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# FOURIER ANALYSIS ON COMPACT GROUPS

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These notes have been written after the talks we gave in the Science Academic Lecture Workshop on Harmonic Analysis, held on 6 and 7 December 2017 in the SS college, Areacode, Malappuram (Kerala). We present a survey of the harmonic analysis on compact groups, Abelian and non Abelian, and explain in detail a few examples.

this workshop, organized by Professor G. Sagith has been attended by students, scholars and faculty members. I have been very pleased to take part in this workshop and I thank very much Professor G. Sagith for the invitation.

# 1 Classical Fourier analysis

A function $f$ defined on $\mathbb{R}$ which is $2\pi$-periodic,

$$f(x+2\pi) = f(x) \quad (x \in \mathbb{R}),$$

can be seen as a function on the group $G = \mathbb{R}/2\pi\mathbb{Z}$, which is homeomorphic to the unit circle $\mathbb{T}$ in $\mathbb{C}$:

$$G = \mathbb{R}/2\pi\mathbb{Z} \to \mathbb{T},\ x \mapsto e^{ix}.$$

Therefore the group $G$ is compact.

The integral of a continuous function $f$ on $G$, identified with a $2\pi$-periodic continuous function on $\mathfrak{R}$, is given by

$$\int_G f(x)\mu(dx) = \frac{1}{2\pi}\int_0^{2\pi} f(x)dx = \frac{1}{2\pi}\int_a^{a+2\pi} fx)dx,$$

for any $a \in \mathbb{R}$.

For $m \in \mathbb{Z}$, the function $\chi_m$ given by $\chi_m(x) = e^{imx}$, is $2\pi$-periodic, and can be seen as a function on $G$. It satisfies

$$\chi_m(x+y) = \chi_m(x)\chi_m(y).$$

The system $\{\chi_m\}_{m\in\mathbb{Z}}$ is orthonormal in $L^2(G,\mu)$:

$$(\chi_m|\chi_n) = 0 \text{ if } m \neq n,\ \|\chi_m\|_2 = 1.$$

Moreover the system $\{\chi_m\}_{m\in\mathbb{Z}}$ is a Hilbert basis of $L^2(G,\mu)$. For a function $f \in L^1(G,\mu)$, the Fourier coefficient $\hat{f}(m)$ is given by

$$\hat{f}(m) = (f|\chi_m) = \int_G f(x)\overline{\chi_m(x)}\mu(dx) = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-imx}dx.$$

For a function $f \in L^2(G,\mu)$,

$$\int_G |f(x)|^2\mu(dx) = \frac{1}{2\pi}\int_0^{2\pi} |f(x)|dx = \sum_{m\in\mathbb{Z}} |\hat{f}(m)|^2.$$

If $f$ is a continuous function such that

$$\sum_{m\in\mathbb{Z}} |\hat{f}(m)| < \infty,$$

then

$$f(x) = \sum_{m \in \mathbb{Z}} \hat{f}(m) e^{imx}.$$

We will see how these classical properties extend to all compact groups $G$. If $G$ is Abelian, then the extension is word for word. In general the extension involves representation theory. We will state without proof the main facts about Fourier analysis on compact groups, and present some examples.

# 2 Compact Abelian groups

A topological group is a group $G$ equiped with a topology such that the maps

$$\begin{array}{l} G \times G \to G, \quad (x, y) \mapsto xy, \\ G \to G, \ x \mapsto x^{-1}, \end{array}$$

are continuous.

Assume $G$ compact. there is on $G$ a unique probability Borel measure $\mu$ which is invariant by right and left translations: for a continuous function $f$ on $G$,

$$\begin{aligned} \int_G f(xg)\mu(dx) &= \int_G f(x)\mu(dx), \\ \int_G f(gx)\mu(dx) &= \int_G f(x)\mu(dx), \end{aligned}$$

for any $g \in G$. the measure $\mu$ is the normalized *Haar measure* of $G$.

If $G$ is finite, $N = \#G$, then

$$\int_G f(x)\mu(dx) = \frac{1}{N} \sum_{x \in G} f(x).$$

Assume $G$ compact and Abelian. In this case we will use the additive notation for the group law. A *character* $\chi$ of $G$ is a continuous group morphism

$$\chi : G \to \mathfrak{T}, \ x \mapsto \chi(x).$$

It satisfies

$$\begin{aligned}\chi(x+y) &= \chi(x)\chi(y),\\ \chi(-x) &= \chi(x)^{-1} = \overline{\chi(x)}.\end{aligned}$$

The product of two characters is a character. Therefore the set of characters is a group: the dual group $\hat{G}$. The identity element in $\hat{G}$ is the trivial character $\chi_0 \equiv 1$

**Proposition 2.1.** *The system $\{\chi\}_{\chi\in\hat{G}}$ is orthonormal in $L^2(G,\mu)$.*

*Proof.* For $\chi \in \hat{G}$ define

$$I(\chi) = \int_G \chi(x)\mu(dx).$$

By the translation invariance of the Haar measure, we get, for any $a \in G$,

$$I(\chi) = \int_G \chi(x+a)\mu(dx) = \chi(a)I(\chi).$$

If $I(\chi) \neq 0$, then $\chi(a) = 1$, and $\chi$ is the trivial character:

$$\int_G \chi(x)\mu(dx) = \begin{cases} 1 & \text{if } \chi = \chi_0,\\ 0 & \text{if } \chi \neq \chi_0.\end{cases}$$

Consider two characters $\chi_1$ and $\chi_2$.

$$(\chi_1|\chi_2) = \int_G \chi_1(x)\overline{\chi_2(x)}\mu(dx) = I(\chi_1\overline{\chi_2}).$$

If $\chi_1 \neq \chi_2$, then $\chi_1\overline{\chi_2}$ is not the trivial character, and $I(\chi_1\overline{\chi_2}) = 0$. ∎

**Theorem 2.2.** *The system $\{\chi\}_{\chi\in\hat{G}}$ is a Hilbert basis of $L^2(G,\mu)$*

The Fourier coefficient of a function $f \in L^1(G,\mu)$ is defined by

$$\hat{f}(\chi) = (f|\chi) = \int_G f(x)\overline{\chi(x)}\mu(dx).$$

**Theorem 2.3.** *(Plancherel) For a function $f \in L^2(G, \mu)$,*

$$f(x) = \sum_{\chi \in \hat{G}} \hat{(}\chi)\chi(x)$$

*in the $L^2$-sense, and*

$$\int_G |f(x)|^2 \mu(dx) = \sum_{\chi \in \hat{G}} |\hat{f}(\chi)|^2.$$

If $f$ is continuous, and if $\sum_{\chi \in \hat{G}} |\hat{f}(\chi)| < \infty$, then

$$f(x) = \sum_{\chi \in \hat{G}} \hat{f}(\chi)\chi(x),$$

uniformly.

# 3 Examples

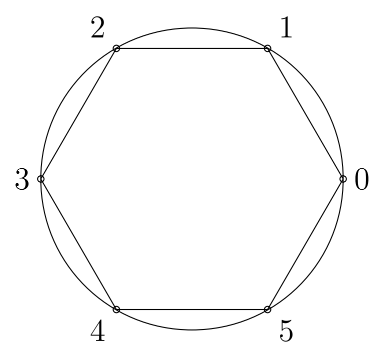
**1) The cyclic group $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$**

The group $\mathbb{Z}_n$ is a finite group with $n$ elements. A system of representatives is $0, 1, \ldots, n-1$. Let $\omega = e^{\frac{2i\pi}{n}}$. Observe that $\omega^n = 1$. The group $G$ can be seen as the group of the $n$-th roots of unity:

$$\{1, \omega, \omega^2, \ldots, \omega^{n-1}\},$$

or the group of the rotaions of the regular polygon with $n$ vertices,

$$\begin{pmatrix} \cos k\frac{2\pi}{n} & -\sin k\frac{2\pi}{n} \\ \sin k\frac{2\pi}{n} & \cos k\frac{2\pi}{n} \end{pmatrix}, \quad (k = 0, 1, \ldots, n-1).$$




The characters of $\mathbb{Z}_n$ are given by

$$\chi_m(x) = \omega^{mx},$$

with $m \in \mathbb{Z}_n$. One checks easily the orthogonality of the characters. Observe that

$$1 + \omega^k + \omega^{2k} + \cdots + \omega^{(n-1)k} = \frac{\omega^{nk} - 1}{\omega^k - 1}.$$

We get

$$(\chi_m|\chi_{m'}) = \frac{1}{n}\sum_{x=0}^{n-1} \omega^{mx}\overline{\omega^{m'x}} = \frac{1}{n}\sum_{x=0}^{n-1} \omega^{kx},$$

with $k = m - m'$, and

$$\sum_{x=0}^{n-1} \omega^{kx} = \begin{cases} n & \text{if } m = m' \bmod n, \\ 0 & \text{if } m \neq m' \bmod n. \end{cases}$$

The Fourier coefficients of a function $f$ on $G$ are given by,

$$\hat{f}(m) = \frac{1}{n}\sum_{x=0}^{n-1} f(x)\omega^{-mx}.$$

Hence the matrix $\mathcal{F}$ of the Fourier transform is the following $n \times n$ matrix:

$$\mathcal{F} = \frac{1}{n} \begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & \omega^{-1} & \dots & \omega^{-(n-1)} \\ 1 & \omega^{-2} & \dots & \omega^{-2(n-1)} \\ \vdots & \vdots & & \vdots \\ 1 & \omega^{-(n-1)} & \dots & \omega^{-(n-1)^2} \end{pmatrix}$$

Observe that $\sqrt{n}\mathcal{F}$ is unitary.

**2) The hypercube** $Q_n = (\mathbb{Z}_2)^n$

An element $x \in Q_n$ is a sequence $(x_1, \dots, x_n)$ where $x_i = 0$ or 1. The addition is given, for $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ by

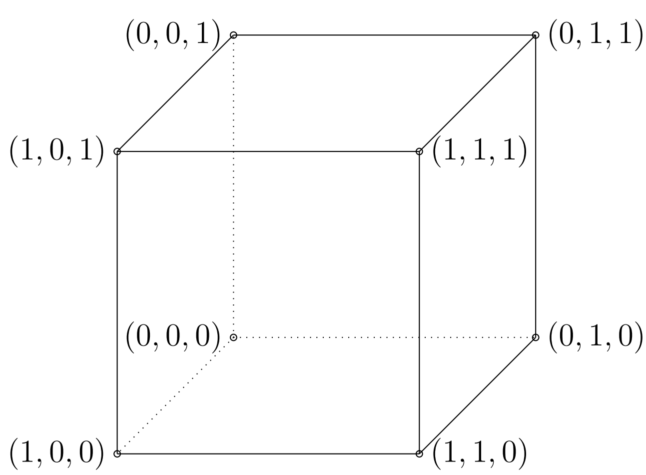
$$x + y = (x_1 + y_1, \dots, x_n + y_n),$$

where the sums $x_i + y_i$ are mod 2. Observe that $-x = x$. The group $Q_n$ has $2^n$ elements.

A character of $Q_n$ is of the form

$$\chi_m(x) = (-1)^{m_1 x_1 + \dots + m_n x_n},$$

with $m = (m_1, \dots, m_n) \in Q_n$.



# 4 The infinite hypercube $Q_\infty = (\mathbb{Z}_2)^{\mathbb{N}}$

An element $x \in Q_\infty$ is an infinite sequence $x = (x_1, \ldots, x_n, \ldots)$ where $x_i = 0$ or 1. The group $Q_\infty$ is equiped with the product topology. For that topology a fundamental system of the identity element $0 = (0, \ldots, 0, \ldots)$ is the following family of subsets $V_A$, where $A$ is a finite subset of $\mathbb{N}$, and

$$V_A = \{x = (x_1, \ldots, x_n, \ldots) \mid x_i = 0 \text{ for } i \in A\}.$$

By Tychonov's theorem, the topological space $Q_\infty$ is compact.

Consider the map

$$q : Q_\infty \to [0, 1], \quad x = (x_1, \ldots, x_n, \ldots) \mapsto t = q(x) = \sum_{n=1}^{\infty} \frac{x_n}{2^n}.$$

Hence $(x_1, \ldots, x_n, \ldots)$ is the dyadic expansion of the real number $t$. The map $q$ is surjective, but not injective. In fact, for $x = (1, 0, \ldots, 0, \ldots)$ and $y = (0, 1, \ldots, 1, \ldots)$, then $q(x) = q(y)$:

$$q(x) = \frac{1}{2}, \quad q(y) = \sum_{n=2}^{\infty} \frac{1}{2^n} = \frac{1}{4} \frac{1}{1 - \frac{1}{2}} = \frac{1}{2}.$$

But if $t$ is not a dyadic number, i.e. a rational number of the form $\frac{k}{2^n}$, then there is a unique $x \in Q_\infty$ such that $t = q(x)$. We will define a map

$$d : [0, 1] \to Q_\infty, \quad t \mapsto d(t),$$

as follows. If $t$ is not a dyadic number, then $d(t) = x$ with $x = q(t)$. If $t$ is a dyadic number, $t = \frac{k}{2^n}$, we choose the finite dyadic expansion:

$$d(t) = (x_1, \ldots, x_n, 0, \ldots), \quad \text{with } x_n = 1.$$

The map $d$ is a right inverse of $q$: $q \circ d = Id$.

Observe that the map $d$ is not continuous. For instance, define

$$t_n = \frac{1}{2} - \frac{1}{2^n},$$

then $t_n \to \frac{1}{2}$, but $x^{(n)} = q(t_n)$ does not converge to $x = q\big(\frac{1}{2}\big)$. In fact $x = (1, 0, \ldots)$, and $x^{(n)} = \big(x_i^{(n)}\big)$, with

$$x_i^{(n)} = \begin{cases} 1 & \text{if } 2 \leq\leq n, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mu$ be the normalized Haar measure of the compact group $Q_\infty$. Then the image by the map $q$ of $\mu$ is the Lebesgue measure on $[0,1]$. This means that, if $f$ is a bounded measurable function on $[0,1]$, then

$$\int_{Q_\infty} f\big(q(x)\big)\mu(dx) = \int_0^1 f(t)dt.$$

Let us give some hints for proving this result. Let $\mu$ be a measure on $Q_\infty$ whose image by $q$ is the Lebesgue measure on $[0,1]$. Take $a = (1, 0, \ldots) \in Q_\infty$. Then one checks that

$$\int_{Q_\infty} f\big(q(x+a)\big)\mu(dx) = \int_0^{\frac{1}{2}} f\Big(t + \frac{1}{2}\Big)dt + \int_{\frac{1}{2}}^1 f\Big(t - \frac{1}{2}\Big)dt$$
$$= \int_0^1 f(t)dt = \int_{Q_\infty} f\big(q(x)\big)\mu(dx).$$

Take now, for $m \in \mathbb{N}$, $a = (a_1, \ldots, a_n, \ldots) \in Q_\infty$ with $a_m = 1$, $a_n = 0$ if $n \neq m$. then one checks that

$$\int_{Q_\infty} f\big(q(x+a)\big)\mu(dx)$$
$$= \sum_{k=0}^{2^{m-1}-1} \Big( \int_{\frac{2k}{2^m}}^{\frac{2k+1}{2^m}} f\Big(t + \frac{1}{2^m}\Big)dt + \int_{\frac{2k+1}{2^m}}^{\frac{2k+2}{2^m}} f\Big(t - \frac{1}{2m}\Big)dt \Big)$$
$$= \int_0^1 f(t)dt.$$

We will determine the characters of $Q_\infty$. Observe first that the hypercube $Q_N$ can be seen as a subgroup of $Q_\infty$, by the embedding

$$Q_N \to Q_\infty, \quad (x_1, \ldots, x_N) \mapsto (x_1, \ldots, x_N, 0, \ldots).$$

Observe that the restriction of a character of $Q_\infty$ to $Q_N$ is a character of $Q_N$.

Denote by $Q_{(\infty)}$ the set of sequences $m = (m_1, m_2, \ldots, m_n, \ldots)$ with $m_i =$ 0 or 1, for which there is only a finite number of indices $i$ such that $m_i = 1$.

**Proposition 4.1.** *A character $\chi$ of $Q_\infty$ is of the form*

$$\chi_m(x) = (-1)^{m_1 x_1 + \cdots + m_n x_n + \cdots},$$

*with $m \in Q_{(\infty)}$.*

Let $\chi$ be a character of $Q_\infty$. For $0 < \varepsilon < 1$ there is a neighborhood $V$ of $0 = (0, \ldots)$ such that, for $x \in V$, $|\chi(x) - 1| \leq \varepsilon$. The neighborhood $V$ contains a neighborhood of the form $V_A$ where $A \subset \mathbb{N}$ is finite. Choose $N$ such that $A \subset \{1, \ldots, N\}$ . The restriction of the character $\chi$ to $Q_N$ is of the form

$$\chi(x) = (-1)^{m_1 x_1 + \cdots + m_N x_N},$$

and takes the values 0 and 1. For $x \in V_A$, $|\chi(x) - 1| \leq \varepsilon < 1$, hence $m_1 x_1 + \cdots + m_N x_N = 0$. This implies that $m_i = 0$ if $i \notin A$.

By Theorem (?) the system $\{\chi_m\}_{m \in Q_{(\infty)}}$ is a Hilbert basis of $L^2(Q_\infty, \mu)$. The characters $\chi_m$ are related to the classical Rademacher and Walsh functions.

The *Rademacher function* $\varphi$ is defined on $[0, 1]$ as folows: $\varphi_0(t) = 1$, and, for $n \geq 1$, if $d(t) = (x_1, \ldots, x_n, \ldots)$ is the dyadic expansion of $t$, then

$$\varphi_n(t) = \begin{cases} 1 & \text{if } x_n = 0, \\ -1 & \text{if } x_n = 1, \end{cases}$$

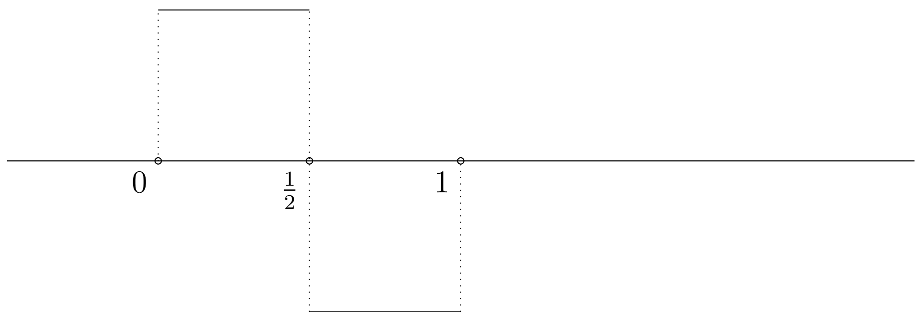
i. e. $\varphi_n(t) = (-1)^{x_n}$. More explicitely,

$$\text{if } \frac{k}{2^n} \leq t < \frac{k+1}{2^n}, \text{ then } \varphi_n(t) = (-1)^k \quad (k = 0, 1, \ldots, 2^n - 1).$$

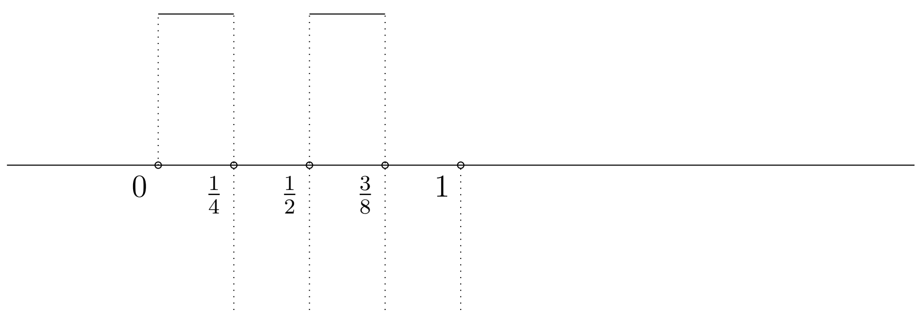
It can also be written $\varphi_n(t) = \operatorname{sign}(\sin 2^n \pi t)$.

For $m = (m_1, \ldots, m_n, \ldots) \in Q_{(\infty)}$, let $n_1, \ldots, n_p$ be the indices for which $m_n = 1$. Then, if $q(x) = t$,

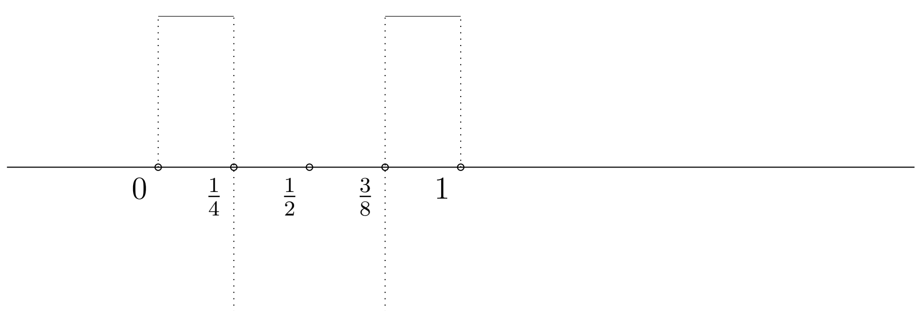
$$\chi_m(x) = \psi_m(t) = \varphi_{n_1}(t) \ldots \varphi_{n_p}(t).$$




Graph of the Rademacher function $\varphi_1$.




Graph of the Rademacher function $\varphi_2$.




Graph of the Walsh function $\psi_{(1,1)} = \varphi_1\varphi_2$

The system $\{\psi_m\}_{m\in Q_{(\infty)}}$ is a Hilbert basis of $L^2(0,1)$. It can be seen as a corollary of Theorem. But it is a classical result whcih can be proven directly as follows:

*The system* $\{\psi_m\}$ *is orthonormal in* $L^2(0,1)$

a) Observe first that $|\psi_m| = 1$, hence $\|\psi_m\|_2 = 1$.

b) *For* $0 \leq n_1 < n_2 < \cdots < n_p$,

$$\int_0^1 \varphi_{n_1}(t)\varphi_{n_2}(t)\ldots\varphi_{n_p}(t)dt = 0.$$

The function $\varphi_{n_1}(t)\varphi_{n_2}(t)\ldots\varphi_{n_{p-1}}(t)$ is constant on the intervals

$$\frac{k}{2^{n_p-1}} \leq t < \frac{k+1}{2^{n_p-1}},$$

and the integral of $\varphi_{n_p}$ on such an interval is equal to 0.

c) For $m \in Q_{(\infty)}$ one defines

$$\mathrm{supp}(m) = \{i \in \mathbb{N} \mid m_i = 1\}.$$

*If* $m \neq m'$, *then* $(\psi_m \mid \psi_{m'}) = 0$. Let $M = \mathrm{supp}(m) \cap \mathrm{supp}(m')$. Then

$$\psi_m(t)\psi_{m'}(t) = \prod_{n \in M} \varphi_n(t).$$

Therefore, by b),

$$\int_0^1 \psi_m(t)\psi_{m'}(t)dt = 0.$$

*The system* $\{\psi_m\}$ *is a Hilbert basis.*

Let $f \in L^2(0,1)$ such that

$$\forall m,\ (f \mid \psi_m) = 0.$$

We will show that $f \equiv 0$. Define

$$F(t) = \int_0^t f(s)ds.$$

Then $F(0) = 0$, and

$$F(1) = \int_0^1 f(s)ds = \int_0^1 \psi_0(s)ds = 0.$$

Take now $\psi_m$, with $m = (1, 0, \ldots)$,

$$\int 0^1 f(s)\varphi_1(s)ds = \int_0^{\frac{1}{2}} f(s)ds - \int_{\frac{1}{2}} f(s)ds = 2F\big(\frac{1}{2}\big),$$

therefore $F\big(\frac{1}{2}\big) = 0$.

Take $\psi_m$ with $m = (0, 1, 0, \ldots)$, and $m = (1, 1, 0, \ldots)$,

$$\begin{aligned} \int_0^1 f(s)\varphi_2(s)ds &= 2\Big(F\big(\frac{1}{4}\big) + F\big(\frac{3}{4}\big)\Big) = 0, \\ \int_0^1 f(s)\varphi_1(s)\varphi_2(s)ds &= 2\Big(F\big(\frac{1}{4}\big) - F\big(\frac{3}{4}\big)\Big) = 0, \end{aligned}$$

therefore $F\big(\frac{1}{4}\big) = 0$, $F\big(\frac{3}{4}\big) = 0$.

Taking $\psi_m$, with

$$m = (0, 0, 1, 0 \ldots),\ (1, 0, 1, 0 \ldots),\ (0, 1, 1, 0, \ldots),\ (1, 1, 1, 0 \ldots),$$

we get

$$\begin{aligned} \int_0^1 f(s)\varphi_3(s)ds &= 2\Big(F\big(\frac{1}{8}\big) + F\big(\frac{3}{8}\big) + F\big(\frac{5}{8}\big) + F\big(\frac{7}{8}\big)\Big) = 0, \\ \int_0^1 f(s)\varphi_1(s)\varphi_3(s)ds &= 2\Big(F\big(\frac{1}{8}\big) + F\big(\frac{3}{8}\big) - F\big(\frac{5}{8}\big) - F\big(\frac{7}{8}\big)\Big) = 0, \\ \int_0^1 f(s)\varphi_2(s)\varphi_3(s)ds &= 2\Big(F\big(\frac{1}{8}\big) - F\big(\frac{3}{8}\big) - F\big(\frac{5}{8}\big) + F\big(\frac{7}{8}\big)\Big) = 0, \\ \int_0^1 f(s)\varphi_1(s)\varphi_2(s)\varphi_3(s)ds &= 2\Big(F\big(\frac{1}{8}\big) - F\big(\frac{3}{8}\big) + F\big(\frac{5}{8}\big) - F\big(\frac{7}{8}\big)\Big) = 0, \end{aligned}$$

therefore

$$F\big(\frac{1}{8}\big) = F\big(\frac{3}{8}\big) = F\big(\frac{5}{8}\big) = F\big(\frac{7}{8}\big) = 0.$$

And by going on one shows that $F$ vanishes at every dyadic rational number. Since the function $F$ is continuous, and the set of dyadic rational numbers is dense in $[0, 1]$, $F$ vanishes identically. This means that $f$ is orthogonal to the chacacteristic functions of the intervals $[0, t]$ $(0 \le t \le 1)$. Since the system of these characteristic functions is total in $L^2(0, 1)$, it follows that $f \equiv 0$. □

See [Higgens,1977], p.45, 2.4 The functions of Rademacher, Walsh and Haar, and also [Alexist,1961], p.51, Ch.1 §7 Rademacher's and Walsh's orthogonal systems. Relations to the theory of probability.

# 5 Representations of compact groups

We give first some definitions. Let $G$ be a topological group and $V$ a normed complex vector space. A *representation* $\pi$ of $G$ on $V$ is a group morphism

$$G \to GL(V),\ g \mapsto \pi(g),$$

i.e.

$$\pi(g_1)g_2) = \pi(g_1\pi(g_2),\ \pi(g^{-1}) = \pi(g)^{-1},$$

such that, for any $v \in V$, the map $g \mapsto \pi(g)v$ is continuous.

A subspace $W \subset V$ is *invariant* if, for any $g \in G$, $\pi(g)W = W$. The representation $\pi$ is *irreducible* if the only closed invariant subspaces are $\{0\}$ and $V$. In particular, if $\dim V = 1$, the $\pi$ is irreducible. Let $(\pi_1, V_1)$, $(\pi_2, V_2)$ two representations of $G$. An it intertwining operator $A$ is an isomorphism $A = V_1 \to V_2$, such that, for any $g \in G$,

$$A\pi_1(g) = \pi_2(g)A.$$

If such an intertwining operator exists, the representations $(\pi_1, V_1)$ and $(\pi_2, V_2)$ are said to be *equivalent*.

Assume that $V = \mathcal{H}$ is a Hilbert space. A representation $\pi$ of $G$ on $\mathcal{H}$ is *unitary* if, for any $g \in G$, $\pi(g)$ is unitary.

We assume now that the group $G$ is compact.

Let $(\pi, V)$ be an irreducible representation of $G$. Then the vector space $V$ is finite dimensional, and there is on $V$ an inner product such that $\pi$ is unitary.

*Schur orthogonality relations*

Let $\pi$ be a irreducible unitary representation of $G$ on a finite dimensional Euclidean vector space, of dimension $d$. Let $\{e_1, \dots, e_d\}$ be an orthonormal basis of $V$, and $\big(\pi_{ij}(g)\big)$ the matrix of $\pi(g)$ with respect to this basis. The functions $\pi_{ij}(g)$ are orthogonal in $L^2(G, \mu)$: if $(i, j) \neq k, \ell)$,

$$\int_G \pi_{ij}(x)\overline{\pi_{k\ell}(x)}\mu(dx) = 0,$$

and

$$\int_G |\pi_{ij}(x)|^2 \mu(dx) = \frac{1}{d}.$$

If $(\pi_1, V_1)$ and $(\pi_2, V_2)$ are irreducible representations of $G$ which are not equivalent, then the space $\mathcal{M}_1$ of matrix coefficients of $(\pi_1, V_1)$ and the space $\mathcal{M}_2$ of matrix coefficients of $(\pi_2, V_2)$ are orthogonal in $L^2(G, \mu)$.

*Peter-Weyl theorem*

The set of equivalence classes of irreducible representations of $G$ is denoted $\hat{G}$. For a class $\lambda \in \hat{G}$, one chooses a representative $(\pi^{(\lambda)}, \mathcal{H}_\lambda)$ and denotes by $d_\lambda$ the dimension of the representation space $\mathcal{H}^\lambda$. By the Schur orthogonality relations, the system

$$\{\sqrt{d_\lambda}\pi_{ij}^{(\lambda)} \mid \lambda \in \hat{G},\ i, j = 1, \ldots, d_\lambda\}.$$

**Theorem 5.1. (Peter-Weyl)** *The system*

$$\{\sqrt{d_\lambda}\pi_{ij}^{(\lambda)} \mid \lambda \in \hat{G},\ i, j = 1, \ldots, d_\lambda\}.$$

*is a Hilbert basis of* $L^2(G, \mu)$.

# 6 Characters and central functions

Let $\pi$ be a representation of $G$ on a finite dimensional vector space $V$. the *character* of the representation $\pi$ is the function $\chi_\pi$ defined on $G$ by

$$\chi_\pi(g) = \operatorname{tr} \pi(g).$$

Observe that $\chi_\pi(e) = \dim V$, and, for $g, x \in G$,

$$\chi_\pi(gxg^{-1}) = \chi_\pi(g).$$

In fact, for two square matrices $A$, $B$, $\operatorname{tr} AB = \operatorname{tr} BA$. Two elements $x, y \in G$ are said to be *conjugate* if there exists $g \in G$ such $y = gxg^{-1}$. this is an equivalence relation. An equivalence class is called a *conjugacy class*. A function $f$ on $G$ is siad to be *central* if

$$f(gxg^{-1}) = f(x) \quad (x, g \in G),$$

i. e. the function $f$ is constant on each conjugacy class. The character of a representation is a central function. By the Schur orthogonality relations, if $\pi$ is irreducible, then

$$\int_G |\chi_\pi(x)|^2 \mu(dx) = \sum_{i=1}^{d_\pi} \int_G |\pi_{ii}(g)|^2 \mu(dx) = 1.$$

If the representations $\pi_1$ and $\pi_2$ are irreducible and non equivalent, then $\chi_{\pi_1}$ and $\chi_{\pi_2}$ are orthogonal in $L^2(G,\mu)$:

$$\int_G \chi_{\pi_1}(x)\overline{\chi_{\pi_2}(x)}\mu(dx) = 0.$$

The characters of two equivalent representations are equal. Therefore we will denote by $\chi_\lambda$ the character of any representation in the class $\lambda \in \hat{G}$. Hence the system $\{\chi_\lambda\}_{\lambda\in\hat{G}}$ is orthogonal in $L^2(G,\mu)$.

As a consequence of the Peter-Weyl theorem (Theorem 5.1) one obtains

**Theorem 6.1.** *The system $\{\chi_\lambda\}_{\lambda\in\hat{G}}$ is a Hilbert basis of the space $L^2(G,\mu)_c$ of central functions in $L^2(G,\mu)$.*

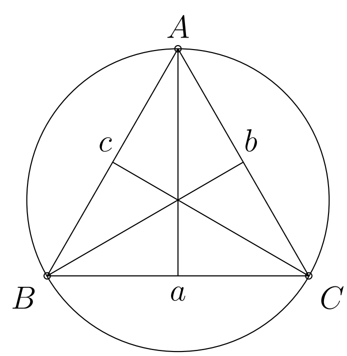
**Proposition 6.2.** *Assume the group $G$ to be finite.*
(i) $\#G = \sum_{\lambda\in\hat{G}} d_\lambda^2$.
(ii) $\#\{\text{conjugacy classes}\} = \#\hat{G}$.

Property (i) follows from the Peter-Weyl theorem (Theorem 5.1), and (ii) from Theorem 6.1.

# 7 Examples

### 1) The group $\mathfrak{S}_3$ of permutations of three elements

The number of elements of $\mathfrak{S}_3$ is $\#\mathfrak{S}_3 = 3! = 6$. The group $\mathfrak{S}_3$ can be seen as the group of isometries of the equilateral triangle.



There are 3 conjugacy classes
- $\mathcal{C}_0 = \{e\}$, where $e$ is the identity

$$e = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}.$$

- $\mathcal{C}_1$: the 3 transpositions

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}.$$

They correspond to the symmetries with repect to the three heights $Aa$, $Bb$, $Bc$.
- $\mathcal{C}_2$: the two circular permutations

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}.$$

They correspond to the rotations with angles $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$.

By property (i) $\#\hat{G} = 3$, $\hat{G} = \{\lambda_0, \lambda_1, \lambda_2\}$. The representation $\pi_{\lambda_0}$ is the trivial representation. $\pi_{\lambda_2}$ is the one dimensional representation given by

$$\pi_{\lambda_1}(g) = \varepsilon(g),$$

the signature of the permutation $g$,

$$\begin{aligned} \varepsilon(g) &= 1 \text{ if } g \in \mathcal{C}_0, \text{ or } g \in \mathcal{C}_2, \\ &= -1 \text{ if } g \in \mathcal{C}_1 \end{aligned}$$

Hence $d_{\lambda_0} = 1$, $d_{\lambda_1} = 1$, and, since by property (ii)

$$d_{\lambda_0}^2 + d_{\lambda_1}^2 + d_{\lambda_2} = \#\mathfrak{S}_3 = 6,$$

a representation in the class $\lambda_2$ is two dimensional: $d_{\lambda_2} = 2$. In fact a representative $\pi_{\lambda_2}$ maps $\mathfrak{S}_3$ to the group of isometries of the equilateral triangle. In particular

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \mapsto \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \mapsto \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}.$$

*Character table*

| | $\mathcal{C}_0$ | $\mathcal{C}_1$ | $\mathcal{C}_2$ |
|---|---|---|---|
| # | 1 | 3 | 2 |
| $\chi_0$ | 1 | 1 | 1 |
| $\chi_1$ | 1 | -1 | 1 |
| $\chi_2$ | 2 | 0 | -1 |

From the character table one can check that the system of the characters is orthonormal in $L^2(G)$ by using the formula

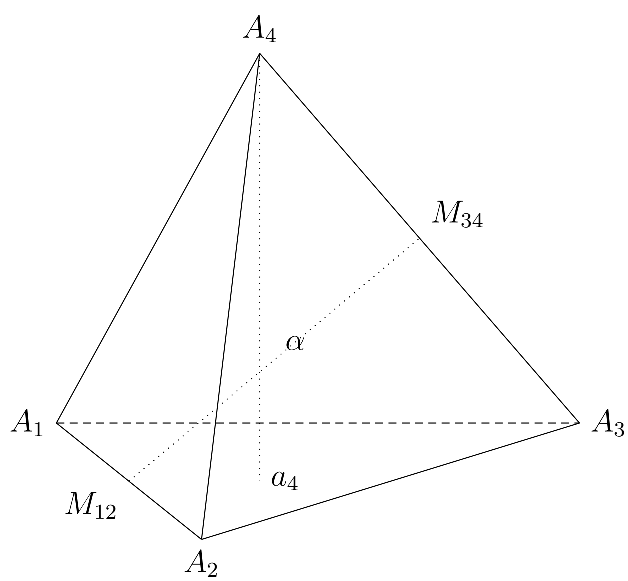
$$(\chi_\lambda \mid \chi_\mu) = \frac{1}{\#G} \sum_i (\#\mathcal{C}_i)\chi_\lambda(\mathcal{C}_i)\overline{\chi_\mu(\mathcal{C}_i)}.$$

**2) The group $A_4$ of even permutations of 4 elements**

The group $G = A_4$ is a subgroup of the permutation group $\mathfrak{S}_4$:

$$G = \{g \in \mathfrak{S}_4 \mid \varepsilon(g) = 1\},$$

and $\#G = \frac{1}{2}\#\mathfrak{S}_4 = \frac{1}{2}4! = 12$. The group $G$ is also the group of rotations of the regular tetrahedron.



There are 4 conjugacy classes:

- $\mathcal{C}_0 = \{e\}$, where $e$ is the identity

$$e = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}.$$

- $\mathcal{C}_1$: the rotation of $\frac{2\pi}{3}$ around the axis $A_4a_4$ and orthogonal to the opposite face oriented toward the vertex $A_4$:

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix},$$

and 3 similar rotations coresponding to the three other vertices. $\#\mathcal{C}_1 = 4$.

- $\mathcal{C}_3$: the rotation of $-\frac{2\pi}{3}$ around the same axis:

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix},$$

and 3 similar rotations. $\#\mathcal{C}_2 = 4$.

- $\mathcal{C}_4$: the rotation of $\pi$ around the line $\alpha$ joining the middles $M_{12}$ and $M_{34}$ of the two opposite edges $A_1A_2$ and $A_3A_4$:

$$a = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix},$$

and two similar rotations around to the lines $\beta$ and $\gamma$,

$$b = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}, \quad c = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}.$$

$\#\mathcal{C}_4 = 3$.

The subset $H = \{e, a, b, c\}$ is an Abelian subgroup:

$$a^2 = b^2 = c^2 = e,\ ab = c,\ ac = b,\ bc = a.$$

It is isomorphic to $\mathfrak{Z}_2 \times \mathfrak{Z}_2$:

$$\begin{array}{cccc} e & a & b & c \\ (0,0) & (1,0) & (0,1) & (1,1) \end{array}.$$

The lines $\alpha, \beta, \gamma$ are 2 by 2 orthogonal and the set $\{\alpha, \beta, \gamma\}$ is invariant under the action of $G = A_4$. The subgroup of the transformations $g \in G$ which fix each of the lines $\alpha, \beta, \beta$ is the subgroup $H$. Hence $H$ is a normal subgroup and the quotient group $G/H$ is isomorphic to $\mathfrak{Z}_3$ acting on the set $\{\alpha, \beta, \gamma\}$ by even permutations. For instance,

$$\text{if } g = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix}, \text{ then } \alpha \mapsto \beta,\ \beta \mapsto \gamma,\ \gamma \mapsto \alpha.$$

Recall that the characters of the Abelian group $\mathfrak{Z}_3$ are given by, if $x \in \{0, 1, 2\}$, then $\chi_\lambda(x) = \omega^{\lambda x}$, where $\lambda \in \{0, 1, 2\}$, and $\omega = e^{\frac{2i\pi}{3}}$. Hence, by composition,

$$G \to G/H \simeq \mathbb{Z}_3 \to \mathbb{T},\ g \mapsto \dot{g} \mapsto \chi_1(\dot{g}),$$

one obtains two one dimensional representations $\pi_1$ and $\pi_2$ of $G$.

By the property (i) of Proposition 6.2, $\#\hat{G} = 4$, $\hat{G} = \{\lambda_0, \lambda_1, \lambda_2, \lambda_3\}$,

$$d_{\lambda_0} = 1,\ d_{\lambda_1} = 1,\ d_{\lambda_2} = 1.$$

By the property (ii) of Proposition 6.2,

$$d_{\lambda_0}^2 + d_{\lambda_1}^2 + d_{\lambda_2}^2 + d_{\lambda_3}^2 = \#G = 12,$$

therefore $d_{\lambda_3} = 3$. A representative of $\lambda_3$ is the representation $\pi_3$ of $G$ which maps an element $g$ to a rotation of the 3-dimensional Euclidean space fixing the regular tetrahedron. For $g \in \mathcal{C}_1$ of $\mathcal{C}_2$, the trace of the rotation $\pi_3(g)$ is

$$\operatorname{tr} \pi_3(g) = \cos\frac{2\pi}{3} + \cos\frac{2\pi}{3} + 1 = 1 + 2\cos\frac{2\pi}{3} = 0.$$

For $g \in \mathcal{C}_3$,

$$\operatorname{tr} \pi_3(g) = -1 - 1 + 1 = -1.$$

*Character table*

| | $\mathcal{C}_0$ | $\mathcal{C}_1$ | $\mathcal{C}_2$ | $\mathcal{C}_3$ |
|---|---|---|---|---|
| # | 1 | 4 | 4 | 3 |
| $\chi_0$ | 1 | 1 | 1 | 1 |
| $\chi_1$ | 1 | $\omega$ | $\omega^2$ | 1 |
| $\chi_2$ | 1 | $\omega^2$ | $\omega$ | 1 |
| $\chi_3$ | 3 | 0 | 0 | -1 |

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