(\xi)\mu(d\xi).$$

Fix N such that $2N > \operatorname{degree} p$. The function f defined by

$$f(\xi) = \frac{p(\xi)}{(1+\|\xi\|^2)^N}$$

is continuous and vanish at infinity. Fix $\varepsilon > 0$. There is a continuous function g with compact support such that, for $\xi \in \mathbb{R}^d$,

$$\Big|\frac{p(\xi)}{(1+\|\xi\|^2)^N} - g(x)\Big| \leq \varepsilon.$$

There is k_0 such that, for $k \geq k_0$,

$$\Big|\int_{\mathbb{R}^d} g(\xi)(1+\|\xi\|^2)^N\mu_{n_k}(dx) - \int_{\mathbb{R}^d} g(\xi)(1+\|\xi\|^2)^N\mu(d\xi)\Big| \leq \varepsilon.$$

By decomposing

$$\begin{aligned}
&\int_{\mathbb{R}^d} p(\xi)\mu_{n_k}(d\xi) - \int_{\mathbb{R}^d} p(\xi)\mu(d\xi)\\
&= \int_{\mathbb{R}^d} \frac{p(\xi)}{(1+\|\xi\|^2)^N}(1+\|\xi\|^2)^N\mu_n(d\xi) - \int_{\mathbb{R}^d} g(\xi)(1+\|\xi\|^2)^N\mu_{n_k}(d\xi)\\
&+ \int_{\mathbb{R}^d} g(\xi)(1+\|\xi\|^2)^N\mu_{n_k}(d\xi) - \int_{\mathbb{R}^d} g(\xi)(1+\|\xi\|^2)^N\mu(d\xi)\\
&+ \int_{\mathbb{R}^d} g(\xi)(1+\|\xi\|^2)^N\mu(d\xi) - \int_{\mathbb{R}^d} \frac{p(\xi)}{(1+\|\xi\|^2)^N}(1+\|\xi\|^2)^N\mu(d\xi),
\end{aligned}$$

one obtains

$$\Big|\int_{\mathbb{R}^d} p(\xi)\mu_{n_k}(d\xi) - \int_{\mathbb{R}^d} p(\xi)\mu(d\xi)\Big| \leq 2\varepsilon C_N + \varepsilon.$$

Observe that

$$\partial\psi_n(0) = \int_{\mathbb{R}^d} (-i\xi)^\alpha\mu_n(d\xi).$$

Therefore

$$\begin{aligned}
\int_{\mathbb{R}^d} (-i\xi)^\alpha\mu(d\xi) &= \lim_{k\to\infty}\int_{\mathbb{R}^d} (-i\xi)^\alpha\mu_{n_k}(d\xi)\\
&= \lim_{k\to\infty}\partial_\alpha\psi_{n_k}(0) = \partial_\alpha\psi(0).
\end{aligned}$$

Since the function ψ is analytic near 0, there is at most one probability measure μ such that

$$\int_{\mathbb{R}^d} (-i\xi)^\alpha \mu(d\xi) = \partial_\alpha \psi(0).$$

Therefore the sequence μ_n itself converges weakly to μ. The Fourier-Laplace transform $\hat{\mu}$ of μ is holomorphic in a tube $\mathbb{R}^d + i\omega$, coincides with ψ in a neighborhood of 0, and ψ_n converges to ψ on compact sets. ∎

3.6 Asymptotics of orbital integrals

For the Gelfand pair $(U(n) \ltimes \mathrm{Herm}(n, \mathbb{C}), U(n))$, the spherical functions are Fourier transforms of orbital measures. More specifically, for a spherical function φ on $\mathrm{Herm}(n, \mathbb{C})$, there exists a diagonal matrix $\lambda = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$, $\lambda_i \in \mathbb{R}$, such that

$$\begin{aligned}\varphi(x) = \varphi_n(\lambda; x) =& I(-ix, \lambda) \\ =& \int_{U(n)} e^{-i\,\mathrm{tr}(xu\lambda u^*)} \alpha_n(du).\end{aligned}$$

By Theorem 3.1 the orbital integral $I(-ix, \lambda)$ can be expanded as follows :

$$\varphi_n(x, \lambda) = I(-ix, \lambda \sum_{m_1 \geq \dots \geq m_n \geq 0} \frac{\delta!}{(\mathbf{m} + \delta)!} s_{\mathbf{m}}(\lambda) s_{\mathbf{m}}(-ix). \qquad (3.7)$$

(see Theorem 3.1.) We will study the asymptotics of $\varphi_n(x, \lambda)$ as n goes to infinity..

In order to study Taylor expansions of the spherical functions which are given by orbital integrals, we introduce an algebra morphism $f \mapsto \tilde{f}$ which maps a symmetric function f onto a continuous function $\tilde{f}$ on Ω:

$$\Lambda \to \mathcal{C}(\Omega), \quad f \mapsto \tilde{f}.$$

Recall that p_m denotes the Newton power sum,

$$p_m(x) = \sum_{k=1}^{\infty} x_k^m.$$

This morphism is such that

$$\tilde{p}_1(\omega) = \beta,$$

$$\tilde{p}_2(\omega) = \gamma + \sum_{k=1}^{\infty} \alpha_k^2,$$

and, for $m \geq 3$,

$$\tilde{p}_m(\omega) = \sum_{k=1}^{\infty} \alpha_k^m.$$

Observe that the functions $\tilde{p}_m$ are continuous on Ω. In fact, for $m \geq 2$, $\tilde{p}_m(\omega) = L_\varphi(\omega)$ with $\varphi(s) = s^{m-2}$.

Proposition 3.12 (i)*For $\omega \in \Omega$, the Taylor expansion of $\Phi(\omega, z)$ is given by:*

$$\Phi(\omega, z) = \sum_{m=0}^{\infty} \widetilde{h}_m(\omega)(-iz)^m.$$

(ii) *For $\omega \in \Omega$ and $z_1, \dots, z_k \in \mathbb{C}$,*

$$\prod_{j=1}^{k} \Phi(\omega, z_j) = \sum_{\mathbf{m}} \widetilde{s}_{\mathbf{m}}(\omega) s_{\mathbf{m}}(-iz_1, \dots, -iz_k),$$

where the sum runs over all partitions $\mathbf{m}$.

Proof

(i) (i) amounts to saying that

$$\widetilde{H}(\omega, -iz) = \Phi(\omega, z).$$

where z is seen as a parameter. Recall the generating function of the complete symmetric functions h_m:

$$H(z; z) = \sum_{m=0}^{\infty} h_m(x) z^m = \prod_{j=1}^{m} \frac{1}{1 - x_j z}.$$

Taking the logarithmic derivatives one obtains:

$$\begin{aligned} \frac{d}{dz} \log H(x; z) &= -\frac{d}{dz} \sum_{j=1}^{n} \log(1 - x_j z) \\ &= \sum_{j=1}^{n} \frac{x_j}{1 - x_j z} \\ &= \sum_{m=0}^{\infty} p_{m+1}(x) z^m. \end{aligned}$$

On the other hand,

$$\frac{d}{dz}\log\Phi(\omega;z) = -i\beta - \Big(\gamma + \sum_{k=1}^{\infty} p_2(\alpha)\Big)z - i\sum_{m=2}^{\infty} p_{m+1}(\alpha)(-iz)^m$$
$$= \frac{d}{dz}\log\widetilde{H}(\omega;-iz).$$

Since $\Phi(\omega,0)=1$ and $\widetilde{H}(\omega,0)=1$, the statement follows. ∎

(ii) Recall the Cauchy identity (Proposition 2.5):

$$\sum_{\mathbf{m}} s_{\mathbf{m}}(x_1,x_2,\dots)s_{\mathbf{m}}(y_1,\dots,y_k) = \prod_{i=1}^{\infty}\prod_{j=1}^{k}\frac{1}{1-x_iy_j} = \prod_{j=1}^{k} H(x,y_j).$$

Apply the morphism $f\mapsto\widetilde{f}$ to both sides of this equality with $y_j=-iz_j$. Then the statement follows by (i). ∎

Define the map

$$T_n:\Omega_n\simeq\mathbb{R}^n\to\Omega,\quad \lambda=(\lambda_1,\dots,\lambda_n)\mapsto\omega=(\alpha,\beta,\gamma)$$

given by

$$\alpha_k=\frac{\lambda_k}{n},\qquad \beta=\frac{\lambda_1+\cdots+\lambda_n}{n},\qquad \gamma=0.$$

Theorem 3.13 *Consider a sequence $(\lambda^{(n)})$ with $\lambda^{(n)}\in\Omega_n\simeq\mathbb{R}^n$. Assume that, for the topology of Ω,*

$$\lim_{n\to\infty} T_n(\lambda^{(n)})=\omega.$$

Then, for every $f\in\Lambda$, homogeneous of degree m,

$$\lim_{n\to\infty}\frac{1}{n^m}f(\lambda^{(n)})=\widetilde{f}(\omega).$$

Proof

It is enough to prove the result for $f=p_m$ since the Newton power sums p_m generate Λ.

For $m=1$,

$$p_1(\lambda^{(n)})=\lambda_1^{(n)}+\cdots+\lambda_n^{(n)},\qquad \widetilde{p}_1(\omega)=\beta.$$

By the assumption, the limit of $\frac{1}{n}p_1(\lambda^{(n)})$ is equal to β so that $\lim_{n\to\infty}\frac{1}{n}p_1(\lambda^{(n)}) = \widetilde{p}_1(\omega)$.

For $m = 2$,

$$p_2(\lambda^{(n)}) = (\lambda_1^{(n)})^2 + \cdots + (\lambda_n^{(n)})^2, \qquad \widetilde{p}_2(\omega) = \gamma + \sum_{k=1}^{\infty} \alpha_k^2.$$

By assumption $(T_n(\lambda^{(n)}) \to \omega$ for the topology of Ω. This means that, for every continuous function φ on $\mathbb{R}$,

$$\lim_{n\to\infty} \sum_{j=1}^{n} \Big(\frac{\lambda_j^{(n)}}{n}\Big)^2 \varphi\Big(\frac{\lambda_j^{(n)}}{n}\Big) = \sum_{k=1}^{\infty} \alpha_k^2 \varphi(\alpha_k) + \gamma\varphi(0).$$

In particular, taking $\varphi \equiv 1$,

$$\lim_{n\to\infty} \sum_{j=1}^{n} \Big(\frac{\lambda_j^{(n)}}{n}\Big)^2 = \sum_{k=1}^{\infty} \alpha_k^2 + \gamma,$$

or

$$\lim_{n\to\infty} \frac{1}{n^2} p_2(\lambda^{(n)}) = \widetilde{p}_2(\omega).$$

For $m \geq 3$ take $\varphi(s) = s^{m-2}$ (note that $\varphi(0) = 0$). This completes the proof of the statement. $\square$

Finally, we state the main theorem of this section about the asymptotics, as the dimension n goes to infinity, of the orbital integrals .

$$\varphi_n(\lambda; x)) = \int_{U(n)} e^{-i\,\mathrm{tr}(xu\lambda u^*)} \alpha_n(du)$$

for a fixed diagonal matrix x in $\mathrm{Herm}(\infty, \mathbb{C})$, i.e., for a point $(x_1, x_2, \dots)$ in $\mathbb{R}^{(\infty)}$.

Theorem 3.14 *As in Theorem 3.13, we consider a sequence $(\lambda^{(n)})$, with $\lambda^{(n)} \in \Omega_n \simeq \mathbb{R}^n$, and assume that, for the topology of Ω,*

$$\lim_{n\to\infty} T_n(\lambda^{(n)}) = \omega.$$

Then, for $x = (x_1, \dots, x_k, 0, 0, \dots)$ in $\mathbb{R}^k \subset \mathbb{R}^{(\infty)}$,

$$\lim_{n\to\infty} \varphi_n(\lambda^{(n)}, x) = \prod_{j=1}^{k} \Phi(\omega, x_j).$$

The convergence is uniform on compact sets of $\mathbb{R}^k$.

Proof

Assume first $k = 1$, i.e., let $x = (z, 0, \dots)$ with $z \in \mathbb{R}$. Then by (3.7), the function $\varphi_n(x, a)$ can be expanded as follows:

$$\varphi_n(\lambda^{(n)}; x) = \sum_{m=0}^{\infty} \frac{(n-1)!}{(m+n-1)!} h_m(\lambda^{(n)})(-iz)^m.$$

For m fixed, since

$$\frac{(n-1)!}{(m+n-1)!} \sim \frac{1}{n^m} \quad (n \to \infty),$$

we get

$$\lim_{n\to\infty} \frac{(n-1)!}{(m+n-1)!} h_m(\lambda^{(n)}) = \widetilde{h}_m(\omega).$$

by Theorem 3.13. Now by applying Proposition 3.11 about the convergence of $\mathcal{C}^\infty$ functions of positive type, we obtain

$$\lim_{n\to\infty} \varphi_n(\lambda^{(n)}; x) = \sum_{m=0}^{\infty} \widetilde{h}_m(\omega)(-iz)^m.$$

Finally, by Theorem 3.13,

$$\sum_{m=0}^{\infty} \widetilde{h}_m(\omega)(-iz)^m = \Phi(\omega, z).$$

Next assume $k \geq 2$. Let $x = (z_1, \dots, z_k, 0, \dots)$. Then

$$\varphi_n(\lambda^{(n)}; x) = \sum_{m_1 \geq \cdots \geq m_n \geq 0} \frac{\delta!}{(\mathbf{m}+\delta)!} s_{\mathbf{m}}(\lambda^{(n)}) s_{\mathbf{m}}(-iz_1, \dots, -iz_k, 0, \dots).$$

For $\mathbf{m}$ fixed,

$$\frac{\delta!}{(\mathbf{m}+\delta)!} \sim \frac{1}{n^{|\mathbf{m}|}} \quad (n \to \infty),$$

with $|\mathbf{m}| = m_1 + m_2 + \cdots$. Hence, by Theorem 3.13,

$$\lim_{n\to\infty} \frac{\delta!}{(\mathbf{m}+\delta)!} s_{\mathbf{m}}(\lambda^{(n)}) = \widetilde{s}_{\mathbf{m}}(\omega).$$

Similarly, by Theorem ??

$$\lim_{n\to\infty} \varphi_n(\lambda^{(n)}; x) = \sum_{\mathbf{m}:\text{partitions}} \widetilde{s}_{\mathbf{m}}(\omega) s_{\mathbf{m}}(-iz_1, \dots, -iz_k, 0, \dots).$$

And finally, by Theorem 3.13,

$$\sum_{\mathbf{m}:\text{partitions}} \widetilde{s}_{\mathbf{m}}(\omega) s_{\mathbf{m}}(-iz_1, \dots, -iz_k, 0, \dots) = \prod_{j=1}^{k} \Phi(\omega, z_j).$$

∎

The converse of Theorem 3.13 has been proven by Olshanski and Vershik [1996]: if, for every $f \in \Lambda$, the sequence

$$\frac{1}{n^m} f(\lambda^{(n)}), \qquad (m \text{ is the degree of } f,$$

has a limit, then the sequence $T_n(\lambda^{(n)})$ converges in Ω.

3.7 Hermitians matrices with entries in $\mathbb{F} = \mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$

More generally let $Herm(n, \mathbb{F})$ denote the space of $n \times n$ Hermitian matrices with entries in $\mathbb{F} = \mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$, the quaternion field. For $\mathbb{F} = \mathbb{R}$, $Herm(n, \mathbb{R}) = Sym(n, \mathbb{R})$, the space of real symmetric matrices. Let also $U(n, \mathbb{F})$ denote the group of unitary matrices with entries in $\mathbb{F}$:

$$U(n, \mathbb{R}) = O(n),\ U(n, \mathbb{C}) = U(n),\ U(n, \mathbb{F}) \simeq Sp(n).$$

We consider in this section the following sequences of Gelfand pairs

$$G(n) = U(n, \mathbb{F}) \ltimes Herm(n, \mathbb{F}),\ K(n) \simeq U(n, \mathbb{F}),$$

and the corresponding Olshanski spherical pairs

$$G = \bigcup_{n=1}^{\infty} G(n) = U(\infty, \mathbb{F}) \ltimes Herm(\infty, \mathbb{F}),$$
$$K = \bigcup_{n=1}^{\infty} K(n) = U(\infty, \mathbb{F}).$$

We will review some results from [Bouali,2007].

The multiplicative property (Theorem 3.5) still holds. Let $d = \dim_{\mathbb{R}} \mathbb{F} = 1, 2,$ or 4.

Theorem 5.1 *The spherical functions of positive type are given by, if* $x = \mathrm{diag}(x_1, \ldots, x_n, 0, \ldots)$, *by*

$$\varphi(x) = \Phi(x_1) \ldots \Phi(x_n),$$

with

$$\Phi(z) = e^{-i\beta z} e^{-\frac{1}{2}\gamma z^2} \prod_{k=1}^{\infty} \frac{e^{i\alpha_k z}}{\left(1 + i\frac{2}{d}\alpha_k z\right)^{\frac{d}{2}}},$$

with $\beta \in \mathbb{R}$, $\gamma \geq 0$, $\alpha = (\alpha_k)$, $\alpha_k \in \mathbb{R}$,

$$\sum_{k=1}^{\infty} \alpha_k^2 < \infty.$$

The spherical dual is hence the same space Ω as in Section 3.

The spherical functions for the Gelfand pair $\big(U(n,\mathbb{F}) \ltimes Herm(n,\mathbb{F}), U(n,\mathbb{F})\big)$ are Fourier transforms of orbital measures:

$$\varphi(\lambda; x) = \int_{U(n,\mathbb{F})} e^{-i\,\mathrm{tr}(xuyu^*)} \alpha_n(du),$$

where λ is a real diagonal matrix: $\lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$. The spherical dual is $\Omega_n \simeq \mathbb{R}^n$ as in the case $\mathbb{F} = \mathbb{C}$. One can consider the same maps $T_n : \Omega_n \to \Omega$ as in Section 3. By using expansions of the orbital integrals in terms of spherical polynomials, one proves:

Theorem 3.15 *Let* $(\lambda^{(n)})$ *be a sequence of diagonal matrices, with* $\lambda^{(n)} \in \Omega_n \simeq \mathbb{R}^n$, *and let* $\omega \in \Omega$. *Assume that, for the topology of* Ω,

$$\lim_{n\to\infty} T_n(\lambda^{(n)}) = \omega.$$

then, for $x \in Herm(\infty, \mathbb{F})$,

$$\lim_{n\to\infty} \varphi_n(\lambda^{(n)}; x) = \det \Phi(\omega; x),$$

uniformly on $Herm(k, \mathbb{F})$.

Chapter 4

Infinite dimensional unitary group

We consider the Olshanski spherical pair $(U(\infty) \times U(\infty), U(\infty))$, where

$$U(\infty) = \bigcup_{n=1}^{\infty} U(n).$$

The story developps similarly to the one in the previous chapter about infinite dimensional Hermitian matrices.

4.1 Gelfand pair associated with a compact group

Consider a compact group U. Let $G = U \times U$ and $K = \{(u, u) \in G \mid u \in U\}$.
If a function f on G is right K-invariant, then

$$f(uu_0, vu_0) = f(u, v), \qquad u_0, u, v \in U.$$

In particular, taking $u_0 = v^{-1}$, we obtain $f(uv^{-1}, e) = f(u, v)$. Therefore we obtain the identification

$$\mathcal{C}(G/K) \simeq \mathcal{C}(U);\ f \longleftrightarrow F; F(u) = f(u, e), \tag{4.1}$$

where $\mathcal{C}(G/K)$ is the space of right K-invariant continuous functions on G and $\mathcal{C}(U)$ is the space of continuous functions on U. Furthermore, if f is left K-invariant, then

$$f(u_0u, u_0v) = f(u, v), \qquad u_0, u, v \in U,$$

so that, under the identification (4.1),

$$F(u_0uu_0^{-1}) = F(u), \qquad u_0, u \in U,$$

i.e. the function F is central. Therefore we obtain the following identification

$$\mathcal{C}(K\backslash G/K) \simeq \mathcal{C}(U)_{\text{central}};\ f \longleftrightarrow F; F(u) = f(u,e), \tag{4.2}$$

where $\mathcal{C}(K\backslash G/K)$ is the space of K-biinvariant continuous functions on G and $\mathcal{C}(U)_{\text{central}}$ is the space of central continuous functions on U. In the same way $L^1(K\backslash G/K) \simeq L^1(U)_{\text{central}}$ as convolution algebras. Hence $L^1(K\backslash G/K)$ is a commutative convolution algebra and so (G,K) is a Gelfand pair.

Let φ be a spherical function on G. Then the function φ satisfies the equation

$$\int_U \varphi(x_1uy_1, x_2uy_2)du = \varphi(x_1,x_2)\varphi(y_1,y_2), \qquad x_1,x_2,y_1,y_2 \in U,$$

where du denote the normalized Haar measure on U. Taking $x_1 = x$, $y_1 = y$, and $x_2 = y_2 = e$, we get

$$\int_U \varphi(xuy, u)du = \varphi(x,e)\varphi(y,e), \qquad x,y \in U.$$

For $\widetilde{\varphi}(x) = \varphi(x,e)$, we obtain the following functional equation

$$\int_U \widetilde{\varphi}(xuyu^{-1})du = \widetilde{\varphi}(x)\widetilde{\varphi}(y), \qquad x,y \in U. \tag{4.3}$$

Conversely, if a function $\widetilde{\varphi} \in L^1(U)_{\text{central}}$ satisfies the equation (4.3), then the function φ defined by $\varphi(u,v) = \widetilde{\varphi}(uv^{-1})$ is a spherical function for the Gelfand pair (G,K). Hence we make the following definition:

A continuous central function φ in U is said to be *spherical* if it satisfies the functional equation

$$\int_U \varphi(xuyu^{-1})du = \varphi(x)\varphi(y), \qquad x,y \in U.$$

Let $\widehat{U}$ be the set of equivalence classes of irreducible representations of U. For each $\lambda \in \widehat{U}$, denote by π_λ a representation of U in the class λ on a vector space $\mathcal{H}_\lambda$. Denote its character by χ_λ:

$$\chi_\lambda(x) = \operatorname{tr}(\pi_\lambda(x)),\ x \in U.$$

Then $d_\lambda = \chi_\lambda(e)$ is the dimension of $\mathcal{H}_\lambda$.

Proposition 4.1 *The characters satisfies the following functional equation:*

$$\int_U \chi_\lambda(xuyu^{-1})du = \frac{1}{d_\lambda}\chi_\lambda(x)\chi_\lambda(y), \qquad x, y \in U. \tag{4.4}$$

Proof

Let $\mathcal{M}_\lambda$ denote the subspace of $\mathcal{C}(U)$ generated by the matrix coefficients of the representation π_λ. Observe that the character χ_λ is a central function which belongs to $\mathcal{M}_\lambda$. As representations spaces for $U \times U$, the space $\mathcal{M}_\lambda$ is isomorphic to the space $\mathcal{L}(\mathcal{H}_\lambda)$ of endomorphisms of $\mathcal{H}_\lambda$. By Schur's Lemma, It follows that the only central functions in $\mathcal{M}_\lambda$ are the functions proportionnal to χ_λ. For x fixed, consider the function ψ defined on U by

$$\psi(y) = \int_U \chi_\lambda(xuyu^{-1})du.$$

The function ψ is central and belongs to $\mathcal{M}_\lambda$. It follows that

$$\psi(y) = C\chi_\lambda(y).$$

Evaluating both sides at e we obtain

$$Cd_\lambda = \chi_\lambda(x).$$

Therefore

$$\int_U \chi_\lambda(xuyu^{-1})du = \frac{1}{d_\lambda}\chi_\lambda(x)\chi_\lambda(y).$$

∎

Hence

$$\varphi(\lambda; u) = \frac{\chi_\lambda(u)}{\chi_\lambda(e)}$$

is a spherical function. One shows that all spherical functions are obtained in that way. Therefore the spherical dual of the Gelfand pair (G, K) is identified with $\widehat{U}$.

In this case the Bochner-Godement theorem (Theorem 1.2) says: the central continuous functions φ of positive type on U are given by

$$\varphi(u) = \sum_{\lambda \in \widehat{U}} a_\lambda \varphi(\lambda; u),$$

with

$$a_\lambda \geq 0, \quad \sum_{\lambda \in \widehat{U}} a_\lambda < \infty.$$

4.2 Unitary groups

We consider the case where U is the unitary group $U(n)$. Let

$$\mathbb{T} = U(1) = \{t \in \mathbb{C} \mid |t| = 1\}.$$

We identify $\mathbb{T}^n$ with the subgroup of $U(n)$ which consists of diagonal matrices $t = \mathrm{diag}(t_1, \ldots, t_n)$. For a diagonal matrix $t = \mathrm{diag}(t_1, \ldots, t_n)$, denote by $V(t)$ the Vandermonde polynomial $V(t) = \prod_{1 \le j < k \le n}(t_j - t_k)$.

First we recall Weyl's integral formula. For any central and integrable function f on $U(n)$,

$$\int_{U(n)} f(x)\alpha_n(dx) = \frac{1}{n!}\int_{\mathbb{T}^n} f(t)|V(t)|^2\beta(dt), \tag{4.5}$$

where α_n is the normalized Haar measure on $U(n)$, and β is the normalized Haar measure on $\mathbb{T}^n$, i.e.,

$$\beta(dt) = \prod_{j=1}^{n} \frac{d\theta_j}{2\pi}, \qquad t_j = e^{i\theta_j},\ 0 \le \theta_j \le 2\pi.$$

Second we recall Weyl's character formula and dimension formula. The set $\widehat{U} = \widehat{U}(n)$ is parameterized by signatures

$$\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n, \qquad \lambda_1 \ge \cdots \ge \lambda_n.$$

The corresponding character χ_λ agrees with the Schur function on $\mathbb{T}^n$:

$$\chi_\lambda(\mathrm{diag}(t_1, \ldots, t_n)) = s_\lambda(t) = \frac{\det(t_i^{\lambda_j+n-j})_{1 \le i,j \le n}}{V(t)}.$$

The dimension of the representation associated to λ is given by

$$d_\lambda = s_\lambda(1, \ldots, 1) = \frac{V(\lambda+\delta)}{V(\delta)},$$

where $\delta = (n-1, \ldots, 1, 0)$. We proved this equality in Proposition 2.2.

The spherical dual Ω_n of the Gelfand pair $\big(U(n)\times U(n), U(n)\big)$ is identified to the set of signatures $\lambda = (\lambda_1, \ldots, \lambda_n)$. We will denote by $\varphi_n(\lambda; u)$ the corresponding spherical function:

$$\varphi_n(\lambda; u) = \frac{\chi_\lambda(u)}{\chi_\lambda(e)}.$$

Its restriction to $\mathbb{T}^n$ is given by

$$\varphi_n\big(\lambda; \mathrm{diag}(t_1, \ldots, t_n)\big) = \frac{s_\lambda(t_1, \ldots, t_n)}{s_\lambda(1, \ldots, 1)}.$$

4.3 Voiculescu functions

We consider now the increasing sequence of Gelfand pairs

$$G(n) = U(n) \times U(n), \quad K(n) = \{(u,u) \mid u \in U(n)\} \simeq U(n),$$

and the inductive limit, the Olshanski spherical pair (G,K):

$$\begin{aligned} G &= \bigcup_{n=1}^{\infty} G(n) = U(\infty) \times U(\infty), \\ K &= \bigcup_{n=1}^{\infty} \{(u,u) \mid u \in U(\infty)\} \simeq U(\infty). \end{aligned} \tag{4.6}$$

A spherical function φ for the pair (G,K) can be seen as a continuous central function φ on $U(\infty)$ such that, for $x, y \in U(\infty)$,

$$\lim_{n\to\infty} \int_{U(n)} \varphi(xuyu^*)\alpha_n(du) = \varphi(x)\varphi(y),$$

where α_n is the normalized Haar mesure on $U(n)$. As in the case of the Olshanski spherical pair we considered in Chapter 3, a spherical function φ is multiplicative in the following sense:

Theorem 4.2 *Let φ be a central bounded continuous function on $U(\infty)$. The function φ is spherical if and only if there exists a continuous function Φ on $\mathbb{T}$ with $\Phi(1) = 1$ such that*

$$\varphi(u) = \det \Phi(u).$$

This means that

$$\varphi\big(\mathrm{diag}(t_1, \ldots, t_n)\big) = \Phi(t_1) \ldots \Phi(t_n).$$

The proof is similar to the one of Theorem 3.5. We will now describe the functions Φ occuring in Theorem 4.2 when the spherical function φ is of positive type. Let us first state an important result by Voiculescu (1976). Let Φ be the following power expansion

$$\Phi(t) = \sum_{m=0}^{\infty} c_m t^m,$$

with

$$c_m \geq 0, \ \sum_{m=0}^{\infty} c_m = 1.$$

The series converges for $|z| \leq 1$, and Φ is a continuous function of positive type on $U(1) = \{t \in \mathbb{C} \mid |t| = 1\}$ and $\Phi(1) = 1$.

We propose to say that the function Φ is a *Voiculescu function* if the function φ defined on $U(\infty)$ by $\varphi(u) = \det \Phi(u)$ is of positive type. Observe that, for $u = \mathrm{diag}(t_1, \ldots, t_n, 1, 1, \ldots)$,

$$\varphi(u) = \Phi(t_1) \ldots \Phi(t_n).$$

Theorem 4.3 *The Voiculescu functions are the following ones:*

$$\Phi(z) = e^{\gamma(z-1)} \prod_{k=1}^{\infty} \frac{1 + \beta_k (z-1)}{1 - \alpha_k (z-1)},$$

with

$$\alpha_k \geq 0,\ 0 \leq \beta_k \leq 1,\ \gamma \geq 0,\ \sum_{k=1}^{\infty} (\alpha_k + \beta_k) < \infty.$$

([Voiculescu,1976], Proposition 1.)

Let us prove the easy part of the theorem: the function Φ given by this formula is a Voiculescu function.

- Consider the function

$$F(t) = \frac{1}{1 - at},$$

with $0 \leq a < 1$. Then

$$\prod_{j=1}^{n} F(t_j) = \prod_{j=1}^{n} \frac{1}{1 - at_j} = \sum_{m=0}^{\infty} h_m(t) a^m,$$

and, for $u \in U(n)$,

$$\det F(u) = \det(I - au)^{-1} = \sum_{m=0}^{\infty} a^m \chi_{(m)}(u).$$

Therefore, $f(u) = \det F(u)$ is of positive type. For $\alpha \geq 0$ the function

$$\Phi(t) = \frac{1}{1 - \alpha(1-t)}$$

is a Voiculescu function. In fact

$$\Phi(t) = \frac{1}{\alpha+1}\frac{1}{1-\frac{\alpha}{1+\alpha}t}, \text{ and } 0 \leq \frac{\alpha}{\alpha+1} < 1.$$

- Put now

$$F(t) = 1 + bt.$$

Then, for $t = (t_1, \ldots, t_n) \in \mathbb{T}^n$,

$$\prod_{j=1}^{n} F(t_j) = \prod_{j=1}^{n}(1+bt_j) = \sum_{k=0}^{n} e_k(t)b^k,$$

and, for $u \in U(n)$,

$$f(u) = \det F(u) = \det(I + bu) = \sum_{k=0}^{n} b^k \chi_{(1^k)}(u)$$

is of positive type for $b \geq 0$. For $0 \leq \beta \leq 1$, the function

$$\Phi(t) = 1 + \beta(t-1) = (1-\beta)\Big(1 + \frac{\beta}{1-\beta}t\Big)$$

is a Voiculescu function.

- The product of two functions of positive type is of positive type, and a limit of functions of positive type is of positive type as well. For $\gamma \geq 0$, put

$$F(t) = e^{\gamma t} = \lim_{k\to\infty}\Big(1 + \frac{\gamma}{k}t\Big)^k.$$

Then, for $u \in U(n)$,

$$\det F(u) = e^{\gamma \operatorname{tr} u} = \lim_{k\to\infty} \det\Big(I + \frac{\gamma}{k}u\Big)^k$$

is of positive type. Therefore, for $\gamma \geq 0$, the function

$$\Phi(z) = e^{\gamma(t-1)}$$

is a Voiculescu function.

It can be shown that

$$e^{\gamma \operatorname{tr} u} = \sum_{m_1 \geq \cdots \geq m_n \geq 0} \frac{\gamma^{|\mathbf{m}|}}{(n)_{\mathbf{m}}} \chi_{\mathbf{m}}(u),$$

with the notation

$$(\alpha)_{\mathbf{m}} = \prod_{i=1}^{n} (\alpha - i + 1)_{m_i}.$$

Finally, by these three examples, the function Φ given by the formula in Theorem 4.3 is a Voiculescu function.

Let Ω_0 denote the set of parameters $\omega = (\alpha, \beta, \gamma)$ with $\alpha = (\alpha_k)$, $\alpha_k \geq 0$, $\beta = (\beta_k)$, $0 \leq \beta_k \leq 1$, $\gamma \geq 0$, and

$$\sum_{k=1}^{\infty} \alpha_k < \infty, \ \sum_{k=1}^{\infty} \beta_k < \infty.$$

We will write

$$\Phi(t) = \Phi(\omega; t) = e^{\gamma(t-1)} \prod_{k=1}^{\infty} \frac{1 + \beta_k(t-1)}{1 - \alpha_k(t-1)}.$$

For a continuous function f on $\mathbb{R}$ we define the function L_f on Ω_0 by

$$L_f(\omega) = \gamma f(0) + \sum_{k=1}^{\infty} \alpha_k f(\alpha_k) + \sum_{k=1}^{\infty} \beta_k f(-\alpha_k),$$

and we consider on the set Ω_0 the initial topology associated to the functions L_f, for $f \in \mathcal{C}(\mathbb{R})$. For t fixed, the function $\omega \mapsto \Phi(\omega, t)$ is continuous on Ω_0. This can be seen by looking at the logarithmic derivative:

$$\frac{d}{dz} \log \Phi(\omega; 1+z) = p_1(\alpha) + p_1(\beta) + \gamma + \sum_{m=1}^{\infty} \big(p_{m+1}(\alpha) + (-1)^m p_{m+1}(\beta)\big) z^m,$$

where $p_m(\alpha)$ is the Newton power sum:

$$p_m(\alpha) = \sum_{k=1}^{\infty} \alpha_k^m.$$

Theorem 4.4 *The spherical functions of positive type for the Olshanski spherical pair*

$$G = U(\infty) \times U(\infty), \quad K = U(\infty)$$

are given by, for $u \in U(\infty)$,

$$\varphi(u) = \det \Phi(\omega^+; u) \det \Phi(\omega^-, u^{-1}),$$

where $\omega^+, \omega^- \in \Omega_0$.

This can be written more explicitely, for $u = \mathrm{diag}(t_1, \ldots, t_n, 1, 1, \ldots)$, $t_j \in \mathbb{T}$,

$$\begin{aligned}
&\varphi\big(\mathrm{diag}(t_1, \ldots, t_n, 1, 1, \ldots)\big) \\
&= \Phi(\omega^+, t_1) \ldots \Phi(\omega^+; t_n) \Phi\big(\omega^-; \frac{1}{t_1}\big) \ldots \Phi\big(\omega^-; \frac{1}{t_n}\big) \\
&= \prod_{j=1}^n \Big(e^{\gamma^+(t_j - 1)} \prod_{k=1}^\infty \frac{1 + \beta_k^-(t_j - 1)}{1 - \alpha_k^+(t_j - 1)} \Big) \Big(\prod_{j=1}^n e^{-\gamma^- \left(\frac{1}{t_j} - 1\right)} \prod_{k=1}^\infty \frac{1 + \beta_k^- \left(\frac{1}{t_j} - 1\right)}{1 - \alpha_k^- \left(\frac{1}{t_j} - 1\right)} \Big),
\end{aligned}$$

with $\omega^+ = (\alpha^+, \beta^+, \gamma^+)$, $\omega^- = (\alpha^-, \beta^-, \gamma^-)$.

([Vershik-Kerov, 1981], [Boyer, 1983])

We will write

$$\varphi(u) = \varphi(\omega^+, \omega^-; u).$$

Hence the spherical dual of the Olshanski spherical pair $\big(U(\infty \times U(\infty), U(\infty)\big)$ can be identified to $\Omega = \Omega_0 \times \Omega_0$.

4.4 Asymptotics of spherical functions of the unitary group

Recall that, for a signature $\lambda = (\lambda_1, \ldots, \lambda_n)$, $\lambda_i \in \mathbb{Z}, \lambda_1 \geq \cdots \geq \lambda_n$,

$$\varphi_n(\lambda; u) = \frac{\chi_\lambda(u)}{\chi_\lambda(e)}, \qquad u \in U(n),$$

where χ_λ is the character of an irreducible representation of $U(n)$ in the class associated to λ. Recall also that

$$\varphi_\lambda(\mathrm{diag}(t_1, \ldots, t_n)) = \frac{s_\lambda(t_1, \ldots, t_n)}{s_\lambda(1, \ldots, 1)}.$$

We will consider the asymptotics of $\varphi_n(\lambda; u)$ as n goes to infinity. This will be parallel to the section 3.5 about the asymptotics of orbital integrals. In a first step we will consider the case where λ is a partition: $\lambda = (\lambda_1, \ldots, \lambda_n)$, $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$.

In order to study the Taylor expansion of the spherical functions, Okunkov and Olshanski introduced an algebra morphism $\Lambda \to \mathcal{C}(\Omega_0)$ which maps a symmetric function to a continuous function $\tilde{f}$ on Ω_0. Since the algebra

Λ is generated by the power Newton sums p_m, this morphism is uniquely determined by their images $\tilde{p}_m$. One puts, for $\omega = (\alpha, \beta, \gamma) \in \Omega_0$,

$$\tilde{p}_1(\omega) = \sum_{k=1}^{\infty} \alpha_k + \sum_{k=1}^{\infty} \beta_k + \gamma,$$
$$\tilde{p}_m(\omega) = \sum_{k=1}^{\infty} \alpha_k^m + (-1)^{m-1} \sum_{k=1}^{\infty} \beta_k^m \ (m \geq 2).$$

The functions $\tilde{p}_m$ are continuous. In fact

$$\tilde{p}_m(\omega) = L_f(\omega),$$

with $f(s) = s^{m-1}$ $(m \geq 1)$.

Proposition 4.5 (i) *For* $z \in \mathbb{C}$, $|z| < r = (1/\sup \alpha_k)$,

$$\Phi(\omega; 1+z) = \sum_{m=0}^{\infty} \tilde{h}_m(\omega) z^m.$$

(ii) *For* $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$, $|z_j| < r$,

$$\prod_{j=1}^{n} \Phi(\omega, 1+z_j) = \sum_{m_1 \geq \cdots \geq m_n \geq 0} \tilde{s}_{\mathbf{m}}(\omega) s_{\mathbf{m}}(z).$$

Proof

Recall the generating function for the complete symmetric functions h_m: for $x = (x_1, \ldots, x_n)$,

$$H(x; z) = \sum_{m=0}^{\infty} h_m(z) z^m = \prod_{j=1}^{n} \frac{1}{1 - x_j z}.$$

Statement (i) can be written

$$\Phi(\omega; 1+z) = \tilde{H}(\omega; z),$$

where z is considdered as a parameter. Since

$$\Phi(\omega; 1) = 1, \ H(x, 0) = 1,$$

it amounts to showing

$$\frac{d}{dz} \log \Phi(\omega; 1+z) = \frac{d}{dz} \log \tilde{H}(\omega, z),$$

and this holds. In fact

$$\frac{d}{dz}\log H(x;z)=\sum_{m=0}^{\infty}p_{m+1}(x)z^m,$$

therefore

$$\frac{d}{dz}\log \tilde{H}(\omega;z)=\sum_{m=0}^{\infty}\tilde{p}_{m+1}(\omega)z^m=\frac{d}{dz}\log\Phi(\omega;1+z).$$

Statement (ii) follows from (i) by applying with respect to x the morphism $f\mapsto\tilde{f}$ to both sides of the Cauchy identity:

$$\prod_{j=1}^{n}H(x;z_j)=\prod_{i,j=1}^{n}\frac{1}{1-x_iz_j}=\sum_{m_1\geq\cdots\geq m_n\geq 0}s_{\mathbf{m}}(x)s_{\mathbf{m}}(z).$$

Let us also observe that, by applying to both sides of the Jacobi-Trudi identy (Proposition 2.7):

$$s_{\mathbf{m}}(x)=\det\big(h_{m_i-i+j}(x)\big)_{1\leq i,j\leq n},$$

one gets

$$\tilde{s}_{(m)}(\omega)=\det\big(\tilde{h}_{m_i-i+j}(\omega)\big)_{1\leq i,j\leq n}.$$

Hence, by Voiculescu' formula (Proposition 2.6), one obtains (ii).

Let $\lambda=(\lambda_1,\lambda_2,\ldots)$ be a partition: the λ_i are in $\mathbb{N}$, and $\lambda 1\geq\lambda_2\geq\ldots\geq 0$. The number λ_i is the number of boxes in the i-th row of the Young diagram of λ. The *conjugate partition* $\lambda'=(\lambda'_1,\lambda'_2,\ldots)$ is associated to the transpose diagram. For instance, if $\lambda=(6,4,4,2,1)$, then $\lambda'=(5,4,3,3,1,1)$. The *Frobenius parameters* $a=(a_1,a_2,\ldots)$ and $b=(b_1,b_2,\ldots)$ of the partition λ are defined by

$$a_i=\lambda_i-i \text{ if } \lambda_i>i,\ a_i=0 \text{ otherwise},$$
$$b_j=\lambda'_j-j+1 \text{ if } \lambda'_j>j-1,\ b_j=0 \text{ otherwise}.$$

For instance, if $\lambda=(6,4,4,2,1)$, then

$$a=(5,2,1,0,\ldots),\ b=(5,3,1,0,\ldots).$$

Observe that

$$\sum\lambda_i=\sum a_i+\sum b_j.$$

Let Ω_n^+ denote the set of partitions of length $\leq n$. Recall that the *length* of a partition $\lambda = (\lambda_1, \lambda_2, \ldots)$ is the largest i with $\lambda_i > 0$. One defines the map

$$T_n : \Omega_n^+ \to \Omega_0$$

as follows. If $\lambda = (\lambda_1, \ldots, \lambda_n) \in \Omega_n^+$ is a partition of lenth $\leq n$, with Frobenius parameters $a = (a_1, a_2, \ldots)$, $b = (b_1, b_2, \ldots)$, then $\omega = T_n(\lambda)$ is the triple $\omega = (\alpha, \beta, \gamma)$ with

$$\alpha_k = \frac{a_k}{n},\ \beta_k = \frac{b_k}{n},\ \gamma = 0.$$

Observe that $0 \leq \beta_k \leq 1$, since $b_k \leq n$.

Recall that Λ^* denote the algebra of shifted symmetric functions.

Theorem 4.6 *Consider a sequence $\lambda^{(n)}$ of partitions with $\lambda^{(n)} \in \Omega_n^+$, and let $\omega \in \Omega_0$. Assume that, for the topology of Ω_0,*

$$\lim_{n\to\infty} T_n(\lambda^{(n)}) = \omega.$$

Then, for every shifted symmetric function $f^* \in \Lambda^*$,

$$\lim_{n\to\infty} \frac{1}{n^m} f^*(\lambda^{(n)}) = \tilde{f}(\omega),$$

where m is the degree of f^*, and $f \in \Lambda$ is the homogeneous part of degree m of f^*.

Proof

We will prove the statement in the special case $f^* = q_m^*$:,

$$q_m^*(\lambda) = \sum_{i\geq 1}([\lambda_i - i + 1]_m - [-i+1]_m).$$

Recall that these functions appeared at the end of Section 2.4. The function q_m^* is of degree m, and the homogeneous part of degree m is equal to the Newton power sum $p_m(\lambda)$. We will show:

$$\lim_{n\to\infty} \frac{1}{n^m} q_m^*(\lambda^{(n)}) = \widetilde{p}_m(\omega). \tag{4.7}$$

Since the functions $q_m^*(\lambda)$ generate Λ^*, the statement of the theorem will be proven.

Lemma 4.7

$$q_m^*(\lambda) = m \sum_{(i,j)\in\lambda} [j-i]_{m-1}.$$

Here (i,j) runs over all squares (i,j) in the Young diagram of λ.

Proof

By the relation $[u+1]_m - [u]_m = m[u]_{m-1}$, one obtains

$$\begin{aligned} m\sum_{(i,j)\in\lambda}[j-i]_{m-1} =& m\sum_{i\geq 1}\sum_{j=1}^{\lambda_i}[j-i]_{m-1} \\ =& \sum_{i\geq 1}\sum_{j=1}^{\lambda_i}([j-i+1]_m - [j-i]_m) \\ =& \sum_{i\geq 1}([\lambda_i - i+1]_m - [-i+1]_m). \end{aligned}$$

∎

Lemma 4.8 *Let $a=(a_i)$ and $b=(b_j)$ be the Frobenius parameters of λ. Then*

$$q^*_m(\lambda) = \sum_{i\geq 1}[a_i+1]_m - \sum_{j\geq 1}[1-b_j]_m.$$

Proof

We first observe that

$$q^*_1(\lambda) = \sum_{i\geq 1}\lambda_i = \sum_{i\geq 1}a_i + \sum_{j\geq 1}b_j.$$

Let $m\geq 2$. One decomposes $q^*_m(\lambda)$ as

$$q^*_m(\lambda) = m\sum_{\substack{(i,j)\in\lambda \\ j>i}}[j-i]_{m-1} + m\sum_{\substack{(i,j)\in\lambda \\ i\geq j}}[j-i]_{m-1}.$$

We see that

$$\begin{aligned} & m\sum_{\substack{(i,j)\in\lambda \\ j>i}}[j-i]_{m-1} = m\sum_{i\geq 1}\sum_{j=i+1}^{\lambda_i}[j-i]_{m-1} = m\sum_{i\geq 1}\sum_{j=1}^{a_i}[j]_{m-1} \\ &= \sum_{i\geq 1}\sum_{j=1}^{a_i}([j+1]_m - [j]_m) = \sum_{i\geq 1}([a_i+1]_m - [1]_m) \\ &= \sum_{i\geq 1}[a_i+1]_m, \qquad \text{since } [1]_m = 0 \text{ for } m\geq 2, \end{aligned}$$

and

$$m\sum_{\substack{(i,j)\in\lambda\\ i\geq j}}[j-i]_{m-1}=m\sum_{j\geq 1}\sum_{i=j}^{\lambda'_j}[j-i]_{m-1}=m\sum_{j\geq 1}\sum_{i=0}^{b_j-1}[-i]_{m-1}$$

$$=\sum_{j\geq 1}\sum_{i=0}^{b_j-1}([1-i]_m-[-i]_m)=-\sum_{j\geq 1}([1-b_j]_m-[1]_m)=-\sum_{j\geq 1}[1-b_j]_m.$$

∎

We complete now the proof of Theorem 4.6.

Proof

Let $a^{(n)}=(a_i^{(n)})$ and $b^{(n)}=(b_j^{(n)})$ be the Frobenius parameters of the partition $\lambda^{(n)}$, and $\omega=(\alpha,\beta,\gamma)\in\Omega_0$ with $\alpha=(\alpha_k)$, $\beta=(\beta_k)$. Consider the measures σ_n and σ on $\mathbb{R}$ given, for a continuous function φ on $\mathbb{R}$, by

$$\int_{\mathbb{R}}\varphi(s)\sigma_n(ds)=\sum_{i\geq 1}\frac{a_i^{(n)}}{n}\varphi\left(\frac{a_i^{(n)}}{n}\right)+\sum_{j\geq 1}\frac{b_j^{(n)}}{n}\varphi\left(-\frac{b_j^{(n)}}{n}\right),$$

$$\int_{\mathbb{R}}\varphi(s)\sigma(ds)=\sum_{k\geq 1}\alpha_k\varphi(\alpha_k)+\sum_{k\geq 1}\beta_k\varphi(-\beta_k)+\gamma\varphi(0).$$

By assumption, for a continuous function φ on $\mathbb{R}$,

$$\lim_{n\to\infty}\int_{\mathbb{R}}\varphi(s)\sigma_n(ds)=\int_{\mathbb{R}}\varphi(s)\sigma(ds). \tag{4.8}$$

Observe that the measures σ_n and σ are positive, and

$$\int_{\mathbb{R}}\sigma_n(ds)=q_1^*(\lambda^{(n)}),\qquad \int_{\mathbb{R}}\sigma(ds)=\widetilde{p}_1(\omega).$$

There exists $A>0$ such that

$$\mathrm{supp}(\sigma_n)\subset[-A,A],\qquad \mathrm{supp}(\sigma)\subset[-A,A].$$

By taking

$$\varphi_n(s)=\frac{1}{n^m s}[ns+1]_m=\frac{1}{n^{m-1}}(ns+1)(ns-1)(ns-2)\cdots(ns-m+2),$$

we obtain

$$\frac{1}{n^m}q_m^*(\lambda^{(n)})=\int_{\mathbb{R}}\varphi_n(s)\sigma_n(ds)$$

Note that

$$\lim_{n\to\infty} \varphi_n(s) = \varphi(s) := s^{m-1}, \tag{4.9}$$

and the convergence is uniform on compact sets.

We can write

$$\begin{aligned}
&\frac{1}{n^m} q_m^*(\lambda^{(n)}) - \widetilde{p}_m(\omega) \\
=&\int_{\mathbb{R}} \varphi_n(s)\sigma_n(ds) - \int_{\mathbb{R}} \varphi(s)\sigma_n(ds) + \int_{\mathbb{R}} \varphi(s)\sigma_n(ds) - \int_{\mathbb{R}} \varphi(s)\sigma(ds).
\end{aligned}$$

Let $\varepsilon > 0$. There is a positive integer n_1 such that, if $n \geq n_1$,

$$\left| \int_{\mathbb{R}} \varphi(s)\sigma_n(ds) - \int_{\mathbb{R}} \varphi(s)\sigma(ds) \right| < \varepsilon \qquad \text{on } [-A, A],$$

and a positive integer n_2 such that, if $n \geq n_2$,

$$|\varphi_n(s) - \varphi(s)| < \varepsilon \qquad \text{on } [-A, A].$$

Furthermore, there exists a constant $c > 0$ such that $\int_{\mathbb{R}} \sigma_n(ds) < c$. Therefore, for $n \geq \max\{n_1, n_2\}$,

$$\begin{aligned}
&\left| \frac{1}{n^m} q_m^*(\lambda^{(n)}) - \widetilde{p}_m(\omega) \right| \\
\leq& \int_{\mathbb{R}} |\varphi_n(s) - \varphi(s)|\sigma_n(ds) + \left| \int_{\mathbb{R}} \varphi(s)\sigma_n(ds) - \int_{\mathbb{R}} \varphi(s)\sigma(ds) \right| \\
\leq& (c+1)\varepsilon.
\end{aligned}$$

Hence we have proven (4.8). ∎

The converse of Theorem **??** has been proven by Okunkov and Olshanski [OKAL-1998c] (Kerov and Vershik [VEKE-1982]): if, for every $f^* \in \Lambda^*$, the sequence

$$\frac{1}{n^m} f^*(\lambda^{(n)}),$$

has a limit, where m is the degree of f^*, then the sequence $T_n(\lambda^{(n)})$ converges in Ω.

Theorem 4.9 *As in Theorem 4.6, consider a sequence $(\lambda^{(n)})$ of partitions, with $\lambda^{(n)} \in \Omega_n^+$, and $\omega \in \Omega_0$. Assume that, for the topology of Ω_0,*

$$\lim_{n\to\infty} T_n(\lambda^{(n)}) = \omega.$$

Then, for $u \in U(\infty)$,

$$\lim_{n\to\infty} \varphi_n(\lambda^{(n)}; u) = \det \Phi(\omega, u),$$

uniformly on each $U(k)$. More explicitely,

$$\lim_{n\to\infty} \varphi_n\big(\lambda^{(n)}; \operatorname{diag}(t_1, \ldots, t_k, 1, \ldots, 1)\big) = \prod_{j=1}^{k} \Phi(\omega, t_j).$$

Proof

By Theorem 2.8,

$$\varphi_n\big(\lambda^{(n)}; \operatorname{diag}(1+z_1, \ldots, 1+z_k, 1, \ldots, 1) = \frac{s_{\lambda^{(n)}}(1+z_1, \ldots, 1+z_k, 1, \ldots, 1)}{s_{\lambda^{(n)}}(1, \ldots, 1)}$$

$$= \sum_{m_1 \geq \cdots \geq m_n \geq 0} \frac{\delta!}{(\mathbf{m}+\delta)!} s^*_{\mathbf{m}}(\lambda^{(n)}) s_{\mathbf{m}}(z_1, \ldots, z_k, 1, \ldots, 1).$$

For $\mathbf{m}$ fixed,

$$\frac{\delta!}{\mathbf{m}+\delta)!} \sim \frac{1}{n^{|\mathbf{m}|}} \quad (n \to \infty),$$

with $|\mathbf{m}| = m_1 + \cdots m_n$. Hence, by Theorem 4.6,

$$\lim_{n\to\infty} \frac{\delta!}{(\mathbf{m}+\delta)!} s^*_{\mathbf{m}}(\lambda^{(n)}) = \tilde{s}_{\mathbf{m}}(\omega).$$

On the other hand, by Proposition 4.5,

$$\sum_{m_1 \geq \cdots \geq m_n \geq 0} \tilde{s}_{\mathbf{m}}(\omega) s_{\mathbf{m}}(z) = \prod_{j=1}^{k} \Phi(\omega; 1+z_j).$$

Proposition 3.11 applies for a sequence of functions of positive type on $\mathbb{T}^k$. It follows that

$$\lim_{n\to\infty} \varphi_n\big(\lambda^{(n)}; \operatorname{diag}(1+z_1, \ldots, 1+z_k, 1, \ldots, 1)\big) = \prod_{j=1}^{k} \Phi(\omega; 1+z_j),$$

and, for $u \in U(\infty)$,

$$\lim_{n\to\infty} \varphi_n(\lambda^{(n)}; u) = \det \Phi(\omega; u),$$

uniformly on $U(k)$. $\square$

We consider now the case of a general signature. To a signature λ one associates two partitions λ^+ and λ^-: if

$$\lambda \geq \cdots \geq \lambda_p \geq 0 \geq \lambda_{p+1} \geq \cdots \geq \lambda_n,$$

then

$$\lambda^+ = (\lambda_1, \ldots, \lambda_p),\ \lambda^- = (-\lambda_n, \ldots, -\lambda_{p+1}).$$

We define the map $T_n : \Omega_n \to \Omega = \Omega_0 \times \Omega_0$, in extending the map previously defined, by putting

$$T_n(\lambda) = \big(T_n(\lambda^+), T_n(\lambda^-)\big).$$

One extends also the map $f \mapsto \tilde{f}$ as an algebra morphism $\Lambda \to \mathcal{C}(\Omega)$. This map is such that

$$H(x;z) = \sum_{m=0}^{\infty} h_m(x) z^m = \prod_{j=1}^{\infty} \frac{1}{1 - x_j z}$$

maps to

$$\Phi(\omega^+; 1+z)\Phi\big(\omega^-; \frac{1}{1+z}\big).$$

In other words the images $\tilde{h}_m(\omega^+, \omega^-)$ of the complete symmetric functions $h_m(x)$ are the coefficients of the following power series:

$$\Phi(\omega^+, 1+z)\Phi\big(\omega^-; \frac{1}{1+z}\big) = \sum_{m=0}^{\infty} \tilde{h}_m(\omega^+, \omega^-) z^m.$$

It follows that

$$\prod_{j=1}^{n} \Phi(\omega^+; 1+z_j)\Phi\big(\omega^-; \frac{1}{1+z}\big) = \sum_{m_1 \geq \ldots \geq m_n \geq 0} \tilde{s}_{\mathbf{m}}(\omega^+, \omega^-) s_{\mathbf{m}}(z_1, \ldots, z_n).$$

We extend now Theorem 1.6:

Theorem 4.10 *Consider a sequence $\lambda^{(n)}$ of signatures, with $\lambda^{(n)} \in \Omega_n$, and let $\omega = (\omega^+, \omega^-) \in \Omega$. Assume that*

$$\lim_{n \to \infty} T_n(\lambda^{(n)}) = \omega = (\omega^+, \omega^-).$$

Then, for every shifted symmetric function $f^* \in \Lambda^*$,

$$\lim_{n \to \infty} \frac{1}{n^m} f^*(\lambda^{(n)}) = \tilde{f}(\omega),$$

where m is the degree of f^*, and $f \in \Lambda$ is the homogeneous part of degree m in f^*.

By theorem 2.10, the generating function $H^*(\lambda, u)$ for the shifted complete symmetric functions factorizes as

$$H^*(\lambda; u) = \prod_{i=1}^{p} \frac{u+i}{u+i-\lambda_i} \prod_{i=p+1}^{n} \frac{u+i}{u+i-\lambda_i},$$

and this can be written:

Lemma 4.11

$$H^*(\lambda; u) = H^*(\lambda^+; u) H^*(\lambda^-; -u-n-1).$$

Proof of Theorem 4.10

By Theorem 4.7,

$$\lim_{n\to\infty} \frac{1}{n^m} h_m^*(\lambda^{(n)+}) = \tilde{h}_m(\omega^+),$$
$$\lim_{n\to\infty} \frac{1}{n^m} h_m^*(\lambda^{(n)-}) = \tilde{h}_m(\omega^-).$$

And by Lemma 4.10 and Theorem 2.10,

$$\sum_{m=0}^{\infty} h_m^*(\lambda^{(n)}) \frac{1}{[u]_m} = \Big(\sum_{p=0}^{\infty} h_p^*(\lambda^{(n)+}) \frac{1}{[u]_p} \Big) \Big(\sum_{q=0}^{\infty} h_q^*(\lambda^{(n)-}) \frac{1}{[-u-n-1]_q} \Big).$$

This formula needs some comment. In fact the series of the left hanside and the first series of the right handside converge for $-\operatorname{Re} u$ large, while the second series of the right handside converges for $\operatorname{Re} u$ large. But each series can be seen as a formal series in $\frac{1}{u}$, and the equality has to be understood as equality of formal series in $\frac{1}{u}$, that is in the algebra $\mathbb{C}[[\frac{1}{u}]]$. In general, to a modified factorial expansion

$$F(u) = \sum_{m=0}^{\infty} a_m \frac{1}{[u]_m},$$

one can associate a formal seires in $\frac{1}{u}$:

$$F(u) = \sum_{k=0}^{\infty} \frac{1}{u^k}.$$

In fact

$$\frac{1}{[u]_m} = \frac{1}{u(u-1)\dots(u-m+1)}$$
$$= \frac{1}{u^m}\frac{1}{\left(1-\frac{1}{u}\right)\dots\left(1-\frac{m-1}{u}\right)} = \sum_{k=m}^{\infty} c_{k,m}\frac{1}{u^k},$$

with $c_{k,k}=1$, and

$$\sum_{m=0}^{\infty}\frac{1}{[u]_m} = \sum_{m=0}^{\infty} a_m\Big(\sum_{k=m}^{\infty} c_{k,m}\frac{1}{u^k}\Big) = \sum_{k=0}^{\infty} c_k\frac{1}{u^k},$$

with

$$c_k = \sum_{m=0}^{k} c_{k,m}a_m.$$

We will consider on the algebra $\mathbb{C}[[\frac{1}{u}]]$ the topology of the coefficient convergence-wise topology.

We continue the proof of Theorem 4.10 and put $u = nv$. Then

$$[u]_m = n^m v\Big(v-\frac{1}{n}\Big)\dots\Big(v-\frac{m-1}{n}\Big).$$

Put also $v' = -v-1$, then

$$[-u-n-1]_q = [nv'-1]_q = n^q\Big(v'-\frac{1}{n}\Big)\dots\Big(v'-\frac{q}{n}\Big).$$

We obtain

$$\sum_{m=0}^{\infty}\frac{1}{n^m}h_m^*(\lambda^{(n)})\frac{1}{v\Big(v-\frac{1}{n}\Big)\dots\Big(v-\frac{m-1}{n}\Big)}$$
$$=\Bigg(\sum_{p=0}^{\infty}\frac{1}{n^p}h_p^*(\lambda^{(n)+})\frac{1}{v\Big(v-\frac{1}{n}\Big)\dots\Big(v-\frac{p-1}{n}\Big)}\Bigg)$$
$$\Bigg(\sum_{q=0}^{\infty}\frac{1}{n^q}h_q^*(\lambda^{(n)-})\frac{1}{\Big(v'-\frac{1}{n}\Big)\dots\Big(v'-\frac{q}{n}\Big)}\Bigg).$$

It follows that

$$a_m := \lim_{n\to\infty}\frac{1}{n^m}h_m^*(\lambda^{(n)}) \text{ exists,}$$

and
$$\sum_{m=0}^{\infty} a_m \frac{1}{v^m} = \Big(\sum_{p=0}^{\infty} \tilde{h}_p(\omega^+)\frac{1}{v^p}\Big)\Big(\sum_{q=0}^{\infty} \tilde{h}_q(\omega^-)\frac{1}{v'^q}\Big).$$
One puts
$$z = \frac{1}{v},\ z' = \frac{1}{v'},$$
then
$$\frac{1}{1+z} = 1 + z'$$
We obtains
$$\sum_{m=0}^{\infty} a_m z^m = \Big(\sum_{p=0}^{\infty} \tilde{h}_p(\omega^+)z^p\Big)\Big(\sum_{q=0}^{\infty} \tilde{h}_q(\omega^-)z'^q\Big) = \Phi(\omega^+, 1+z)\Phi(\omega^-, \frac{1}{1+z}).$$
Hence, finally $a_m = \tilde{h}_m(\omega)$, and we have proven
$$\lim_{n\to\infty} \frac{1}{n^m} h_m^*(\lambda^{(n)}) = \tilde{h}_m(\omega^+, \omega^-).$$
∎

We can now extend Theorem 4.9.

Theorem 4.12 *Consider a sequence $(\lambda^{(n)})$ of signatures, with $\lambda^{(n)} \in \Omega_n$, and let $\omega = (\omega^+, \omega^-) \in \Omega$. Assume that*
$$\lim_{n\to\infty} T_n(\lambda^{(n)}) = \omega = (\omega^+, \omega^-).$$
Then, for $u \in U(\infty)$,
$$\lim_{n\to\infty} \varphi_n(\lambda^{(n)}; u) = \det \Phi(\omega^+, u) \det \Phi(\omega^-; u^{-1}),$$
uniformy on each $U(k)$.

Proof The proof is the same as the one of Theorem 4.9. On one hand
$$\varphi_n\big(\lambda^{(n)}; \mathrm{diag}(1+z_1, \ldots, 1+z_k, 1, \ldots, 1)\big) = \sum_{m_1 \geq \cdots \geq m_n \geq 0} \frac{\delta!}{(\mathbf{m}+\delta)!} s_{\mathbf{m}}^*(\lambda^{(n)}) s_{\mathbf{m}}(z_1, \ldots, z_k),$$
and, by Theorem 4.10,
$$\lim_{n\to\infty} \frac{\delta!}{\mathbf{m}+\delta!} s_{\mathbf{m}}^*(\lambda^{(n)}) = \tilde{s}_{\mathbf{m}}(\omega^+, \omega^-).$$

On the other hand we saw that

$$\sum_{m_1\geq\cdots\geq m_n\geq 0} \widetilde{s}_{\mathbf{m}}(\omega^+,\omega^-)s_{\mathbf{m}}(z_1,\ldots,z_k) = \prod_{j=1}^{k} \Phi(\omega^+;1+z_j)\Phi\big(\omega^-;\frac{1}{1+z_j}\big).$$

By Proposition 3.11, it follows that, uniformly on $\mathbb{T}^k$,

$$\lim_{n\to\infty} \varphi_n\big(\lambda^{(n)};\mathrm{diag}(t_1,\ldots,t_k,1,\ldots,1))\big) = \prod_{j=1}^{k} \Phi(\omega^+;t_j)\Phi\big(\omega_-,\frac{1}{t_j}\big).$$

Hence, uniformly on $U(k)$,

$$\lim_{n\to\infty} \varphi_n(\lambda^{(n)};u) = \det\Phi(\omega^+;u)\det\Phi(\omega^-;u^{-1}).$$

□

Theorems 4.10 and 4.12 are consistent with the following property of the spherical functions of the unitary group $U(n)$:

$$\varphi_n(\lambda;u^{-1}) = \varphi_n(\overline{\lambda};u),$$

where, if $\lambda=(\lambda_1,\ldots,\lambda_n)$, then $\overline{\lambda}=(-\lambda_n,\ldots,-\lambda_1)$. In fact $\overline{\lambda}$ is the highest weight of the contragredient representation of the representation with highest weight λ.

Chapter 5

Inductive limits of compact symmetric spaces

In this last chapter we will present without proof the main results by Okunkov and Olshanski about the spherical functions of inductive limits of compact Riemannian symmetric spaces.

5.1 Inductive limits of compact Riemannian symmetric spaces of type A

We consider one of the following sequences of compact Riemannian pairs :

$$\begin{aligned}
&G(n) = U(n), K(n) = O(n), d = 1\\
&G(n) = U(n) \times U(n), d = 2\\
&G(n) = U(2n), K(n) = Sp(n), d = 4
\end{aligned}$$

n is the rank, d is the multiplicity, the system of restricted roots is of type A_{n-1}, $A(n) = \exp \mathfrak{a}(n) \simeq \mathbb{T}^n$ is a Cartan subgroup.

Ω_n is the spherical unitary dual for the pair $(G(n), K(n))$, parametrized by signatures $\lambda_1 \leq \lambda_n$.

$\Omega = \Omega_0 \times \Omega_0$

Define

$$\Phi(\omega; z) = e^{\gamma(z-1)} \prod_{k=1}^{\infty} \frac{1 + \beta_k(z-1)}{\left(1 - \frac{2}{d}\alpha_k(z-1)\right)^{\frac{d}{2}}}.$$

Theorem 5.1 *The spherical functions for the pair* (G, K) *are given, for* $a = (z_1, \ldots, z_n, 1, \ldots) \in A$*, by*

$$\varphi(\omega; a) = \prod_{j=1}^{n} \Phi(\omega^+; z_j) \Phi(\omega^-; \frac{1}{z_j}).$$

One defines the map T_n as in the case of the unitary group ($d = 2$).

The spherical functions for the pair $\big(G(n), K(n)\big)$ are given by Jack polynomials: for $\lambda = (\lambda_1, \ldots, \lambda_n) \in \Omega_n$, $a = (z_1, \ldots, z_n) \in A(n)$,

$$\varphi(\lambda; z_1, \ldots, z_n) = \frac{P_\lambda(z_1, \ldots, z_n; \theta)}{P_\lambda(1, \ldots, 1; \theta)} \quad \big(\theta = \frac{d}{2}\big).$$

Theorem 5.2 *Let* $\lambda^{(n)} = \lambda_1^{(n)}, \ldots, \lambda_n^{(n)})$ *be a sequence of signatures, with* $\lambda^{(n)} \in \Omega_n$*. If*

$$\lim_{n\to\infty} T(\lambda^{(n)}) = \omega = (\omega^+, \omega^-),$$

then

$$\lim_{n\to\infty} \varphi(\lambda^{(n)}; z_1, \ldots, z_k, 1, \ldots) = \prod_{j=1}^{k} \Phi(\omega^+, z_j) \Phi(\omega^-; \frac{1}{z_j}).$$

The proof uses a binomial formula for Jack polynomials.

5.2 Inductive limits of compact Riemannian symmetric spaces of type BC

n is the rank, the system of rerestricted roots is of type BC_n

roots : $\pm\varepsilon_i \pm \varepsilon_j, \quad \pm\varepsilon_i, \quad \pm 2\varepsilon_i,$

multiplicities : d, p, q.

Let us assume that the multiplicities don't depend on n.

The spherical unitary dual Ω_n is parametrized by positive signatures (or partitions): $\lambda = (\lambda_1, \ldots, \lambda_n)$,with

$$\lambda 1 \geq \ldots, \lambda_n \geq 0, \quad \lambda_i \in \mathbb{N}.$$

The spherical functions are given by multivariate Jacobi polynomials:

$$\varphi(\lambda; z_1, \ldots, z_n) = \frac{\mathcal{J}_\lambda(z_1, \ldots, z_n; \theta, a, b)}{\mathcal{J}_\lambda(1, \ldots, 1; \theta, a, b)}.$$

The spherical unitary dual is $\Omega = \Omega_0$.

As in Section 5.1, one defines

$$\Phi(\omega; z) = e^{\gamma(z-1)} \prod_{k=1}^{\infty} \frac{1 + \beta_k(z-1)}{\left(1 - \frac{2}{d}\alpha_k(z-1)\right)^{\frac{d}{2}}}.$$

Theorem 5.3 *The spherical functions for the pair (G, K) are given, for $a = (z_1, \ldots, z_n, 1, \ldots) \in A$, by*

$$\varphi(\omega; a) = \prod_{j=1}^{n} \Phi(\omega; z_j)\Phi(\omega; \frac{1}{z_j}).$$

Observe that the formula does not depend on p and q.

One defines the map T_n as in the case of the unitary group for positive signatures.

Theorem 5.4 *Let $\lambda^{(n)} = \lambda_1^{(n)}, \ldots, \lambda_n^{(n)})$ be a sequence of signatures, with $\lambda^{(n)} \in \Omega_n$. If*

$$\lim_{n\to\infty} T(\lambda^{(n)}) = \omega,$$

then

$$\lim_{n\to\infty} \varphi(\lambda^{(n)}; z_1, \ldots, z_k, 1, \ldots) = \prod_{j=1}^{k} \Phi(\omega, z_j)\Phi(\omega; \frac{1}{z_j}).$$

Observe that the limit does not depend on p and q.

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