ASYMPTOTICS OF SPHERICAL FUNCTIONS FOR LARGE RANK, AN INTRODUCTION

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This paper has been written following a talk given as an introduction to the work of Okounkov and Olshanski about asymptotics of spherical functions for compact symmetric spaces as the rank goes to infinity. This topic belongs to the asymptotic harmonic analysis, *i.e.* the study of the asymptotics of functions related to the harmonic analysis on groups or homogeneous spaces as the dimension goes to infinity. Such questions have been considered long time ago, for instance by Krein and Schoenberg for Euclidean spaces, spheres and real hyperbolic spaces, which are Riemannian symmetric spaces of rank one. The behaviour is very different when the rank is unbounded, and new phenomenons arise in that case.

In this introductory paper we present the scheme developped by Okounkov and Olshanski for studying limits of spherical functions on a compact symmetric space $G(n)/K(n)$ as the rank n goes to infinity. These limits are identified as spherical functions for the Olshanski spherical pair (G, K), with

$$G = \bigcup_{n=1}^{\infty} G(n), \quad K = \bigcup_{n=1}^{\infty} K(n).$$

We will explain results and methods in the special case of the unitary groups $U(n)$. This amounts to studying asymptotics of Schur functions. The proof uses a binomial formula for Schur functions involving shifted Schur functions. This presentation is based on two papers: [Okounkov-Olshanski, 1998c], for the type A, and [Okounkov-Olshanski,2006], for the type BC. The case of the unitary groups have been considered by Vershik and Kerov, following a slightly different method ([1982]).

In Section 5 we present without proof general results by Okounkov and Olshanski for series of classical compact symmetric spaces, and finally in Section 6 we consider the cases for which there is a determinantal formula for the spherical functions.

1. Olshanski spherical pairs. — Let us recall first what is a spherical function for a Gelfand pair. A pair (G, K), where G is a locally compact group, and K a compact subgroup, is said to be a *Gelfand pair* if the convolution algebra $L^1(K\backslash G/K)$ of K-biinvariant integrable functions on G is commutative. Fix now a Gelfand pair (G, K). A *spherical function* is a continuous function φ on G which is K-biinvariant, $\varphi(e) = 1$, and satisfies the functional equation

$$\int_K \varphi(xky)\alpha(dk) = \varphi(x)\varphi(y) \quad (x, y \in G),$$

where α is the normalized Haar measure on the compact group K. The characters χ of the commutative Banach algebra $L^1(K\backslash G/K)$ are of the form

$$\chi(f) = \int_G f(x)\varphi(x)m(dx),$$

where φ is a bounded spherical function (m is a Haar measure on the group G, which is unimodular since (G, K) is a Gelfand pair).

If the spherical function φ is of positive type (*i.e* positive definite), there is an irreducible unitary representation $(\pi, \mathcal{H})$ with $\dim \mathcal{H}^K = 1$, where $\mathcal{H}^K$ denotes the subspace of K-invariant vectors in $\mathcal{H}$, such that

$$\varphi(x) = \big(u|\pi(x)u\big),$$

with $u \in \mathcal{H}^K$, $\|u\| = 1$. The representation $(\pi, \mathcal{H})$ is unique up to equivalence. An irreducible unitary representation $(\pi, \mathcal{H})$ with $\dim \mathcal{H}^K = 1$ is said to be *spherical* , and the set Ω of equivalence classes of spherical representations will be called the *spherical dual* for the pair (G, K). Equivalently Ω is the set of spherical functions of positive type. We will denote the spherical functions of positive type for the Gelfand pair (G, K) $\varphi(\lambda; x)$ ($\lambda \in \Omega$, $x \in G$).

Consider now an increasing sequence of Gelfand pairs $\big(G(n), K(n)\big)$:

$$G(n) \subset G(n+1),\ K(n) \subset K(n+1),\ K(n) = G(n) \cap K(n+1),$$

and define

$$G = \bigcup_{n=1}^{\infty} G(n), \quad K = \bigcup_{n=1}^{\infty} K(n).$$

We say that (G, K) is an *Olshanski spherical pair.* A *spherical function* for the Olshanski spherical pair (G, K) is a continuous function φ on G, $\varphi(e) = 1$, which is K-biinvariant and satisfies

$$\lim_{n\to\infty} \int_{K(n)} \varphi(xky)\alpha_n(dk) = \varphi(x)\varphi(y) \quad (x, y \in G),$$

where α_n is the normalized Haar measure on $K(n)$. As in the case of a Gelfand pair, if φ is a spherical function of positive type, there exists a spherical representation $(\pi, \mathcal{H})$ of G (*i.e* irreducible, unitary, with $\dim \mathcal{H}^K = 1$) such that

$$\varphi(x) = \big(u|\pi(x)u\big),$$

with $u \in \mathcal{H}^K$, $\|u\| = 1$. In the same way the spherical dual Ω is identified with the set of spherical functions of positive type. Such a function will be written $\varphi(\omega; x)$ ($\omega \in \Omega$, $x \in G$).

On Ω, seen as the set of spherical functions of positive type, we will consider the topology of uniform convergence on compact sets.

We will consider the following question. Let Ω_n be the spherical dual for the Gelfand pair $\big(G(n), K(n)\big)$, and let us write a spherical function of positive type for $\big(G(n), K(n)\big)$ as $\varphi_n(\lambda, x)$ ($\lambda \in \Omega_n$, $x \in G(n)$). For which sequences $(\lambda^{(n)})$, with $\lambda^{(n)} \in \Omega_n$, does there exist $\omega \in \Omega$ such that

$$\lim_{n\to\infty} \varphi_n(\lambda^{(n)}; x) = \varphi(\omega; x) \quad (x \in G) \ ?$$

In the cases we will consider, there is, for each n, a map

$$T_n : \Omega_n \to \Omega,$$

such that, if

$$\lim_{n\to\infty} T_n(\lambda^{(n)}) = \omega,$$

for the topology of Ω, then

$$\lim_{n\to\infty} \varphi_n(\lambda^{(n)}; x) = \varphi(\omega; x).$$

It is said that $(\lambda^{(n)})$ is a *Vershik-Kerov sequence.*

2. The unitary group. — For a compact group U, we consider the pair

$$G = U \times U,\ K = \{(u,u) \mid u \in U\} \simeq U.$$

Then $G/K \simeq U$. A K-biinvariant function f on G is identified to a central function φ on U by

$$f(x,y) = \varphi(xy^{-1}).$$

The convolution algebra $L^1(K\backslash G/K)$ is isomorphic to the convolution algebra $L^1(U)_{\mathrm{central}}$ of central integrable functions on U, which is commutative. Hence (G,K) is a Gelfand pair. We will say that a continuous central function φ is spherical if $\varphi(e)=1$, and

$$\int_U \varphi(xuyu^{-1})\alpha(du) = \varphi(x)\varphi(y) \quad (x,y \in U),$$

where α is the normalized Haar measure on U. In fact it amounts to saying that the corresponding function f on G is spherical for the Gelfand pair (G,K).

If $(\pi,\mathcal{H})$ is an irreducible representation of U, then the normalized character

$$\varphi(u) = \frac{\chi_\pi(u)}{\chi_\pi(e)}, \quad \chi_\pi(u) = \mathrm{tr}\big(\pi(u)\big),$$

is a spherical function, and all spherical functions are of that form. Hence the spherical dual Ω for the pair (G,K) is the dual $\hat{U}$ of the compact group U.

For $U = U(n)$, the unitary group, the spherical dual $\Omega_n = \widehat{U(n)}$ is identified to the set of signatures

$$\lambda = (\lambda_1, \ldots, \lambda_n),\ \lambda_i \in \mathbb{Z},\ \lambda_1 \geq \cdots \geq \lambda_n.$$

The character χ_λ of an irreducible representation in the class λ is given by a Schur function. Define, for $t = (t_1, \ldots, t_n) \in (\mathbb{C}^*)^n$, $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n$,

$$A_\alpha(t) = \det(t_j^{\alpha_i}).$$

For a signature λ, the Schur function s_λ is given by

$$s_\lambda(t) = \frac{A_{\lambda+\delta}(t)}{V(t)},$$

where $\delta = (n-1, n-2, \ldots, 1, 0)$, $V(t) = A_\delta(t)$ is the Vandermonde determinant:

$$V(t) = \prod_{i<j}(t_i - t_j).$$

For a diagonal matrix $u = \mathrm{diag}(t_1, \ldots, t_n)$,

$$\chi_\lambda(u) = s_\lambda(t).$$

3. The infinite dimensional unitary group. — The infinite dimensional unitary group $U(\infty)$ is defined as

$$U(\infty) = \bigcup_{n=1}^{\infty} U(n).$$

One associates to $U(\infty)$ the following inductive limit of Gelfand pairs:

$$\begin{aligned} G(n) &= U(n) \times U(n), \ K(n) = \big\{(u,u) \mid u \in U(n)\big\}, \\ G &= \bigcup_{n=1}^{\infty} G(n) = U(\infty) \times U(\infty), \\ K &= \bigcup_{n=1}^{\infty} K(n) = \big\{(u,u) \mid u \in U(\infty)\big\}. \end{aligned}$$

Let us first state the following result by Voiculescu [1976]. Consider a power series

$$\Phi(t) = \sum_{m=0}^{\infty} c_m t^m,$$

with

$$c_m \geq 0, \ \Phi(1) = \sum_{m=0}^{\infty} c_m = 1, \ |t| \leq 1.$$

Define the function φ on $U(\infty)$ by

$$\varphi(g) = \det \Phi(g).$$

This means that the function φ is central, and, if $g = \operatorname{diag}(t_1, \ldots, t_n, 1, \ldots)$, then

$$\varphi(g) = \Phi(t_1) \ldots \Phi(t_n).$$

THEOREM 3.1 (VOICULESCU, 1976). — *The function φ is of positive type if and only if Φ has the following form*

$$\Phi(t) = e^{\gamma(t-1)} \prod_{k=1}^{\infty} \frac{1 + \beta_k(t-1)}{1 - \alpha_k(t-1)},$$

with

$$\alpha_k \geq 0, \ 0 \leq \beta_k \leq 1, \ \gamma \geq 0, \ \sum_{k=1}^{\infty} (\alpha_k + \beta_k) < \infty.$$

We propose to call such a function a *Voiculescu function.* Let Ω_0 be the set of triples $\omega = (\alpha, \beta, \gamma)$ as above. We will write

$$\Phi(t) = \Phi(\omega; t),$$

and consider on Ω_0 the topology corresponding to the uniform convergence of the functions $\Phi(\omega; \cdot)$ on the unit circle. This topology can be expressed in terms of the parameters α, β, γ as follows: for a continuous function u on $\mathbb{R}$, put

$$L_u(\omega) = \sum_{k=1}^{\infty} \alpha_k u(\alpha_k) + \sum_{k=1}^{\infty} \beta_k u(-\beta_k) + \gamma u(0).$$

Then the topology of Ω_0 coincides with the initial topology defined by the functions L_u (*i.e.* the coarser topology for which all the functions L_u are continuous).

The Voiculescu function $\Phi(\omega; t)$ is meromorphic in t, with poles $1 + \frac{1}{\alpha_k}$. It is holomorphic in the disc $|t| < r$, with $r = 1 + \inf \frac{1}{\alpha_k}$. Its logarithmic derivative is holomorphic near 1:

$$\frac{d}{dz} \log \Phi(\omega; 1 + z) = \sum_{m=0}^{\infty} a_m z^m,$$

with

$$a_0 = \gamma + \sum_{k=1}^{\infty} \alpha_k + \sum_{k=1}^{\infty} \beta_k,$$

$$a_m = \sum_{k=1}^{\infty} \alpha_k^{m+1} + (-1)^m \sum_{k=1}^{\infty} \beta_k^{m+1}, \ m \geq 1.$$

Observe that

$$a_m = L_{u_m}(\omega) \text{ with } u_m(s) = s^m.$$

THEOREM 3.2. — *The spherical functions of positive type on $U(\infty)$ are the following ones*

$$\varphi(\omega^+, \omega^-; g) = \det \Phi(\omega^+; g) \det \Phi(\omega^-; g^{-1}),$$

with ω^+, $\omega^- \in \Omega_0$.

[Vershik-Kerov,1982], [Boyer,1983].

Hence the spherical dual of the Olshanski spherical pair (G, K) associated to $U(\infty)$ is the set $\Omega = \Omega_0 \times \Omega_0$ of pairs (ω^+, ω^-).

We will now describe the sequences of signatures $(\lambda^{(n)})$ with

$$\lambda^{(n)} = (\lambda_1^{(n)}, \ldots, \lambda_n^{(n)}) \in \Omega_n,$$

for which there exists $\omega = (\omega^+, \omega^-)$ such that

$$\lim_{n\to\infty} \varphi_n(\lambda^{(n)}; g) = \varphi(\omega^+, \omega^-; g).$$

We will first consider the case of positive signatures. We say that a signature λ is positive if the numbers λ_i are ≥ 0, and we will denote by Ω_n^+ the set of positive signatures in Ω_n. One defines the Frobenius parameters $a = (a_i)$ and $b = (b_i)$ of a positive signature λ as follows

$$\begin{aligned} &a_i = \lambda_i - i \text{ if } \lambda_i > i,\ a_i = 0 \text{ otherwise},\\ &b_j = \lambda'_j - j + 1 \text{ if } \lambda'_j > j - 1,\ b_j = 0 \text{ otherwise}, \end{aligned}$$

where λ' is the transpose signature. For instance, if $\lambda = (6, 4, 4, 2, 1)$, then $a = (5, 2, 1, 0, 0)$, $b = (5, 3, 1, 0, 0)$.

We define the map

$$T_n : \Omega_n^+ \to \Omega_0,\ \lambda \mapsto \omega = (\alpha, \beta, \gamma),$$

by

$$\alpha_k = \frac{a_k}{n},\ \beta_k = \frac{b_k}{n},\ \gamma = 0.$$

Theorem 3.3. — *Let $\lambda^{(n)} = (\lambda_1^{(n)}, \ldots, \lambda_n^{(n)})$ be a sequence of positive signatures. Assume that*

$$\lim_{n\to\infty} T_n\big(\lambda^{(n)}\big) = \omega,$$

for the topology of Ω_0. Then, for $g \in U(\infty)$,

$$\lim_{n\to\infty} \varphi_n(\lambda^{(n)}; g) = \det \Phi(\omega; g),$$

uniformly on each $U(k)$.

[Vershik-Kerov,1982], [Okounkov-Olshanski,1998c].

Example

For two numbers $p, k \in \mathbb{N}$ with $p \geq k$, consider the positive signature

$$\lambda = (p, \ldots, p, 0, \ldots),$$

where p is repeated k times. The Young diagram of λ is a rectangle with sides p and k. The Frobenius parameters are $a = (a_i)$ with

$$a_i = p - i \text{ if } i \leq k, \ a_i = 0 \text{ if } i > k,$$

and $b = (b_j)$ with

$$b_j = k - j + 1 \text{ if } j \leq k, \ b_j = 0 \text{ if } j > k.$$

Observe that

$$\sum a_i + \sum b_j = kp.$$

For a continuous function u on $\mathbb{R}$,

$$L_u\big(T_n(\lambda)\big) = \sum_{i=1}^{k} \frac{p-i}{n} u\Big(\frac{p-i}{n}\Big) + \sum_{j=1}^{k} \frac{k-j+1}{n} u\Big(-\frac{k-j+1}{n}\Big).$$

Consider now two sequences $(p^{(n)})$ and $(k^{(n)})$, and let $(\lambda^{(n)})$ be the corresponding sequence of signatures. Assume that

$$p^{(n)} \sim \sqrt{n}, \ k^{(n)} \sim \sqrt{n}.$$

Then

$$\lim_{n \to \infty} L_u\big(T_n(\lambda^{(n)})\big) = u(0).$$

This means that

$$\lim_{n \to \infty} T_n\big(\lambda^{(n)}\big) = \omega,$$

with $\omega = (0, 0, 1)$, *i.e.* $\alpha_k = 0$, $\beta_k = 0$, $\gamma = 1$. Therefore

$$\lim_{n \to \infty} \varphi_n(\lambda^{(n)}; g) = \det\big(\exp(g - I)\big) = e^{\operatorname{tr}(g-I)}.$$

We consider now the general case. To a signature λ one associates two positive signatures λ^+ and λ^-: if

$$\lambda_1 \geq \cdots \lambda_p \geq 0 \geq \lambda_{p+1} \geq \cdots \geq \lambda_n,$$

then

$$\lambda^+ = (\lambda_1, \dots, \lambda_p, 0, \dots), \ \lambda^- = (-\lambda_n, \dots, -\lambda_{p+1}, 0, \dots).$$

One adds as many zeros as necessary to get positive signatures λ^+, λ^- in Ω_n^+ Then we define the map

$$T_n : \Omega_n \to \Omega = \Omega_0 \times \Omega_0$$

by extending the map T_n previously defined:

$$T_n(\lambda) = \big(T_n(\lambda^+), T_n(\lambda^-)\big).$$

THEOREM 3.4. — *Let $(\lambda^{(n)})$ be a sequence of signatures, with $\lambda^{(n)} \in \Omega_n$. Assume that*

$$\lim_{n\to\infty} T_n(\lambda^{(n)}) = \omega = (\omega^+, \omega^-).$$

Then, for $g \in U(\infty)$,

$$\lim_{n\to\infty} \varphi_n(\lambda^{(n)}; g) = \det \Phi(\omega^+; g) \det \Phi(\omega^-; g^{-1})$$

uniformly on each $U(k)$.

We will prove Theorem 3.3 in Section 5. For the proof of Theorem 3.4 see [Okounkov-Olshanski,1998c], and also [Faraut,2008]. The proof of Theorem 3.3 will involve a binomial formula for Schur functions.

4. Binomial formula for Schur functions. — We will use a formula for Schur expansions due to Hua ([Hua,1963], Theorem 1.2.1).

PROPOSITION 4.1 (HUA'S FORMULA). — *Consider n power series:*

$$f_i(w) = \sum_{m=0}^{\infty} c_m^{(i)} w^m,$$

which are convergent for $|w| < r$ for some $r > 0$. Define the function F on $\mathbb{C}^n$ by

$$F(z) = F(z_1, \dots, z_n) = \frac{\det\big(f_i(z_j)\big)}{V(z)} \quad |z_j| < r.$$

Then F admits the following Schur expansion:

$$F(z) = \sum_{m_1 \geq \cdots \geq m_n \geq 0} a_{\mathbf{m}} s_{\mathbf{m}}(z),$$

with

$$a_{\mathbf{m}} = \det\big(c^{(i)}_{m_j+n-j}\big).$$

In particular

$$\lim_{z_1,\ldots,z_n \to 0} \frac{\det f_i(z_j)}{V(z)} = F(0) = a_{\mathbf{0}} = \det\big(c^{(i)}_{n-j}\big).$$

For a positive signature $\mathbf{m} = (m_1, \ldots, m_n)$, the *shifted Schur function* $s^*_{\mathbf{m}}$ is defined, for a signature $\lambda = (\lambda_1, \ldots, \lambda_n)$ by

$$s^*_{\mathbf{m}} = \frac{\det\big([\lambda_i + \delta_i]_{m_j+\delta_j}\big)}{\det\big([\lambda_i + \delta_i]_{\delta_j}\big)},$$

where $\delta_i = n - i$, and

$$[a]_k = a(a-1)\ldots(a-k+1).$$

The functions $s^*_{\mathbf{m}}(\lambda)$ are shifted symmetric functions. The ordinary Schur function $s_{\mathbf{m}}(x)$ is symmetric, *i.e.*

$$s_{\mathbf{m}}(\ldots, x_i, x_{i+1}, \ldots) = s_{\mathbf{m}}(\ldots, x_{i+1}, x_i, \ldots),$$

while the shifted Schur function $s^*_{\mathbf{m}}(\lambda)$ satisfies:

$$s^*_{\mathbf{m}}(\ldots, \lambda_i, \lambda_{i+1}, \ldots) = s^*_{\mathbf{m}}(\ldots, \lambda_{i+1} - 1, \lambda_i + 1, \ldots).$$

The algebra of symmetric functions is denoted by Λ, and the algebra of shifted symmetric functions will be denoted by Λ^*. (See [Okounkov-Olshanski,1998a] and [1998b].)

Theorem 4.2 (Binomial formula).

$$\frac{s_\lambda(1+z_1, \ldots, 1+z_n)}{s_\lambda(1, \ldots, 1)} = \sum_{m_1 \geq \cdots \geq m_n \geq 0} \frac{\delta!}{(\mathbf{m}+\delta)!} s^*_{\mathbf{m}}(\lambda) s_{\mathbf{m}}(z).$$

For $n = 1$ this is nothing but the classical binomial formula:

$$(1+z)^\lambda = \sum_{m=0}^{\infty} \frac{1}{m!} [\lambda]_m w^m.$$

Proof. The theorem is a straightforward application of Hua's formula (Proposition 4.1) in the case

$$f_i(w) = (1+w)^{\lambda_i+\delta_i} = \sum_{m=0}^{\infty} \frac{1}{m!}[\lambda_i+\delta_i]_m w^m.$$

One observes that

$$s_\lambda(1,\dots,1) = \frac{V(\lambda+\delta)}{V(\delta)} = \frac{\det\big([\lambda_i+\delta_i]_{\delta_j}\big)}{\delta!}. \qquad \square$$

If λ is a positive signature, then $s^*_{\mathbf{m}}(\lambda) = 0$ if $\mathbf{m} \not\subseteq \lambda$, and

$$\frac{s_\lambda(1+z_1,\dots,1+z_n)}{s_\lambda(1,\dots,1)} = \sum_{\mathbf{m}\subseteq\lambda} \frac{\delta!}{(\mathbf{m}+\delta)!} s^*_{\mathbf{m}}(\lambda) s_{\mathbf{m}}(z).$$

If, in Theorem 4.2, one takes $z_1 = z$, $z_2 = 0, \dots, z_n = 0$, then one obtains Lemma 3 in [Vershik-Kerov,1982]:

$$\frac{s_\lambda(1+z,1,\dots,1)}{s_\lambda(1,\dots,1)} = 1 + \sum_{m=1}^{\infty} \frac{1}{n(n+1)\dots(n+m-1)} h^*_m(\lambda) z^m.$$

The shifted complete symmetric function $h^*_m(\lambda)$ is denoted by $\Phi_m(\lambda)$ in [Vershik-Kerov,1982]. By using the fact that the value of a determinant does not change when adding to a column a linear combination of the other ones, one obtains, with $\ell_i = \lambda_i + n - i$,

$$\begin{aligned} h^*_m(\lambda) &= \frac{1}{V(\ell)} \begin{vmatrix} [\ell_1]_{m+n-1} & [\ell_1]_{n-2} & \dots & 1 \\ [\ell_2]_{m+n-1} & [\ell_2]_{n-2} & \dots & 1 \\ \vdots & \vdots & & \vdots \\ [\ell_n]_{m+n-1} & [\ell_n]_{n-2} & \dots & 1 \end{vmatrix} \\ &= \frac{1}{V(\ell)} \begin{vmatrix} [\ell_1]_{m+n-1} & \ell_1^{n-2} & \dots & 1 \\ [\ell_2]_{m+n-1} & \ell_2^{n-2} & \dots & 1 \\ \vdots & \vdots & & \vdots \\ [\ell_n]_{m+n-1} & \ell_n^{n-2} & \dots & 1 \end{vmatrix}. \end{aligned}$$

By expanding now $[x]_{m+n-1}$ in powers of x:

$$\begin{aligned} &[x]_{m+n-1} = x(x-1)\dots(x-m-n+2) \\ &= \sum_{k=0}^{m} e_{m-k}\big(0,-1,\dots,-(m+n-2)\big)x^{k+n-1} \\ &+ \text{terms of degree} < n-1, \end{aligned}$$

where e_k is the k-th elementary symmetric function, one obtains the formula from Lemma 3 in [Vershik-Kerov,1982]:

$$h_m^*(\lambda) = \sum_{k=0}^{m} e_{m-k}\big(0, -1, \ldots, -(m+n-2)\big) h_k(\ell).$$

5. Proof of Theorem 3.3. — We follow the method of proof of [Okounkov-Olshanski,1998c].

a) *The morphism* $\Lambda \to \mathcal{C}(\Omega_0)$

One defines an algebra morphism $\Lambda \to \mathcal{C}(\Omega_0)$ which maps a symmetric function f to a continuous function $\tilde{f}$ on Ω_0. Since the power sums

$$p_m(x_1, \ldots, x_n, \ldots) = \sum_i x_i^m$$

generate Λ as an algebra, this morphism is uniquely determined by their images $\widetilde{p_m}$. One puts, for $\omega = (\alpha, \beta, \gamma) \in \Omega_0$, with $\alpha = (\alpha_k)$, $\beta = (\beta_k)$,

$$\widetilde{p_1}(\omega) = \sum_{k=1}^{\infty} \alpha_k + \sum_{k=1}^{\infty} \beta_k + \gamma,$$

$$\widetilde{p_m}(\omega) = \sum_{k=1}^{\infty} \alpha_k^m + (-1)^{m-1} \sum_{k=1}^{\infty} \beta_k^m \quad (m \geq 2).$$

The functions $\widetilde{p_m}$ are continuous on Ω_0. In fact, as we saw above, $\widetilde{p_m}(\omega) = L_u(\omega)$, with $u(s) = s^{m-1}$ $(m \geq 1)$.

Proposition 5.1. — *The functions* $\widetilde{h_m}(\omega)$ *are the Taylor coefficients of the Voiculescu function* $\Phi(\omega; t)$ *at* $t = 1$*: for* $z \in \mathbb{C}$, $|z| < r = \inf \frac{1}{\alpha_k}$,

$$\Phi(\omega; 1 + z) = \sum_{m=0}^{\infty} \widetilde{h_m}(\omega) z^m.$$

Proof. One starts from the generating function of the complete symmetric functions h_m:

$$H(x; z) = \sum_{m=0}^{\infty} h_m(x) z^m = \prod_{j=1}^{n} \frac{1}{1 - x_j z}.$$

Its logarithmic derivative is given by

$$\frac{d}{dz} \log H(x; z) = \sum_{m=0}^{\infty} p_{m+1}(x) z^m.$$

On the other hand, as we saw in Section 3,

$$\frac{d}{dz} \log \Phi(\omega; 1+z) = \sum_{m=0}^{\infty} \widetilde{p_{m+1}}(\omega) z^m.$$

Therefore the coefficients $c_m(\omega)$ defined by

$$\Phi(\omega; 1+z) = \sum_{m=0}^{\infty} c_m(\omega) z^m,$$

are images, by the morphism $f \mapsto \tilde{f}$, of the complete symmetric functions h_m: $c_m(\omega) = \widetilde{h_m}(\omega)$. ∎

Corollary 5.2. — *For $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$, $|z_j| < r$,*

$$\prod_{j=1}^{n} \Phi(\omega; 1+z_j) = \sum_{m_1 \geq \ldots \geq m_1 \geq 0} \widetilde{s_{\mathbf{m}}}(\omega) s_{\mathbf{m}}(z).$$

Proof. Observe that the statement of Proposition 5.1 can be written:

$$\tilde{H}(\omega; z) = \Phi(\omega; 1+z),$$

and apply the morphism $f \mapsto \tilde{f}$ to both sides of the Cauchy identity

$$\prod_{j=1}^{n} H(x; z_j) = \prod_{i,j=1}^{n} \frac{1}{1 - x_i z_j} = \sum_{m_1 \geq \ldots \geq m_n \geq 0} s_{\mathbf{m}}(x) s_{\mathbf{m}}(z).$$

b) *Asymptotics of shifted symmetric functions*

Proposition 5.3. — *Consider a sequence $(\lambda^{(n)})$ of positive signatures with $\lambda^{(n)} \in \Omega_n^+$, and let $\omega \in \Omega_0$. Assume that, for the topology of Ω_0,*

$$\lim_{n \to \infty} T_n(\lambda^{(n)}) = \omega.$$

Then, for every shifted symmetric function $f^* \in \Lambda^*$,

$$\lim_{n \to \infty} \frac{1}{n^m} f^*(\lambda^{(n)}) = \tilde{f}(\omega),$$

where m is the degree of f^*, and f is the homogeneous part of degree m in f^*.

We will prove the statement in the special case $f^* = q_m^*$:

$$q_m^*(\lambda) = \sum_{i\geq 1}([\lambda_i - i + 1]_m - [-i+1]_m).$$

The function $q_m^*(\lambda)$ is shifted symmetric of degree m and the homogeneous part of degree m is equal to the Newton power sum $p_m(\lambda)$. Since the functions $q_m^*(\lambda)$ generate Λ^* as an algebra, the statement of the proposition will be proven.

LEMMA. — *Let $a = (a_i)$, $b = (b_j)$ be the Frobenius parameters of the positive signature λ. Then*

$$q_m^*(\lambda) = \sum_{i\geq 1}[1+a_i]_m - \sum_{j\geq 1}[1-b_j]_m.$$

Proof of Proposition 5.3

Let $a^{(n)} = (a_i^{(n)})$ and $b^{(n)} = (b_j^{(n)})$ be the Frobenius parameters of the positive signature $\lambda^{(n)}$, and $\omega = (\alpha, \beta, \gamma) \in \Omega_0$, with $\alpha = (\alpha_k)$, $\beta = (\beta_k)$. By assumption, for every continuous function u on $\mathbb{R}$,

$$\lim_{n\to\infty} L_u\big(T_n(\lambda^{(n)})\big) = L_u(\omega),$$

or

$$\lim_{n\to\infty}\Big(\sum_{i\geq 1}\frac{a_i^{(n)}}{n}u\Big(\frac{a_i^{(n)}}{n}\Big) + \sum_{j\geq 1}\frac{b_j^{(n)}}{n}u\Big(-\frac{b_j^{(n)}}{n}\Big)\Big)$$
$$= \sum_{k=1}^{\infty}\alpha_k u(\alpha_k) + \sum_{k=1}^{\infty}\beta_k u(-\beta_k) + \gamma u(0).$$

Consider the sequence of the functions

$$u_n(s) = \frac{1}{n^m s}[ns+1]_m.$$

Then

$$L_{u_n}\big(T_n(\lambda^{(n)})\big) = \frac{1}{n^m}q_m^*(\lambda^{(n)}).$$

On the other hand the sequence $u_n(s)$ converges to the function $u(s) = s^{m-1}$ uniformly on compacts sets in $\mathbb{R}$, and

$$L_u(\omega) = \widetilde{p_m}(\omega).$$

It follows that

$$\lim_{n\to\infty}\frac{1}{n^m}q_m^*(\lambda^{(n)}) = \widetilde{p_m}(\omega). \qquad \square$$

c) *End of the proof of Theorem 3.3*

To finish the proof one applies the following:

PROPOSITION 5.4. — *Let ψ_n be a sequence of $\mathcal{C}^\infty$-functions on the torus $\mathbb{T}^k$ of positive type, with $\psi_n(0) = 1$, and ψ an analytic function in a neighborhood of 0. Assume that, for every $\alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbb{N}^k$,*

$$\lim_{n\to\infty} \partial^\alpha \psi_n(0) = \partial^\alpha \psi(0).$$

Then ψ has an analytic extension to $\mathbb{T}^k$, and ψ_n converges to ψ uniformly on $\mathbb{T}^k$.

For the proof, see for instance [Faraut,2008], Proposition 3.11.

We consider a sequence of positive signatures $(\lambda^{(n)})$ such that

$$\lim_{n\to\infty} T_n(\lambda^{(n)}) = \omega.$$

Put, with $t_j = e^{i\theta_j}$,

$$\psi_n(t_1, \ldots, t_k) = \varphi_n\big(\lambda^{(n)}; \mathrm{diag}(t_1, \ldots, t_k, 1, \ldots)\big),$$

$$\psi(t_1, \ldots, t_k) = \prod_{j=1}^k \Phi(\omega; t_j).$$

By Theorem 4.2,

$$\psi_n(1 + z_1, \ldots, 1 + z_k) = \sum_{m_k \geq \cdots \geq m_1 \geq 0} \frac{\delta!}{(\mathbf{m} + \delta)!} s^*_{\mathbf{m}}(\lambda^{(n)}) s_{\mathbf{m}}(z_1, \ldots, z_k).$$

Then, by Proposition 5.3,

$$\lim_{n\to\infty} \frac{1}{n^{|\mathbf{m}|}} s^*_{\mathbf{m}}(\lambda^{(n)}) = \widetilde{s_{\mathbf{m}}}(\omega),$$

and, by Corollary 5.2,

$$\sum_{m_1 \geq \cdots \geq m_k} \widetilde{s_{\mathbf{m}}}(\omega) s_{\mathbf{m}}(z_1, \ldots, z_k) = \prod_{j=1}^k \Phi(\omega; 1 + z_k) = \psi(1 + z_1, \ldots, 1 + z_k).$$

Finally, observing that

$$\frac{(\mathbf{m} + \delta)!}{\delta!} \sim n^{|\mathbf{m}|} \quad (n \to \infty),$$

we obtain, by Proposition 5.4,

$$\lim_{n\to\infty} \psi_n(t_1, \ldots, t_k) = \psi(t_1, \ldots, t_k),$$

uniformly on $\mathbb{T}^k$. In fact, the Taylor coefficients of ψ_n, as a function on $\mathbb{T}^k$, are finite linear combinations of the coefficients in the Schur expansion of $\psi_n(1 + z_1, \ldots, 1 + z_n)$. ∎

6. Inductive limits of compact symmetric spaces. — One knows that, if G/K is a Riemannian symmetric space, then (G,K) is a Gelfand pair. Let $G(n)/K(n)$ be a compact symmetric space of rank n, and

$$\mathfrak{g}(n) = \mathfrak{k}(n) + \mathfrak{p}(n)$$

be a Cartan decomposition of the Lie algebra $\mathfrak{g}(n)$ of $G(n)$. Fix a Cartan subspace $\mathfrak{a}(n) \subset \mathfrak{p}(n)$, $\mathfrak{a}(n) \simeq \mathbb{R}^n$, and put $A(n) = \exp \mathfrak{a}(n) \simeq \mathbb{T}^n$. Let $\mathcal{R}_n$ denote the system of restricted roots for the pair $\big(\mathfrak{a}(n)_{\mathbb{C}}, \mathfrak{g}(n)_{\mathbb{C}}\big)$.

a) *Classical series of type A*

We consider one of the following series of compact symmetric spaces.

$G(n)$	$K(n)$	d
$U(n)$	$O(n)$	1
$U(n) \times U(n)$	$U(n)$	2
$U(2n)$	$Sp(n)$	4

The system $\mathcal{R}_n$ of restricted roots is of type A_{n-1}. For a suitable basis $(e_1, \ldots, e_n)$ of $\mathfrak{a}(n)$, the restricted roots are

$$\alpha_{ij} = \varepsilon_i - \varepsilon_j \quad (i \neq j)$$

($(\varepsilon_1, \ldots, \varepsilon_n)$ is the dual basis), with multiplicities $d = 1, 2, 4$.

These symmetric spaces appear as Shilov boundaries of bounded symmetric domains of tube type. In particular the symmetric space $U(n)/O(n)$ can be seen as the space of symmetric unitary $n \times n$ matrices. The subgroup $A(n)$ can be taken as the subgroup of unitary diagonal matrices.

The space $U(n)/O(n)$ can also be seen as the Lagrangian manifold $\Lambda(n)$, the manifold of n-Lagrangian subspaces in $\mathbb{R}^{2n}$.

The spherical dual Ω_n of the Gelfand pair $\big(G(n), K(n)\big)$ is parametrized by signatures

$$\lambda = (\lambda_1, \ldots, \lambda_n),\ \lambda_i \in \mathbb{Z},\ \lambda_1 \geq \cdots \geq \lambda_n.$$

The restricted highest weight of the spherical representation corresponding to λ is $\sum_{i=1}^{n} \lambda_i \varepsilon_i$.

The restriction to $A(n) \simeq \mathbb{T}^n$ of the spherical function $\varphi_n(\lambda; x)$ is a normalized Jack function: for $a = (t_1, \ldots, t_n)$,

$$\varphi_n(\lambda; a) = \frac{J_\lambda(t_1, \ldots, t_n; \alpha)}{J_\lambda(1, \ldots, 1; \alpha)},$$

with $\alpha = \frac{2}{d}$. For $d = 2$ it is a Schur function. (See [Stanley,1989] for definition and properties of Jack functions, and also [Macdonald,1995], Section VI.10.)

The Jack functions are orthogonal with respect to the following inner product:

$$(P|Q) = \int_{\mathbb{T}^n} P(t)\overline{Q(t)}|V(t)|^d \beta(dt),$$

where β is the normalized Haar measure on $\mathbb{T}^n$. With $t_j = e^{i\theta_j}$,

$$|V(t)|^d = \prod_{j<k} 4\Big| \sin \frac{\theta_j - \theta_k}{2}\Big|^d,$$

$$\beta(dt) = \frac{1}{(2\pi)^n} d\theta_1 \dots d\theta_n.$$

We consider now the Olshanski spherical pair (G, K) with

$$G = \bigcup_{n=1}^{\infty} G(n), \quad K = \bigcup_{n=1}^{\infty} K(n).$$

We state without proof the main results by Okounkov and Olshanski ([1998c]). The spherical dual for the pair (G, K) is, as in the case of the infinite dimensional unitary group, parametrized by a pair $\omega = (\omega^+, \omega^-)$, *i.e.* $\Omega = \Omega_0 \times \Omega_0$. For $\omega \in \Omega_0$, $\omega = (\alpha, \beta, \gamma)$, with $\alpha = (\alpha_k)$, $\beta = (\beta_k)$, define

$$\Phi^{(d)}(\omega; t) = e^{\gamma(t-1)} \prod_{k=1}^{\infty} \frac{1 + \beta_k(t-1)}{\Big(1 - \frac{2}{d}\alpha_k(t-1)\Big)^{\frac{d}{2}}} \quad (t \in \mathbb{T}).$$

For $d = 2$, it is the Voiculescu function we considered in Section 3.

THEOREM 6.1. — *The spherical functions of positive type, for the Olshanski spherical pair (G, K), are given, for $a = (t_1, \dots, t_n, 1, \dots) \in A \simeq \mathbb{T}^{(\infty)}$, by*

$$\varphi(\omega; a) = \prod_{j=1}^{n} \Phi^{(d)}(\omega^+; t_j)\Phi^{(d)}(\omega^-; \frac{1}{t_j}),$$

with $\omega = (\omega^+, \omega^-) \in \Omega$.

One defines the map $T_n : \Omega_n \to \Omega = \Omega_0 \times \Omega_0$ as in the case of the unitary groups (see Section 3).

THEOREM 6.2. — *Let $(\lambda^{(n)})$ be a sequence of signatures with $\lambda^{(n)} \in \Omega_n$. If*

$$\lim_{n\to\infty} T_n(\lambda^{(n)}) = \omega = (\omega^+, \omega^-),$$

then, with $a = (t_1, \ldots, t_k, 1, \ldots) \in A$*,*

$$\lim_{n \to \infty} \varphi_n(\lambda^{(n)}; a) = \prod_{j=1}^{k} \Phi^{(d)}(\omega^+; t_j)\Phi^{(d)}(\omega^-; \frac{1}{t_j}).$$

Since there is no simple formula for the Jack functions for $\alpha \neq 1$, the proof for $d \neq 2$ is more difficult than in the case of the unitary groups. However it follows the same lines. The first step is a binomial formula for the normalized Jack functions.

b) *Classical series of type BC*

We consider the following series of compact symmetric spaces.

	$G(n)$	$K(n)$	$\mathcal{R}_n$	d	p	q
1	$O(2n) \times O(2n)$	$O(2n)$	D_n	2	0	0
2	$O(2n+1) \times O(2n+1)$	$O(2n+1)$	B_n	2	2	0
3	$Sp(n) \times Sp(n)$	$Sp(n)$	C_n	2	0	2
4	$Sp(n)$	$U(n)$	C_n	1	0	1
5	$O(4n)$	$U(2n)$	C_n	4	0	1
6	$O(4n+2)$	$U(2n+1)$	BC_n	4	4	1
7	$O(2n+k)$	$O(n) \times O(n+k)$	BC_n	1	k	0
8	$U(2n+k)$	$U(n) \times U(n+k)$	BC_n	2	2k	1
9	$Sp(2n+k)$	$Sp(n) \times Sp(n+k)$	BC_n	4	4k	3

The possible roots and multiplicities are

α	$\pm\varepsilon_i \pm \varepsilon_j$	ε_i	$2\varepsilon_i$
m_α	d	p	q

Series 1, 2, and 3 are compact groups seen as symmetric spaces.
Series 4, 5, and 6 are compact Hermitian symmetric spaces.
Series 7, 8, and 9 are Grassmann manifolds: spaces of n-subspaces in $\mathbb{F}^{2n+k}$, with $\mathbb{F} = \mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$, $d = \dim_{\mathbb{R}}\mathbb{F}$, $p = dk$, $q = d-1$. If $k = 0$, the root system $\mathcal{R}_n$ is of type C_n. The symmetric space $U(2n+k)/U(n) \times U(n+k)$ is Hermitian as well.

For series 7, 8, and 9 the Cartan subgroup $A(n)$ can be taken as the group of the following matrices

$$a(\theta) = \begin{pmatrix} \cos\frac{\theta}{2} & 0 & -\sin\frac{\theta}{2} \\ 0 & I_k & 0 \\ \sin\frac{\theta}{2} & 0 & \cos\frac{\theta}{2} \end{pmatrix},$$

with $\theta = (\theta_1, \ldots, \theta_n)$, and

$$\cos\frac{\theta}{2} = \mathrm{diag}\big(\cos\frac{\theta_1}{2}, \ldots, \cos\frac{\theta_n}{2}\big), \quad \sin\frac{\theta}{2} = \mathrm{diag}\big(\sin\frac{\theta_1}{2}, \ldots, \sin\frac{\theta_n}{2}\big)$$

We assume that the multiplicities d, p, q don't depend on n. The spherical dual Ω_n is parametrized by positive signatures:

$$\lambda = (\lambda_1, \ldots, \lambda_n),\ \lambda_i \in \mathbb{N},\ \lambda_1 \geq \cdots \geq \lambda_n \geq 0.$$

The restriction to $A(n) \simeq \mathbb{T}^n$ of the corresponding spherical function is a normalized Jacobi polynomial. (See Hypergeometric and Special Functions, by G. Heckman, in [Heckman-Schichtkrull,1994], for definition and properties of Jacobi polynomials associated to a root system.) For $a = (t_1, \ldots, t_n) \in A(n)$,

$$\varphi_n(\lambda; a) = \frac{\mathfrak{P}_\lambda(t_1, \ldots, t_n)}{\mathfrak{P}_\lambda(1, \ldots, 1)}.$$

The polynomials $\mathfrak{P}_\lambda$ are orthogonal with respect to the inner product

$$(P|Q) = \int_{\mathbb{T}^n} P(t)\overline{Q(t)}|D(t)|\beta(dt),$$

with, if $t_j = e^{i\theta_j}$,

$$D(t) = \prod_{i<j}\Big(\sin\frac{\theta_i+\theta_j}{2}\Big)^d\Big(\sin\frac{\theta_i-\theta_j}{2}\Big)^d\prod_{i=1}^n\Big(\sin\frac{\theta_i}{2}\Big)^p(\sin\theta_i)^q.$$

By putting $x_i = \cos\theta_i = \frac{1}{2}(t_i + t_i^{-1})$, the inner product is carried over an integral on $[-1,1]^n$ with the weight

$$\prod_{i<j}|x_i - x_j|^d\prod_{i=1}^n(1-x_i)^\alpha(1+x_i)^\beta,$$

with $\alpha = \frac{1}{2}(p+q-1)$, $\beta = \frac{1}{2}(q-1)$. We will write P_λ for the Jacobi polynomial in the variables x_i:

$$P_\lambda(x_1, \ldots, x_n) = \mathfrak{P}_\lambda(t_1, \ldots, t_n), \quad x_i = \tfrac{1}{2}(t_i + t_i^{-1}).$$

As in 6.a), we define, for $\omega \in \Omega_0$,

$$\Phi^{(d)}(\omega; t) = e^{\gamma(t-1)} \prod_{k=1}^{\infty} \frac{1+\beta_k(t-1)}{\left(1-\frac{2}{d}\alpha_k(t-1)\right)^{\frac{d}{2}}} \quad (t \in \mathbb{T}).$$

THEOREM 6.3. — *The spherical dual for the pair (G,K) is parametrized by Ω_0. The spherical functions are given, for $a = (t_1, \ldots, t_n, 1, \ldots) \in A \simeq \mathbb{T}^{(\infty)}$, by*

$$\varphi(\omega; a) = \prod_{j=1}^{n} \Phi^{(d)}(\omega; t_j)\Phi^{(d)}(\omega; \frac{1}{t_j}),$$

with $\omega \in \Omega_0$.

One defines the map $T_n : \Omega_n \to \Omega_0$ as in the case of the unitary groups for positive signatures.

THEOREM 6.4. — *Let $(\lambda^{(n)})$ be a sequence of signatures, with $\lambda^{(n)} \in \Omega_n$. If*

$$\lim_{n\to\infty} T_n(\lambda^{(n)}) = \omega,$$

then, for $a = (t_1, \ldots, t_k, 1, \ldots)$,

$$\lim_{n\to\infty} \varphi_n(\lambda^{(n)}; a) = \prod_{j=1}^{k} \Phi^{(d)}(\omega; t_j)\Phi^{(d)}(\omega; \frac{1}{t_j}).$$

7. The case $d = 2$. Determinantal formula, binomial formula for multivariate Jacobi polynomials. — In this last section, we will present, in case $d = 2$, a determinantal formula for the multivariate Jacobi polynomials, and then a binomial formula.

In their paper, Berezin and Karpelevič gave a determinantal formula for the spherical functions on the Grassmann manifolds $U(p+q)/U(p) \times U(q)$ ([1958], see also [Takahashi, 1977], [Hoogenboom,1982]). In fact such a determinantal formula exists in all cases with $d = 2$.

Let μ be a positive measure on $\mathbb{R}$ with infinite support and finite moments: for all $m \in \mathbb{N}$,

$$\int_{\mathbb{R}} |t|^m \mu(dt) < \infty.$$

By orthogonalizing the monomials t^m, one obtains a sequence of orthogonal polynomials $p_m(t)$:

$$\int_{\mathbb{R}} p_\ell(t)p_m(t)\mu(dt) = 0 \text{ if } \ell \neq m.$$

For a positive signature λ, define the multivariate polynomials P_λ

$$P_\lambda(x_1,\ldots,x_n) = \frac{\det\bigl(p_{\lambda_i+\delta_i}(x_j)\bigr)}{V(x)},$$

where λ is a positive signature, and, as above, $\delta = (n-1,\ldots,1,0)$. The symmetric polynomials P_λ are orthogonal with respect to the inner product

$$(P|Q) = \int_{\mathbb{R}^n} P(x_1,\ldots,x_n)\overline{Q(x_1,\ldots,x_n)}V(x_1,\ldots,x_n)^2\mu(dx_1)\ldots\mu(dx_n).$$

If the polynomials p_m are normalized such that

$$p_m(t) = t^m + \text{ lower order terms},$$

then

$$P_\lambda(x_1,\ldots,x_n) = s_\lambda(x_1,\ldots,x_n) + \text{ lower order terms},$$

Consider now the measure μ on $\mathbb{R}$ given by

$$\int_{\mathbb{R}} f(t)\mu(dt) = \int_{-1}^{1} f(t)(1-t)^\alpha(1+t)^\beta dt,$$

with $\alpha, \beta > -1$. Then the orthogonal polynomials with respect to this measure are the Jacobi polynomials $p_m(t) = p_m^{(\alpha,\beta)}(t)$. The multivariable polynomials $P_\lambda^{(\alpha,\beta)}$ given by, for $x = (x_1,\ldots,x_n)$,

$$P_\lambda^{(\alpha,\beta)}(x) = \frac{\det\bigl(p^{(\alpha,\beta)}_{\lambda_i+\delta_i}(x_j)\bigr)}{V(x)},$$

are orthogonal for the inner product

$$(P|Q) = \int_{[-1,1]^n} P(x)\overline{Q(x)}\prod_{i<j}(x_i-x_j)^2\prod_{i=1}^{n}(1-x_i)^\alpha(1+x_i)^\beta dx_1\ldots dx_n,$$

and are, up to a constant factor, the Jacobi polynomials associated with the root system of type BC_n and the multiplicity (d,p,q), with $d=2$.

Normalized by the condition $p_m^{(\alpha,\beta)}(1) = 1$, the Jacobi polynomials $p_m^{(\alpha,\beta)}$ admit the following hypergeometric representation:

$$\begin{aligned} p_m^{(\alpha,\beta)}(t) &= {}_2F_1\Bigl(-m, m+\alpha+\beta+1; \alpha+1; \frac{1-t}{2}\Bigr) \\ &= \sum_{k=0}^{m} \frac{(-m)_k(m+\alpha+\beta+1)_k}{(\alpha+1)_k}\frac{1}{k!}\Bigl(\frac{1-t}{2}\Bigr)^k. \end{aligned}$$

Let us introduce the notation:

$$\sigma = \frac{\alpha+\beta+1}{2},\ \ell = m+\sigma,$$
$$[\ell,\sigma]_k = \big(\ell^2-\sigma^2\big)\dots\big(\ell^2-(\sigma+k-1)^2\big).$$

The binomial formula for the Jacobi polynomial $p_m^{(\alpha,\beta)}$ can be written:

$$p_m^{(\alpha,\beta)}(1+w) = \sum_{k=0}^{m} a_k^{(m)} w^k = \sum_{k=0}^{m} \frac{1}{k!}\frac{[\ell,\sigma]_k}{(\alpha+1)_k}\Big(\frac{w}{2}\Big)^k.$$

By Hua's formula,

$$P_\lambda^{(\alpha,\beta)}(1,\dots,1) = \det\big(a_{\delta_j}^{(\lambda_i+\delta_i)}\big) = 2^{-\frac{n(n-1)}{2}}\frac{1}{\delta!}\prod_{i=1}^{n}\frac{1}{(\alpha+1)_{\delta_i}}V(\ell_1^2,\dots,\ell_n^2),$$

with $\ell_i = \lambda_i+\delta_i+\sigma$. Since

$$\det\big([\ell_i,\sigma]_{\delta_j}\big) = V(\ell_1^2,\dots,\ell_n^2).$$

THEOREM 7.1.

$$\frac{P_\lambda^{(\alpha,\beta)}(1+z_1,\dots,1+z_n)}{P_\lambda^{(\alpha,\beta)}(1,\dots,1)}$$
$$= \sum_{\mu\subseteq\lambda} 2^{-|\mu|}\frac{\delta!}{(\mu+\delta)!}\frac{\prod_{i=1}^n(\alpha+1)_{\delta_i}}{\prod_{i=1}^n(\alpha+1)_{\mu_i+\delta_i}}S_\mu^*(\lambda)s_\mu(z_1,\dots,z_n),$$

with

$$S_\mu^*(\lambda) = \frac{\det\big([\ell_i,\sigma]_{\mu_j+\delta_j}\big)}{V(\ell_1^2,\dots,\ell_n^2)},\quad \ell_i = \lambda_i+\delta_i+\sigma.$$

Proof. This is once more an application of Hua's formula (Proposition 4.1). In the present case

$$f_i(w) = p_{\lambda_i+\delta_i}^{(\alpha,\beta)}(1+w) = \sum_{k=0}^{\lambda_i+\delta_i} a_k^{(\lambda_i+\delta_i)}w^k = \sum_{k=0}^{\lambda_i+\delta_i}\frac{1}{k!}\frac{[\ell_i,\sigma]_k}{(\alpha+1)_k}2^{-k}w^k,$$

with $\ell_i = \lambda_i+\delta_i+\sigma$. Then we get

$$P_\lambda^{(\alpha,\beta)}(1+z_1,\dots,1+z_n) = \sum_{\mu_1\geq\ldots\geq\mu_n\geq 0} a_\mu s_\mu(z_1,\dots,z_n),$$

with

$$a_\mu = \det\big(c_{\mu_j+\delta_j}^{(\lambda_i+\delta_i)}\big) = \frac{1}{(\mu+\delta)!}\frac{1}{\prod_{i=1}^n(\alpha+1)_{\mu_i+\delta_i}}\det\big([\ell_i,\sigma]_{\mu_j+\delta_j}\big).$$

Observe that, if $\mu\not\subseteq\lambda$, then $a_\mu = 0$. ∎

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