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Dedicated to Grigori Olshanski
on the occasion of his 60th birthday

OLSHANSKI SPHERICAL PAIRS RELATED TO THE HEISENBERG GROUP

Abstract. — An Olshanski spherical pair (G,K) is the inductive limit of a sequence of Gelfand pairs $\big(G(n),K(n)\big)$. A natural question arises: how a spherical function for (G,K) can be obtained as limit of spherical functions for $\big(G(n),K(n)\big)$. In this paper we consider a sequence of Gelfand pairs $\big(G(n),K(n)\big)$ related to the Heisenberg group.

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1. Introduction. — For a locally compact group G and a compact subgroup K, $L^1(K\backslash G/K)$ is the convolution algebra of K-biinvariant integrable functions on G. Assume that (G,K) is a Gelfand pair, i.e. the algebra $L^1(K\backslash G/K)$ is commutative. A spherical function for the Gelfand pair (G,K) is a continuous K-biinvariant function φ on G with $\varphi(e)=1$, and

$$\int_K \varphi(xky)\alpha(dk)=\varphi(x)\varphi(y)\quad (x,y\in G),$$

(α denotes the normalized Haar measure on K). A character χ of the commutative Banach algebra $L^1(K\backslash G/K)$ has the form

$$\chi(f)=\int_G f(x)\varphi(x)m(dx),$$

where φ is a bounded spherical function (m is a left Haar measure on G, which is a right Haar measure as well since G is unimodular). The Gelfand spectrum Σ of $L^1(K\backslash G/K)$ can be identified with the set of bounded spherical functions. We will write $\varphi(\sigma;x)$ for the spherical function associated to $\sigma \in \Sigma$. The Gelfand spectrum is a locally compact topological space.

Assume that G is a connected Lie group, and let $\mathbb{D}(G/K)$ denote the algebra of invariant differential operators on the quotient space G/K. A spherical function is of $\mathcal{C}^\infty$ class, and $\varphi(\sigma;x)$ is an eigenfunction of every $D \in \mathbb{D}(G/K)$:

$$D\varphi(\sigma;x) = \hat{D}(\sigma)\varphi(\sigma;x).$$

The function $\hat{D}$ is continuous on Σ. Moreover the topology on Σ coincide with the initial topology with respect to the set of functions $\{\hat{D} \mid D \in \mathbb{D}(G/K)\}$ ([Ferrari-Ruffino,2007]).

An Olshanski spherical pair (G,K) is the inductive limit of an increasing sequence of Gelfand pairs $\big(G(n),K(n)\big)$:

$$G = \bigcup_{n=1}^{\infty}, G(n) \quad K = \bigcup_{n=1}^{\infty} K(n),$$

and a spherical function for the Olshanski spherical pair (G,K) is a K-biinvariant continuous function φ on G, with $\varphi(e)=1$, and such that

$$\lim_{n\to\infty} \int_{K(n)} \varphi(xky)\alpha_n(dk) = \varphi(x)\varphi(y),$$

where α_n denotes the normalized Haar measure on $K(n)$. Let Σ_n denote the Gelfand spectrum of the Gelfand pair $\big(G(n),K(n)\big)$, and write $\phi_n(\sigma;x)$ for the spherical function associated to $\sigma \in \Sigma_n$. We consider the following question: for which sequences $(\sigma^{(n)})$, with $\sigma^{(n)} \in \Sigma_n$, does the sequence $\varphi_n(\sigma^{(n)};x)$ converge as n goes to infinity ? Such a sequence is called a Vershik-Kerov sequence. This question has been solved in several cases. Kerov and Vershik have considered the case of the infinite symmetric group: $G(n) = \mathfrak{S}_n \times \mathfrak{S}_n$, $K(n) \simeq \mathfrak{S}_n$ [1981], and the case of the infinite dimensional unitary group: $G(n) = U(n) \times U(n)$, $K(n) \simeq U(n)$ [1982]. The case of the generalized motion group: $G(n) = U(n) \ltimes Herm(n,\mathbb{C})$, $K(n) = U(n)$ is the subject of [Olshanski-Vershik,1996]. The papers [Okounkov-Olshanski,1998 and 2006] are related to the case of sequences $G(n)/K(n)$ of compact symmetric spaces. We will consider in this paper an Olshanski spherical pair associated to the infinite dimensional Heisenberg group.

Let us say in which terms the Vershik-Kerov sequences can be described in each of these cases. One introduces a topological space Σ, the 'spectrum' of the Olshanski spherical pair (G,K), which parametrizes a family $\varphi(\sigma;x)$ of spherical functions for (G,K). The topology of Σ corresponds to the convergence of the spherical functions $\varphi(\sigma;x)$ uniformly on compact sets in $G(n)$. For each n one defines an injective map $T_n : \Sigma_n \to \Sigma$. Let $(\sigma^{(n)})$ be a sequence with $\sigma^{(n)} \in \Sigma_n$. Then $(\sigma^{(n)})$ is a Vershik-Kerov sequence if and only if the sequence $T_n(\sigma^{(n)})$ converges for the topology of Σ:

$$\lim_{n\to\infty} T_n(\sigma^{(n)}) = \sigma.$$

In such a case,

$$\lim_{n\to\infty} \varphi_n(\sigma^{(n)}; x) = \varphi(\omega; x).$$

(See the survey [Faraut,2008], and further examples [Rabaoui,2008], [Faraut,2010].)

(1) To prove the convergence one establishes generalized Taylor expansions for $\varphi_n(\sigma; x)$ and $\varphi(\sigma; x)$ at the identity element of $G(n)$ and G. One shows that the convergence of $T_n(\sigma^{(n)})$ to $\sigma \in \Sigma$ implies the convergence of the coefficients in the expansions, and further the convergence of $\varphi_n(\sigma^{(n)}; x)$ to $\varphi(\sigma; x)$.

(2) For the converse one assumes that $\varphi_n(\sigma^{(n)}; x)$ converges to a continuous function φ on G. One looks at the restriction of these functions to $G(1)$, and gets that the sequence $T_n(\sigma^{(n)})$ is relatively compact in Σ. Therefore there is a subsequence $(\sigma^{(n_j)})$ such that $T_n(\sigma^{(n_j)})$ converges to σ_0 in Σ. By the step (1)

$$\lim_{j\to\infty} \varphi_n(\sigma^{(n_j)}; x) = \varphi(\sigma_0; x),$$

and $\varphi(\omega_0; x) = \varphi(x)$. Hence there is only one possible limit for a subsequence. Therefore the sequence $T_n(\sigma^{(n)})$ itself converges.

In this paper we will establish such a result for an Olshanski spherical pair related to the infinite dimensional Heisenberg group. These pairs are inductive limits of Gelfand pairs $\big(G(n), K(n)\big)$ where $G(n)$ is the semi-direct product $K(n) \ltimes H(n)$, $H(n) = W(n) \times \mathbb{R}$ is a Heisenberg group, $W(n)$ is a complex Euclidean vector space, and $K(n)$ is a group of automorphisms of $H(n)$. In [Faraut,2010] we have considered the case of $W(n) = M(n, n+q; \mathbb{C})$, a space of complex rectangular matrices, and $K(n) = U(n) \times U(n+q)$ acting on both sides on $M(n, n+q; \mathbb{C})$. In the present paper we consider the three cases $W(n) = Sym(n, \mathbb{C})$, $M(n, \mathbb{C})$, and $Skew(2n, \mathbb{C})$.

In Section 2 we recall the definition of shifted symmetric polynomials and some of their properties. Then in Section 3 we introduce the three sequences of Gelfand pairs and establish in Section 4 series expansion for their bounded spherical functions. In next section, by using a result by Ferrari-Ruffino we determine the Gelfand spectrum of these Gelfand pairs. Then we define in Section 6 the Olshanski spherical pairs, inductive limits of the Gelfand pairs which were considered in Section 3. In Sections 7 and

8 we determine the Vershik-Kerov sequences relative to these Olshanski spherical pairs. In Section 9 are some remarks about multivariate Laguerre polynomials.

2. Spherical polynomials, shifted spherical polynomials. — We consider the symmetric cone Ω of positive definite Hermitian matrices in the Euclidean vector space $V = Herm(n, \mathbb{F})$ where $\mathbb{F} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$, the field of quaternions, with the inner product $(x|y) = \mathrm{tr}(xy)$. The cone Ω is a Riemannian symmetric space, $\Omega = L/K_0$, where L is the connected component of the group of linear automorphisms of Ω, and $K_0 \subset L$ is the isotropy subgroup of the identity matrix e. The spherical functions for the Gelfand pair (L, K_0) are given by

$$\varphi(\mathbf{s}; x) = \int_{K_0} \Delta_{\mathbf{s}}(k \cdot x)\alpha(dk) \quad (\mathbf{s} \in \mathbb{C}^n),$$

where $\Delta_{\mathbf{s}}$ is the power function

$$\Delta_{\mathbf{s}}(x) = \Delta_1(x)^{s_1 - s_n}\Delta_2(x)^{s_2 - s_3} \ldots \Delta_n(x)^{s_n},$$

and $\Delta_1, \Delta_2, \ldots, \Delta_n$ are the principal minors, $\mathbf{s} = (s_1, \ldots, s_n)$.

To a K_0-invariant polynomial P on V one associates an invariant differential operator $D_P = p\big(x, \frac{\partial}{\partial x}\big)$ on $V \times V$ such that

$$p(g \cdot x, \xi) = p(x, g'\xi) \quad (g \in L)$$
$$p(e, \xi) = P(\xi).$$

The spherical function $\varphi(\mathbf{s}; x)$ is an eigenfunction of D_P,

$$D_P\varphi(\mathbf{s}; x) = P^*(\mathbf{s})\varphi(\mathbf{s}; x).$$

The function

$$P^*(\mathbf{s}) = P\Big(\frac{\partial}{\partial x}\Big)\varphi(\mathbf{s}; x)\big|_{x=e}$$

is a shifted symmetric polynomial in the sense that

$$\gamma(\lambda) = P^*(\lambda + \rho)$$

is symmetric, with

$$\rho_j = \frac{d}{4}(2j - n - 1), \quad d = \dim_{\mathbb{R}}\mathbb{F} = 1, 2, 4.$$

In fact γ corresponds to D_P via the Harish-Chandra isomorphism

$$\mathbb{D}(G/K) \simeq S(\mathbb{C}^n)^{\mathfrak{S}_n}.$$

In other words P^* is symmetric in the variables $s_j - \theta j$, $\theta = \frac{d}{2}$. We will say that P^* is θ-shifted symmetric.

Under the action of L, the space $\mathcal{P}(V)$ of polynomial functions on V decomposes multiplicity free as

$$\mathcal{P}(V) = \bigoplus_{\mathbf{m}} \mathcal{P}_{\mathbf{m}},$$

where the summation is over the set of partitions $\mathbf{m} = (m_1, \ldots, m_n)$, $m_j \in \mathbb{N}$, $m_1 \geq \cdots \geq m_n \geq 0$ of length $\ell(\mathbf{m}) \leq n$ ([Schmid,1969], see also [Faraut-Korányi,1994], XI.2). The space $\mathcal{P}_{\mathbf{m}}^{K_0}$ of K_0-invariant polynomials in $\mathcal{P}_{\mathbf{m}}$ is one dimensional, generated by the spherical polynomial $\Phi_{\mathbf{m}}$, which is normalized by $\Phi_{\mathbf{m}}(e) = 1$. Furthermore $\Phi_{\mathbf{m}}(x) = \varphi(\mathbf{m}; x)$. The shifted spherical polynomial is given by

$$\Phi^*_{\mathbf{m}}(\mathbf{s}) = \Phi_{\mathbf{m}}\Big(\frac{\partial}{\partial x}\Big)\varphi(\mathbf{s}; x)\big|_{x=e}.$$

Observe that, for $n = 1$,

$$\varphi(s; x) = x^s, \ \Phi_m(s) = x^m, \ \Phi^*_m(s) = s(s-1)\ldots(s-m+1) = [s]_m.$$

A K_0-invaraint function f on V is of the form

$$f(x) = F(x_1, \ldots, x_n),$$

where $x_1, \ldots, x_n$ are the eigenvalues of x, and F is a symmetric function, i.e. invaraint under the symmetric group $\mathfrak{S}_n$. The spherical polynomials are related to the Jack polynomials as follows

$$\Phi_{\mathbf{m}}(x) = \frac{P_{\mathbf{m}}(x_1, \ldots, x_n; \theta)}{P_{\mathbf{m}}(1, \ldots, 1; \theta)},$$

with the notation of [Okounkov-Olshanski,1997], $\theta = \frac{d}{2}$ (or [Macdonald,1995], where the parameter is $\alpha = \frac{2}{d}$ insead θ), and also

$$\Phi^*_{\mathbf{m}}(s_1, \ldots, s_n) = \frac{P^*_{\mathbf{m}}(s_1, \ldots, s_n; \theta)}{P_{\mathbf{m}}(1, \ldots, 1; \theta)}.$$

By [Knop-Sahi,1996], for partitions $\mathbf{m}$ and $\mathbf{p}$,

$$\Phi^*_{\mathbf{m}}(\mathbf{p}) = 0, \text{ if } \mathbf{m} \not\subset \mathbf{p},$$
$$\Phi^*_{\mathbf{m}}(t\mathbf{s}) \sim t^{|\mathbf{m}|}\Phi_{\mathbf{m}}\big(\mathrm{diag}(s_1, \ldots, s_n)\big) \quad (t \to \infty).$$

Proposition 1 (Binomial formula). — *For* $\mathbf{s}\in\mathbb{C}^n$, $x\in V$, $\|x\|_{\rm op}<1$,

$$\varphi(\mathbf{s};e+x)=\sum_{\mathbf{m}}\frac{d_{\mathbf{m}}}{\big(1+\theta(n-1)\big)_{\mathbf{m}}}\Phi^*_{\mathbf{m}}(\mathbf{s})\Phi_{\mathbf{m}}(x),$$

where $d_{\mathbf{m}}=\dim\mathcal{P}_{\mathbf{m}}$, *and, for* $u\in\mathbb{C}$,

$$(u)_{\mathbf{m}}=\prod_{j=1}^{n}\big(u-\theta(j-1)\big)_{m_j}.$$

Observe that $\theta=\frac{d}{2}=\frac{1}{2},1$ or 2, $N=\dim V=n+n(n-1)\theta$, and $\frac{N}{n}=1+(n-1)\theta$.

If $\mathbf{s}=\mathbf{p}$ is a partition, then the sum is finite:

$$\varphi(\mathbf{p};e+x)=\Phi_{\mathbf{p}}(e+x)=\sum_{\mathbf{m}\subset\mathbf{p}}\frac{d_{\mathbf{m}}}{\big(1+\theta(n-1)\big)_{\mathbf{m}}}\Phi^*_{\mathbf{m}}(\mathbf{p})\Phi_{\mathbf{m}}(x),$$

Proof. The spherical function $\varphi(\mathbf{s};x)$ admits a holomorphic continuation in the tube $V+i\Omega\subset V_{\mathbb{C}}$, and the ball $\{z\in V_{\mathbb{C}}\mid \|z-e\|_{\rm op}<1\}$ is contained in $V+i\Omega$. Therefore the spherical expansion of $\varphi(\mathbf{s};z)$ at $z=e$, converges in the ball $\{z\in V_{\mathbb{C}}\mid \|z-e\|_{\rm op}<1\}$. This follows from Theorem XII.3.1 in [Faraut-Korányi,1994]. $\square$

The binomial formula has been established for Jack polynomials $P_{\mathbf{m}}(x_1,\ldots,x_n;\theta)$ for all $\theta>0$ in [Okounkov-Olshanski,1997].

3. Gelfand pairs associated with the Heisenberg group. — For a Euclidean complex vector space W we consider the Heisenberg group $H=W\times\mathbb{R}$ with the product

$$(z,t)(z',t')=\big(z+z',t+t'+{\rm Im}(z'|z)\big).$$

The unitary group $U(W)$ acts on H by automorphisms:

$$u\cdot(z,t)=(u\cdot z,t).$$

Let $K\subset U(W)$ be a closed subgroup, and $G=K\ltimes H$.

Theorem 2 ([Carcano,1987]. — *(G,K) is a Gelfand pair if and only if K acts multiplicity free on $\mathcal{P}(W)$, the space of holomorphic polynomial functions on W.*

These Gelfand pairs and the associated spherical functions have been studied by C. Benson, J. Jenkins, and G. Ratcliff in a series of papers ([1992],[1996],[1998]); see also [Dib,1990], and the book by J. Wolf [2007], chapter 13. In the rest of the paper the space W will be the complexification $W = V_{\mathbb{C}}$ of one of the real Euclidean vector spaces $Herm(n, \mathbb{F})$ we considered in Section 3, with the action of the compact group K of complex linear automorphisms of the bounded symmetric domain of tube type $\mathcal{D} = \{z \in W \mid \|z\|_{\mathrm{op}} < 1\}$.

W	K	d
$Sym(n, \mathbb{C})$	$U(n)$	1
$M(n, \mathbb{C})$	$U(n) \times U(n)$	2
$Skew(2n, \mathbb{C})$	$U(2n)$	4

In the first case $k \in K = U(n)$ acts on W by $k \cdot z = kzk'$, where k' denotes the transpose of k. In the second case $k = (k_1, k_2) \in K = U(n) \times U(n)$ acts by $k \cdot z = k_1 z k_2^{-1}$, and in the third case the action is the same as in the first case. A K-invariant function f on W can be written $f(z) = F(r_1, \ldots, r_n)$ where $r_1, \ldots, r_n$ are the eigenvalues of zz^*. Notice that in the third case, $W = Skew(2n, \mathbb{C})$, generically the eigenvalues $r_1, \ldots, r_n$ have multiplicity 2. By the Schmid decomposition, the multiplicity free condition is satisfied, and (G, K) is a Gelfand pair.

4. Bounded spherical functions. — There are two kinds of spherical functions. The spherical functions of first kind are associated to the Bargmann representation of H, and the ones of second kind to one dimensional representations of H.

a) *Bounded spherical functions of first kind.*

For $\lambda \in \mathbb{R}^*$ one considers the Fock space $\mathcal{F}_\lambda(W)$ of holomorphic functions ψ on W such that

$$\|\psi\|_\lambda^2 = \Big(\frac{|\lambda|}{\pi}\Big)^N \int_W |\psi(\zeta)|^2 e^{-|\lambda| \|\zeta\|^2} m(d\zeta) < \infty,$$

and the representation π_λ of the Heisenberg group $H = W \times \mathbb{R}$ on $\mathcal{F}_\lambda(W)$ is defined, if $\lambda > 0$, by

$$\big(\pi_\lambda(z,t)\psi\big)(\zeta) = e^{\lambda\left(it - \frac{1}{2}\|z\|^2 - (\zeta|z)\right)} \psi(\zeta + z),$$

and $\pi_\lambda(z,t) = \pi_{-\lambda}(\bar{z}, -t)$, for $\lambda < 0$. The group K acts on $\mathcal{F}_\lambda(W)$:

$$\big(\tau(k)\psi\big)(\zeta) = \psi(k^{-1} \cdot \zeta),$$

and

$$\tau(k)\pi_\lambda(z,t)\tau(k^{-1}) = \pi_\lambda(k\cdot z, t).$$

For the action of K, the Fock space decomposes multiplicity free:

$$\mathcal{F}_\lambda(W) = \widehat{\bigoplus_{\mathbf{m}}} \mathcal{P}_{\mathbf{m}}.$$

If f in $L^1(H)$ is K-invariant, then the operator

$$T_\lambda(f) = \int_H T_\lambda(z,t)f(z,t)m(dz)dt$$

commutes with the K-action. Therefore, by Schur's lemma, for every $\mathbf{m}$, $\mathcal{P}_{\mathbf{m}}$ is an eigenfunction of $T_\lambda(f)$: for $\psi \in \mathcal{P}_{\mathbf{m}}$,

$$T_\lambda(f)\psi = \hat{f}(\lambda,\mathbf{m})\psi.$$

The character $f \mapsto \hat{f}(\lambda,\mathbf{m})$ of the commutative convolution algebra $L^1(H)^K$ can be written

$$\hat{f}(\lambda,\mathbf{m}) = \int_H f(z,t)\varphi(\lambda,\mathbf{m};z,t)m(dz)dt,$$

with a bounded spherical function $\varphi(\lambda,\mathbf{m};z,t)$. Suppose first $\lambda > 0$. For $\psi \in \mathcal{P}_{\mathbf{m}}$,

$$\int_H e^{\lambda\left(it-\frac{1}{2}\|z\|^2-(\zeta|z)\right)}\psi(\zeta+z)f(z,t)m(dz)dt = \hat{f}(\lambda,\mathbf{m})\psi(\zeta).$$

Taking for ψ the spherical polynomial $\Phi_{\mathbf{m}}$, and $\zeta = e$, we obtain

$$\varphi(\lambda,\mathbf{m};z,t) = e^{i\lambda t}e^{-\frac{1}{2}\lambda\|z\|^2}\int_K e^{-\lambda(e|k\cdot z)}\Phi_{\mathbf{m}}(e+k\cdot z)\alpha(dk).$$

Theorem 3. — *The bounded spherical functions of first kind admit the following expansion:*

$$\varphi(\lambda,\mathbf{m};z,t)$$
$$= e^{i\lambda t}e^{-\frac{1}{2}|\lambda|\|z\|^2}\sum_{\mathbf{p}\subset\mathbf{m}} d_{\mathbf{p}}\frac{1}{\left(\left(1+(n-1)\theta\right)_{\mathbf{p}}\right)^2}(-|\lambda|)^{|\mathbf{p}|}\Phi^*_{\mathbf{p}}(\mathbf{m})\Phi_{\mathbf{p}}(r),$$

where $r = \operatorname{diag}(r_1,\ldots,r_n)$, *and* $r_1,\ldots,r_n$ *are the eigenvalues of* zz^*.

[Dib,1990], Théorème 3.1.

Proof. Assume first $\lambda > 0$. The integral over K can be written as

$$\int_K e^{-\lambda(e|k\cdot z)}\Phi_{\mathbf{m}}(e+k\cdot z)\alpha(dk) = \int_K f_1(k\cdot z)\overline{f_2(k\cdot z)}\alpha(dk),$$

with $f_1(z) = \Phi_{\mathbf{m}}(e+z)$, $f_2(z) = e^{-\lambda \operatorname{tr} z}$. Let us expand both functions. By Proposition 1,

$$f_1(z) = \Phi_{\mathbf{m}}(e+z) = \sum_{\mathbf{p}\subset\mathbf{m}} d_{\mathbf{p}} \frac{1}{\big(1+(n-1)\theta\big)_{\mathbf{p}}} \Phi^*_{\mathbf{p}}(\mathbf{m})\Phi_{\mathbf{p}}(z),$$

and, by Proposition XII.1.3 in [Faraut-Korányi,1994],

$$f_2(z) = e^{-\lambda \operatorname{tr} z} = \sum_{\mathbf{p}} d_{\mathbf{p}}(-\lambda)^{|\mathbf{p}|} \frac{1}{\big(1+(n-1)\theta\big)_{\mathbf{p}}} \Phi_{\mathbf{p}}(z).$$

By orthogonality

$$\int_K f_1(k\cdot z)\overline{f_2(k\cdot z)}\alpha(dk)$$
$$= \sum_{\mathbf{p}\subset\mathbf{m}} (d_{\mathbf{p}})^2(-\lambda)^{|\mathbf{p}|} \frac{1}{\Big(\big(1+(n-1)\theta\big)_{\mathbf{p}}\Big)^2} \Phi^*_{\mathbf{p}}(\mathbf{m}) \int_K |\Phi_{\mathbf{p}}(k\cdot z)|^2\alpha(dk).$$

By Proposition XI.4.1 and Corollary XI.4.2 in [Faraut-Korányi,1994],

$$\int_K |\Phi_{\mathbf{p}}(k\cdot z)|^2\alpha(dk) = \frac{1}{d_{\mathbf{p}}}\Phi_{\mathbf{p}}(r).$$

For $\lambda < 0$, one uses the relation

$$\varphi(-\lambda,\mathbf{m};z,t) = \varphi(\lambda,\mathbf{m};z,-t). \qquad \square$$

b) *Bounded spherical functions of second kind.*

For $w \in W$ let η_w be the one dimensional unitary representation of H given by

$$\eta_w(z,t) = e^{2i\mathrm{Im}\,(z|w)}.$$

The character $f \mapsto \eta_w(f)$ of the commutative Banach algebra $L^1(H)^K$ can be written

$$\eta_w(f) = \int_H f(z,t)\psi(\rho;z)m(dz)dt,$$

with the bounded spherical function

$$\psi(\rho; z) = \int_K e^{2i\mathrm{Im}\,(z|k\cdot w)}\alpha(dk),$$

where $\rho = \mathrm{diag}(\rho_1, \ldots, \rho_n)$, $\rho_1, \ldots, \rho_n$ are the eigenvalues of ww^*.

THEOREM 4. — *The bounded spherical functions of second kind admit the following expansion*

$$\psi(\rho; z) = \sum_{\mathbf{p}} d_{\mathbf{p}} \frac{1}{\Big(\big(1+(n-1)\theta\big)_{\mathbf{p}}\Big)^2} (-1)^{|\mathbf{p}|} \Phi_{\mathbf{p}}(\rho)\Phi_{\mathbf{p}}(r),$$

where $r = \mathrm{diag}(r_1, \ldots, r_n)$, *and* $r_1, \ldots, r_n$ *are the eigenvalues of* zz^*.

Proof. Let $\mathcal{K}_{\mathbf{p}}$ denote the reproducing kernel of $\mathcal{P}_{\mathbf{p}}$ in the Fock space $\mathcal{F}_1(W)$. Since $e^{(z|w)}$ is the reproducing kernel of $\mathcal{F}_1(W)$,

$$e^{(z|w)} = \sum_{\mathbf{p}} \mathcal{K}_{\mathbf{p}}(z, w).$$

Observing that

$$e^{2i(z|k\cdot w)} = e^{(z|k\cdot w)}\overline{e^{-(z|k\cdot w)}},$$

we obtain, by orthogonality,

$$\psi(\rho; z) = \sum_{\mathbf{p}} (-1)^{|\mathbf{p}|} \int_K |\mathcal{K}_{\mathbf{p}}(z, k\cdot w)|^2 \alpha(dk).$$

We use now the relation (see Section XI.4 in [Faraut-Korányi,1994]):

$$\int_K |\mathcal{K}_{\mathbf{p}}(z, k\cdot w)|^2 \alpha(dk)$$
$$= \frac{1}{d_{\mathbf{p}}} \mathcal{K}_{\mathbf{p}}(z, z)\mathcal{K}_{\mathbf{p}}(w, w) = \frac{d_{\mathbf{p}}}{\Big(\big(1+(n-1)\theta\big)_{\mathbf{p}}\Big)^2} \Phi_{\mathbf{p}}(r)\Phi_{\mathbf{p}}(\rho). \qquad \square$$

Let Σ^1 be the part of the spectrum Σ of the commutative Banach algebra $L^1(H)^K$ corresponding to the bounded spherical functions of first kind. The set Σ^1 is parametrized by pairs $(\lambda, \mathbf{m})$ with $\lambda \in \mathbb{R}^*$, and $\mathbf{m}$ is a partition of length $\ell(\mathbf{m}) \leq n$. Let also Σ^2 denote the part of Σ corresponding to the bounded spherical functrions of second kind. The set Σ^2 is parametrized by $\rho \in \mathbb{R}^n$, with $\rho_1 \geq \cdots \geq \rho_n \geq 0$. By [Benson-Jenkins-Ratcliff,1992], the spectrum is the disjoint union $\Sigma = \Sigma^1 \cup \Sigma^2$. Furthermore the bounded spherical functions are of positive type.

We will write $\varphi(\sigma; z, t)$ for the bounded spherical function associated to σ:

$$\begin{aligned}\varphi(\sigma; z, t) &= \varphi(\lambda, \mathbf{m}; z, t) \text{ if } \sigma = (\lambda, \mathbf{m}) \in \Sigma^1, \\ &= \psi(\rho; z) \text{ if } \sigma = (\rho) \in \Sigma^2.\end{aligned}$$

These expansions can also be written in terms of Jack polynomials $P_{\mathbf{m}}(x_1, \ldots, x_n; \theta)$. This will be convenient for studying the asymptotics of the spherical functions as n goes to infinity.

We use the same notation as in [Okounkov-Olshanski,1997]: let $\mathbf{m} = (m_1, \ldots, m_n)$ be a partition viewed as a diagram. Fix a box $s = (i, j) \in \mathbf{m}$. One defines

$$\begin{aligned}a(s) &= m_i - j, \quad a'(s) = j - 1, \\ \ell(s) &= m'_j - i, \quad \ell'(s) = i - 1,\end{aligned}$$

where $\mathbf{m}'$ is the transpose diagram, and

$$\begin{aligned}H(\mathbf{m}; \theta) &= \prod_{s \in \mathbf{m}} \big(a(s) + \theta\ell(s) + 1\big) \\ H'(\mathbf{m}; \theta) &= \prod_{s \in \mathbf{m}} \big(a(s) + \theta\ell(s) + \theta\big).\end{aligned}$$

Observe that the generalized Pochhammer symbol can be written, for $u \in \mathbb{C}$,

$$(u)_{\mathbf{m}} = \prod_{s \in \mathbf{m}} \big(u + a'(s) - \theta\ell'(s)\big).$$

Recall also the notation $Q_{\mathbf{m}}(x_1, \ldots, x_n; \theta)$ for the modified Jack polynomials:

$$Q_{\mathbf{m}}(x_1, \ldots, x_n; \theta) = \frac{H'(\mathbf{k}; \theta)}{H(\mathbf{k}; \theta)} P_{\mathbf{m}}(x_1, \ldots, x_n; \theta).$$

By using the relation

$$\Phi_{\mathbf{m}}(x) = \frac{H'(\mathbf{m}; \theta)}{(n\theta)_{\mathbf{m}}} P_{\mathbf{m}}(x_1, \ldots, x_n; \theta),$$

for $x = \mathrm{diag}(x_1, \ldots, x_n)$, and the formula

$$d_{\mathbf{m}} = \frac{\big(1 + (n-1)\theta\big)_{\mathbf{m}} (n\theta)_{\mathbf{m}}}{H(\mathbf{m}; \theta) H'(\mathbf{m}; \theta)},$$

one obtains

$$\varphi(\lambda, \mathbf{m}; z, t) = e^{i\lambda t} e^{-\frac{1}{2}|\lambda| \|z\|^2}$$
$$\sum_{\mathbf{k} \subset \mathbf{m}} (-1)^{|\mathbf{k}|} \frac{1}{(n\theta)_{\mathbf{k}} \big(1 + (n-1)\theta\big)_{\mathbf{k}}} |\lambda^{|\mathbf{k}|} P^*_{\mathbf{k}}(\mathbf{m}; \theta) Q_{\mathbf{k}}(r; \theta),$$

and

$$\psi(\rho; z) = \sum_{\mathbf{k}} (-1)^{|\mathbf{k}|} \frac{1}{(n\theta)_{\mathbf{k}} \big(1 + (n-1)\theta\big)_{\mathbf{k}}} P_{\mathbf{k}}(\rho; \theta) Q_{\mathbf{k}}(r; \theta).$$

The spherical function $\varphi(\sigma; z, t)$ can be written

$$\begin{aligned}\varphi(\sigma; z, t) = e^{i\lambda t} e^{-\frac{1}{2}|\lambda| \|z\|^2} \\ \sum_{\mathbf{k}} (-1)^{|\mathbf{k}|} \frac{1}{(n\theta)_{\mathbf{k}} \big(1 + (n-1)\theta)_{\mathbf{k}}} a_{\mathbf{k}}(\sigma) Q_{\mathbf{k}}(r; \theta),\end{aligned}$$

with

$$\begin{aligned} a_{\mathbf{k}}(\sigma) &= |\lambda|^{|\mathbf{k}|} P^*_{\mathbf{k}}(\mathbf{m}; \theta) \text{ if } \sigma = (\lambda, \mathbf{m}) \in \Sigma^1_n, \\ &= P_{\mathbf{k}}(\rho; \theta) \text{ if } \sigma = (\rho) \in \Sigma^2_n. \end{aligned}$$

We will need in Section 8 the following expansions of the function $\varphi(\sigma; xE_{11}, 0)$ $(x \in \mathbb{R})$. We use the notation $[m]$ $(m \in \mathbb{N})$ for the partition $(m, 0, \ldots)$.

LEMMA 5.

$$\varphi(\sigma; xE_{11}, 0) = 1 - A_n(\sigma)x^2 - B_n(\sigma)x^4 + \cdots$$

where, for $\sigma = (\lambda, \mathbf{m}) \in \Sigma^1_n$*,*

$$A_n(\sigma) = |\lambda| \Big(\frac{1}{2} + \frac{\theta}{(n\theta)\big(1 + (n-1)\theta\big)} P^*_{[1]}(\mathbf{m}; \theta) \Big),$$

and, for $\sigma = (\rho) \in \Sigma^2_n$*,*

$$\begin{aligned} B_n(\sigma) = \lambda^2 \Big(\frac{1}{8} &+ \frac{\theta}{2(n\theta)\big(1 + (n-1)\theta\big)} P^*_{[1]}(\mathbf{m}; \theta) \\ &+ \frac{\theta(\theta+1)}{2(n\theta)(n\theta+1)\big(1 + (n-1)\theta\big)\big(2 + (n-1)\theta\big)} P^*_{[2]}(\mathbf{m}; \theta) \Big). \end{aligned}$$

Furthermore, there are constants D_1 *and* D_2*, which do not depend on* n *and* σ*, such that*

$$B_n(\sigma) \leq D_1 \big(A_n(\sigma)\big)^2,$$

and

$$\begin{aligned} A_n(\sigma) &\geq D_2 \frac{|\lambda||\mathbf{m}|}{n^2}, \text{ if } \sigma = (\lambda, \mathbf{m}) \in \Sigma^1_n, \\ A_n(\sigma) &\geq D_2 \frac{\rho_1 + \cdots + \rho_n}{n^2}, \text{ if } \sigma = (\rho) \in \Sigma^2_n. \end{aligned}$$

Proof.

From Theorem 3, one gets

$$\begin{aligned}
&\varphi(\lambda, \mathbf{m}; xE_{11}, 0) \\
&= e^{-\frac{1}{2}|\lambda|x^2} \sum_{k=0}^{m_1} (-1)^k \frac{(\theta)_k}{k!} \frac{1}{(n\theta)_k \big(1+(n-1)\theta\big)_k} |\lambda|^k P^*_{[k]}(\mathbf{m};\theta) x^{2k} \\
&= \Big(1 - \frac{1}{2}|\lambda|x^2 + \frac{1}{8}\lambda^2 x^4 + \cdots\Big) \\
&\Big(1 - \frac{\theta|\lambda|}{(n\theta)\big(1+(n-1)\theta\big)} P^*_{[1]}(\mathbf{m};\theta)x^2 \\
&+ \frac{\theta(\theta+1)\lambda^2}{2(n\theta)(n\theta+1)\big(1+(n-1)\theta\big)\big(2+(n-1)\theta\big)} P^*_{[2]}(\mathbf{m};\theta)x^4 + \cdots\Big) \\
&= 1 - |\lambda| \Big(\frac{1}{2} + \frac{\theta}{(n\theta)\big(1+(n-1)\theta\big)} P^*_{[1]}(\mathbf{m};\theta)\Big) x^2 \\
&+ \Big(\frac{1}{8} + \frac{\theta}{2(n\theta)\big(1+(n-1)\theta\big)} P^*_{[1]}(\mathbf{m};\theta) \\
&+ \frac{\theta(\theta+1)}{2(n\theta)(n\theta+1)\big(1+(n-1)\theta\big)\big(2+(n-1)\theta\big)} P^*_{[2]}(\mathbf{m};\theta\Big) x^4 + \cdots,
\end{aligned}$$

and, from Theorem 4,

$$\begin{aligned}
&\psi(\rho; xE_{11}) = \sum_{k=0}^{\infty} (-1)^k \frac{(\theta)_k}{k!} \frac{1}{(n\theta)_k \big(1+(n-1)\theta\big)_k} P_{[k]}(\rho;\theta) x^{2k} \\
&= 1 - \frac{\theta}{(n\theta)\big(1+(n-1)\theta\big)} P_{[1]}(\rho;\theta) x^2 \\
&+ \frac{\theta(\theta+1)}{2(n\theta)(n\theta+1)\big(1+(n-1)\theta\big)\big(2+(n-1)\theta\big)} P_{[2]}(\mathbf{m};\theta) x^4 + \cdots
\end{aligned}$$

One uses furthermore the formulae:

$$P_{[1]}(x;\theta) = x_1 + x_2 + \cdots, \quad P^*_{[1]}(\mathbf{s};\theta) = s_1 + s_2 + \cdots,$$

and

$$\begin{aligned}
P_{[2]}(x;\theta) &= \sum_i x_i^2 + \frac{2\theta}{\theta+1} \sum_{i<j} x_i x_j, \\
P^*_{[2]}(\mathbf{s};\theta) &= \sum_i s_i(s_i - 1) + \frac{2\theta}{\theta+1} \sum_{i<j} (s_i - 1)s_j. \qquad \square
\end{aligned}$$

5. Invariant differential operators, and topology of the spectrum. — The following left-invariant vector fields on H form a basis of the complexified Lie algebra $\mathfrak{h}^{\mathbb{C}}$ of $\mathfrak{h} = Lie(H)$:

$$T = \frac{\partial}{\partial t},\ Z_\alpha = \frac{\partial}{\partial z_\alpha} + \frac{1}{2i}\bar{z}_\alpha \frac{\partial}{\partial t},\ \bar{Z}_\alpha = \frac{\partial}{\partial \bar{z}_\alpha} - \frac{1}{2i} z_\alpha \frac{\partial}{\partial t},$$

where the coordinates z_α are relative to an orthonormal basis of W. For the Bargmann representation π_λ, with $\lambda > 0$,

$$d\pi_\lambda(T) = i\lambda,\ d\pi_\lambda(Z_\alpha) = \frac{\partial}{\partial \zeta_\alpha},\ d\pi_\lambda(\bar{Z}_\alpha) = -\lambda\zeta_\alpha,$$

and, for the one-dimensional representation η_w,

$$d\eta_w(T) = 0,\ d\eta_w(Z_\alpha) = \bar{w}_\alpha,\ d\eta_w(\bar{Z}_\alpha) = -w_\alpha.$$

To a polynomial $p(\bar{z}, z)$ on W we associate the left-invariant differential operator $\mathcal{D}_p = p(\bar{Z}, Z)$ on H. We mean that the Z_α's are applied first, and then the $\bar{Z}_\alpha$'s. Hence

$$d\pi_\lambda(\mathcal{D}_p) = p\big(-\lambda\zeta, \frac{\partial}{\partial\zeta}\big),\ d\eta_w(\mathcal{D}_p) = p(-w, \bar{w}).$$

A K-invariant polynomial $p(\bar{z}, z)$ can be written

$$p(\bar{z}, z) = P(r_1, \ldots, r_n),$$

where P is a symmetric polynomial in n variables, and $r_1, \ldots, r_n$ are the eigenvalues of zz^*. In such a case the operator $\mathcal{D}_p$ commutes with the K-action on $\mathcal{F}_\lambda(W)$. Therefore, by Schur's Lemma, the subspaces $\mathcal{P}_{\mathbf{m}}$ are eigenspaces of $\mathcal{D}_p$.

Theorem 6. — *Assume that $p(\bar{z}, z)$ is K-invariant and homogeneous of degree ℓ.*

(i) *For $\psi \in \mathcal{P}_{\mathbf{m}}$,*

$$d\pi_\lambda(\mathcal{D}_p)\psi = (-\lambda)^\ell P^*(m_1, \ldots, m_n)\psi,$$

where P^* is the θ-shifted symmetric polynomial associated to P as in Section 2. Furthermore

$$d\eta_w(\mathcal{D}_p) = (-1)^\ell P(\rho_1, \ldots, \rho_n),$$

where $\rho_1, \ldots, \rho_n$ are the eigenvalues of ww^*.

(ii) *The spherical functions are eigenfunctions of $\mathcal{D}_p$:*

$$\mathcal{D}_p\varphi(\lambda,\mathbf{m};z,t) = (-\lambda)^\ell P^*(m_1,\dots,m_n)\varphi(\lambda,\mathbf{m};z,t),$$
$$\mathcal{D}_p\psi(\rho;z,t) = (-1)^\ell P(\rho_1,\dots,\rho_n)\psi(\rho;z,t).$$

By [Ferrari-Ruffino,2007] one deduces the topology of the spectrum (see Section 1 of the present paper):

COROLLARY 7. — *The map $\Sigma \to \mathbb{R}^{n+1}$ defined by*

$$(\lambda,\mathbf{m}) \in \Sigma^1 \mapsto (\lambda, |\lambda| m_1,\dots,|\lambda| m_n),$$
$$(\rho) \in \Sigma^2 \mapsto (0,\rho_1,\dots,\rho_n),$$

is a homeomorphism of the spectrum Σ onto its image, a multi-dimensional Heisenberg fan.

This means in particular that

$$\lim_{\lambda\to 0,\lambda m_i\to\rho_i} \varphi(\lambda,\mathbf{m};z,t) = \psi(\rho,z),$$

uniformly on compact sets in H. This can also be obtained from the expansions of $\varphi(\lambda,\mathbf{m};z,t)$ and $\psi(\rho;z)$ (Theorems 3 and 4).

6. An Olshanski spherical pair. — We consider the increasing sequences

$W(n)$	$K(n)$	d
$Sym(n,\mathbb{C})$	$U(n)$	1
$M(n,\mathbb{C})$	$U(n)\times U(n)$	2
$Skew(2n,\mathbb{C})$	$U(2n)$	4

Furthermore we consider the sequence $H(n) = W(n)\times\mathbb{R}$ of Heisenberg groups, and the infinite dimensional Heisenberg group

$$H = \bigcup_{n=1}^\infty H(n),$$

and also the Olshanski spherical pair (G,K),

$$G = \bigcup_{n=1}^\infty G(n), \quad K = \bigcup_{n=1}^\infty K(n).$$

A spherical function can be seen as a K-invariant function $\varphi(z,t)$ on H with $\varphi(0,0)=1$ such that

$$\lim_{n\to\infty}\int_{K(n)} \varphi\big(z+k\cdot z', t+t'+\mathrm{Im}\,(k\cdot z'|z)\big)\alpha_n(dk)=\varphi(z,t)\varphi(z',t).$$

THEOREM 8. — *Let φ be a K-invariant continuous function on H. Then φ is spherical if and only if there exists $\lambda\in\mathbb{C}$, and a continuous function Φ on $[0,\infty[$ such that*

$$\varphi(z,t)=e^{\lambda t}\prod_i \Phi(r_i),$$

where the numbers r_i are the eigenvalues of zz^*.

The proof is the same as for Theorem 6.1 in [Faraut,2010].

7. The topological space Ξ and extended symmetric functions. In the last sections of the paper we will study the limits of the spherical functions as n goes to infinity, following the method used in [Okounkov-Olshanski,1998]. As in [Faraut,2010], we consider the topological space

$$\Xi=\{\xi=(\alpha,\gamma)\mid \alpha=(\alpha_j),\ \alpha_j\geq 0,\ \sum_{j=1}^{\infty}\alpha_j<\infty,\ \gamma\geq 0\},$$

equipped with the initial topology with respect to the functions L_h, where h is a continuous function on $[0,\infty[$, defined by

$$L_h(\xi)=\gamma h(0)+\sum_{j=1}^{\infty}\alpha_j h(\alpha_j)\quad\big(\xi=(\alpha,\gamma)\big).$$

For $C>0$, the set

$$\Xi_C=\{\xi=(\alpha,\gamma)\mid \sum_{j=1}^{\infty}\alpha_j+\gamma\leq C\}$$

is compact. The Pólya type function

$$\Phi(\xi;x)=e^{-\gamma x}\prod_{j=1}^{\infty}\frac{1}{1+\alpha_j x}$$

is continuous on $\Xi\times[0,\infty[$. In fact

$$-\log\Phi(\xi;x)=L_h(\xi),$$

with

$$h(t) = \frac{1}{t}\log(1+tx)\ (t>0),\ h(0)=x.$$

Let Λ denote the algebra of symmetric functions. Recall that a symmetric function is a polynomial function on $\mathbb{C}^{(\infty)} = \cup_{n=1}^{\infty}\mathbb{C}^n$ which is invariant under the infinite symmetric group $\mathfrak{S}_{(\infty)} = \cup_{n=1}^{\infty}\mathfrak{S}_n$. We consider an algebra morphism from Λ into the algebra $\mathcal{C}(\Xi)$ of continuous functions on Ξ:

$$\Lambda \to \mathcal{C}(\Xi), \quad f \mapsto \tilde{f},$$

such that the images of the Newton power sums p_m are given by

$$\tilde{p}_1(\xi) = \gamma + \sum_{j=1}^{\infty}\alpha_j,$$

and, for $m \geq 2$,

$$\tilde{p}_m(\xi) = \sum_{j=1}^{\infty}\alpha_j^m.$$

Since the functions p_m generate Λ as an algebra, the morphism is uniquely determined by these conditions. The function $\tilde{f}$ is said to be the extended symmetric function of f. The Jack polynomial $P_{\mathbf{m}}(x;\theta)$ is a symmetric function, and, according to the definition, $\tilde{P}_{\mathbf{m}}(\xi;\theta)$ will denote the extended symmetric function of $P_{\mathbf{m}}(x;\theta)$.

Proposition 9. — (i) *The power series expansion of $\Phi(\xi;x)^\theta$ near 0 is given by*

$$\Phi(\xi;x)^\theta = \sum_{m=0}^{\infty}\frac{(\theta)_m}{m!}\tilde{P}_{[m]}(\xi;\theta)(-x)^m,$$

where, for $m \in \mathbb{N}$, $[m]$ denotes the partition $(m,0,\ldots)$.

(ii) *More generally*

$$\prod_i \Phi(\xi;x_i)^\theta = \sum_{\mathbf{m}}\tilde{P}_{\mathbf{m}}(\xi;\theta)Q_{\mathbf{m}}(-x;\theta).$$

Proof. Recall the Cauchy identity

$$\prod_{i,j}(1-x_iy_j)^{-\theta} = \sum_{\mathbf{m}}P_{\mathbf{m}}(x;\theta)Q_{\mathbf{m}}(y;\theta),$$

which, in case of $y=(z,0,\ldots)$, reduces to

$$\prod_i(1-x_iz)^{-\theta} = \sum_{m=0}^{\infty}\frac{(\theta)_m}{m!}P_{[m]}(x;\theta)z^m,$$

Essentially the proof amounts to applying the morphism $f \mapsto \tilde{f}$ to the Cauchy identity, in the variable $y = (y_1, y_2, \ldots)$. $\square$

8. Asymptotics of the spherical functions. — We saw in Section 4 that the spectrum Σ_n for the Gelfand pair $\big(G(n), K(n)\big)$ decomposes as $\Sigma_n = \Sigma_n^1 \cap \Sigma_n^2$, with

$$\Sigma_n^1 = \{(\lambda, \mathbf{m}) \mid \lambda \in \mathbb{R}^*,\ \mathbf{m} \text{ is a partition},\ \ell(\mathbf{m}) \leq n\},$$
$$\Sigma_n^2 = \{\rho \in \mathbb{R}^n \mid \rho_1 \geq \cdots \geq \rho_n \geq 0\}.$$

For $(\lambda, \xi) \in \mathbb{R} \times \Xi$, and $(z, t) \in H$, define

$$\varphi(\lambda, \xi; z, t) = e^{i\lambda t} e^{-\frac{1}{2}|\lambda| \|z\|^2} \prod_i \Phi(\xi; \theta^{-2} r_i)^\theta,$$

where $r_1, \ldots, r_n$ are the eigenvalues of zz^*. For every n we define the map

$$T_n : \Sigma_n \to \mathbb{R} \times \Xi,\ \sigma \mapsto (\lambda, \xi) = (\lambda, \alpha, \gamma),$$

with, if $\sigma = (\lambda, \mathbf{m}) \in \Sigma_n^1$,

$$\alpha_j = \frac{1}{n^2} |\lambda| m_j\ (1 \leq j \leq n),\ \alpha_j = 0\ (j > n),\ \gamma = 0,$$

and, if $\sigma = (\rho) \in \Sigma_n^2$,

$$\lambda = 0,\ \alpha_j = \frac{1}{n^2} \rho_j\ (1 \leq j \leq n),\ \alpha_j = 0\ (j > n),\ \gamma = 0.$$

THEOREM 10. — *Let $(\sigma^{(n)})$ be a sequence with $\sigma^{(n)} \in \Sigma_n$. Assume that*

$$\lim_{n \to \infty} T_n(\sigma^{(n)}) = (\lambda, \xi)$$

for the topology of $\mathbb{R} \times \Xi$. Then

$$\lim_{n \to \infty} \varphi_n(\sigma^{(n)}; z, t) = \varphi(\lambda, \xi; z, t),$$

uniformly on compact sets in H.

The proof is the similar to the one of Theorem 6.5 in [Faraut,2010]. Let Λ^θ denote the algebra of θ-shifted symmetric functions. Let $P^* \in \Lambda^\theta$ of degree ℓ, and P the homogeneous part of P^* of degree ℓ. Then P is symmetric. For $\sigma \in \Sigma_n$, define

$$Q(P^*, \sigma) = |\lambda|^\ell P^*(\mathbf{m}) \text{ if } \sigma = (\lambda, \mathbf{m}) \in \Sigma_n^1,$$
$$= P(\rho) \text{ if } \sigma = (\rho) \in \Sigma_n^2.$$

Proposition 11. — *Let* $(\sigma^{(n)})$ *be a sequence with* $\sigma^{(n)} \in \Sigma_n$. *Assume that*

$$\lim_{n\to\infty} T_n(\sigma^{(n)}) = (\lambda, \xi),$$

for the topology of $\mathbb{R} \times \Xi$. *Then, for every* $P^* \in \Lambda^\theta$ *of degree* ℓ,

$$\lim_{n\to\infty} \frac{1}{n^{2\ell}} Q(P^*, \sigma^{(n)}) = \tilde{P}(\xi),$$

the extended symmetric function of P *introduced in Section 6.*

This is proved in the same way as Proposition 6.6 in [Faraut,2010]. Instead of the shifted power functions one considers the θ-shifted power functions

$$p_\ell^*(x) = \sum_i \big((x_i - i\theta)^\ell - (-i\theta)^\ell\big).$$

Proof of Theorem 8. The spherical function $\varphi_n(\sigma; z, t)$ can be written

$$\varphi_n(\lambda, \mathbf{m}; z, t) = e^{i\lambda t} e^{-\frac{1}{2}|\lambda|\|z\|^2}$$
$$\sum_{\mathbf{k}} (-1)^{|\mathbf{k}|} \frac{1}{(n\theta)_{\mathbf{k}}\big(1 + (n-1)\theta\big)_{\mathbf{k}}} a_{\mathbf{k}}(\sigma) Q_{\mathbf{k}}(r; \theta),$$

with

$$a_{\mathbf{k}}(\sigma) = |\lambda|^{|\mathbf{k}|} P_{\mathbf{k}}^*(\mathbf{m}; \theta) \text{ if } \sigma = (\lambda, \mathbf{m}) \in \Sigma_n^1,$$
$$= P_{\mathbf{k}}(\rho; \theta) \text{ if } \sigma = (\rho) \in \Sigma_n^2.$$

By Proposition 11,

$$\lim_{n\to\infty} \frac{1}{n^{2|\mathbf{k}|}} a_{\mathbf{k}}(\sigma^{(n)}) = \tilde{P}_{\mathbf{k}}(\xi; \theta).$$

Since, for $\mathbf{k}$ fixed,

$$(n\theta)_{\mathbf{k}}\big(1 + (n-1)\theta\big)_{\mathbf{k}} \sim \theta^{2|\mathbf{k}|} n^{2|\mathbf{k}|} \quad (n \to \infty),$$

it follows by Lemma 3.4 in (Faraut,2010], that

$$\lim_{n\to\infty} \sum_{\mathbf{k}} (-1)^{|\mathbf{k}|} \frac{1}{(n\theta)_{\mathbf{k}}\big(1 + (n-1)\theta\big)_{\mathbf{k}}} a_{\mathbf{k}}(\sigma^{(n)}) Q_{\mathbf{k}}(r; \theta)$$
$$= \sum_{\mathbf{k}} (-1)^{|\mathbf{k}|} \theta^{-2|\mathbf{k}|} \tilde{P}_{\mathbf{k}}(\xi; \theta) Q_{\mathbf{k}}(r; \theta) = \prod_i \Phi(\xi, \theta^{-2} r_i)^\theta,$$

by Proposition 9. ∎

THEOREM 12. — *If $(\sigma^{(n)})$ is a sequence with $\sigma^{(n)} \in \Sigma_n$, and such that*

$$\lim_{n\to\infty} \varphi_n(\sigma^{n)}; z, t) = \varphi(z,t),$$

uniformly on compact sets in H, where φ is a continuous function on H, then the sequence $T_n(\sigma^{(n)})$ converges in $\mathbb{R} \times \Xi$,

$$\lim_{n\to\infty} T_n(\sigma^{(n)}) = (\lambda, \xi),$$

and

$$\varphi(z,t) = \varphi(\lambda, \xi; z, t).$$

By Theorems 10 and 12, a sequence $\sigma^{(n)}$ is a Vershik-Kerov sequence if and only if the sequence $T_n(\sigma^{(n)})$ converges in $\mathbb{R} \times \Xi$.

Proof. For $z = 0$,

$$\varphi(0,t) = \lim_{n\to\infty} \varphi_n(\sigma^{(n)}; 0, t) = \lim_{n\to\infty} e^{i\lambda^{(n)}t},$$

uniformly on compact sets in $\mathbb{R}$, with $\sigma^{(n)} = (\lambda^{(n)}, \mathbf{m}^{(n)})$ if $\sigma^{(n)} \in \Sigma_n^1$, and $\lambda^{(n)} = 0$ if $\sigma^{(n)} \in \Sigma_n^2$. Hence the sequence $\lambda^{(n)}$ converges and $\varphi(0,t) = e^{i\lambda t}$, with $\lambda = \lim_{n\to\infty} \lambda^{(n)}$. Put, for $z = xE_{11}$, with $x \in \mathbb{R}$,

$$\psi_n(x) = \varphi_n(\sigma^{(n)}; xE_{11}, 0).$$

The function ψ_n is continuous of positive type on $\mathbb{R}$, with $\psi_n(0) = 1$, hence is the Fourier transform of a probability measure ν_n on $\mathbb{R}$,

$$\psi_n(x) = \int_{\mathbb{R}} e^{ixy} \nu_n(dy).$$

By Lemma 5, the function ψ_n has the following expansion

$$\psi_n(x) = 1 - A_n(\sigma^{(n)})x^2 + B_n(\sigma^{(n)})x^4 + \cdots,$$

and the moments of order 2 and 4 of the measure ν_n are

$$\mathfrak{M}_2(\nu_n) = 2A_n(\sigma^{(n)}), \quad \mathfrak{M}_4(\nu_n) = 24B_n(\sigma^{(n)}).$$

Also by Lemma 5, there is a constant A, which does not depend on n, such that

$$\mathfrak{M}_4(\nu_n) \leq \big(\mathfrak{M}_2(\nu_n)\big)^2.$$

Since the sequence (ψ_n) converges uniformly on compact sets, the sequence (ν_n) converges weakly, hence is relatively compact for the weak topology.

By Lemma 5.2 in [Okounkov-Olshanski,1998] (see also Lemma 4.3 in [Faraut,2010]), there is a constant C such that

$$A(\sigma^{(n)}) \leq C.$$

Form this inequality together with the last one in Lemma 5, it follows that the sequence $T_n(\sigma^{(n)})$ is relatively compact in $\mathbb{R} \times \Xi$. $\square$

9. Multivariate Laguerre polynomials. — The bounded spherical functions of first kind can be expressed in terms of multivariate Laguerre polynomials. Following [Muirhead,1982], [Dib,1990] (see also [Lassalle,1991], [Faraut-Korányi,1994], [Baker-Forrester,1997]) the multivariate Laguerre polynomials $L^{\alpha}_{\mathbf{m}}(x_1, \ldots, x_n; \theta)$ are defined, for $x \in Herm(n, \mathbb{F})$, by

$$L^{\alpha}_{\mathbf{m}}(x) = \big(\alpha + 1 + (n-1)\theta\big)_{\mathbf{m}} \sum_{\mathbf{k} \subset \mathbf{m}} \frac{(-1)^{|\mathbf{k}|}}{\big(\alpha + 1 + (n-1)\theta\big)_{\mathbf{k}}} \binom{\mathbf{m}}{\mathbf{k}} \Phi_{\mathbf{k}}(x).$$

(There are slight variations regarding the parameter α in the above references.) The generalized binomial coefficients are defined by the relation

$$\Phi_{\mathbf{m}}(e + x) = \sum_{\mathbf{k} \subset \mathbf{m}} \binom{\mathbf{m}}{\mathbf{k}} \Phi_{\mathbf{k}}(x).$$

It follows that, with the notation of [Okounkov-Olshanski,1997],

$$\binom{\mathbf{m}}{\mathbf{k}} = \frac{P^*_{\mathbf{k}}(\mathbf{m}; \theta)}{H(\mathbf{k}; \theta)}.$$

Therefore, for $x = \mathrm{diag}(x_1, \ldots, x_n)$,

$$L^{\alpha}_{\mathbf{m}}(x; \theta) = \big(\alpha + 1 + (n-1)\theta\big)_{\mathbf{m}} \sum_{\mathbf{k} \subset \mathbf{m}} (-1)^{|\mathbf{k}|} \frac{1}{(n\theta)_{\mathbf{k}} \big(\alpha + 1 + (n-1)\theta\big)_{\mathbf{k}}} P^*_{\mathbf{k}}(\mathbf{m}; \theta) Q_{\mathbf{k}}(x; \theta).$$

The bounded spherical functions of first kind can be written

$$\varphi(\lambda, \mathbf{m}; z, t) = e^{i\lambda t} e^{-\frac{1}{2}|\lambda| \|z\|^2} \frac{L^0_{\mathbf{m}}\big(|\lambda| r; \theta\big)}{L^0_{\mathbf{m}}(0; \theta)},$$

with $r = (r_1, \ldots, r_n)$, and $r_1, \ldots, r_n$ are the eigenvalues of zz^*.

Let us define, for $\alpha \in \mathbb{C}$, $\mathbf{s} \in \mathbb{C}^n$, $x = (x_1, \ldots, x_n) \in \mathbb{C}^n$, $\theta = \frac{1}{2}, 1, 2$.

$$F^*(\alpha, \mathbf{s}; x) = \sum_{\mathbf{k}} (-1)^{|\mathbf{k}|} \frac{1}{(n\theta)_{\mathbf{k}} \big(\alpha + 1 + (n-1)\theta\big)_{\mathbf{k}}} P^*_{\mathbf{k}}(\mathbf{s}; \theta) Q_{\mathbf{k}}(x; \theta).$$

We assume that $\big(\alpha + (n-1)\theta\big)_{\mathbf{k}} \neq 0$ for every partition $\mathbf{k}$. The series converges for all $\mathbf{s}$ and x. To show the convergence one can use the following Cauchy inequality which follows from Proposition 1: for every r with $0 < r < 1$,

$$|\Phi^*_{\mathbf{k}}(\mathbf{s})| \leq \big(1 + (n-1)\theta\big)_{\mathbf{k}} r^{-|\mathbf{k}|} M(r, \mathbf{s}),$$

where

$$M(\mathbf{s}, r) = \sup_{\|z\|_{\mathrm{op}} \leq r} |\varphi(\mathbf{s}; e + z)|.$$

Observe that, if $\mathbf{s} = \mathbf{m}$ is a partition, the series is a finite sum and

$$F^*(\alpha, \mathbf{m}; x) = \frac{L^\alpha_{\mathbf{m}}(x; \theta)}{L^\alpha_{\mathbf{m}}(0; \theta)}.$$

Define also, for $\alpha \in \mathbb{C}$, $x, y \in \mathbb{C}^n$,

$$F(\alpha; x, y) = \sum_{\mathbf{k}} (-1)^{|\mathbf{k}|} \frac{1}{(n\theta)_{\mathbf{k}} \big(\alpha + 1 + (n-1)\theta\big)_{\mathbf{k}}} P_{\mathbf{k}}(x; \theta) Q_{\mathbf{k}}(y; \theta).$$

The series converge for all x and y.

Proposition 13. — *The following confluence property holds:*

$$\lim_{t \to \infty} F^*(\alpha, t\mathbf{s}; \frac{x}{t}) = F(\alpha; \mathbf{s}; x).$$

Proof. This follows from

$$\lim_{t \to \infty} t^{-|\mathbf{k}|} P^*_{\mathbf{k}}(t\mathbf{s}) = P_{\mathbf{k}}(\mathbf{s}).$$ ∎

In case $n = 1$, these properties are classical. In fact, noticing that $[s]_k = (-1)^k (-s)_k$,

$$F^*(\alpha, s; x) = \sum_{k=0}^{\infty} (-1)^k \frac{1}{k!(\alpha+1)_k} [s]_k x^k = {}_1F_1(-s, \alpha + 1; x)$$

$$F^*(\alpha, m; x) = {}_1F_1(-m, \alpha + 1; x) = \frac{L^\alpha_m(x)}{L^\alpha_m(0)},$$

and

$$F(\alpha; x, y) = \sum_{k=0}^{\infty} (-1)^k \frac{1}{k!(\alpha+1)_k} x^k y^k = {}_0F_1(\alpha+1; -xy).$$

For a partition $\mathbf{m}$ with length $\ell(\mathbf{m}) \leq n$, define $\mathcal{T}_n : \mathbf{m} \mapsto \xi = (\alpha, \gamma) \in \Xi$ by

$$\alpha_j = \frac{m_j}{n^2} \ (1 \leq j \leq n), \ \alpha_j = 0 \ (j > n), \ \gamma = 0.$$

From Theorem 10, with $\lambda = 1$, one obtains:

Proposition 14. — *Let $\theta = \frac{1}{2}, 1$ or 2. Let $\mathbf{m}^{(n)}$ be a sequence of partitions with $\ell(\mathbf{m}^{(n)}) \leq n$. Assume that*

$$\lim_{n \to \infty} \mathcal{T}_n(\mathbf{m}^{(n)}) = \xi,$$

for the topology of Ξ. Then

$$\lim_{n \to \infty} \frac{L_{\mathbf{m}^{(n)}}(x_1, x_2, \ldots, x_k, 0, \ldots)}{L_{\mathbf{m}^{(n)}}(0, \ldots; \theta)} = \prod_i \Phi(\xi, \theta^{-2} x_i).$$

We don't now whether this statement holds for all $\theta > 0$.

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