RANDOM MATRIX THEORY, AN INTRODUCTION

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High dimensional phenomena in probability - Fluctuations and discontinuity

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In Random Matrix Theory, one considers a probability measure on the space of $n \times n$ real symmetric matrices, or Hermition matrices. One studies the eigenvalues which are random variables, and their distribution. The main problems are about the asymptotics of the distribution of the eigenvalues as $n \to \infty$.

We will consider the following questions

- Density of the statistical distribution of the eigenvalues
- Limit of the statistical distribution of the eigenvalues
- Probability, for a matrix, to be positive definite
- Conditional statistical distribution of the eigenvalues under the condition that the matrix is positive definite
- The Sylvester index of a random matrix

We will present two analytic method:

Orthogonal polynomials

The method gives explicit results, but is only valid for Hermitian matrices.

Logarithmic Potential Theory, or Log-Gas method

It is called Log-Gas method, because of the analogy with Thermodynamics, and is related to the Electrostatistic Model of Stieltjes.

Main sources

M. L. Mehta, Random matrices, Academic Press, 1991

P. Deift, Orthogonal polynomials and random matrices: a Riemann-Hilbert approach, A. M. S. & Courant Institute, 1998.

L. Pastur & M. Shesherbina, eigenvalues distribution of large random matrices, A. M. S. 2010.

S. M. Majumdar, C. Nadal, A. Scardicchio, & P. Vivo, How many eigenvalues of a Gaussian random matrix are positive ? Physical Review, 83 (2011), 041105.

I have been very pleased to take part in this Summer School. I thank Peter Eichelsbacher and Michael Voit for inviting me to give this minicourse.

1 The probability space $(H_n, \mathbb{P}_n)$

For $\mathbb{F} = \mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$, $H_n = Herm(n, \mathbb{F})$ denotes the space of $n \times n$ Hermitian matrices with entries in $\mathbb{F}$. On $Herm(n, \mathbb{F})$ we consider a probability measure of the form

$$\mathbb{P}(dx) = \frac{1}{C_n} e^{-\operatorname{tr} Q(x)} m(dx),$$

Q is real valued function defined on $\mathbb{R}$, and, for a matrix x, $Q(x)$ is defined via the functional calculus. m is the Euclidean measure on H_n associated to the inner product $(x \mid y) = \operatorname{tr}(xy)$, and C_n is a normalization constant:

$$C_n = \int_{H_n} e^{-\operatorname{tr} Q(x)} m(dx).$$

Main example: Gaussian probability

$Q(t) = \gamma t^2$ $(\gamma > 0)$. Then

$$C_n = \int_{H_n} e^{-\gamma \operatorname{tr}(x^2)} m(dx) = \left(\sqrt{\frac{\pi}{\gamma}}\right)^N,$$

where

$$N = \dim_{\mathbb{R}} H_n = n + \frac{\beta}{2} n(n-1), \quad \beta = \dim_{\mathbb{R}} \mathbb{F} = 1, 2, 4.$$

This probability is invariant under the group $U_n = U(n, \mathbb{F})$ of $n \times n$ unitary matrices with entries in $\mathbb{F}$, acting on H_n by the transformations

$$x \mapsto uxu^* \quad (u \in U(n, \mathbb{F}).$$

- $U(n, \mathbb{R}) = O(n)$, the orthogonal group,
- $U(n, \mathbb{C}) = U(n)$, the unitary group,
- $U(n, \mathbb{H}) \simeq Sp(n)$, the compact symplectic group.

For $Q(t) = t^2$, and $\mathbb{F} = \mathbb{R}$, the probability space $(H_n, \mathbb{P}_n)$ is called the Gaussian Orthogonal Ensemble: GOE. For $\mathbb{F} = \mathbb{C}$, the Gaussian Unitary Ensemble: GUE, and for $\mathbb{F} = \mathbb{H}$, the Gaussian Symplectic Ensemble, GSE.

Empirical eigenvalue distribution

The *empirical eigenvalue distribution* of a matrix $x \in H_n$ is the random measure $M_n^{(x)}$ on $\mathbb{R}$ defined by

$$M_n^{(x)} = \frac{1}{n}\sum_{i=1}^{n} \delta_{\lambda_i^{(x)}},$$

where $\lambda_1^{(x)}, \dots \lambda_n^{(x)}$ are the eigenvalues of x. Observe that, for $B \subset \mathbb{R}$,

$$M_n^{(x)}(B) = \frac{1}{n}\#\{\text{eigenvalues of } x \text{ in } B\}.$$

For a function φ on $\mathbb{R}$,

$$\int_{\mathbb{R}} \varphi(t) M_n^{(x)}(dt) = \frac{1}{n}\sum_{i=1}^{n} \varphi(\lambda_i^{(x)}) = \frac{1}{n}\mathrm{tr}\left(\varphi(x)\right).$$

Statistical distribution of the eigenvalues

The *statistical distribution of the eigenvalues* is the expectation of the empirical distribution of the eigenvalues, it is the measure M_n on $\mathbb{R}$ defined by, for a Borel set $B \subset \mathbb{R}$,

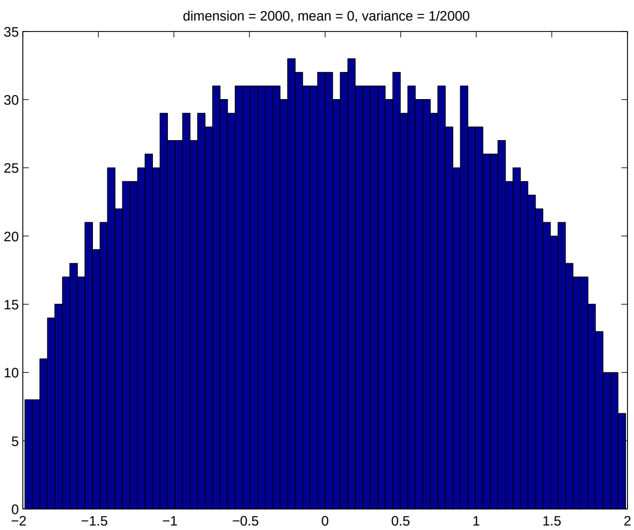
$$M_n(B) = \mathbb{E}_n\big(M_n^{(x)}(B)\big).$$

For a bounded measurable function φ,

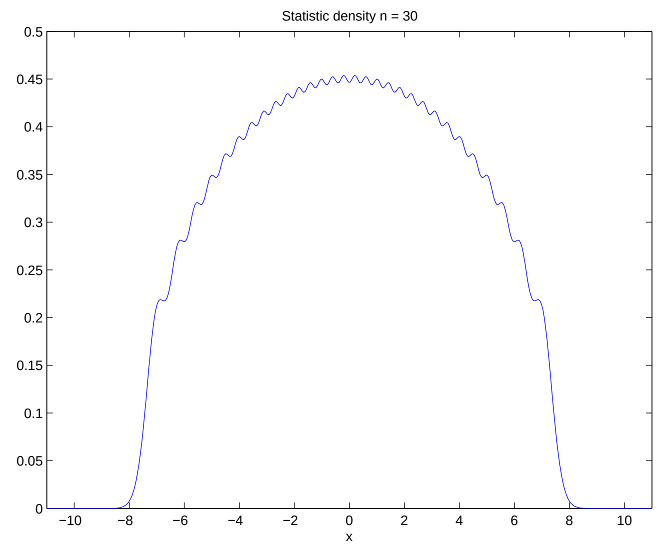
$$\begin{aligned}
\int_{\mathbb{R}} \varphi(t) M_n(dt) &= \mathbb{E}_n\Big(\int_{\mathbb{R}} \varphi(t) M_n^{(x)}(dt)\Big) \\
&= \frac{1}{n}\int_{H_n} \mathrm{tr}\left(\varphi(x)\right)\mathbb{P}_n(dx).
\end{aligned}$$

One of the main problems in Random Matrix Theory is to determine the asymptotic of the statistical distribution of the eigenvalues M_n as $n \to \infty$.

Empirical distribution of the eigenvalues, GUE, n=2000



Statistical distribution of the eigenvalues, GUE, n=30



Weyl integration formula

We recall the notation:

$H_n = Herm(n, \mathbb{F})$, $\mathbb{F} = \mathbb{R}, \mathbb{C}$, or $\mathbb{H}$, $U_n = U(n, \mathbb{F})$.

By the classical spectral theorem, every matrix $x \in H_n$ can be diagonalized in an orthogonal basis, and the eigenvalues are real. This can be said as follows: the map

$$U_n \times D_n \to H_n, \quad (u, t) \mapsto utu^*,$$

is surjective, where D_n denotes the space of real diagonal matrices.

Theorem 1.1. *If f is an integrable function on H_n, then*

$$\int_{H_n} f(x)m(dx) = c_n \int_{D_n} \int_{U_n} f(utu*)\alpha_n(du)|V_n(t)|^\beta dt_1 \ldots dt_n,$$

where $t = \operatorname{diag}(t_1, \ldots, t_n)$,

$$V_n(t) = \prod_{j<k} (t_k - t_j)$$

is the Vandermonde polynomial, α_n is the normalized Haar measure of the compact group U_n, c_n is a positive constant, $\beta = \dim_{\mathbb{R}} \mathbb{F} = 1, 2,$ or 4.

If the function f is U_n-invariant:

$$f(uxu^*) = f(x) \quad (u \in U_n),$$

then f only depends on the eigenvalues $\lambda_1, \ldots, \lambda_n$ of x:

$$f(x) = F(\lambda_1, \ldots, \lambda_n),$$

where the function F is defined on $\mathbb{R}^n$, and symmetric:

$$F(\lambda_{\sigma(1)}, \ldots, \lambda_{\sigma(n)}) = F(\lambda_1, \ldots, \lambda_n),$$

for $\sigma \in \mathfrak{S}_n$, the symmetric group. In that case the Weyl integration formula simplifies:

$$\int_{H_n} f(x)m(dx) = c_n \int_{\mathbb{R}^n} F(\lambda_1, \ldots, \lambda_n)|V_n(\lambda)|^\beta d\lambda_1 \ldots d\lambda_n.$$

Let us come back to the probability measure on H_n of the form

$$\mathbb{P}_n(dx) = \frac{1}{C_n} e^{-\operatorname{tr}\big(Q(x)\big)} m(dx).$$

If the function f on H_n is U_n-invariant, then

$$\int_{H_n} f(x)\mathbb{P}_n(dx) = \int_{\mathbb{R}^n} F(\lambda) q_n(\lambda) d\lambda_1 \ldots \lambda_n,$$

with

$$q_n(\lambda) = \frac{1}{Z_n} e^{-\big(Q(\lambda_1)+\cdots+Q(\lambda_n)\big)} |V_n(\lambda|^\beta,$$

and

$$Z_n = \int_{\mathbb{R}^n} e^{-\big(Q(\lambda_1+\cdots+Q(\lambda_n)\big)} |V_n(\lambda)|^\beta.$$

Z_n is sometimes called the partition function, in analogy to the partition function which is defined in Thermodynamics.

For $Q(t) = \frac{1}{2}t^2$, Z_n is the *Mehta integral*:

$$Z_n = \int_{\mathbb{R}^n} e^{-\frac{1}{2}\|x\|^2} |V_n(x)|^\beta dx_1 \ldots dx_n = (2\pi)^{\frac{n}{2}} \prod_{j=1}^n \frac{\Gamma\Big(j\frac{\beta}{2}+1\Big)}{\Gamma\Big(\frac{\beta}{2}+1\Big)}.$$

In general there is no explicit evaluation of the integral Z_n. However we will see that it is possible to determine asymptotics for Z_n by using logarithmic potential theory.

2 Point process (or Point field)

Consider n particules in a space X. An n-point configuration is a possible position $(x_1, \ldots, x_n)$ of the particules. The set $\operatorname{Conf}_n(X)$ of the n-point configurations can be identified with $X^n/\mathfrak{S}_n$. An n-point process (or n-point field) is the probability space $(\operatorname{Conf}_n(X), \mathbb{P}_n)$, where $\mathbb{P}_n$ is a probability measure on $\operatorname{Conf}_n(X)$. This probability measure can be seen as a probability measure on X^n which is symmetric, i.e. invariant under permutations.

Correlation functions

The correlation measure ρ_m ($1 \leq m \leq n$) is the symmetric measure on X^m defined by, if φ is a function on X^m,

$$\langle \rho_m, \varphi \rangle = \mathbb{E}\bigl(\sum \varphi(x_1, \ldots, x_m)\bigr),$$

where the summation is taken over all ordered m-tuples of particules chosen from the point configuration.

If a reference measure μ is given on X, the density R_m of ρ_m, if it exists, is called the correlation function:

$$\langle \rho, \varphi \rangle = \int_{X^m} \varphi(x_1, \ldots, x_m) R_m(x_1, \ldots, x_m) \mu(dx_1) \ldots \mu_m(dx_m).$$

Assume that

$$\mathbb{P}_n(dx) = p_n(x_1, \ldots, x_n)\mu(dx_1) \ldots \mu(dx_n),$$

where p_n is a symmetric function on X^n. Then

$$\begin{aligned} &R_m(x_1, \ldots, x_m) \\ &= \frac{n!}{(n-m)!} \int_{X^{n-m}} p_n(x_1, \ldots, x_m, x_{m+1}, \ldots, x_n)\mu(dx_{m+1}) \ldots \mu(dx_n). \end{aligned}$$

In particular, for $m = n$,

$$R_n(x_1, \ldots, x_n) = n! p_n(x_1, \ldots, x_n),$$

and, for $n = 1$,

$$R_1(x_1) = n \int_{\mathbb{X}^{n-1}} p_n(x_1, x_2, \ldots, x_n)\mu(dx_2) \ldots \mu(dx_n).$$

Determinantal point process

Recall the Fredholm notation. If $K(x, y)$ is a kernel on X, then, for $x_1, \ldots, x_m, y_1, \ldots, y_m \in X$,

$$K \begin{pmatrix} x_1 & \ldots & x_m \\ y_1 & \ldots & y_m \end{pmatrix} = \det_{1 \leq i,j \leq m}\bigl(K(x_i, y_j)\bigr).$$

The point process $(\mathrm{Conf}_n(X), \mathbb{P}_n)$ is said to be determinantal if there is a kernel $K_n(x, y)$ such that, for $1 \le m \le n$,

$$R_m(x_1, \ldots, x_m) = K_n \begin{pmatrix} x_1 & \ldots & x_m \\ x_1 & \ldots & x_m \end{pmatrix}.$$

The n-point process of the eigenvalues

Recall the probability space $(H_n, \mathbb{P}_n)$, with

$$\mathbb{P}_n(dx) = \frac{1}{C_n} e^{-\operatorname{tr} Q(x)} m(dx).$$

By the Weyl integration formula, and considering the n eigenvalues as particules on the line we get an n-point process with $X = \mathbb{R}$, and the probability

$$\mathbb{P}_n(d\lambda_1, \ldots, \lambda_n) = q_n(\lambda_1, \ldots, \lambda_n) d\lambda_1 \ldots d\lambda_n,$$

with

$$q_n(\lambda_1, \ldots, \lambda_n) = \frac{1}{Z_n} e^{-\left(Q(\lambda_1)+\cdots+Q(\lambda_n)\right)} . |V_n(\lambda)|^\beta.$$

We will see in next section that, for $\beta = 2$ ($\mathbb{F} = \mathbb{C}$), and if the function $e^{-Q(t)}$ admits moments of all order: for all $m \ge 0$,

$$\int_{\mathbb{R}} |t|^m e^{-Q(t)} dt < \infty,$$

the n-point process of the eigenvalues is determinantal. Therefore, in that case, it will be possible to evaluate the density $w_n(t)$ of the statistical eigenvalue distribution $M_n(dt)$, because of the following proposition.

Proposition 2.1. *The statistical distribution M_n of the eigenvalues has a density w_n,*

$$w_n(t) = \frac{1}{n} R_1(t).$$

Proof.

Let φ be a bounded measurable function on $\mathbb{R}$. then

$$f(x) := \frac{1}{n} \operatorname{tr}\big(\varphi(x)\big) = \frac{1}{n}\big(\varphi(\lambda_1) + \cdots + \varphi(\lambda_n)\big),$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of x. Apply the Weyl integration formula to the $U(n, \mathbb{F})$-invariant function f:

$$\begin{aligned}
\int_{\mathbb{R}} \varphi(t) M_n(dt) &= \int_{H_n} f(x) \mathbb{P}_n(dx) \\
&= \frac{1}{n} \sum_{i=1}^{n} \int_{\mathbb{R}^n} \varphi(\lambda_i) q_n(\lambda) d\lambda_1 \ldots d\lambda_n \\
&= \int_{\mathbb{R}^n} \varphi(\lambda_1) q_n(\lambda) d\lambda_1 \ldots d\lambda_n = \int_{\mathbb{R}} \varphi(t) w_n(t) dt,
\end{aligned}$$

with

$$w_n(t) = \int_{\mathbb{R}^{n-1}} q_n(t, \lambda_2, \ldots, \lambda_n) = \frac{1}{n} R_1(t).$$

3 Orthogonal polynomials

Let μ be a positive measure on $\mathbb{R}$. We assume that the support of μ is infinite, and, for all $m \geq 0$,

$$\int |t|^m \mu(dt) < \infty.$$

Consider a probability measure on $\mathbb{R}^n$ of the form

$$\mathbb{P}_n(dx) = \frac{1}{Z_n} \prod_{i<j} (x_j - x_i)^2 \mu(dx_1) \ldots \mu(dx_n),$$

where Z_n is a normalization constant:

$$Z_n = \int_{\mathbb{R}^n} \prod_{1<j} (x_j - x_i)^2 \mu(dx_1) \ldots \mu(dx_n).$$

We will see in this section that the n-point process $(\mathrm{Conf}_n(\mathbb{R}), \mathbb{P}_n)$ is determinantal.

On the space $\mathcal{P}$ of polynomials in one variable with real coefficients one considers the inner product

$$(p|q) = \int_{\mathbb{R}} p(t) q(t) \mu(dt),$$

for which $\mathcal{P}$ is a pre-Hilbert space. The monomials $1, t, \dots, t^m, \dots$ are independent, and, by the Gram-Schmidt orthogonalization, one gets a sequence $\{p_m\}$ of orthogonal polynomials: p_m is of degree m, and

$$\int_{\mathbb{R}} p_m(t)q_n(t)\mu(dt) = 0 \text{ if } m \neq n.$$

We assume that

$$p_m(t) = t^m + \cdots ,$$

and we define

$$d_m = \int_{\mathbb{R}} p_m(t)^2 \mu(dt.$$

Example: Hermite polynomials

Let μ be the Gaussian measure

$$\mu(dt) = e^{-t^2} dt.$$

The Hermite polynomials H_m are orthogonal in $L^2(\mathbb{R}, \mu)$.

$$H_m(t) = (-1)^m e^{t^2} \left(\frac{d}{dt}\right)^m e^{-t^2}.$$

Since $H_m(t) = 2^m t^m + \cdots$, we define

$$p_m(t) = 2^{-m} H_m(t),$$

and one gets

$$d_m = 2^{-m} m! \sqrt{\pi}.$$

Evaluation of Z_n

Proposition 3.1. *Recall that*

$$Z_n = \int_{\mathbb{R}^n} \prod_{1 \leq i < j \leq n} (x_j - x_i)^2 \mu(dx_1) \dots \mu(dx_n).$$

Then

$$Z_n = n! d_0 d_1 \dots d_{n-1}.$$

Proof.
Notation: for $\mathbf{m} = (m_1, \dots, m_n)$ the polynom $p_{\mathbf{m}}$ in n variables is given by

$$p_{\mathbf{m}}(x_1, \dots, x_n) = p_{m_1}(x_1) \dots p_{m_n}(x_n).$$

The polynoms $p_{\mathbf{m}}$ are orthogonal for the inner product

$$(p \mid q) = \int_{\mathbb{R}^n} p(x)q(x)\mu(dx_1) \dots \mu(dx_n),$$

and

$$\|p\|^2 = d_{m_1} \dots d_{m_n}.$$

For a permutation $\sigma \in \mathfrak{S}_n$,

$$\sigma \cdot \mathbf{m} = (m_{\sigma(0)}, m_{\sigma(1)}, \dots, m_{\sigma(n-1)}).$$

Define also

$$\delta = (0, 1, \dots, n-1).$$

Recall the Vandermonde determinant

$$V_n(x) = \prod_{1 \le i < j \le n} (x_j - x_i)^2 = \begin{vmatrix} 1 & 1 & \dots & 1 \\ x_1 & x_2 & \dots & 1 \\ \vdots & \vdots & & \vdots \\ x_1^{n-1} & x_2^{n-1} & \dots & x_n^{n-1} \end{vmatrix}$$
$$= \begin{vmatrix} p_0(x_1) & p_0(x_2) & \dots & p_0(x_n) \\ p_1(x_1) & p_1(x_2) & \dots & p_1(x_n) \\ \vdots & \vdots & & \vdots \\ p_{n-1}(x_1) & p_{n-1}(x_2) & \dots & p_{n-2}(x_n) \end{vmatrix} = \sum_{\sigma \in \mathfrak{S}_n} \varepsilon(\sigma) p_{\sigma \cdot \delta}(x).$$

The third equality comes from the following observation: one does not change the value of a determinant if one adds to a column a linear combination of the other ones. Therefore

$$\int_{\mathbb{R}^n} V_n(x)^2 \mu(dx_1) \dots \mu(dx_n) = n! d_0 d_1 \dots d_{n-1}.$$

Example: the Gaussian case

The measure μ is Gaussian:

$$\mu(dt) = e^{-t^2} dt.$$

Therefore

$$Z_n = n! d_0 d_1 \dots d_{n-1} = n! \prod_{j=0}^{n-1} (2^{-j} j! \sqrt{\pi}) = \pi^{\frac{n}{2}} 2^{-\frac{n(n-1)}{2}} \prod_{j=0}^{n} j!.$$

This is in agreement with the evaluation of Mehta's integral, for $\beta = 2$:

$$Z_n = \int_{\mathbb{R}^n} e^{-\frac{1}{2}\|t\|^2} |V_n(x)|^\beta dx_1 \dots dx_n = (2\pi)^{\frac{n}{2}} \prod_{j=1}^{n} \frac{\Gamma\Big(j\frac{\beta}{2} + 1\Big)}{\Gamma\Big(\frac{\beta}{2} + 1\Big)}.$$

Christoffel-Darboux kernel

Let S_n be the orthogonal projection of $L^2(\mathbb{R}, \mu)$ onto the space $\mathcal{P}_{n-1}$ of polynomials of degree $\leq n-1$.

$$(S_n f)(x) = \sum_{k=0}^{n-1} \frac{1}{d_k} (f \mid p_k) p_k(x) = \int_{\mathbb{R}} K_n(x, y) f(y) \mu(dy),$$

where K_n is the following kernel, called the Christoffel-Darboux kernel,

$$K_n(x, y) = \sum_{k=0}^{n-1} \frac{1}{d_k} p_k(x) p_k(y).$$

Proposition 3.2. *For $x \neq y$,*

$$K_n(x, y) = \frac{1}{d_{n-1}} \frac{p_n(x) p_{n-1}(y) - p_{n-1}(x) p_n(y)}{x - y},$$

and

$$K_n(x, x) = \frac{1}{d_{n-1}} \big(p_n'(x) p_{n-1}(x) - p_n(x) p_{n-1}'(x) \big).$$

The proof uses the three terms recursion formula. Define b_m:

$$p_m(x) = x^m + b_m x^{m-1} + \cdots$$

and

$$\beta_m = b_m - b_{m+1}, \quad \gamma_m = \frac{d_m}{d_{m-1}}.$$

With this notation

$$xp_m(x) = p_{m+1}(x) + \beta_m p_m(x) + \gamma_m p_{m-1}(x).$$

Mehta's formulas

Recall the n-point process $\big(\mathrm{Conf}_n(\mathbb{R}), \mathbb{P}_n\big)$, with

$$P_n(dx) = \frac{1}{Z_n} V_n(x)^2 \mu(dx_1) \ldots \mu_n(dx_n).$$

We will now prove that the n-point process $(\mathrm{Conf}_n(\mathbb{R}), \mathbb{P}_n)$, is determinantal:

$$\begin{aligned} R_m(x_1, \ldots, x_m) &= K_n \begin{pmatrix} x_1 & x_2 & \ldots & x_m \\ y_1 & y_2 & \ldots & y_m \end{pmatrix} \\ &:= \det\big(K_n(x_i, y_j)\big)_{1 \le i,j \le n}. \end{aligned}$$

where K_n is the Christoffel-Darboux kernel.

Lemma 3.3.

$$\frac{1}{Z_n} V_n(x)^2 = \frac{1}{n!} K_n \begin{pmatrix} x_1 & \ldots & x_n \\ x_1 & \ldots & x_n \end{pmatrix}.$$

Proof.

Consider the matrix $A = (a_{ij})$ with

$$a_{ij} = \frac{1}{\sqrt{d_j}} p_j(x_i), \quad i = 1, \ldots, n, \quad j = 0, \ldots n-1.$$

Then

$$\det A = \frac{1}{\sqrt{d_0 \ldots d_{n-1}}} V_n(x).$$

The entries of $B = AA^T$ are

$$b_{ij} = \sum_{k=0}^{n-1} \frac{1}{\sqrt{d_k}} p_k(x_i) \frac{1}{\sqrt{d_k}} p_k(x_j) = K_n(x_i, x_j).$$

Recall that $Z_n = n! d_0 \dots d_{n-1}$. The formula follows.

For $1 \leq m \leq n$, the correlation function R_m is defined on $\mathbb{R}^m$ by

$$R_m(x_1, \dots, x_m) = \frac{n!}{(n-m)!} \frac{1}{Z_n} \int_{\mathbb{R}^{n-m}} V_n(x_1, \dots, x_m, x_{m+1}, \dots, x_n)^2 \mu(dx_{m+1}) \dots \mu(dx_n).$$

Theorem 3.4. (Mehta) *The n-point process of the eigenvalues is determinantal:*

$$R_m(x_1, \dots, x_m) = K_n \begin{pmatrix} x_1 & \dots & x_m \\ x_1 & \dots & x_m \end{pmatrix}.$$

In particular, for $n = 1$, $R_1(x) = K_n(x, x)$.

We will prove the proposition by a backwards recursion on m. By the lemma, the formula holds for $m = n$. For the recursion we will use the following lemma.

Lemma 3.5. *Let K be the kernel of the orthogonal projection P of $L^2(\mathbb{R}, \mu)$ onto a subspace of dimension n,*

$$(Pf)(x) = \int_{\mathbb{R}} K(x, y) f(y) \mu(dy).$$

Then

$$\int_{\mathbb{R}} K \begin{pmatrix} x_1 & \dots & x_m \\ x_1 & \dots & x_m \end{pmatrix} \mu(dx_m) = (n - m + 1) K \begin{pmatrix} x_1 & \dots & x_{m-1} \\ x_1 & \dots & x_{m-1} \end{pmatrix}.$$

Proof.

The kernel K satisfies the following properties

- Since $P^* = P$, then

$$K(y, x) = \overline{K(x, y)}$$

- Since $P \circ P = P$, then

$$\int_{\mathbb{R}} K(x, u) K(u, y) \mu(du) = K(x, y).$$

- Since $\operatorname{tr} P = n$, then

$$\int_{\mathbb{R}} K(x,x)\mu(dx) = n.$$

With entries $a_{ij} = K(x_i, x_j)$, consider the matrix $A_m = (a_{ij})_{1\le i,j\le m}$, and write

$$A_m = \begin{pmatrix} A_{m-1} & \alpha \\ \alpha^* & \gamma \end{pmatrix}.$$

Then

$$\det A_m = \det A_{m-1} \cdot \gamma - \alpha^* \tilde{A}_{m-1} \alpha,$$

where $\tilde{A}_{m-1}$ is the matrix of the cofactors $\tilde{a}_{ij}$ of A_{m-1}.

Therefore

$$\begin{aligned} \int_{\mathbb{R}} K\begin{pmatrix} x_1 & \dots & x_m \\ x_1 & \dots & x_m \end{pmatrix} \mu(dx_m) &= K\begin{pmatrix} x_1 & \dots & x_{m-1} \\ x_1 & \dots & x_{m-1} \end{pmatrix} \int_{\mathbb{R}} K(x_m, x_m) dx_m \\ &\quad - \sum_{i,j=1}^{m-1} a_{ij} \int_{\mathbb{R}} K(x_j, x_m) K(x_m, x_i) \mu(dx_m) \\ &= nK\begin{pmatrix} x_1 & \dots & x_{m-1} \\ x_1 & \dots & x_{m-1} \end{pmatrix} - \sum_{i,j=1}^{m-1} \tilde{a}_{ij} K(x_j, x_i) \\ &= (n-m+1) K\begin{pmatrix} x_1 & \dots & x_{m-1} \\ x_1 & \dots & x_{m-1} \end{pmatrix}. \end{aligned}$$

since

$$\sum_{i,j=1}^{m-1} \tilde{a}_{ij} a_{ji} = \det A_{m-1}.$$

Density of the statistical eigenvalue distribution

We apply the preceding results to the case where

$$\mu(dt) = e^{-Q(t)} dt.$$

By defining

$$\varphi_m(t) = \frac{1}{\sqrt{d_m}} e^{-\frac{1}{2}Q(t)} p_m(t),$$

we get an orthonormal basis of $L^2(\mathbb{R})$. We define also a modified Christoffel-Darboux kernel

$$\begin{aligned}
\mathcal{K}_n(s,t) &= \sum_{k=0}^{n-1} \varphi_k(s)\varphi_k(t) \\
&= e^{\frac{1}{2}\left(Q(s)+Q(t)\right)} K_n(s,t) \\
&= \sqrt{\frac{d_n}{d_{n-1}}}\,\frac{\varphi_n(s)\varphi_{n-1}(t)-\varphi_{n-1}(s)\varphi_n(t)}{s-t},
\end{aligned}$$

and

$$\mathcal{K}_n(t,t) = \sqrt{\frac{d_n}{d_{n-1}}}\big(\varphi_n'(t)\varphi_{n-1}(t)-\varphi_{n-1}'(t)\varphi_n(t)\big).$$

The statistical distribution of the eigenvalues has the following density

$$w_n(t) = \frac{1}{n}\mathcal{K}_n(t,t) = \frac{1}{n}\sqrt{\frac{d_n}{d_{n-1}}}\big(\varphi_n'(t)\varphi_{n-1}(t)-\varphi_{n-1}'(t)\varphi(t)\big).$$

Formula for the density $w_n(t)$ in the GUE case

In the GUE case $Q(t)=t^2$.

$$p_m(t) = 2^{-m}H_m(t), \quad d_m = 2^{-m}m!\sqrt{\pi}, \quad \frac{d_n}{d_{n-1}} = \frac{n}{2}.$$

The functions φ_m are the following Hermite functions

$$\varphi_m(t) = \frac{1}{\sqrt{d_m}}e^{-t^2}H_m(t),$$

and

$$\begin{aligned}
\mathcal{K}_n(t,t) &= \sqrt{\frac{n}{2}}\big(\varphi'(t)\varphi_{n-1}(t)-\varphi_{n-1}'(t)\varphi_n(t)\big), \\
w_n(t) &= \frac{1}{\sqrt{2n}}\big(\varphi'(t)\varphi_{n-1}(t)-\varphi_{n-1}'(t)\varphi_n(t)\big).
\end{aligned}$$

Taking the derivative

$$w_n'(t) = \frac{1}{\sqrt{2n}}\big(\varphi_n''(t)\varphi_{n-1}(t)-\varphi_{n-1}''(t)\varphi_n(t)\big).$$

The Hermite polynomial H_n is a solution of the following differential equation

$$y'' - 2ty' + 2ny = 0,$$

from which one deduces

$$\varphi_n''(t) + (2n+1)\varphi_n(t) = t^2\varphi_n(t).$$

We get

$$w_n'(t) = -\sqrt{\frac{2}{n}}\varphi_n(t)\varphi_{n-1}(t).$$

Proposition 3.6. *The density w_n is the Schwartz function given by*

$$w_n(t) = -\sqrt{\frac{2}{n}}\int_{-\infty}^{t} \varphi_n(s)\varphi_{n-1}(s)ds.$$

Remark. Let f be a Schwartz function on $\mathbb{R}$. If

$$\int_{\mathbb{R}} f(s)ds = 0,$$

then the function F defined by

$$F(t) = \int_{-\infty}^{t} f(s)ds,$$

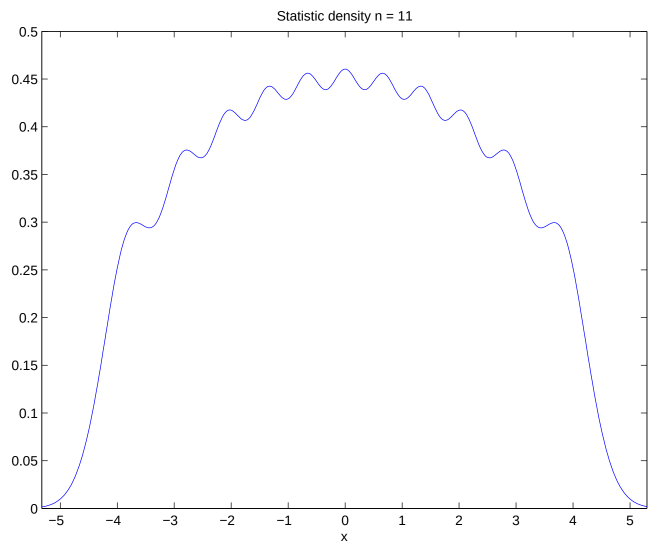
is a Schwartz function.

Let $\alpha_1^{(n)} < \alpha_2^{(n)} < \cdots < \alpha_n^{(n)}$ denote the zeros of the Hermite polynomial H_n. The derivative $w_n'(t)$ vanishes at $\alpha_1^{(n)}, \ldots, \alpha_n^{(n)}$, and $\alpha_1^{(n-1)}, \ldots \alpha_{n-1}^{(n-1)}$. The zeros of H_{n-1} interlace the zeros of H_n:

$$\alpha_1^{(n)} < \alpha_1^{(n-1)} < \alpha_2^{(n)} \leq \cdots \alpha_{n-1}^{(n-1)} < \alpha_n^{(n)}.$$

The function w_n admits a local maximum at each zero $\alpha_i^{(n)}$ of H_n and a local minimum at each zero $\alpha_i^{(n-1)}$ of H_{n-1}.

Graph of the function w_n, for $n = 11$



4 Theorem of Wigner

The Fourier transform of the statistical distribution of the eigenvalues

We assume that $Q(t) = t^2$. Recall the Laguerre polynomials:

$$L_m^\alpha(x) = e^x \frac{x^{-\alpha}}{n!} \left(\frac{d}{dx}\right)^m (e^{-x} x^{m+\alpha}).$$

Proposition 4.1. *The Fourier transform of the statistical distribution of the eigenvalues*

$$\widehat{M_n}(\tau) = \int_{\mathbb{R}} e^{-it\tau} M_n(dt),$$

is given by

$$\widehat{M_n}(\tau) = \frac{1}{n} e^{-\frac{\tau^2}{4}} L^1_{n-1}\Big(\frac{\tau^2}{4}\Big).$$

Proof.

Recall that

$$w_n(t) = \frac{1}{n}\mathcal{K}_n(t,t) = \frac{1}{n}\sum_{k=0}^{n1} \varphi_k(t)^2.$$

We determine first the Fourier transform of

$$W_r(t) = \sum_{k=0}^{\infty} r^k \varphi_k(t)^2. \quad (0 < r < 1).$$

By using the classical formula of Mehler:

$$\sum_{k=0}^{\infty} \frac{1}{2^k k!} H_k(x)^2 = \frac{1}{\sqrt{1-r^2}} e^{2x^2 \frac{r}{1+r}},$$

one gets

$$W_r(t) = \frac{1}{\sqrt{\pi}} \frac{1}{\sqrt{1-r^2}} e^{-\frac{1-r}{1+r}t^2}.$$

This is a Gaussian function, whose Fourier transform is given by

$$\widehat{W_r}(\tau) = \frac{1}{1-r} e^{-\frac{1+r}{1-r}\frac{\tau^2}{4}}.$$

We consider the product of two power series:

$$\begin{aligned}\Big(\sum_{k=0}^{\infty} r^k\Big)\Big(\sum_{k=0}^{\infty} \varphi_k(t)^2 r^k\Big) &= \sum_{k=0}^{\infty}\Big(\sum_{k=0}^{n} \varphi_k(t)^2\Big) r^n \\ &= \sum_{n=0}^{\infty} \mathcal{K}_{n+1}(t,t) r^n.\end{aligned}$$

or

$$\frac{1}{1-r} W_r(t) = \sum_{n=0}^{\infty} \mathcal{K}(t,t) r^n.$$

Therefore

$$\frac{1}{1-r}\widehat{W_r}(\tau) = e^{\frac{\tau^2}{4}} \frac{1}{(1-r)^2} e^{-\frac{r}{1-r}\frac{\tau^2}{2}}.$$

and one recognizes the generating function of the Laguerre polynomials L_n^1:

$$= e^{-\frac{\tau^2}{4}} \sum_{n=0}^{\infty} r^n L_n^1\Big(\frac{\tau^2}{2}\Big).$$

Recall the Lévy-Kramér Theorem: let μ_n be a sequence of probability measures on $\mathbb{R}$ such that, for every $\tau \in \mathbb{R}$,

$$\lim_{n\to\infty} \widehat{\mu_n}(\tau) = \varphi(\tau),$$

the function φ being continuous at 0. Then the sequence μ_n converges for the tight topology to a probability measure μ, whose Fourier transform is equal to φ: for every continuous bounded function f on $\mathbb{R}$,

$$\lim_{n\to\infty} \int_{\mathbb{R}} f(t)\mu_n(dt) = \int_{\mathbb{R}} f(t)\mu(dt).$$

Semi-circle law σ_a: for $a > 0$,

$$\int_{\mathbb{R}} f(t)\sigma_a(dt) = \frac{2}{\pi a^2}\int_{-a}^{a} f(t)\sqrt{a^2-t^2}\,dt.$$

Theorem 4.2. *(Theorem of Wigner) After scaling, the statistical distribution of the eigenvalues M_n converges to the semi-circle law σ_a with $a = \sqrt{2}$, for the tight topology. Precisely, for every continuous bounded function f on $\mathbb{R}$,*

$$\lim_{n\to\infty} \int_{\mathbb{R}} f\Big(\frac{t}{\sqrt{n}}\Big) M_n(dt) = \frac{1}{\pi}\int_{-\sqrt{2}}^{\sqrt{2}} f(u)\sqrt{2-u^2}\,du.$$

Proof.

By the Lévy-Cramér Theorem it amounts to showing that

$$\lim_{n\to\infty} \widehat{M_n}\Big(\frac{\tau}{\sqrt{n}}\Big) = \widehat{\sigma_a}(\tau).$$

Introduce the function

$$F(\tau) = \frac{2}{\pi}\int_{-1}^{1} e^{-it\tau}\sqrt{1-t^2}\,dt.$$

Up to a simple factor it is a Bessel function:

$$J_1(\tau) = \frac{\tau}{2} F(\tau).$$

One uses power series expansions:

$$\widehat{M_n}\Big(\frac{\tau}{\sqrt{n}}\Big) = e^{-\frac{\tau^2}{4n}} \sum_{k=0}^{n-1} (-1)^k c_k(n) \frac{1}{k!(k+1)!} \Big(\frac{\tau^2}{2}\Big)^k,$$

with

$$c_k(n) = \frac{(n-1)(n-2)\dots(n-k)}{n^k},$$

and

$$F(\tau\sqrt{2}) = \sum_{k=0}^{\infty} (-1)^k \frac{1}{k!(k+1)!} \Big(\frac{\tau^2}{2}\Big)^k.$$

It follows that

$$\lim_{n\to\infty} \widehat{M_n}\Big(\frac{\tau}{\sqrt{n}}\Big) = F(\sqrt{2}\tau) = \widehat{\sigma_{\sqrt{2}}}(\tau).$$

5 The probabilities $A_n(B)$

For a Borel set $B \subset \mathbb{R}$, one denotes by $A_n(B)$ the probability, for a random matrix x, to have no eigenvalues in B: the set B is a gap in the spectrum of x. Let $\lambda_{\max}$ be the largest eigenvalue of the random matrix x. Then

$$\mathbb{P}_n(\{\lambda_{\max} \leq \alpha\}) = A_n(]\alpha, \infty[).$$

The probability for a random matrix x to be positive definite is given by

$$A_n(]-\infty, 0[) = \mathbb{P}_n(\overline{\Omega_n}),$$

where $\Omega_n \subset Herm(n, \mathbb{F})$ denote the cone of positive definite Hermitian matrices. We will see that, using the fact that the n-process of the eigenvalues is determinantal, the probabilities $A_n(B)$ can be evaluated in terms of Fredholm determinants.

Fredholm determinant

Let (X, μ) be a measured space with $\mu(X) < \infty$, and $K(x, y)$ a bounded measurable kernel on X. For $z \in \mathbb{C}$, the Fredholm determinant $D(z) = \mathrm{Det}(I - zK)$ is defined by the series

$$
\begin{aligned}
D(z) &= \mathrm{Det}(I - zK) = 1 - z \int_X K(x, x)\mu(dx) + \cdots \\
&\quad + \frac{(-1)^n}{n!} z^n \int_{X^n} K \begin{pmatrix} x_1 & \dots & x_n \\ x_1 & \dots & x_n \end{pmatrix} \mu(dx_1) \dots \mu(dx_n) + \cdots .
\end{aligned}
$$

One shows that this series converges for all $z \in \mathbb{C}$, and define an entire function of z. Define the integral operator

$$(Lf)(x) = \int_X K(x, y) f(y) \mu(dy).$$

If L is of finite rank, then

$$\mathrm{Det}(I - zK) = \det(I - \lambda L).$$

Theorem 5.1. *(Mercer) Assume X to be a compact topological space, and* $\mathrm{supp}(\mu) = X$. *If the kernel K is continuous, Hermitian, of positive type, then L is nuclear, and*

$$\mathrm{Det}(I - zK) = \prod_k (1 - z\alpha_k),$$

where α_k are the positive eigenvalues of L.

Recall the modified Christoffel-Darboux kernel

$$K_n(s, t) = \sum_{k=0}^{n-1} \varphi_k(s) \varphi_k(s).$$

Proposition 5.2. *Assume $B \subset \mathbb{R}$ to be of finite measure $\mu(B) < \infty$. Then*

$$A_n(B) = \mathrm{Det}_B(I - K_n).$$

The index B means that the kernel $K_n(s, t)$ is restricted to B.

Let χ be the characteristic function of the set B. Then the characteristic function of the set $\{\forall j,\ x_j \notin B\}$ is

$$\prod_{j=1}^{n} \big(1-\chi(x_j)\big).$$

Therefore

$$A_n(B) = \int_{\mathbb{R}^n} \prod_{j=1}^{n} \big(1-\chi(x_j)\big) q_n(x_1,\dots,x_n) dx_1 \dots dx_n.$$

More generally we will compute

$$A(z) = \int_{\mathbb{R}^n} \prod_{j=1}^{n} \big(1-z\chi(x_j)\big) q_n(x_1,\dots,x_n) dx_1 \dots dx_n.$$

Recall the formulas for the elementary symmetric functions

$$\begin{array}{rcl} \sigma_1(\alpha_1,\dots,\alpha_n) & = & \alpha_1 + \cdots \alpha_n, \\ \sigma_2(\alpha_1,\dots,\alpha_n) & = & \sum_{i<j} \alpha_i \alpha_j, \\ \dots & & \dots \\ \sigma_n(\alpha_1,\dots,\alpha_n) & = & \alpha_1 \dots \alpha_n, \end{array}$$

and

$$\prod_{j=1}^{n} (1-z\alpha_j) = 1 - \sigma_1 z + \sigma_2 z^2 - \cdots + (-1)^n \sigma_n z^n.$$

Therefore

$$\prod_{j=1}^{n} \big(1-z\chi(x_j)\big) = \sum_{k=0}^{n} (-1)^k z^k \sigma_k\big(\chi(x_1),\dots,\chi(x_n)\big).$$

We compute now the integral of each term. By using the symmetry of the function q we obtain

$$\begin{array}{rl} & \int_{\mathbb{R}^n} \sigma_k\big(\chi(x_1),\dots,\chi(x_n)\big) q_n(x_1,\dots,x_n) dx_1 \dots dx_n \\ = & \binom{n}{k} \int_{\mathbb{R}^n} \chi(x_1)\dots\chi(x_k) q_n(x_1,\dots,x_n) dx_1 \dots dx_n \end{array}$$

$$= \frac{1}{k!}\int_{B^k} R_k(x_1,\dots,x_k)dx_1\dots dx_k,$$

where R_k is the k-th correlation function.

We get finally

$$A(z)=\sum_{k=0}^{n}\frac{(-1)^k}{k!}z^k\int_{B^k} R_k(x_1,\dots,x_k)dx_1\dots dx_k.$$

We use now the fact that the n-point process of the eigenvalues is determinantal and get

$$\begin{aligned} A(z) &= \sum_{k=0}^{n}\frac{(-1)^k}{k!}z^k\int_{B^k} K_n\begin{pmatrix} x_1 & \dots & x_k \\ x_1 & \dots & x_k\end{pmatrix} dx_1\dots dx_k \\ &= Det_B(I-zK_n). \end{aligned}$$

Asymptotics of the probability $A_n(B)$

Define the kernel $\mathcal{K}$,

$$\mathcal{K}(\xi,\eta)=\frac{1}{\pi}\frac{\sin(\xi-\eta)}{\xi-\eta}.$$

Theorem 5.3. *Let $B\subset\mathbb{R}$ be a Borel set. Then*

$$\lim_{n\to\infty} A_n\Big(\frac{1}{\sqrt{2n}}B\Big)=\mathrm{Det}_B(I-\mathcal{K}).$$

Define

$$\tilde{K}_n(\xi,\eta)=K_n\Big(\frac{1}{\sqrt{2n}}\xi,\frac{1}{\sqrt{2n}}\eta\Big)\frac{1}{\sqrt{2n}}.$$

By using asymptotics of the Hermite functions φ_n, on shows that

$$\lim_{n\to\infty}\tilde{K}_n(\xi,\eta)=\mathcal{K}(\xi,\eta).$$

In case $B=[-a,a]$, consider the integral operator L with kernel $\mathcal{K}$:

$$(Lf)(\xi)=\int_{-a}^{a}\mathcal{K}(\xi,\eta)f(\eta)d\eta.$$

The eigenfunctions ψ_j of L are prolate spheroidal wave functions: $L\psi_j = \alpha_j\psi_j$, and

$$\mathrm{Det}_B(I-\mathcal{K}) = \prod_{j=1}^{\infty}(1-\alpha_j).$$

One can show that the eigenvalues α_j go to zero very rapidly (See [Mehta,1991]).

6 Logarithmic potential theory

Energy, equilibrium measure

Let $\Sigma \subset \mathbb{R}$ be a interval and Q a function defined on Σ with values on $]-\infty,\infty]$, continuous on $\mathrm{int}(\Sigma)$. If Σ is unbounded, it is assumed that

$$\lim_{|x|\to\infty}\Big(Q(x)-\log(x^2+1)\Big)=\infty.$$

Some examples

- $\Sigma = \mathbb{R}$, $Q(x) = x^2$.
- $\Sigma = [-1,1]$, $Q(x) = \alpha\log\frac{1}{1-x} + \beta\log\frac{1}{1+x}$.
- $\Sigma = [0,\infty[$, $Q(x) = x + \alpha\log\frac{1}{x}$.

If μ is a probability measure supported by Σ, the **energy** $E(\mu)$ of μ is defined by

$$\begin{aligned} E(\mu) &= \int_{\Sigma\times\Sigma}\log\frac{1}{|x-y|}\mu(dx)\mu(dy) + \int_{\Sigma}Q(x)\mu(dx) \\ &= \int_{\Sigma}U^{\mu}(x)\mu(dx) + \int_{\Sigma}Q(x)\mu(dx), \end{aligned}$$

where U^{μ} is the **logarithmic potential** of the measure μ:

$$U^{\mu}(x) = \int_{\Sigma}\log\frac{1}{|x-y|}\mu(dy).$$

One shows that the energy $E(\mu)$ is bounded from below: $E(\mu) \geq m$, where

$$m = \inf_{x\in\Sigma}\big(Q(x)-\log(x^2+1)\big).$$

We define

$$E^* = \inf\{E(\mu) \mid \mu \in \mathrm{Prob}(\Sigma)\}.$$

Theorem 6.1. *There is a unique probability measure* $\mu^* \in \mathrm{Proba}(\Sigma)$ *such that*

$$E(\mu^*) = E^*.$$

The support of μ^* *is compact.*

μ^* is called the **equilibrium measure**.

Prokhorov's criterium

Consider on $\mathrm{Prob}(\mathbb{R})$ the tight topology. For a set of measures $M \subset \mathrm{Prob}(\mathbb{R})$ to be relatively compact it is necessary and sufficient that, for any $\varepsilon > 0$, there is a compact set $K \subset \mathbb{R}$ such that, for all measures μ in M,

$$\mu(K) \geq 1 - \varepsilon.$$

As a consequence: let $M \subset \mathrm{Prob}(\mathbb{R})$. Assume that there is a measurable function $h \geq 0$ such that

$$\lim_{|x|\to\infty} h(x) = \infty,$$

and a constant C such that, for all $\mu \in M$,

$$\int_{\mathbb{R}} h(x)\mu(dx) \leq C.$$

Then M is relatively compact.

a) Existence One shows that the map

$$\mu \mapsto E(\mu)$$

is lower semi-continuous for the tight topology:

If μ_n is a sequence in $\mathrm{Prob}(\mathbb{R})$ supported by Σ which converges to the measure μ for the tight topology, one shows that

$$E(\mu) \leq \liminf_{n\to\infty} E(\mu_n).$$

It follows that, for $C > E^*$, the set

$$M_C = \{\mu \in \mathrm{Proba}(\Sigma) \mid E(\mu) \leq C\}$$

is closed. We will prove that this set is relatively compact by using Prokhorov's criterium. Define

$$k(x,y) = \log\frac{1}{|x-y|} + \frac{1}{2}Q(x) + \frac{1}{2}Q(y).$$

Then

$$E(\mu) = \int_{\Sigma\times\Sigma} k(x,y)\mu(dx)\mu(dy).$$

Define

$$h(x) = Q(x) - \log(x^2+1).$$

Then

$$k(x,y) \geq \frac{1}{2}h(x) + \frac{1}{2}h(y)$$

and, for $\mu \in M_C$,

$$\int_\Sigma h(x)\mu(dx) \leq C.$$

We have proved that M_C is compact. Therefore there exists $\mu = \mu^* \in \mathrm{Prob}(\Sigma)$ such that

$$E(\mu^*) = E^* := \inf\{E(\mu) \mid \mu \in \mathrm{Prob}(\Sigma)\}.$$

b) One shows that a measure $\mu \in \mathrm{Proba}(\Sigma)$ with $E(\mu) = E^*$ is compactly supported.

c) Uniqueness One shows that the map

$$\mu \mapsto E(\mu),$$

restricted to the set of compactly supported measures in $\mathrm{Prob}(\Sigma)$, is strictly convex. Consider $\mu_1 \neq \mu_2 \in \mathrm{Prob}(\Sigma)$, and, for $0 \leq t \leq 1$, the energy of $(1-t)\mu_1 + t\mu_2$:

$$E\big((1-t)\mu_1 + t\mu_2\big) = at^2 + bt + c,$$

with

$$a = \int_\mathbb{R} \log\frac{1}{|x-y|}\nu(dx)\nu(dy), \quad \nu = \mu_1 - \mu_2.$$

The coefficient a is > 0 by the following Fourier analysis lemma:

Lemma 6.2. *Let ν be a signed measure on $\mathbb{R}$ with compact support and zero integral. Then*

$$\int_{\mathbb{R}^2} \log \frac{1}{|x-y|} \nu(dx)\nu(dy) = \int_0^\infty \frac{|\widehat{\nu}(t)|^2}{t} dt,$$

where $\widehat{\nu}$ is the Fourier transform of ν:

$$\widehat{\nu}(t) = \int_{\mathbb{R}} e^{itx} \nu(dx).$$

The following statement, which is not the best possible, works for the examples we have in mind.

Proposition 6.3. *Let $\mu \in \mathrm{Proba}(\Sigma)$ with compact support. Assume that the potentiel U^μ of μ is continuous and that there is a constant C such that*
(i) $U^\mu(x) + \frac{1}{2}Q(x) \geq C$ *on* Σ,
(ii) $U^\mu(x) + \frac{1}{2}Q(x) = C$ *on* $\mathrm{supp}(\mu)$.
Then μ is the equilibrium measure: $\mu = \mu^*$.

The constant C is called the *(modified) Robin constant*. Observe that

$$E^* = C + \frac{1}{2} \int_\Sigma Q(x)\mu^*(dx).$$

The idea of the proof is that μ^* should be a critical point of the energy under the condition $\mu(\Sigma) = 1$. The differential of the energy should be proportional to the linear form $\mu \to \mu(\Sigma)$. For two measures μ and ν,

$$\begin{aligned} E(\mu+\nu) = E(\mu) &+ 2\int_\Sigma \Big(U^\mu(x) + \frac{1}{2}Q(x)\Big)\nu(dx) \\ &+ \int_{\Sigma^2} \log \frac{1}{|x-y|} \nu(dx)\nu(dy). \end{aligned}$$

This gives $U^{\mu^*}(x) + \frac{1}{2}Q(x) = C$. This argument is not correct, because one has to take into account that the minimum is relative to the set of positive measures μ with $\mu(\Sigma) = 1$.

Examples

a) $\Sigma = [-1,1]$, $Q(x) = 0$. Then the equilibrium measure μ^* is the arcsinus law:

$$\int_{[-1,1]} \varphi(x)\mu^*(dx) = \frac{1}{\pi} \int_{-1}^1 \varphi(x) \frac{dx}{\sqrt{1-x^2}}.$$

b) $\Sigma = \mathbb{R}$, $Q(x) = x^2$. Then the equilibrium measure μ^* is the semi-circle law:

$$\int_{\mathbb{R}} \varphi(x)\mu^*(dx) = \frac{1}{\pi}\int_{-\sqrt{2}}^{\sqrt{2}} \varphi(x)\sqrt{2-x^2}dx.$$

c) $\Sigma =]0,\infty[$, $Q(x) = x + (c-1)\log\frac{1}{x}$ $(c > 1)$. Then the equilibrium measure is the Marchenko-Pastur law

$$\int_{]0,\infty[} \varphi(x)\mu^*(dx) = \frac{1}{2\pi}\int_a^b \varphi(x)\sqrt{(x-a)(b-x)}\frac{dx}{x},$$

with $a = (\sqrt{c}-1)^2$, $b = (\sqrt{c}+1)^2$.

7 Pastur's formula

Theorem 7.1. *Let $\Sigma = \mathbb{R}$, and Q a polynomial of even degree $2k$ $(k \geq 1)$, convex. Then the equilibrium measure μ^* is given by*

$$\int_{\mathbb{R}} f(x)\mu^*(dx) = \frac{1}{\pi}\int_a^b f(x)q(x)\sqrt{(x-a)(b-x)}dx,$$

where q is the polynomial of degree $2k-2$ given by

$$q(x) = \frac{1}{2\pi}\int_a^b \frac{Q'(x)-Q'(t)}{x-t}\frac{dt}{\sqrt{t-a)(b-t)}}.$$

The numbers a and b are determined by the conditions

$$\int_a^b \frac{Q'(t)}{\sqrt{t-a)(b-t)}}dt = 0, \qquad \int_a^b \frac{tQ'(t)}{\sqrt{(t-a)(b-t)}}dt = 2\pi.$$

Example

For $Q(x) = x^2$,

$$\frac{Q'(x)-Q'(t)}{x-t} = 2.$$

Hence

$$q(z) = \frac{1}{\pi}\int_a^b \frac{dt}{\sqrt{(t-a)(b-t)}} = 1.$$

The numbers a and b are determined by

$$\int_a^b \frac{2t}{\sqrt{(t-a)(b-t)}}dt = 0, \quad \int_a^b \frac{2t^2}{\sqrt{(t-a)(b-t)}}dt = 2\pi.$$

The first equation gives $a+b=0$, and the second $a^2=b^2=2$. Therefore μ^* is the semi-circle law of radius $\sqrt{2}$.

For the proof of Pastur's formula we will use some complex analysis. The **Cauchy-Stieltjes transform** of a bounded measure μ on $\mathbb{R}$ is the function defined on $\mathbb{C} \setminus \operatorname{supp}(\mu)$ by

$$G_\mu(z) = \int_{\mathbb{R}} \frac{1}{z-t}\mu(dt).$$

The Cauchy-Stieltjes transform is holomorphic.

We will use some properties of the boundary value distribution of a holomorphic function. Let f be holomorphic in $\mathbb{C} \setminus \mathbb{R}$. It is said of moderate growth near $\mathbb{R}$ if, for every compact set $K \subset \mathbb{R}$, there are $\varepsilon > 0$, $N > 0$, and $C > 0$ such that

$$|f(x+iy)| \leq \frac{C}{|y|^N} \quad (x \in K,\ 0 < |y| \leq \varepsilon).$$

Then the formula, with $\varphi \in \mathcal{D}(\mathbb{R})$,

$$\langle T, \varphi \rangle = \lim_{\varepsilon \to 0, \varepsilon > 0} \int_{\mathbb{R}} \varphi(t)\big(f(t+i\varepsilon) - f(t-i\varepsilon)\big)dt,$$

defines a distribution on $\mathbb{R}$ which is denoted by $T = [f]$, and called the difference of boundary values of f. One shows that the function extends as a holomorphic function in $\mathbb{C} \setminus \operatorname{supp}([f])$. In particular, if $[f] = 0$, then f extends as a holomorphic function in $\mathbb{C}$.

Theorem 7.2. *Let μ be a bounded positive measure on $\mathbb{R}$.*

(i) *The Cauchy-Stieltjes transform G_μ is of moderate growth near $\mathbb{R}$, and*

$$[G_\mu] = -2i\pi\mu.$$

(ii) *Assume that the support of μ is compact. Let F be holomorphic in $\mathbb{C} \setminus \mathbb{R}$, of moderate growth near $\mathbb{R}$, such that*

$$[F] = -2i\pi\mu, \text{ and } \lim_{|z| \to \infty} F(z) = 0,$$

then $F = G_\mu$.

Example 1

Consider the probability measure μ on $\mathbb{R}$ defined, for $a < b$, by

$$\int_{\mathbb{R}} f(t)\mu(dt) = \frac{1}{\pi}\int_a^b f(t)\frac{dt}{\sqrt{(t-a)(b-t)}}.$$

The function F, defined, for $z \notin [a,b]$, by

$$F(z) = \frac{1}{\sqrt{z-a)(z-b)}},$$

satisfies

$$[F] = -2i\pi\mu, \qquad \lim_{|z|\to\infty} F(z) = 0.$$

Therefore $G_\mu = F$.

Example 2

The semi-circle law σ_a of radius a is defined by

$$\int_{\mathbb{R}} f(t)\sigma_a(dt) = \frac{2}{\pi a^2}\int_{-a}^{a} f(t)\sqrt{a^2-t^2}dt.$$

The function f defined, for $z \notin [-a,a]$, by

$$F(z) = \sqrt{z^2-a^2},$$

satisfies $[f] = i\pi a^2\mu$. Consider the Laurent expansion of f at infinity:

$$f(z) = z\sqrt{1-\frac{a^2}{z^2}} = z - \frac{a^2}{2}\frac{1}{z} + \cdots .$$

The function F, defined for $z \notin [-a,a]$, by

$$F(z) = -\frac{2}{a^2}f(z) + \frac{a^2}{2}z,$$

satisfies

$$[F] = -2i\pi\sigma_a, \qquad \lim_{|z|\to\infty} F(z) = 0.$$

Therefore

$$G_{\sigma_a}(z) = \frac{a^2}{2}z - \frac{2}{a^2}\sqrt{z^2-a^2}.$$

Proof of Pastur's formula

Taking the derivatives in the relation (ii) of Proposition 6.3 one gets

$$vp \int \frac{1}{x-y}\mu(dy) = \frac{1}{2}Q'(x) \text{ on } \operatorname{supp}(\mu)$$

(vp : valeur principale). Such an equation has been considered by Tricomi. The boundary value of G_μ is as follows:

$$\lim_{\varepsilon \to 0, \varepsilon > 0} G_\mu(x + i\varepsilon) = vp \int \frac{1}{x-y}\mu(dy) - i\pi\mu.$$

Therefore (ii) of Proposition 6.3 leads to the relation

$$\operatorname{Re} G_\mu(x) = \frac{1}{2}\, Q'(x) \quad \bigl(x \in \operatorname{supp}(\mu)\bigr).$$

For instance, for $a = \sqrt{2}$, the semi-circle law σ_a satisfies the relation

$$\operatorname{Re} G_{\sigma_a}(x) = \frac{1}{2}\, Q'(x) = x,$$

if $Q(x) = x^2$.

Let us assume that the equilibrium measure is of the form

$$\mu(dt) = u(t)dt,$$

where u is an integrable function with support $[a, b]$. We will determine the Cauchy-Stieltjes transform of μ,

$$G(z) = \int_a^b \frac{u(t)}{z-t}dt,$$

by using the relation

$$\operatorname{Re} G(x) = \frac{1}{2}Q'(x), \text{ for } a \leq x \leq b.$$

Put

$$\tilde{G}(z) = \frac{G(z)}{\sqrt{(z-a)(z-b)}}.$$

The function $\tilde{G}$ is holomorphic in $\mathbb{C} \setminus [a,b]$, and

$$[\tilde{G}] = -i\frac{Q'(t)}{\sqrt{(t-a)(b-t)}}\chi(t),$$

where χ is the charasteristic function of $[a,b]$. Furthermore

$$\tilde{G}(z) \sim \frac{1}{z^2} \quad (|z| \to \infty).$$

By the previous theorem

$$G(z) = \frac{1}{2\pi}\int_a^b \frac{1}{z-t}\frac{Q'(t)}{\sqrt{(t-a)(b-t)}}dt,$$

which can be written

$$\tilde{G}(z) = -\frac{1}{2\pi}\frac{Q'(z)-Q'(t)}{z-t}\frac{dt}{\sqrt{t-a)(b-t)}}dt+Q'(z)\frac{1}{2\pi}\int_a^b \frac{1}{z-t}\frac{dt}{\sqrt{(t-a)(b-t)}}dt.$$

We have seen that

$$\frac{1}{\pi}\int_a^b \frac{1}{z-t}\frac{dt}{\sqrt{(t-a)(b-t)}}. = \frac{1}{\sqrt{(z-a)(z-b)}}.$$

Therefore

$$\tilde{G}(z) = -q(z) + \frac{1}{2}Q'(z)\frac{1}{\sqrt{(z-a)(z-b)}},$$

and

$$G(z) = -q(z)\sqrt{(z-a)(z-b)} + \frac{1}{2}Q'(z).$$

Let us take the difference of the boundary values:

$$[G] = -2iq(t)\sqrt{(t-a)(b-t)}\chi(t).$$

Since $[G] = -2i\pi\mu$, we get

$$u(t) = \frac{1}{\pi}q(t)\sqrt{(t-a)(b-t)}\chi(t).$$

Consider the Laurent development at infinity of $\tilde{G}$:

$$\tilde{G}(z) = \frac{1}{2\pi}\frac{1}{z-t}\frac{Q'(t)}{\sqrt{(t-a)(b-t)}}dt = \frac{a_0}{z} + \frac{a_1}{z^2} + \cdots$$

with

$$a_0 = \frac{1}{2\pi} \int_a^b Q'(t)\sqrt{(t-a)(b-t)}dt, \quad a_1 = \frac{1}{2\pi} \frac{tQ'(t)}{\sqrt{(t-a)(b-t)}}dt.$$

From this one gets the Laurent development of G:

$$\begin{aligned} G(z) &= \tilde{G}(z)\sqrt{(z-a)(b-z)} \\ &= \Big(\frac{a_0}{z} + \frac{a_1}{z^2} + \cdots\Big)\Big(z - \frac{a+b}{2} - \frac{(a-b)^2}{8}\frac{1}{z} + \cdots\Big) \\ &= a_0 + \Big(a_1 - a_0\frac{a+b}{2}\Big)\frac{1}{z} + \cdots \end{aligned}$$

Observing that $G(z) \sim \frac{1}{z}$, we get the conditions $a_0 = 0$, $a_1 = 1$. Under these conditions $\mu(dt) = u(t)dt$ is a probability measure. In fact, the function q is positive since the function Q is convex. One shows that

$$\begin{aligned} \frac{d}{dx}U^\mu(x) + \frac{1}{2}Q'(x) &= -q(x)\sqrt{(a-x)(b-x)}, \text{ if } x < a, \\ &= 0, \text{ if } a \le x \le b, \\ &= q(x)\sqrt{(x-a)(x-b)}, \text{ if } x \ge b. \end{aligned}$$

Therefore there is a constant C such that

$$\begin{aligned} U^\mu(x) + \frac{1}{2}Q(x) &= C, \text{ if } a \le x \le b, \\ &\ge C \text{ everywhere.} \end{aligned}$$

This establish that μ is actually the equilibrium measure.

8 Generalized theorem of Wigner

We come back to the n-point process of the eigenvalues. $\Sigma \subset \mathbb{R}$ is an interval. Q is a continuous function on Σ such that

$$\lim_{t\to\infty} Q(t) - \log(1+t^2) = \infty,$$

and $\beta > 0$.

On $\mathrm{Conf}_n(\Sigma)$ we consider the probability

$$\mathcal{P}_n(dx) = \frac{1}{\mathcal{Z}_n} \int_{\Sigma^n} \exp\Big(-n \sum_{i=1}^n Q(x_i)\Big) |V(x)|^\beta dx_1 \ldots dx_n.$$

The partition function $\mathcal{Z}_n$ is given by

$$\mathcal{Z}_n = \int_{\Sigma^n} \exp\Big(-n \sum_{i=1}^n Q(x_i)\Big) |V(x)|^\beta dx_1 \ldots dx_n.$$

and the statistical distribution of the eigenvalues $\mathcal{M}_n$ by

$$\int_\Sigma f(t)\mathcal{M}_n(dt) = \int_{\Sigma^n} \Big(\frac{1}{n}\sum_{i=1}^n f(x_i)\Big)\mathcal{P}_n(dx).$$

We have modified the definitions by a different scaling. This makes possible to treat more general functions Q. In the Gaussian case: $\Sigma = \mathbb{R}$, and $Q(t) = t^2$, the function Q being homogeneous, we get the simple relations

$$\begin{aligned}
&\int_{\mathbb{R}^n} \varphi(\frac{x}{\sqrt{n}})\mathbb{P}_n(dx) = \int_{\mathbb{R}^n} \varphi(y)\mathcal{P}_n(dy), \\
&\int_{\mathbb{R}} f\big(\frac{t}{\sqrt{n}}\big) M_n(dt) = \int_{\mathbb{R}} f(u)\mathcal{M}_n(du). \\
&Z_n = n^{N(\beta)}\mathcal{Z}_n,
\end{aligned}$$

with $N(\beta) = n + \frac{\beta}{2}n(n-1)$.

The asymptotic of the partition function $\mathcal{Z}_n$ and the limit of the statistical distribution $\mathcal{M}_n$ of the eigenvalues are related to the following problem in logarithmic potential theory:

Energy on $\mathrm{Prob}(\Sigma)$:

$$E(\mu) = \frac{\beta}{2}\int_{\Sigma^2} \log\frac{1}{|s-t|}\mu(ds)\mu(dt) + \int_\Sigma Q(t)\mu(dt).$$

Equilibrium energy:

$$E^* = \inf\{E(\mu) \mid \mu \in \mathrm{Prob}(\Sigma)\},$$

and μ^* is the equilibrium measure.

Theorem 8.1. (i) *The asymptotic of the partition function is given by*

$$\lim_{n\to\infty} \frac{1}{n^2} \log \mathcal{Z}_n = -E^*.$$

(ii) *For a bounded continuous function* f

$$\lim_{n\to\infty} \int_\Sigma f(t)\mathcal{M}_n(dt) = \int_\Sigma f(t)\mu^*(dt).$$

The idea of the proof comes from Laplace integrals, i.e. integrals of the form

$$Z(\lambda) = \int_U e^{-\lambda\varphi(x)} a(x)m(dx),$$

where U is an open set in $\mathbb{R}^n$, φ is a continuous function on U, a is continuous, positive and integrable, m is the Lebesgue measure. Furthermore one assumes that

$$\lim_{\|x\|\to\infty} \varphi(x) = \infty,$$

and φ attains its infimum in only one point x_0. One considers also the integral, where f is continuous and bounded:

$$I(\lambda; f) = \frac{1}{Z(\lambda)} \int_U f(x)e^{-\lambda\varphi(x)} a(x)m(dx).$$

Proposition 8.2.

$$\begin{aligned} \lim_{\lambda\to\infty} \frac{1}{\lambda} \log Z(\lambda) &= -\varphi(x_0), \\ \lim_{\lambda\to\infty} I(\lambda; f) &= f(x_0). \end{aligned}$$

However, in the situation of the integral defining the partition function $\mathcal{Z}_n$, the situation is less simple because the number n of integration variables goes to infinity. The integrant can be written

$$\exp -n^2 \Big(\frac{\beta}{2} \sum_{i\neq j} \log \frac{1}{|x_i - x_j|} \frac{1}{n^2} + \sum_{i=1}^n Q(x_i)\frac{1}{n}.$$

Heuristacally

$$\mathcal{Z}_n = \int_{\Sigma^n} \exp\Big(-n^2 E\big(\mu^{(x)}\big)\Big) dx_1 \dots dx_n,$$

with

$$\mu^{(x)} = \frac{1}{n}\sum_{i=1}^{n} \delta_{x_i}.$$

But this is not correct since $E(\mu^{(x)}) = \infty$.

The proof of Theorem 8.1 is somewhat sophisticated and we will not give it (see [Deift,1998], also [Faraut,2014]).

9 Electrostatistic model of Stieltjes

In the proof of Theorem 8.1 one has to consider points $x^{(n)} = (x_1^{(n)}, \ldots, x_n^{(n)})$ in Σ^n where the function

$$\exp\bigl(-n\sum_{i=1}^{n} Q(x_i)\bigr)|V(x)|^{\beta}$$

attains its maximum. In this section we look at the Gaussian case, and see that such a point $x^{(n)}$ is related to the zeros of the Hermite polynomials. This has been observed by Stieltjes as he was studying the distribution of the zeros of the classical orthogonal polynomials.

Consider the function F on $\mathbb{R}^n$:

$$F(x) = e^{-\|x\|^2} V_n(x)^2.$$

The function F is ≥ 0, continuous and

$$\lim_{\|x\|\to\infty} F(x) = 0.$$

We will determine the points where F attains its maximum.

Let $E = -\log F$,

$$E(x) = 2\sum_{i<j} \log\frac{1}{|x_i - x_j|} + \sum_{i=1}^{n} x_i^2.$$

Stieltjes considers that it is the energy of a system of n particules in $\mathbb{R}$. We will determine the points where the function E attains its minimum.

Proposition 9.1. *The function E attains its minimum at the $n!$ points whose coordinates are the n zeros of the Hermite polynomial H_n.*

Proof.

Such a point is a critical point.

$$\frac{\partial E}{\partial x_j} = -2 \sum_{i \neq j} \frac{1}{x_j - x_i} + 2x_j.$$

$x = (x_1, \ldots, x_n)$ is critical for E if

$$\sum_{i \neq j} \frac{1}{x_j - x_i} = x_j.$$

To a critical point x we associate the polynomial

$$p_x(t) = (t - x_1)(t - x_2) \ldots (t - x_n).$$

Consider the logarithmic derivative

$$\frac{p_x'(t)}{p_x(t)} = \sum_{i=1}^{n} \frac{1}{t - x_i}.$$

Fix j and write,

$$\frac{p_x'(t)}{p_x(t)} - \frac{1}{t - x_j} = \sum_{i \neq j} \frac{1}{t - x_i}.$$

Lemma 9.2. *f is of class $\mathcal{C}^2$ on $\mathbb{R}$, $f(t_0) = 0$, $f'(t_0) \neq 0$. Then*

$$\lim_{t \to t_0} \Big(\frac{f'(t)}{f(t)} - \frac{1}{t - t_0} \Big) = \frac{f''(t_0)}{2f'(t_0)}.$$

By the lemma, as $t \to x_j$, we get

$$\frac{p_x''(x_j)}{2p_x'(x_j)} = x_j, \text{ or } p_x''(x_j) - 2x_j p_x'(x_j) = 0.$$

Hence the polynomial $p_x''(t) - 2tp_x'(t)$ vanish at the n points $x_1, \ldots, x_n$, therefore is proportional to p_x:

$$p_x''(t) - 2tp_x'(t) = Cp_x(t).$$

Looking at the coefficient of t^n, we get $C = -2n$: the polynomial p_x satisfies the differential equation

$$p_x''(t) - 2tp_x'(t) + 2np_x(t) = 0.$$

A polynomial which is solution of this differential equation is proportional to the Hermite polynomial H_n.

10 Wishart Unitary Ensemble or Laguerre Unitary Ensemble

Ω_n : cone of positive definite $n \times n$ Hermitian matrices in the real vector space $H_n = Herm(n, \mathbb{C})$. For $p > n-1$, the Wishart law W_n^p is the probability measure on Ω_n defined by

$$\int_{\Omega_n} f(x) W_n^p(dx) = \frac{1}{Z_n(p)} \int_{\Omega_n} f(x) e^{-\operatorname{tr} x} (\det x)^{n-p} m(dx).$$

The function $Z_n(p)$ is the gamma function of the cone Ω_n:

$$\begin{aligned} Z_n(p) &= \int_{\Omega_n} e^{-\operatorname{tr} x} (det)^{p-n} m(dx) \\ &= (2\pi)^{\frac{n(n-1)}{2}} \prod_{j=1}^{n} \Gamma(p-j+1). \end{aligned}$$

The n-point process of the eigenvalues is given by the following measure on $\mathbb{R}^n$:

$$\mathbb{P}_n(dx) = \frac{1}{Z_n(p)} e^{-(x_1+\ldots+x_n)} \prod_{j=1}^{n} x_j^{n-p} V_n(x)^2.$$

Let M_n^p denote the statistical distribution of the eigenvalues.

Consider $\Sigma =]0, \infty[$, and

$$Q(t) = t + (c-1)\log\frac{1}{t}.$$

Recall the energy of a probability measure μ

$$E(\mu) = \int_{\Sigma^2} \log \frac{1}{|s-t|} \mu(ds)\mu(dt) + \int_{\Sigma} Q(t)\mu(dt).$$

It can be shown that the equilibrium measure is given by the Pastur's formula. It is the Marchenko-Pastur law μ_c:

$$\int_{\Sigma} f(t)\mu_c(dt) = \frac{1}{2\pi} \int_a^b f(t)\sqrt{(t-a)(b-t)}\frac{dt}{t},$$

where $a = (\sqrt{c}-1)^2$, $b = (\sqrt{c}+1)^2$.

We can apply the generalized Wigner Theorem. Assume that p depends on n: $p = p(n)$.

Theorem 10.1. *Assume that*

$$\lim_{n\to\infty} \frac{p(n)}{n} = c \geq 1.$$

Then, for a bounded continuous function f on $\mathbb{R}$,

$$\lim_{n\to\infty} \int_{\Sigma} f\Big(\frac{t}{n}\Big) M_n^p(dt) = \int_{\Sigma} f(t)\mu_c(dt).$$

For $0 < c < 1$, The Marchenko-Pastur has an atom at 0:

$$\int_{\Sigma} f(t)\mu_c(dt) = (1-c)f(0) + \frac{1}{2\pi} \int_a^b f(t)\sqrt{(t-a)(b-t)}\frac{dt}{t}.$$

The proof has to be modified in that case.

11 The probability for a matrix to be positive

Let $(H_n, \mathbb{P}_n)$ be a Gaussian ensemble, and $\Omega_n \subset H_n$ the cone of positive definite matrices. What can be said about the numbers $p_n = \mathbb{P}_n(\Omega_n)$, the probability for a matrix x, to be positive positive ?

It is the probability that all eigenvalues are > 0. If the eigenvalues were independant random variables, this probability would be $\frac{1}{2^n}$. But it is not the case. For $n = 2$, one computes easily

$$p_2 = \frac{2-\sqrt{2}}{4} \simeq 0.14,$$

much less that $\frac{1}{4} = 0.25$.

For $n \leq 5$, the numbers p_n can be obtained from computations which can be found in Kuriki'thesis (1992). In case of $V_n = Sym(n, \mathbb{R})$ $(\beta = 1)$:

$$\begin{aligned}
p_1 &= & & 0,5 \\
p_2 &= & \frac{2-\sqrt{2}}{4} \simeq & & 0,14 \\
p_3 &= & \frac{\pi - 2\sqrt{2}}{4\pi} \simeq & & 0,023 \\
p_4 &= & \frac{(4-\sqrt{2})\pi - 8}{16\pi} \simeq & & 0,002 \\
p_5 &= & \frac{3\pi - 8 - \sqrt{2}}{24\pi} \simeq & & 0,00014
\end{aligned}$$

Actually p_n goes to 0 very rapidly.

The probability $p_n = \mathbb{P}_n(\Omega_n)$ is given by the integral

$$p_n = \frac{1}{C_n} \int_{\Omega_n} \exp - \mathrm{tr}\,(x^2) m(dx),$$

with

$$C_n = \int_{H_n} \exp - \mathrm{tr}\,(x^2) m(dx).$$

By using the Weyl integration formula,

$$p_n = \frac{Z_n^+}{Z_n},$$

with

$$Z_n = \int_{\mathbb{R}^n} e^{-\frac{\beta}{2}(x_1^2 + \cdots + x_n^2)} |V_n(x)|^\beta dx_1 \ldots dx_n,$$

and

$$Z_n^+ = \int_{\mathbb{R}_+^n} e^{-\frac{\beta}{2}(x_1^2 + \cdots + x_n^2)} |V_n(x)|^\beta dx_1 \ldots dx_n,$$

For simplicity we will only consider the case of GUE: $\beta = 2$.

Let E^* be the equilibrium energy for $\Sigma = \mathbb{R}$, $Q(t) = t^2$: the minimum of

$$E(\mu) = \int_{\mathbb{R}^2} \log \frac{1}{|s-t|} \mu(ds)\mu(dt) + \int_{\mathbb{R}} Q(t)\mu(dt),$$

for all $\mu \in \mathrm{Prob}(\mathbb{R})$,

Then

$$\lim_{n\to\infty} \frac{1}{n^2} \log Z_n = -E^*.$$

Let also E_+^* be the equilibrium energy for $\Sigma = [0, \infty[$, $Q(t) = t^2$: the minimum of

$$E(\mu) = \int_{\mathbb{R}_+^2} \log \frac{1}{|s-t|} \mu(ds)\mu(dt) + \int_{\mathbb{R}_+} Q(t)\mu(dt),$$

for all $\mu \in \mathrm{Prob}(\mathbb{R}_+)$,

Then

$$\lim_{n\to\infty} \frac{1}{n^2} \log Z_n^+ = -E_+^*.$$

In the case $(\mathbb{R}, Q(t) = t^2)$, the equilibrium measure is the semi-circle law, and the equilibrium energy is

$$E^* = \frac{3}{4} + \frac{1}{2}\log 2.$$

In the case $(\mathbb{R}_+, Q(t) = t^2)$, the equilibrium measure has been determined by Dean and Majumdar, and the equilibium energy is

$$E_+^* = \frac{3}{4} + \frac{1}{2}\log 2 + \frac{1}{2}\log 3.$$

Theorem 11.1. *(Dean & Majumdar)*

$$\lim_{n\to\infty} \log p_n = -\frac{1}{2}\log 3.$$

For the probability space $(H_n, \mathbb{P}_n)$, consider the conditionnal statistical distribution of the eigenvalues with the condition :$x \in \Omega_n$, i.e. the matrice x is positive definite, or the eigenvalues of x are > 0: for a function φ on $\mathbb{R}$,

$$\int_{\mathbb{R}} \varphi(t) M_n^+(dt) = \mathbb{E}_n\Big(\mathrm{tr}\,\varphi(x) \mid x \in \Omega_n\Big).$$

Let $\mu{*}_+$ be the equilibrium measure in case $(\mathbb{R}_+, Q(t) = t^2)$

Theorem 11.2. *(Dean & Majumdar)*
The conditional statistical distribution $\mathcal{M}_n^+$ converges to μ_+^*, as $n \to \infty$.

By using a formula similar to Pastur's formula, one obtains the density of the Dean-Majumdar law: for a function f on $\mathbb{R}$,

$$\int_{\mathbb{R}} f(t)\mu_+^*(dt) = \frac{1}{\pi}\int_0^b f(t)\Big(t + \frac{b}{2}\Big)\sqrt{\frac{b-t}{t}}dt,$$

with $b = \frac{2}{3}\sqrt{6}$.

As we did in Section 8 we rescale the Gaussian probability:

$$\mathcal{P}_n(dx) = \frac{1}{C_n}\exp\big(-n\mathrm{tr}(x^2)\big)V_n(x)^2 m(dx).$$

Consider the condition, for a matrix x, that its eigenvalues are all $> \sigma$. The conditional statistical distribution $\mathcal{M}_n^\sigma$ of the eigenvalues is given by, for a Borel ser $B \subset \mathbb{R}$,

$$\mathcal{M}_n^\sigma(B) = \mathcal{E}_n\big(M_n^{(x)}(B) \mid \text{spectrum}(x) \subset [\sigma, \infty[\big),$$

and for a function φ defined on $\mathbb{R}$,

$$\int \varphi(t)\mathcal{M}_n^\sigma(dt) = \mathcal{E}_n\big(\mathrm{tr}\,(x) \mid \text{spectrum}(x) \subset [\sigma, \infty[\big).$$

For the associated problem in Logarithmic Potential Theory, $\Sigma = [\sigma, \infty[$, $Q(t) = t^2$. Let μ_σ^* be the equilibrium measure. For a continuous bounded function f on $\mathbb{R}$,

$$\lim_{n\to\infty} \int f(t)\mathcal{M}_n^\sigma(dt) = \int f(t)\mu_\sigma^*(dt).$$

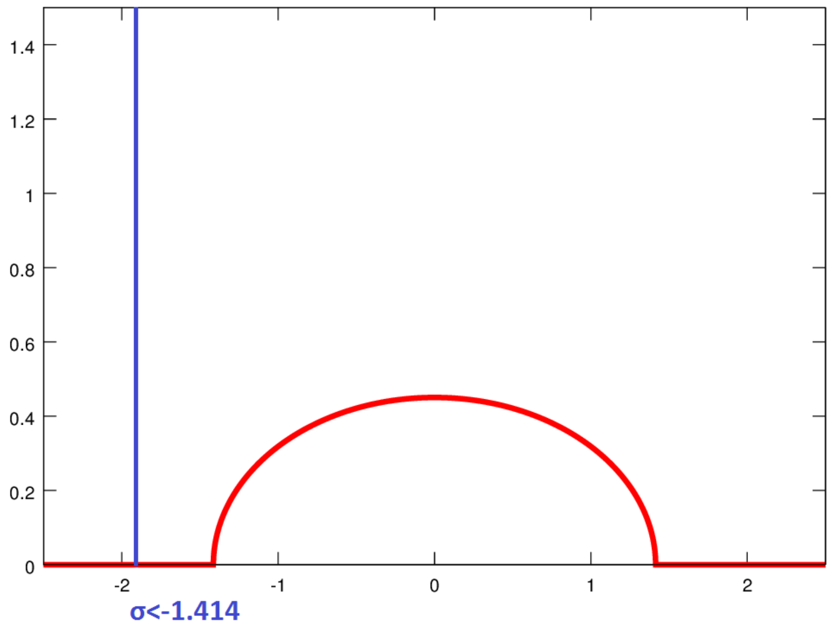
For $\sigma \leq \sqrt{2}$, the conditional statistical distribution of the eigenvalues is the semi-circle law.

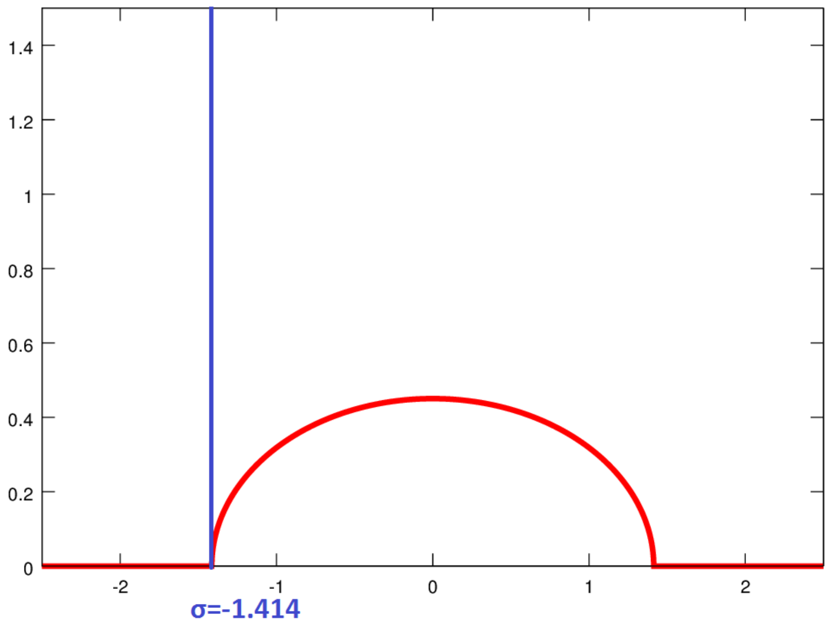
For $\sigma = 0$, the conditional statistical distribution of the eigenvalues is the Dean-Majumdar law.

For $\sigma \geq -\sqrt{2}$, the equilibrium measure μ_σ^* has been determined in [Dean-Majumdar,2008]:

$$\int f(t)\mu_\sigma^*(dt) = \frac{1}{\pi}\int_a^b f(t)\sqrt{\frac{b-t}{t-q}}\Big(t + \frac{b-a}{2}\Big)dt,$$

with $a = \sigma$, $b = \frac{1}{3}\sigma + \frac{2}{3}\sqrt{\sigma^2 + 6}$. ONe checks that, for σ^*_σ is nothing but the semi-circle law, and that, for $\sigma = 0$, $\mu^*_0$is the Dean-Majumdar law.

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0.8
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0.4
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-2
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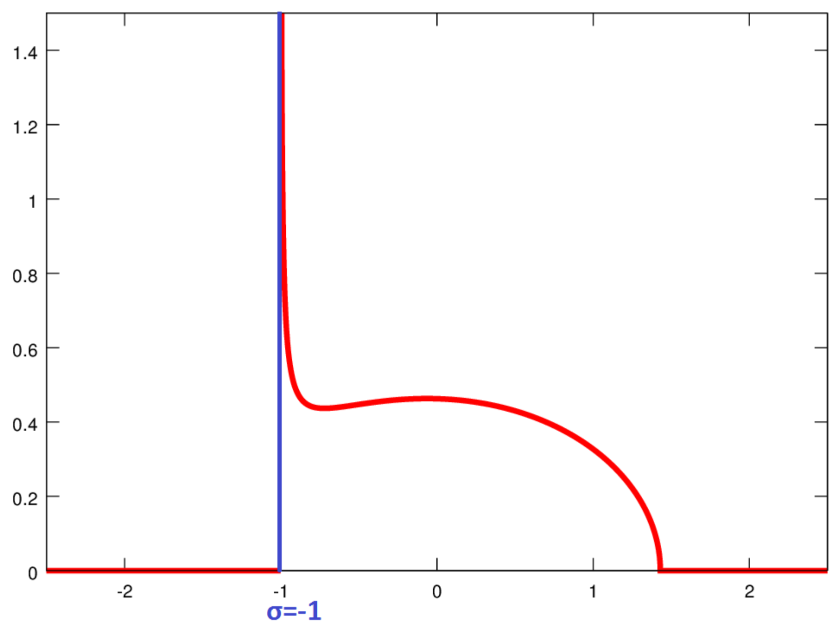
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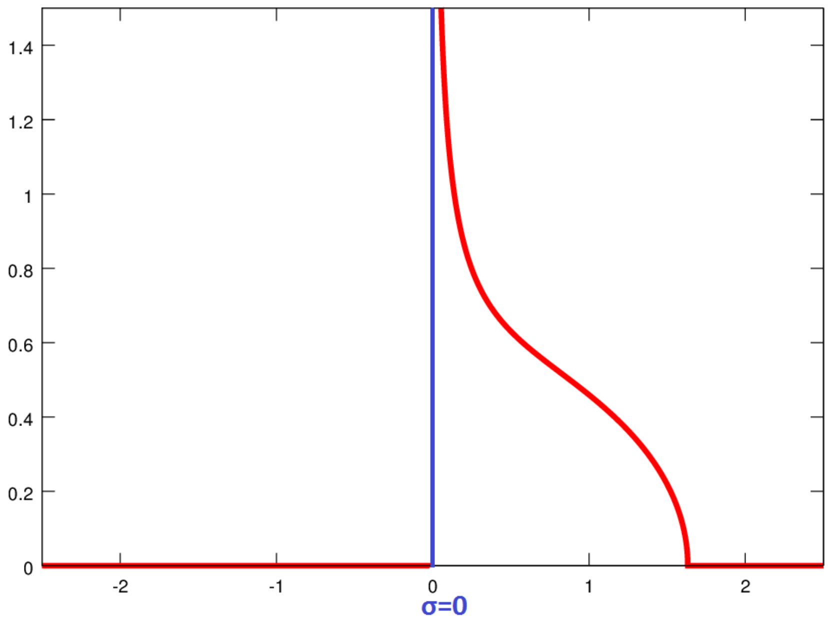
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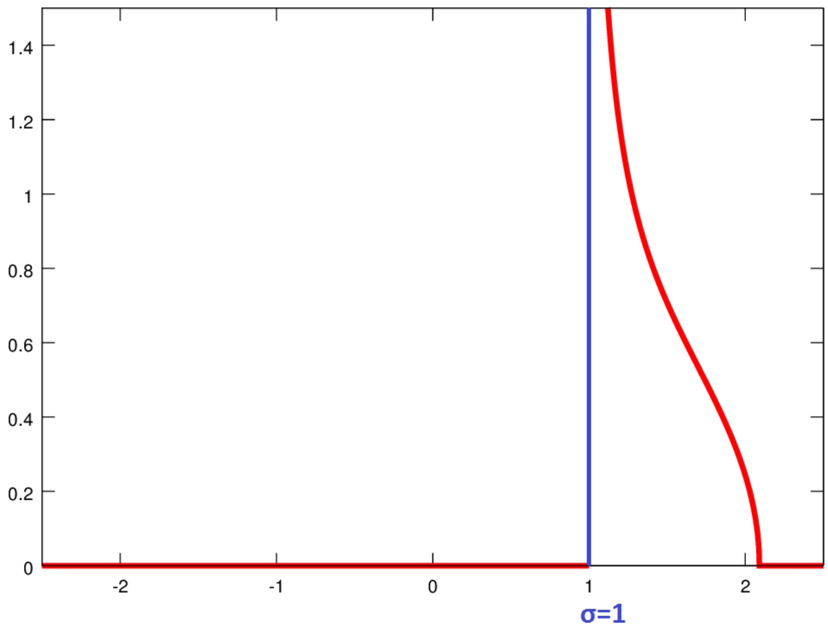


12 The Sylvester index of a random Hermitian matrix

For an $n \times n$ Hermitian matrix x let k be the number of eigenvalues which are ≥ 0, and ℓ the number of negative eigenvalues $(k+\ell = n)$. The pair (k, ℓ) is the Sylvester index of the matrix x. We consider the following conditional statistical distribution of the eigenvalues: for a Borel set $B \subset \mathbb{R}$,

$$\mathcal{M}_n^{(k,\ell)}(B) = \mathcal{E}_n\big(M_n^{x)}(B) \mid \mathrm{index}(x) = (k, \ell)\big).$$

One is interested in the asymptotic of $\mathcal{M}_n^{(k,\ell)}$ as $n \to \infty$ and $\frac{k}{\ell} \to \delta$, for $0 \leq \delta \leq 1$.

The associated problem in Logarithmic Potential Theory is as follows: the conditional equilibrium energy is defined by

$$E_\delta^* = \inf\{E(\mu) \mid \mu \in \mathrm{Proba}(\mathbb{R}),\ \mu([0, \infty[) = \delta\}.$$

It is shown in [Majumdar et al.,2011] that there is a unique probability measure μ_δ^* on $\mathbb{R}$ such that $\mu_\delta^*([0,\infty[) = \delta$ and $E(\mu_\delta^*) = E_\delta^*$. Furthermore, for a continuous bounded function f on $\mathbb{R}$,

$$\lim_{n\to\infty, \frac{k}{\ell}\to\delta} \int f(t)\mathcal{M}_n^{(k,\ell)}(dt) = \int f(t)\mu_\delta^*(dt.$$

Consider the following family of probability measures on $\mathbb{R}$. Let a, b, c be real numbers such that $a < b < 0 < c$. Define the measure $\mu^{(a,b,c)}$ on $\mathbb{R}$, supported by $[a,b]\cup[0,c]$, by

$$\int f(t)\mu^{(a,b,c)}(dt) = \frac{1}{\pi}\int_{[a,b]\cup[0,c]} f(t)\sqrt{\frac{t-a)(t-b)(c-t)}{t}}dt.$$

If

$$a+b+c=0,\ a^2+b^2+c^2=4,$$

then $\mu^{(a,b,c)}$ is a probability measure and $\mu = \mu^{(a,b,c)}$ satisfies

$$\operatorname{Re} G_\mu(x) = x \text{ on } \operatorname{supp}(\mu).$$

Furthermore if $\mu^{(a,b,c)}([0,c]) = \delta$, then $\mu^{(a,b,c)}$ is the equilibrium measure,

$$\mu^{(a,b,c)} = \mu_\delta^*.$$

Observe the limit cases: if

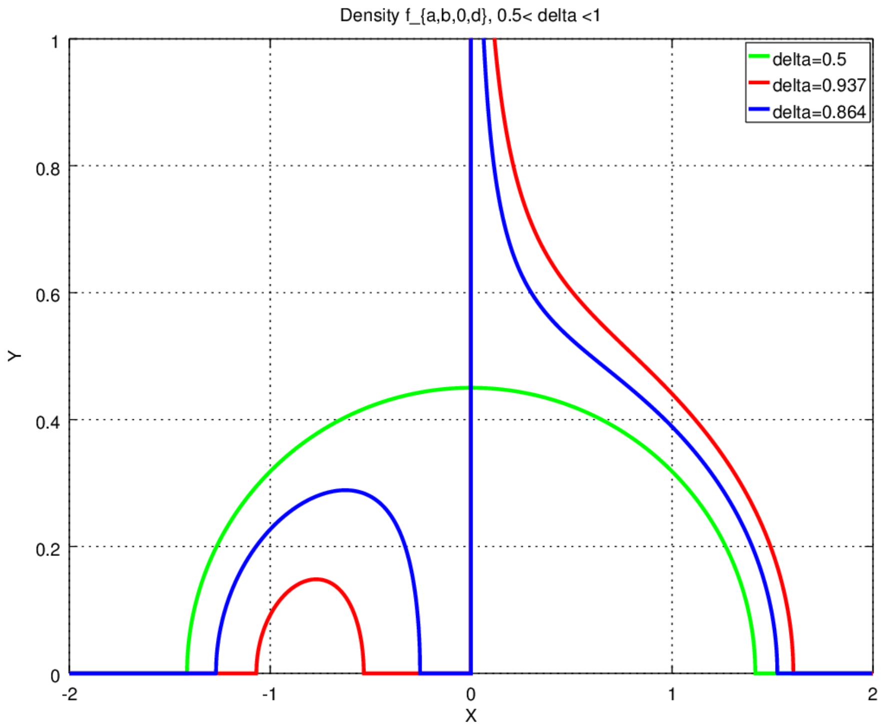
$$a = -\sqrt{2},\ b=0,\ c=\sqrt{2},$$

then $\delta = \frac{1}{2}$ and $\mu^{(a,b,c)}$ is the semi-circle law. If

$$a = b = -\frac{1}{3}\sqrt{6},\ c = \frac{2}{3}\sqrt{6},$$

then $\delta = 1$, and $\mu^{(a,b,c)}$ is the Dean-Majumdar law.

The condition $\mu^{(a,b,c)}([0,c]) = \delta$ is difficult to handle. It involves elliptic integrals. See [Pérez Castillo,2016].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delta=0.937
delta=0.864
Y
X
0
0.2
0.4
0.6
0.8
1
-2
-1
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1
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