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To the memory of Andrzej Hulanicki

ASYMPTOTIC SPHERICAL ANALYSIS ON THE HEISENBERG GROUP

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Asymptotics of spherical functions for large dimensions are related to spherical functions for Olshanski spherical pairs. In this paper we consider inductive limits of Gelfand pairs associated to the Heisenberg group. The group $K = U(n) \times U(p)$ acts multiplicity free on $\mathcal{P}(V)$, the space of polynomials on $V = M(n, p; \mathbb{C})$, the space of $n \times p$ complex matrices. The group K acts also on the Heisenberg group $H = V \times \mathbb{R}$. By a result of Carcano, the pair (G, K), with $G = K \ltimes H$ is a Gelfand pair. The main results of the paper are asymptotics of the spherical functions related to the pair (G, K) for large n and p. This analysis involves asymptotics of shifted Schur functions.

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1. Introduction. To a finite dimensional Euclidean complex vector space V one associates the Heisenberg group $H = V \times \mathbb{R}$ with the product

$$(z, t)(z', t') = \big(z + z', t + t' + \mathrm{Im}(z|z')\big).$$

Let K be a closed subgroup of the unitary group $U(V)$. An element $k \in K$ defines an automorphism of H: $k \cdot (z, t) = (k \cdot z, t)$. In 1980, A. Hulanicki and F. Ricci have considered the case of $V = \mathbb{C}^n$, and $K = \mathbb{T}^n$ acting diagonaly. They observed that (G, K) with $G = K \ltimes H$ is a Gelfand pair, and determine the corresponding spherical functions. These functions can be expressed in terms of Laguerre polynomials. By using a Tauberian theorem for the associated spherical Fourier transform, A. Hulanicki and F. Ricci established a tangential convergence theorem for harmonic functions on balls in $\mathbb{C}^n$. In 1980, in a conference in Wisła organized by A. Hulanicki, A. Korányi gave a talk about the following result: let $\mathbf{G}/\mathbf{K}$ be a Riemannian symmetric space of rank one, with the Iwasawa decomposition $\mathbf{G} = \mathbf{K}AN$, M being the centralizer of A in $\mathbf{K}$, then (MN, M) is a Gelfand pair. In particular, if $\mathbf{G} = SU(1, n + 1)$, then N is the Heisenberg group $H = \mathbb{C}^n \times \mathbb{R}$, and $M = U(n)$. A natural question arised: for which subgroup $K \subset U(V)$ is the pair (G, K) a Gelfand pair ? $(G = K \ltimes H)$. The answer has been given by Carcano [1987]: (G, K) is a Gelfand pair if and only if K acts on the space of polynomials $\mathcal{P}(V)$

multiplicity free. These Gelfand pairs and the associated spherical functions have been studied in a series of papers by C. Benson, J. Jenkins, G. Ratcliff (among them [1992], [1998]). The subgroups $K \subset U(V)$ acting irreducibly on V and on $\mathcal{P}(V)$ multiplicity free have been classified by Kac [1980]. (A complete classification of linear multiplicity free actions, extending [Kac,1980], is given in [Benson-Racliff,1996], and [Leahy,1998].) This subject is now available in a book form: [Wolf,2007], Chapter 13.

If K acts multiplicity free on $\mathcal{P}(V)$, hence (G,K) is a Gelfand pair. A spherical function for (G,K) can be seen as a K-invariant function on H. That is why we make the following definition: a continuous complex valued function φ on H is said to be *spherical* if it satisfies the following functional equation:

$$\int_K \varphi\big(z + k\cdot z', t + t' + \mathrm{Im}(z|k\cdot z')\big)\alpha(dk) = \varphi(z,t)\varphi(z',t'),$$

for $(z,t),(z',t') \in H$ (α denotes the normalized Haar measure on K). The bounded spherical functions are the characters of the commutative convolution Banach algebra $L^1(H)^K$ of K-invariant integrable functions on H. We will consider the case of $V = M(n,p;\mathbb{C})$ $(p \geq n)$, with the inner product given by $(z|w) = \mathrm{tr}\,(zw^*)$, and $K = U(n)\times U(p)$ acting on V by: $k\cdot z = uzv^*$ if $k = (u,v)$. We will determine the spherical functions in that case, and then study the asymptotics of these spherical functions for large n and p.

In Section 2 we establish some series expansions in the Fock space $\mathcal{F}(V)$ with a subgroup $K \subset U(V)$ acting on V multiplicity free. In case of $V = M(n,p;\mathbb{C})$ and $K = U(n)\times U(p)$, we describe explicitely the decomposition of the Fock space. Then, in Section 3, in that case we determine the spherical functions of positive type of the associated Gelfand pair (G,K), with $G = K \ltimes H$, $H = V \times \mathbb{R}$.

In Section 4 we recall the notion of Olshanski spherical pair. Such a pair (G,K) is the inductive limit of an increasing sequence of Gelfand pairs $\big(G(n),K(n)\big)$. We recall the method, due to Okounkov and Olshanski, for studying the asymptotics of the spherical functions for the pair $\big(G(n),K(n)\big)$, and the convergence to the spherical functions for the pair (G,K).

Then we come back to the special case of the Gelfand pairs associated to the Heisenberg group $H = V \times \mathbb{R}$, with $V = M(n,p;\mathbb{C})$, and $K = U(n)\times U(p)$. In Section 5, n is kept fixed and p goes to infinity. The Olshanski spherical pair (G,K) is of finite rank: the spherical functions for the pair (G,K) depend on a finite number of real parameters. In Section 6, we consider $V = M(n,n+q)$, with q fixed, and n goes to infinity. Then the rank of the Olshanski spherical pair (G,K) is infinite: the spherical functions for the pair (G,K) depend on an infinite number of real parameters.

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2. Preliminaries about the Fock space. Let Z be a finite dimensional complex manifold, and K a compact group acting on Z by holomorphic automorphisms. Then K acts on the space $\mathcal{O}(Z)$ of holomorphic functions:

$$\big(\pi(k)f\big)(z) = f(k^{-1}\cdot z).$$

Let $\mathcal{H} \subset \mathcal{O}(Z)$ be a K-invariant Hilbert subspace: there is a K-invariant Hilbert structure on $\mathcal{H}$, and the injection $\mathcal{H} \hookrightarrow \mathcal{O}(Z)$ is continuous.

Lemma 2.1. *Assume $\mathcal{H}$ irreducible. Then, for $f \in \mathcal{H}$,*

$$\int_K |f(k \cdot z)|^2 \alpha(dk) = \frac{1}{d} \mathcal{K}(z, z) \|f\|^2,$$

where $\mathcal{K}$ denotes the reproducing kernel of $\mathcal{H}$, α the normalized Haar measure of K, and $d = \dim \mathcal{H}$.

Proof. Let $\mathcal{K}_z$ denote the coherent state $\mathcal{K}_z(w) = \mathcal{K}(w, z)$. By the reproducing property, $f(z) = (f|\mathcal{K}_z)$, hence

$$f(k \cdot z) = (f|\mathcal{K}_{k \cdot z}) = (f|\pi(k)\mathcal{K}_z).$$

Therefore

$$\int_K |f(k \cdot z)|^2 \alpha(dk) = \int_K |\big(\pi(k)\mathcal{K}_z|f\big)|^2 \alpha(dk) = \frac{1}{d} \|\mathcal{K}_z\|^2 \|f\|^2,$$

by the Schur orthogonality relations. Furthermore $\|\mathcal{K}_z\|^2 = \mathcal{K}(z, z)$. ∎

More generally, for $f_1, f_2 \in \mathcal{H}$,

$$\int_K f_1(k \cdot z) \overline{f_2(k \cdot z)} \alpha(dk) = \frac{1}{d} \mathcal{K}(z, z)(f_1|f_2).$$

We assume that K acts multiplicity free on $\mathcal{O}(Z)$: every K-invariant Hilbert subspace $\mathcal{H} \subset \mathcal{O}(Z)$ decomposes multilicity free:

$$\mathcal{H} = \widehat{\bigoplus_{\mu \in \mathfrak{M}}} \mathcal{H}_\mu,$$

into a Hilbert sum of irreducible K-invariant Hilbert subspaces $\mathcal{H}_\mu$.

Proposition 2.2. *For $f \in \mathcal{H}$, then*

$$f(z) = \sum_{\mu \in \mathfrak{M}} f_\mu(z) \quad (f_\mu \in \mathcal{H}_\mu)$$

(the series converges in $\mathcal{H}$, and uniformly on compact sets), and

$$\int_K |f(k \cdot z)|^2 \alpha(dk) = \sum_{\mu \in \mathfrak{M}} \frac{1}{d_\mu} \mathcal{K}_\mu(z, z) \|f_\mu\|^2 \quad (d_\mu = \dim \mathcal{H}_\mu).$$

For $f_1, f_2 \in \mathcal{H}$,

$$\int_K f_1(k\cdot z)\overline{f_2(k\cdot z)}\alpha(dk) = \sum_{\mu\in\mathfrak{M}} \frac{1}{d_\mu}\mathcal{K}_\mu(z,z)(f_{1,\mu}|f_{2,\mu}).$$

Let $Z = V$ be a finite dimensional Euclidean complex vector space, and K a closed subgroup of the unitary group $U(V)$ acting multiplicity free on $\mathcal{O}(V)$, or equivalently acting multiplicity free on the space $\mathcal{P}(V)$ of polynomials on V. The Fock space $\mathcal{F}(V)$ is the space of holomorphic functions f on V such that

$$\|f\|^2 = \frac{1}{\pi^N}\int_V e^{-\|z\|^2}|f(z)|^2 m(dz) < \infty$$

(m denotes the Euclidean measure on V, $N = \dim V$.) The reproducing kernel of $\mathcal{F}(V)$ is

$$\mathcal{K}(z,w) = e^{(z|w)}.$$

The Fock space decomposes multiplicity free:

$$\mathcal{F}(V) = \widehat{\bigoplus_{\mu\in\mathcal{M}}}\mathcal{H}_\mu.$$

Let $\mathcal{K}_\mu$ denotes the reproducing kernel of $\mathcal{H}_\mu$. Then

$$e^{(z|w)} = \sum_{\mu\in\mathfrak{M}}\mathcal{K}_\mu(z,w).$$

By Proposition 2.2, for $f(z) = e^{(z|w)}$ (w fixed), we get

Proposition 2.3.

$$\int_K e^{2\mathrm{Re}(k\cdot z|w)}\alpha(dk) = \sum_{\mu\in\mathfrak{M}}\frac{1}{d_\mu}\mathcal{K}_\mu(z,z)\mathcal{K}_\mu(w,w).$$

Consider first $V = \mathbb{C}^p$, $K = U(p)$. Let $\mathcal{H}_m$ denote the space of polynomials homogeneous of degree m. The irreducible K-invariant Hilbert subspaces are the spaces $\mathcal{H}_m$,

$$\mathcal{K}_m(z,w) = \frac{1}{m!}(z|w)^m, \quad \dim\mathcal{H}_m = \frac{(m+p-1)!}{m!(p-1)!} = \frac{(p)_m}{m!}.$$

(Recall the Pochhammer symbol: $(\alpha)_m = \alpha(\alpha+1)\ldots(\alpha+m-1)$.) Hence $\mathfrak{M} = \mathbb{N}$. Then, for $f \in \mathcal{F}(\mathbb{C}^p)$,

$$f(z) = \sum_{m=0}^{\infty} f_m(z) \quad (f_m \in \mathcal{P}_m),$$

we obtain

$$\int_{U(p)} |f(k\cdot z)|^2 \beta_p(dk) = \sum_{m=0}^{\infty} \frac{1}{(p)_m} \|z\|^{2m} \|f_m\|^2$$

(β_p is the normalized Haar measure on $U(p)$), and

$$\int_{U(p)} e^{2\mathrm{Re}(k\cdot z|w)} \beta_p(dk) = \sum_{m=0}^{\infty} \frac{1}{(p)_m} \frac{1}{m!} \|z\|^{2m} \|w\|^{2m}.$$

More generally, consider $V = M(n,p;\mathbb{C})$ $(n \leq p)$, with the inner product $(z|w) = \mathrm{tr}\,(zw^*)$, and $K = U(n) \times U(p)$ acting on $M(n,p;\mathbb{C})$ by $k\cdot z = uzv^*$ $(u \in U(n),\, v \in U(p))$. For a partition $\mathbf{m}$ of length $\ell(\mathbf{m}) \leq n$: $\mathbf{m} = (m_1, \ldots, m_n)$ with $m_i \in \mathbb{N}$, $m_1 \geq \ldots \geq m_n \geq 0$, let $\mathcal{H}_{\mathbf{m}}$ denote the space of polynomials on V generated by the polynomials $\pi(k)\Delta_{\mathbf{m}}$ $(k \in K)$, where

$$\Delta_{\mathbf{m}}(z) = \Delta_1(z)^{m_1-m_2} \Delta_2(z)^{m_2-m_3} \ldots \Delta_n(z)^{m_n},$$

with

$$\Delta_k(z) = \det(z_{ij})_{1\leq i,j\leq k} \quad (k \leq n).$$

The polynomial $\Delta_{\mathbf{m}}$ is a highest weight vector for the restriction $\pi_{\mathbf{m}}$ of π to $\mathcal{H}_{\mathbf{m}}$ with respect to the subgroup $T_n^- \times T_p^+$, where T_n^- is the group of lower triangular matrices in $GL(n,\mathbb{C})$, and T_p^+ of upper triangular matrices in $GL(p,\mathbb{C})$. The representation $\pi_{\mathbf{m}}$ is equivalent to the tensor product of the irreducible representation of $U(n)$ with highest weight $(m_1, \ldots, m_n)$ with the irreducible representation of $U(p)$ with highest weight $(m_1, \ldots, m_n, 0, \ldots, 0)$. Recall that the character $\chi_{\mathbf{m}}$ of the representation of $U(n)$ with highest weight $\mathbf{m}$ can be expressed in terms of the Schur function $s_{\mathbf{m}}$:

$$\chi_{\mathbf{m}}\big(\mathrm{diag}(t_1, \ldots, t_n)\big) = s_{\mathbf{m}}(t_1, \ldots, t_n).$$

Hence the dimension $d_{\mathbf{m}}$ of $\mathcal{H}_{\mathbf{m}}$ is given by

$$d_{\mathbf{m}} = s_{\mathbf{m}}(1^n) s_{\mathbf{m}}(1^p),$$

where $1^q = (1, \ldots, 1, 0, \ldots)$ (1 is repeated q times). We will use the following expansion, for $x \in M(n,\mathbb{C})$,

$$e^{\mathrm{tr}\,x} = \sum_{\mathbf{m}} \frac{1}{h(\mathbf{m})} \chi_{\mathbf{m}}(x),$$

where $h(\mathbf{m})$ is the product of the hook-lengths of the partition $\mathbf{m}$ ([Macdonald,1995],p.66). This number $h(\mathbf{m})$ does not depend on n. It can also be written, if $\ell(\mathbf{m}) \leq n$,

$$\frac{1}{h(\mathbf{m})} = \frac{s_{\mathbf{m}}(1^n)}{(n)_{\mathbf{m}}}.$$

where $(\alpha)_{\mathbf{m}}$ denotes the generalized Pochhammer symbol:

$$(\alpha)_{\mathbf{m}} = \prod_{j=1}^{n} (\alpha - j + 1)_{m_j}.$$

(Observe that the definition of $(\alpha)_{\mathbf{m}}$ depends on n.)

Proposition 2.4. (i) *The reproducing kernel* $\mathcal{K}_{\mathbf{m}}$ *of* $\mathcal{H}_{\mathbf{m}}$ *is given by*

$$\mathcal{K}_{\mathbf{m}}(z,w)=\frac{1}{h(\mathbf{m})}\chi_{\mathbf{m}}(zw^*).$$

(ii) *Let*

$$f(z)=\sum_{\mathbf{m}} f_{\mathbf{m}}(z) \quad (f\in\mathcal{H}_{\mathbf{m}}),$$

be the expansion of a function $f\in\mathcal{F}(V)$. *Then*

$$\int_{U(n)\times U(p)}|f(uzv^*)|^2\beta_n(du)\beta_p(dv)=\sum_{\mathbf{m}}\frac{1}{(n)_{\mathbf{m}}}\frac{1}{(p)_{\mathbf{m}}}h(\mathbf{m})\chi_{\mathbf{m}}(zz^*)\|f_{\mathbf{m}}\|^2.$$

(iii) *For* $f(z)=e^{(z|w)}$ *one obtains*

$$\int_{U(n)\times U(p)}e^{2\mathrm{Re}(uzv^*|w)}\beta_n(du)\beta_p(dv)=\sum_{\mathbf{m}}\frac{1}{(n)_{\mathbf{m}}}\frac{1}{(p)_{\mathbf{m}}}\chi_{\mathbf{m}}(zz^*)\chi_{\mathbf{m}}(ww^*).$$

Note that, for $n=1$, these formulae agree with the formulae given above in case $V=\mathbb{C}^p$.

Proof of (ii). By Proposition 2.2,

$$\int_{U(n)\times U(p)}|f(uzv^*)|^2\alpha_n(du)\alpha_p(dv)=\sum_{\mathbf{m}}\frac{1}{d_{\mathbf{m}}}\frac{1}{h(\mathbf{m})}\chi_{\mathbf{m}}(zz^*)\|f_{\mathbf{m}}\|^2,$$

and

$$d_{\mathbf{m}}=s_{\mathbf{m}}(1^n)s_{\mathbf{m}}(1^p)=\frac{(n)_{\mathbf{m}}}{h(\mathbf{m})}\frac{(p)_{\mathbf{m}}}{h(\mathbf{m})}.$$ ∎

3. Spherical functions. We come back to the Gelfand pair we introduced in Section 1. There are two kinds of spherical functions. The spherical functions of first kind are associated to the Bargmann representation, and the ones of second kind to characters.

a) *Spherical functions of positive type and first kind*

We recall first the Fock realization of the Bargmann representation. (See for instance [Faraut,1987].) For $\lambda\in\mathbb{R}^*$ one considers the Fock space $\mathcal{F}_\lambda(V)$ of holomorphic functions ψ on V such that

$$\|\psi\|_\lambda^2=\left(\frac{|\lambda|}{\pi}\right)^N\int_V|\psi(\zeta)|^2e^{-|\lambda|\|\zeta\|^2}m(d\zeta)<\infty,$$

and the representation T_λ of the Heisenberg $H = V \times \mathbb{R}$ on $\mathcal{F}_\lambda(V)$ defined by, if $\lambda > 0$,

$$\big(T_\lambda(z,t)\psi\big)(\zeta) = e^{\lambda\left(it - \frac{1}{2}\|z\|^2 - (\zeta|z)\right)}\psi(\zeta + z),$$

and $T_\lambda(z,t) = T_{-\lambda}(\bar{z}, -t)$ for $\lambda < 0$.

We consider a closed subgroup $K \subset U(V)$. The group K acts on $\mathcal{F}_\lambda(V)$:

$$\pi(k)\psi(\zeta) = \psi(k^{-1}\zeta),$$

and

$$\pi(k)T_\lambda(z,t)\pi(k^{-1}) = T_\lambda(k \cdot z, t).$$

We assume that K acts multiplicity free on $\mathcal{P}(V)$. Then the Fock space $\mathcal{F}_\lambda(V)$ decomposes multiplicity free:

$$\mathcal{F}_\lambda(V) = \bigoplus_{\mu \in \mathfrak{M}} \mathcal{H}_\mu.$$

Define as usual, for $f \in L^1(H)$,

$$T_\lambda(f) = \int_H T_\lambda(z,t) f(z,t) m(dz) dt.$$

If the function f is K-invariant, then, for every $\mu \in \mathfrak{M}$, the subspace $\mathcal{H}_\mu$ is an eigenspace of $T_\lambda(f)$: for $\psi \in \mathcal{H}_\mu$,

$$T_\lambda(f)\psi = \hat{f}(\lambda, \mu)\psi.$$

The map $f \mapsto \hat{f}(\lambda, \mu)$ is a character of the commutative convolution algebra $L^1(H)^K$. It can be written

$$\hat{f}(\lambda, \mu) = \int_H f(z,t)\varphi(\lambda, \mu; z, t) m(dz) dt,$$

with a spherical function $\varphi(\lambda, \mu; \cdot)$. Suppose first $\lambda > 0$. We can write, if $\psi \in \mathcal{H}_\mu$,

$$\int_H e^{\lambda\left(it - \frac{1}{2}\|z\|^2 - (\zeta|z)\right)}\psi(\zeta + z) f(z,t) m(dz) dt = \hat{f}(\lambda, \mu)\psi(\zeta).$$

Fix $e \in V$, and $\psi \in \mathcal{H}_\mu$ such that $\psi(e) = 1$. Then,

$$\varphi(\lambda, \mu; z, t) = e^{i\lambda t} e^{-\frac{1}{2}\lambda\|z\|^2} \int_K e^{-\lambda(e|k \cdot z)}\psi(e + k \cdot z)\alpha(dk).$$

By using Proposition 2.2, it is possible to obtain an expansion of the spherical function $\varphi(\lambda, \mu; \cdot)$

We consider the case of $V = M(n,p;\mathbb{C})$, $K = U(n) \times U(p)$. We choose $e = \begin{pmatrix} I_n & 0 \end{pmatrix}$, and $\psi(z) = \Phi_{\mathbf{m}}(z_0)$, with $z_0 = ze^*$, the projection of z onto $M(n;\mathbb{C})$. Here $\Phi_{\mathbf{m}}$ is the spherical polynomial, which is the normalized character:

$$\Phi_{\mathbf{m}}(x) = \frac{\chi_{\mathbf{m}}(x)}{\chi_{\mathbf{m}}(I_n)} \quad \big(x \in M(n;\mathbb{C})\big).$$

(See [Faraut-Korányi,1994], Chapter XI.) Then we can write

$$\int_K e^{-\lambda(e|k\cdot z)}\psi(e+k\cdot z)\alpha(dk) = \int_K f_1(k\cdot z)\overline{f_2(k\cdot z)}\alpha(dk),$$

with

$$f_1(z) = \Phi_{\mathbf{m}}(I_n + z_0),$$
$$f_2(z) = e^{-\lambda \operatorname{tr}(z_0)}.$$

We will expand both functions. We saw that

$$e^{-\lambda \operatorname{tr}(z_0)} = \sum_{\mathbf{k}} (-\lambda)^{|\mathbf{k}|} \frac{1}{(n)_{\mathbf{k}}} s_{\mathbf{k}}(1^n)\chi_{\mathbf{k}}(z_0).$$

For expanding $\Phi_{\mathbf{m}}(I_n + z_0)$, we need the binomial formula for the Schur functions, which can be written

$$\frac{s_{\mathbf{m}}(1+z_1,\ldots,1+z_n)}{s_{\mathbf{m}}(1^n)} = \sum_{\mathbf{k}\subset\mathbf{m}} \frac{1}{(n)_{\mathbf{k}}} s^*_{\mathbf{k}}(\mathbf{m}) s_{\mathbf{k}}(z_1,\ldots,z_n),$$

where $s^*_{\mathbf{m}}$ is the shifted Schur function ([Okounkov-Olshanski,1998a] Theorem 5.1, see also [Faraut,2008] Theorem 2.8). Hence we get the following expansion

$$\Phi_{\mathbf{m}}(I_n + z_0) = \sum_{\mathbf{k}\subset\mathbf{m}} \frac{1}{(n)_{\mathbf{k}}} s^*_{\mathbf{k}}(\mathbf{m})\chi_{\mathbf{k}}(z_0).$$

By Proposition 2.4 we obtain

$$\int_K e^{-\lambda(e|k\cdot z)}\psi(e+k\cdot z)\alpha(dk) = \sum_{\mathbf{k}\subset\mathbf{m}} (-\lambda)^{|\mathbf{k}|} \frac{1}{(p)_{\mathbf{k}}}\frac{1}{(n)_{\mathbf{k}}} s^*_{\mathbf{k}}(\mathbf{m})\chi_{\mathbf{k}}(zz^*),$$

since

$$\|\chi_{\mathbf{k}}(z_0)\|^2 = (n)_{\mathbf{k}}.$$

Proposition 3.1. *The spherical functions of positive type of first kind admit the following expansion:*

$$\varphi(\lambda,\mathbf{m};z,t) = e^{i\lambda t}e^{-\frac{1}{2}\lambda\|z\|^2} \sum_{\mathbf{k}\subset\mathbf{m}} (-\lambda)^{|\mathbf{k}|} \frac{1}{(p)_{\mathbf{k}}}\frac{1}{(n)_{\mathbf{k}}} s^*_{\mathbf{k}}(\mathbf{m})\chi_{\mathbf{k}}(zz^*).$$

and, for $\lambda < 0$, $\varphi(\lambda,\mathbf{m};z,t) = \overline{\varphi(-\lambda,\mathbf{m};z,t)}$.

The relation $\varphi(-\lambda,\mathbf{m};z,t) = \overline{\varphi(\lambda,\mathbf{m};z,t)}$ comes from the fact that z and $\bar{z}$ are in the same K-orbit. By this relation it suffices to consider the case $\lambda > 0$, and in further proofs we will assume $\lambda > 0$.

The spherical functions $\varphi(\lambda,\mathbf{m};z,t)$ can be expressed in terms of the multivariate Laguerre polynomials $L_{\mathbf{m}}^{(\nu-1)}$ as defined in [Faraut-Korányi,1994] p.343 with a slightly different parametrisation (see also [Faraut-Wakayama,2008], p.10). In case $d=2$ (with the notation of [Faraut-Korányi,1994]), for $x\in M(n;\mathbb{C})$,

$$\frac{L_{\mathbf{m}}^{(\nu-1)}(x)}{L_{\mathbf{m}}^{(\nu-1)}(0)}=\sum_{\mathbf{k}\subset\mathbf{m}}\frac{1}{(n)_{\mathbf{k}}}\frac{1}{(\nu)_{\mathbf{k}}}s_{\mathbf{k}}^{*}(\mathbf{m})\chi_{\mathbf{k}}(-x).$$

Therefore

$$\begin{aligned}\varphi(\lambda,\mathbf{m};z,t)&=e^{i\lambda t}e^{-\frac{1}{2}\lambda\|z\|^2}\frac{L_{\mathbf{m}}^{(p-1)}(\lambda zz^*)}{L_{\mathbf{m}}^{(\nu-1)}(0)}\\&=e^{i\lambda t}\frac{\Psi_{\mathbf{m}}^{(p)}\left(\frac{1}{2}\lambda zz^*\right)}{\Psi_{\mathbf{m}}^{(p)}(0)}.\end{aligned}$$

where $\Psi_{\mathbf{m}}^{(\nu)}$ is the multivariate Laguerre function,

$$\Psi_{\mathbf{m}}^{(\nu)}(x)=e^{-\operatorname{tr}(x)}L_{\mathbf{m}}^{(\nu-1)}(2x).$$

(There are similar results and proofs in [Dib,1990], Section III.)

2) *Spherical functions of positive type and second kind*

These functions are obtained by averaging Euclidean characters: For $w\in V$, let $\rho_1\geq\ldots\geq\rho_n\geq 0$ denote the eigenvalues of the positive Hermitian matrix ww^*. We define

$$\Psi(\rho;z)=\int_{U(n)\times U(p)}e^{2i\Re\operatorname{tr}(uzv^*w^*)}\beta_n(du)\beta_p(dv).$$

By Proposition 2.4 one obtains:

Proposition 3.2.

$$\psi(\rho;z)=\sum_{\mathbf{k}}(-1)^{|\mathbf{k}|}\frac{1}{(p)_{\mathbf{k}}}\frac{1}{(n)_{\mathbf{k}}}s_{\mathbf{k}}(\rho)\chi_{\mathbf{k}}(zz^*).$$

The set Σ of spherical functions of positive type will be called the spherical dual of the Gelfand pair (G,K). In fact, in our case, it coincides with the set of bounded spherical functions (see [Benson-Ratcliff,1992]). One considers on Σ the topology of uniform convergence on compact sets. Let Σ^1 be the set of spherical functions of the first kind, and Σ^2 of the second kind. Hence $\Sigma=\Sigma^1\cup\Sigma^2$. We embed Σ in $\mathbb{R}^{n+1}$ as follows: to the spherical function of the first kind $\varphi(\lambda,\mathbf{m};z,t)$ corresponds the point $(\lambda,|\lambda|m_1,\ldots,|\lambda|m_n)$, and to the spherical function of the second kind $\psi(\rho;z)$ corresponds $(0,\rho_1,\ldots,\rho_n)$. Then this map is a homeomorphism from Σ onto its image (a multidimensional Heisenberg fan). This is a consequence of:

Proposition 3.3. *As $\lambda \to 0$, $\lambda m_i \to \rho_i$,*

$$\lim \varphi(\lambda, \mathbf{m}; z, t) = \psi(\rho; z),$$

uniformly on compact sets in H.

Proof. Recall that, by Propositions 3.1,

$$\varphi(\lambda, \mathbf{m}; z, t) = e^{i\lambda t} e^{-\frac{1}{2}\lambda\|z\|^2} \sum_{\mathbf{k} \subset \mathbf{m}} (-\lambda)^{|\mathbf{k}|} \frac{1}{(p)_{\mathbf{k}}} \frac{1}{(n)_{\mathbf{k}}} s^*_{\mathbf{k}}(\mathbf{m}) \chi_{\mathbf{k}}(zz^*),$$

and by Proposition 3.2,

$$\psi(\rho; z) = \sum_{\mathbf{k}} (-1)^{|\mathbf{k}|} \frac{1}{(p)_{\mathbf{k}}} \frac{1}{(n)_{\mathbf{k}}} s_{\mathbf{k}}(\rho) \chi_{\mathbf{k}}(zz^*).$$

Since

$$s^*_{\mathbf{k}}(\mathbf{m}) = s_{\mathbf{k}}(\mathbf{m}) + \textit{terms of degree } < |\mathbf{k}|,$$

the statement follows from:

Lemma 3.4. *Let (ψ_n) be a sequence of $\mathcal{C}^\infty$ functions on $\mathbb{R}^d$ of positive type, and ψ an analytic function on a neighborhood of 0. Assume that, for every $k = (k_1, \dots, k_d) \in \mathbb{N}^n$,*

$$\lim_{n \to \infty} \partial^k \psi_n(0) = \partial^k \psi(0).$$

Then ψ has an analytic extension to $\mathbb{R}^d$, and ψ_n converges to ψ uniformly on compact sets in $\mathbb{R}^d$.

(Lemma 4.2 in [Okounkov-Olshanski,1998b]. See also Proposition 3.10 in [Faraut,2008].) $\square$

We will use the following notation: for $\sigma \in \Sigma$,

$$\varphi(\sigma; z, t) = e^{i\lambda t} e^{-\frac{1}{2}|\lambda|\|z\|^2} \sum_{\mathbf{k}} (-1)^{|\mathbf{k}|} \frac{1}{(n)_{\mathbf{k}}} \frac{1}{(p)_{\mathbf{k}}} a_{\mathbf{k}}(\sigma) \chi_{\mathbf{k}}(zz^*).$$

If $\sigma = (\lambda, \mathbf{m}) \in \Sigma^1$, then

$$a_{\mathbf{k}}(\sigma) = \lambda^{|\mathbf{k}|} s^*_{\mathbf{k}}(\mathbf{m})$$

if $\mathbf{k} \subset \mathbf{m}$, and $a_{\mathbf{k}}(\mathbf{m}) = 0$ otherwise. If $\sigma = \rho \in \Sigma^2$, then λ is taken to be 0, and

$$a_{\mathbf{k}}(\sigma) = s_{\mathbf{k}}(\rho).$$

Observe that, for a partition $\mathbf{k}$, $a_{\mathbf{k}}(\sigma)$ is a continuous function on Σ.

We will need in Sections 5 and 6 the expansion at order 4 with respect to x of the function given on $\Sigma \times \mathbb{R}$ by $\varphi(\sigma; xE_{11}, 0)$ where E_{11} denotes the matrix with entry 1 at the place (1,1) and 0 elsewhere.

Recall that, for $\mathbf{k} = [k] := (k, 0, \ldots)$ ($k \in \mathbb{N}$), $s_{\mathbf{k}}(x_1, \ldots, x_n) = h_k(x_1, \ldots, x_n)$ is the complete symmetric function. In particular

$$h_1(\rho_1, \ldots, \rho_n) = \rho_1 + \cdots + \rho_n,$$
$$h_2(\rho_1, \ldots, \rho_n) = \sum_{i<j} \rho_i \rho_j.$$

Also, for $\mathbf{k} = [k]$, $s^*_{\mathbf{k}}(m_1, \ldots, m_n)$ is the shifted complete symmetric function,

$$h^*_1(m_1, \ldots, m_n) = m_1 + \cdots + m_n,$$
$$h^*_2(m_1, \ldots, m_n) = \sum_{i<j} (m_i - 1) m_j.$$

Recall that, if $x = (x_1, 0, \ldots, 0)$, then $s_{\mathbf{k}}(x_1, 0, \ldots, 0) = 0$ except if $\mathbf{k} = [k]$ ($k \in \mathbb{N}$), and then
$s_{[k]}(x_1, 0, \ldots, 0) = x_1^k$.

Lemma 3.5.

$$\varphi(\sigma; xE_{11}, 0) = 1 - A_{n,p}(\sigma)x^2 + B_{n,p}(\sigma)x^4 + \cdots,$$

with, if $\sigma = (\lambda, \mathbf{m}) \in \Sigma^1$,

$$A_{n,p}(\sigma) = \lambda\Big(\frac{1}{2} + \frac{1}{np} h^*_1(\mathbf{m})\Big),$$
$$B_{n,p}(\sigma) = \lambda^2\Big(\frac{1}{8} + \frac{1}{2np} h^*_1(\mathbf{m}) + \frac{1}{n(n+1)p(p+1)} h^*_2(\mathbf{m})\Big),$$

and, if $\sigma = \rho \in \Sigma^2$,

$$A_{n,p}(\sigma) = \frac{1}{np} h_1(\rho),$$
$$B_{n,p}(\sigma) = \frac{1}{n(n+1)p(p+1)} h_2(\rho).$$

Proof. If $\sigma = (\lambda, \mathbf{m}) \in \Sigma^1$, then

$$\begin{aligned}
\varphi(\sigma; xE_{11}, 0) &= e^{-\frac{1}{2}\lambda x^2} \sum_{k=0}^{m_1} (-\lambda)^k \frac{1}{(n)_k} \frac{1}{(p)_k} h^*_k(\mathbf{m}) x^{2k} \\
&= \Big(1 - \frac{1}{2}\lambda x^2 + \frac{1}{8}\lambda^2 x^4 + \cdots\Big) \\
&\quad \Big(1 - \frac{\lambda}{np} h^*_1(\mathbf{m}) x^2 + \frac{\lambda^2}{n(n+1)p(p+1)} h^*_2(\mathbf{m}) x^4 + \cdots\Big) \\
&= 1 - \lambda\Big(\frac{1}{2} + \frac{1}{np} h^*_1(\mathbf{m})\Big) x^2 + \\
&\quad + \lambda^2\Big(\frac{1}{8} + \frac{1}{2np} h^*_1(\mathbf{m}) + \frac{1}{n(n+1)p(p+1)} h^*_2(\mathbf{m})\Big) x^4 + \cdots,
\end{aligned}$$

and, if $\sigma = \rho \in \Sigma^2$,

$$\begin{aligned}\varphi(\sigma, xE_{11}, 0) &= \sum_{k=0}^{\infty}(-1)^k \frac{1}{(n)_k}\frac{1}{(p)_k} h_k(\rho)x^{2k} \\ &= 1 - \frac{1}{np}h_1(\rho)x^2 + \frac{1}{n(n+1)p(p+1)}h_2(\rho)x^4 + \cdots\end{aligned}$$
∎

4. Olshanski spherical pairs. Let $\big(G(n), K(n)\big)$ be an increasing sequence of Gelfand pairs,

$$G(n) \subset K(n), \quad K(n) = G(n) \cap K(n+1),$$

and define

$$G = \bigcup_{n=1}^{\infty} G(n), \quad K = \bigcup_{n=1}^{\infty} K(n).$$

We consider on G the inductive limit topology; then K is a closed subgroup of G. Such pairs (G, K) have been introduced and studied in [Olshanski,1990], and we call them Olshanski spherical pairs. In general G is not locally compact, and K is not compact. A K-biinvariant continuous function φ on G is said to be spherical if

$$\lim_{n\to\infty} \int_{K(n)} \varphi(xky)\alpha_n(dk) = \varphi(x)\varphi(y),$$

where α_n denotes the normalized Haar measure on $K(n)$.

Let $\mathcal{P}$ denote the set of K-biinvariant continuous functions φ on G which are of positive type, with $\varphi(e) = 1$.

Theorem 4.1. *For $\varphi \in \mathcal{P}$ the following properties are equivalent*
- φ is spherical,
- φ is extremal in the convex set $\mathcal{P}$,
- φ is pure. This means that the unitary representation associated to φ by the Gelfand-Naimark-Segal construction is irreducible.

([Olshanski,1990], §23, see also [Faraut,2008], Chapter 1.)

Recall that, by the Gelfand-Naimark-Segal construction, given a K-biinvariant continuous function φ of positive type, one obtains a unitary representation π of G on a Hilbert space $\mathcal{H}$, such that

$$\varphi(x) = \big(u|\pi(x)u\big),$$

where u is a K-invariant cyclic vector in $\mathcal{H}$.

In the case of a Gelfand pair, the equivalence of these properties are classical. It has been proven in [Olshanski,1990] that it holds for an Olshanski spherical pair as well.

Theorem 4.2. *Let φ be a spherical function of positive type for the pair (G, K). There exists a sequence (φ_n), for which φ_n is a spherical function of positive type for the pair $\big(G(n), K(n)\big)$, such that*

$$\varphi(x) = \lim_{n\to\infty} \varphi_n(x),$$

uniformly on compact sets of G.

([Olshanski,1990], Theorem 22.10.)

Given an Olshanski spherical pair (G, K), there is a natural program:

- Determine the spherical functions,

- Determine the set Ω of spherical functions of positive type. We will call Ω the spherical dual of the pair (G, K).

In several examples we know, this is done by obtaining the functions in Ω as limits of spherical functions for the Gelfand pairs $\big(G(n), K(n)\big)$, according to Theorem 4.2.

- A further point which will not be considered in the present paper is: For $\varphi \in \Omega$ describe a realization of the irreducible representation of G associated to φ by the Gelfand-Naimark-Segal construction.

Let us give a general scheme for studying spherical functions for the pair (G, K) as limits of spherical functions for the pairs $\big(G(n), K(n)\big)$. This scheme is essentially the one used in [Okounkov-Olshanski,1998b] (See also [Olshanski-Vershik,1996]). Notice that the restriction to $G(n)$ of a spherical function for the pair (G, K) is not spherical in general.

Let Ω_n denote the spherical dual of the pair $\big(G(n), K(n)\big)$, and Ω the one of (G, K). For $\mu \in \Omega_n$ we write $\varphi_n(\mu; x)$ the corresponding spherical function, and, for $\omega \in \Omega$, we write $\varphi(\omega; x)$.

For each n one defines an injective map $T_n : \Omega_n \to \Omega$. Let $(\mu^{(n)})$ be a sequence with $\mu^{(n)} \in \Omega_n$.

(1) In a first step, one shows that, if $\lim_{n\to\infty} T_n(\mu^{(n)}) = \omega$ for the topology of Ω, then

$$\lim_{n\to\infty} \varphi_n(\mu^{(n)}; x) = \varphi(\omega; x),$$

uniformly on compact subsets of G. For this step one uses Lemma 3.4.

(2) The second step is as follows: Assume that

$$\lim_{n\to\infty} \varphi_n(\mu^{(n)}; x) = \varphi(x),$$

where φ is a continuous function on G. The aim of this step is to show that there is $\omega \in \Omega$ such that $\lim_{n\to\infty} T_n(\mu^{(n)}) = \omega$. For that it is enough to show that the sequence $\big(T_n(\mu^{(n)})\big)$ is relatively compact in Ω. In fact, if $(\mu^{(n_j)})$ is a subsequence such that $\lim_{j\to\infty} T_{n_j}(\mu^{n_j}) = \omega_0$, then, by (1),

$$\lim_{j\to\infty} \varphi_{n_j}(\mu^{(n_j)}; x) = \varphi(\omega_0; x),$$

and $\varphi(x) = \varphi(\omega_0; x)$. Hence there is only one possible limit for a subsequence. Therefore the sequence $T_n(\mu^{(n)})$ itself converges. In the examples we know, it is enough to only consider elements x in $G(1)$ for showing that the sequence $\big(T_n(\mu^{(n)})\big)$ is relatively compact. For this second step we will use:

Lemma 4.3. *Let M be a set of probability measures on $\mathbb{R}^d$, relatively compact for the weak topology (tight topology). Assume that, for every $\mu \in M$ and $k \leq 4$,*

$$\mathfrak{M}_k(\mu) := \int_{\mathbb{R}^d} \|x\|^k \mu(dx) < \infty,$$

and that there is a constant $A > 0$ such that, for every $\mu \in M$,

$$\mathfrak{M}_4(\mu) \leq A \ \mathfrak{M}_2(\mu)^2.$$

Then there is a constant $C > 0$ such that, for every $\mu \in M$,

$$\mathfrak{M}_2(\mu) \leq C.$$

([Okounkov-Olshanski,1998b], Lemma 5.2.)

Proof. Since M is relatively compact, for $0 < \varepsilon < \frac{1}{A}$, there is $R > 0$ such, for every $\mu \in M$,

$$\mu\big(\{\|x\| > R\}\big) \leq \varepsilon.$$

By the Schwarz inequality,

$$\Big(\int_{\|x\|>R} \|x\|^2 \mu(dx)\Big)^2 \leq \varepsilon \mathfrak{M}_4(\mu) \leq \varepsilon A \ \mathfrak{M}_2(\mu)^2.$$

Therefore

$$\mathfrak{M}_2(\mu) \leq R^2 + \int_{\|x\|>R} \|x\|^2 \mu(dx) \leq R^2 + \sqrt{\varepsilon A} \ \mathfrak{M}_2(\mu),$$

or

$$\mathfrak{M}_2(\mu) \leq \frac{R^2}{1-\sqrt{\varepsilon A}}.$$

∎

5. An Olshanski spherical pair with finite rank. In this section n is fixed. We consider the increasing sequences,

$$\begin{aligned} V(p) &= M(n,p;\mathbb{C}), \quad H(p) = V(p) \times \mathbb{R}, \\ K(p) &= U(n) \times U(p), \quad G(p) = K(p) \ltimes H(p), \end{aligned}$$

the infinite dimensional Heisenberg group

$$H = \bigcup_{p=1}^{\infty} H(p),$$

and the Olshanski spherical pair

$$G = \bigcup_{p=1}^{\infty} G(p), \quad K = \bigcup_{p=1}^{\infty} K(p).$$

a) *Spherical functions*

Let φ be a K-invariant continuous function on H. It can be written

$$\varphi(z,t) = \Phi(zz^*, t),$$

where Φ is a continuous function on $Herm(n,\mathbb{C}) \times \mathbb{R}$ which is $U(n)$-invariant:

$$\Phi(uwu^*, t) = \Phi(w,t) \quad \big(w \in Herm(n,\mathbb{C}),\ u \in U(n)\big).$$

Then

$$\begin{aligned} &\varphi\big(z + k\cdot z', t + t' + \mathrm{Im}(z|k\cdot z')\big) \\ &= \Phi\big((z+uz'v^*)(z+uz'v^*)^*, t+t' + \mathrm{Im}(z|uz'v^*)\big), \end{aligned}$$

For $v \in U(p)$, let $[v]_m$ denote the upper left $m\times m$-block. If $z, z' \in V(m) = M(n,m;\mathbb{C})$, then
$(z+uz'v^*)(z+uz'v^*)^*$ and $(z|uz'v^*)$, as functions of v, only depend on $[v]_m$:

$$\begin{aligned} (z+uz'v^*)(z+uz'v^*)^* &= zz^* + zvz'^*u^* + uz'v^*z^* + uz'z'^*u^* \\ &= zz^* + z[v]_m z'^*u^* + uz'[v]_m^* z^* + uz'z'^*u^*, \\ (z|uz'v^*) &= \mathrm{tr}\,(zvz'^*u^*) = \mathrm{tr}\,(z[v]_m z'^*u^*). \end{aligned}$$

We will use the following lemma about the asymptotics of the normalized Haar measure of $U(p)$.

Lemma 5.1. *Assume $p \geq 2m$. The image of the normalized Haar measure β_p of $U(p)$ under the projection $v \mapsto [v]_m$ is given as follows: If f is a continuous function on the unit ball B_m of $M(m,\mathbb{C})$ (with respect to the operator norm), then*

$$\int_{U(p)} f([v]_m)\beta_p(dv) = c_{p,m} \int_{B_m} f(w) \det(I_m - ww^*)^{p-2m} m(dw),$$

where m is the Euclidean measure, and $c_{p,m}$ is the normalization constant. Furthermore, for m fixed,

$$\lim_{p\to\infty} \int_{U(p)} f([v]_m)\beta_p(dv) = f(0).$$

Proof. a) This integration formula can be obtained from the Weyl integration formula for the compact symmetric space $U(p)/U(m)\times U(p-m)$ related to the Cartan decomposition.

The specialization of this Weyl integration formula to our present case is written down in [Faraut,2006], Section 5.2. We obtain

$$\int_{U(p)} f([v]_m)\beta_p(dv) = \int_{[0,\frac{\pi}{2}]^m} \int_{U(m)\times U(m)} f\big((u_1 \mathrm{diag}(\cos\theta_1,\ldots,\cos\theta_m)u_2\big)$$
$$\beta_m(du_1)\beta_m(du_2)D_{m,p}(\theta)d\theta_1\ldots d\theta_m,$$

with

$$D_{m,p}(\theta) = a_{m,p}\Big|\prod_{1\le i<j\le m}\sin^2(\theta_i+\theta_j)\sin^2(\theta_i-\theta_j)\prod_{i=1}^m(\sin 2\theta_i)(\sin\theta_i)^{2(p-2m)}\Big|.$$

On the other hand recall the integration formula on $M(m;\mathbb{C})$ related to the polar decomposition: for an integrable function on $M(m;\mathbb{C})$,

$$\int_{M(m;\mathbb{C})} f(w)m(dw) = \int_{\mathbb{R}_+^m} f\big(u_1\mathrm{diag}(a_1,\ldots,a_m)u_2\big)\beta_m(du_1)\beta_m(du_2)\Delta_m(a)da_1\ldots da_m,$$

with

$$\Delta_m(a) = c_m\prod_{1\le i<j\le m}(a_i^2-a_j^2)^2\prod_{i=1}^m a_i.$$

(See for instance [Faraut-Korányi,1994], Proposition X.3.4.) By the change of variables given by $a_i=\cos\theta_i$, and the identity

$$\sin(\theta_i+\theta_j)\sin(\theta_i-\theta_j) = \cos^2\theta_j - \cos^2\theta_i,$$

noticing moreover that, if $w = u_1\mathrm{diag}(a_1,\ldots,a_m)u_2$,

$$\prod_{i=1}^m(\sin\theta_i)^{2(p-2m)} = \prod_{i=1}^m(1-\cos^2\theta_i)^{p-2m} = \det(I_m - ww^*)^{p-2m},$$

we get

$$\int_{U(p)} f([v]_m)\beta_p(dv) = c_{m,p}\int_{B_m} f(w)\det(I_m-ww^*)^{p-2m}m(dw).$$

b) The proof of the second part of the lemma is standard. (See for instance Lemma 5.4 in [Faraut, 2006].) ∎

From Lemma 5.1 it follows that

$$\lim_{p\to\infty}\int_{U(n)\times U(p)}\varphi\big(z+uz'v^*, t+t'+\mathrm{Im}(z|uz'v^*)\big)\beta_n(du)\beta_p(dv)$$
$$= \int_{U(n)}\Phi(zz^*+uz'z'^*u^*, t+t')\beta_n(du).$$

Therefore the function φ is spherical for the Olshanski spherical pair (G,K), *i.e.*

$$\lim_{p\to\infty}\int_{K(p)}\varphi\big(z+k\cdot z',t+t'+\mathrm{Im}(z|k\cdot z')\big)\alpha_p(dk)=\varphi(z,t)\varphi(z',t'),$$

if and only if the function Φ satisfies the following functional equation:

$$\int_{U(n)}\Phi(x+ux'u^*,t+t')\beta_n(du)=\Phi(x,t)\Phi(x',t')\quad\big(x,x'\in Herm(n,\mathbb{C})\big).$$

This equation means that, as a function of t, it is an exponential, and, as a function of $x\in Herm(n,\mathbb{C})$, it is a spherical function for the motion group $U(n)\ltimes Herm(n,\mathbb{C})$. Such a spherical function Ψ has the form

$$\Psi(\alpha;x)=\int_{U(n)}e^{-\mathrm{tr}\,(uxu^*\alpha)}\beta_n(du),$$

with $\alpha\in M(n;\mathbb{C})$. This integral can be evaluated if α is diagonalizable. It only depends on the eigenvalues $x_1,\ldots,x_n$ of x, and on those $\alpha_1,\ldots,\alpha_n$ of α. We can assume that $\alpha=\mathrm{diag}(\alpha_1,\ldots,\alpha_n)$.

$$\Psi(\alpha;x)=(\delta_n)!\frac{\det\Big((e^{-\alpha_ix_j})_{1\le i,j\le n}\Big)}{V(\alpha)V(x)},$$

with $\delta_n=(n-1,\ldots,1,0)$, $(\delta_n)!=1!2!\ldots(n-1)!$, and where V denotes the Vandermonde polynomial. This function admits the following expansion

$$\Psi(\alpha;x)=\sum_{\mathbf{m}}(-1)^{|\mathbf{m}|}\frac{1}{(n)_{\mathbf{m}}}s_{\mathbf{m}}(\alpha)\chi_{\mathbf{m}}(x).$$

Theorem 5.2. *The spherical function for the Olshanski spherical pair (G,K) are given by the following formulae*

$$\begin{aligned}\varphi(\lambda,\alpha;z,t)&=e^{i\lambda t}\int_{U(n)}e^{-\mathrm{tr}\,(uzz^*u^*\alpha)}\alpha_n(du)\\&=e^{i\lambda t}(\delta_n)!\frac{\det\Big((e^{-\alpha_ix_j})_{1\le i,j\le n}\Big)}{V(\alpha)V(x)},\\&=e^{i\lambda t}\sum_{\mathbf{m}}(-1)^{|\mathbf{m}|}\frac{1}{(n)_{\mathbf{m}}}s_{\mathbf{m}}(\alpha)\chi_{\mathbf{m}}(zz^*),\end{aligned}$$

with $\lambda\in\mathbb{C}$, $\alpha=\mathrm{diag}(\alpha_1,\ldots,\alpha_n)$, and $x_1,\ldots,x_n$ are the eigenvalues of $x=zz^*$.

b) *Asymptotics of spherical functions*

In order to state the result we need some notation. As we saw in Section 3 the spherical dual Σ_p of the pair $\big(G(p), K(p)\big)$ can be identified to $\Sigma_p = \Sigma_p^1 \cup \Sigma_p^2$, with

$$\Sigma_p^1 = \{(\lambda, \mathbf{m}) \mid \lambda \in \mathbb{R}^*,\ \mathbf{m} \text{ is a partition},\ \ell(\mathbf{m}) \leq n\},$$
$$\Sigma_p^2 = \{\rho \in \mathbb{R}^n \mid \rho_1 \geq \cdots \geq \rho_n \geq 0\}.$$

We also saw that the spherical function corresponding to $\sigma \in \Sigma_p$ can be written

$$\varphi_p(\sigma; z, t) = e^{i\lambda t} e^{-\frac{1}{2}|\lambda| \|z\|^2} \sum_{\mathbf{k}} \frac{1}{(n)_{\mathbf{k}}} \frac{1}{(p)_{\mathbf{k}}} a_{\mathbf{k}}(\sigma) \chi_{\mathbf{k}}(zz^*).$$

Define

$$\Omega = \{(\lambda, \alpha) \mid \lambda \in \mathbb{R},\ \alpha \in \mathbb{R}^n,\ \alpha_1 \geq \cdots \geq \alpha_n \geq 0\},$$

and, for $\omega = (\lambda, \alpha)$,

$$\varphi(\lambda, \alpha; z, t) = e^{i\lambda t} e^{-\frac{1}{2}|\lambda||z|^2} \int_{U(n)} e^{-\operatorname{tr}(uzz^* u^* \alpha)} \beta_n(du).$$

Finally define the map $T_p : \Sigma_p \to \Omega$ by,

$$\text{for } \sigma = (\lambda, \mathbf{m}) \in \Sigma_p^1,\ T_p(\sigma) = \big(\lambda, \frac{1}{p}|\lambda|\mathbf{m}\big),$$
$$\text{for } \sigma = \rho \in \Sigma_p^2,\ T_p(\sigma) = \big(0, \frac{1}{p}\rho\big).$$

Theorem 5.3. *If $(\sigma^{(p)})$ is a sequence with $\sigma^{(p)} \in \Sigma_p$ such that*

$$\lim_{p\to\infty} T_p(\sigma^{(p)}) = \omega = (\lambda, \alpha),$$

then

$$\lim_{p\to\infty} \varphi_p(\sigma^{(p)}; z, t) = \varphi(\lambda, \alpha; z, t),$$

uniformly on compact sets in H.

Proof. We will use

$$s_{\mathbf{k}}^*(x) = s_{\mathbf{k}}(x) + \text{terms of order } \leq |\mathbf{k}|,$$
$$(p)_{\mathbf{k}} \sim p^{|\mathbf{k}|} \quad (p \to \infty).$$

Hence,

$$\text{if } \lim_{p\to\infty} \frac{1}{p}|\lambda^{(p)}|\mathbf{m}^{(p)} = \alpha, \text{ then } \lim_{p\to\infty} \frac{1}{(p)_{\mathbf{k}}} |\lambda^{(p)}|^{|\mathbf{k}|} s_{\mathbf{k}}^*(\mathbf{m}^{(p)}) = s_{\mathbf{k}}(\alpha),$$
$$\text{if } \lim_{p\to\infty} \frac{1}{p}\rho^{(p)} = \alpha, \text{ then } \lim_{p\to\infty} \frac{1}{(p)_{\mathbf{k}}} s_{\mathbf{k}}(\rho^{(p)}) = s_{\mathbf{k}}(\alpha).$$

Therefore,

$$\text{if } \lim_{p\to\infty} T_p(\sigma^{(p)}) = (\lambda,\alpha), \text{ then } \lim_{p\to\infty} \frac{1}{(p)_{\mathbf{k}}} a_{\mathbf{k}}(\sigma^{(p)}) = s_{\mathbf{k}}(\alpha).$$

It follows that, by Lemma 3.4,

$$\begin{aligned}\lim_{p\to\infty} \varphi_p(\sigma^{(p)}; z,t) &= e^{i\lambda t} e^{-\frac{1}{2}|\lambda|\|z\|^2} \sum_{\mathbf{k}} (-1)^{|\mathbf{k}|} \frac{1}{(n)_{\mathbf{k}}} s_{\mathbf{k}}(\alpha)\chi_{\mathbf{k}}(zz^*) \\ &= \varphi(\lambda,\alpha; z,t).\end{aligned}$$

Corollary 5.4. *For $(\lambda,\alpha)\in\Omega$, the spherical function $\varphi(\lambda,\alpha;z,t)$ is of positive type.*

We will see that we obtain in that way all the spherical functions of positive type for the pair (G,K).

Theorem 5.5. *If $(\sigma^{(p)})$ is a sequence with $\sigma^{(p)}\in\Sigma_p$ such that*

$$\lim_{p\to\infty} \varphi_p(\sigma^{(p)}; z,t) = \varphi(z,t),$$

uniformly on compact sets in H, where φ is a continuous function on H, then the sequence $T_p(\sigma^{(p)})$ converges in Ω,

$$\lim_{p\to\infty} T_p(\sigma^{(p)}) = (\lambda,\alpha),$$

and

$$\varphi(z,t) = \varphi(\lambda,\alpha; z,t).$$

Proof. For $z=0$,

$$\varphi(0,t) = \lim_{p\to\infty} \varphi_p(\sigma^{(p)}; 0,t) = \lim_{p\to\infty} e^{i\lambda^{(p)}t},$$

uniformly on compact sets in $\mathbb{R}$, with $\sigma^{(p)} = (\lambda^{(p)}, \mathbf{m}^{(p)})$ if $\sigma^{(p)}\in\Sigma_p^1$, and $\lambda^{(p)}=0$ if $\sigma^{(p)}\in\Sigma_p^2$. Hence the sequence $\lambda^{(p)}$ converges, and $\varphi(0,t)=e^{i\lambda t}$, with $\lambda = \lim_{p\to\infty}\lambda^{(p)}$.

For $t=0$, $z=xE_{11}$, with $x\in\mathbb{R}$, put

$$\psi_p(x) = \varphi_p(\sigma^{(p)}; xE_{11}, 0).$$

The function ψ_p is continuous and of positive type on $\mathbb{R}$, with $\psi_p(0)=1$. By Bochner's Theorem, ψ_p is the Fourier transform of a probability measure ν_p on $\mathbb{R}$,

$$\psi_p(x) = \int_{\mathbb{R}} e^{ixy}\nu_p(dy).$$

As in Lemma 3.5, we write the expansion at order 4 of the function ψ_p as

$$\psi_p(x) = 1 - A_{n,p}(\sigma^{(p)})x^2 + B_{n,p}(\sigma^{(p)})x^4 + \cdots.$$

The moments of order 2 and 4 of the measure ν_p are given by

$$\mathfrak{M}_2(\nu_p) = 2\,A_{n,p}(\sigma), \quad \mathfrak{M}_4(\nu_p) = 24\,B_{n,p}(\sigma).$$

By Lemma 3.5 there is a constant D, which does not depend on p, such that

$$B_{n,p}(\sigma^{(p)}) \leq D\,\left(A_{n,p}(\sigma^{(p)})\right)^2.$$

If the sequence (ψ_p) converges uniformly on compact sets, then the sequence (ν_p) converges for the weak topology, hence is relatively compact. Therefore, by Lemma 4.3, there is a constant C such that

$$A_{n,p}(\sigma^{(p)}) \leq C.$$

This shows that the sequence $\big(T_p(\sigma^{(p)})\big)$ is relatively compact in Ω. By what has been said at the end of Section 4, this proves the statement. ∎

By Corollary 5.4, and Theorem 5.5 with Theorem 4.2 we obtain:

Corollary 5.6. *The spherical functions of positive type for the pair (G,K) are the functions $\varphi(\lambda,\alpha;z,t)$, with $(\lambda,\alpha)\in\Omega$.*

6. An Olshanski spherical pair with infinite rank. We consider now the following increasing sequences, for q fixed:

$$V(n) = M(n,n+q;\mathbb{C}), \quad H(n) = V(n)\times\mathbb{R},$$
$$K(n) = U(n)\times U(n+q), \quad G(n) = K(n)\ltimes H(n),$$

with the inductive limit

$$H = \bigcup_{n=1}^{\infty} H(n),$$

and the Olshanski spherical pair

$$G = \bigcup_{n=1}^{\infty} G(n), \quad K = \bigcup_{n=1}^{\infty} K(n).$$

a) *Spherical functions*

Theorem 6.1. *Let φ be a continuous function on H which is K-invariant. Then φ is spherical if and only there exist $\lambda\in\mathbb{C}$, and a continuous function Φ defined on $[0,\infty[$ with $\Phi(0)=1$, such that*

$$\varphi(z,t) = e^{\lambda t}\det\Phi(zz^*).$$

The matrix $\Phi(zz^*)$ is defined via the functional calculus. We will use an asymptotic property of the Haar measure of the unitary group $U(n)$. Let L_m denote the following subgroup of $U(n)$:

$$L_m = \Big\{ \begin{pmatrix} I_m & 0 \\ 0 & v \end{pmatrix} \mid v \in U(n-m) \Big\},$$

and $w_m \in U(2m)$:

$$w_m = \begin{pmatrix} 0 & -I_m \\ I_m & 0 \end{pmatrix}.$$

Lemma 6.2. *Let f be a continuous function on $U(\infty)$ which is L_m-biinvariant. Then*

$$\lim_{n\to\infty} \int_{U(n)} f(u)\beta_n(du) = \int_{U(m)\times U(m)} f(v_1 w_m v_2)\beta_m(dv_1)\beta_m(dv_2).$$

(See [Olshanski,1990], p.449-452, [Faraut,2006], Theorem 5.3, or [Faraut,2008], Proposition 3.3.)

Proof of Theorem 6.1. By Lemma 6.2, if $z, z' \in V(m)$, then

$$\lim_{n\to\infty} \int_{K(n)} \varphi\big(z + k\cdot z', t + t' + \mathrm{Im}(z|k\cdot z')\big)\alpha_n(dk)$$
$$= \int_{U(m)\times U(m)\times U(m+q)\times U(m+q)}$$
$$\varphi\big(z + u_1 w_m u_2 z' v_2^* w_{m+q}^* v_1^*, t + t' + \mathrm{Im}(z|u_1 w_m u_2 z' v_2^* w_{m+q}^* v_1^*)\big)$$
$$\beta_m(du_1)\beta_m(du_2)\beta_{m+q}(dv_1)\beta_{m+q}(dv_2).$$

Since $K(m)$ acts trivially on the space $w_m V(m) w_{m+q}^*$, the integrant does not depend on u_1, v_1, and since φ is K invariant, its does not depend on u_2, v_2 either. Furthermore the spaces $V(m)$ and $w_m V(m) w_{m+q}^*$ are orthogonal. Therefore

$$\lim_{n\to\infty} \int_{K(n)} \varphi\big(z + k\cdot z', t + t' + \mathrm{Im}(z|k\cdot z')\big)\alpha_n(dk) = \varphi(z + w_m z' w_{m+q}^*, t + t'),$$

Hence the function φ is spherical if and only if it satisfies the following multiplicative property: for $z, z' \in V(m)$,

$$\varphi((z + w_m z' w_{m+q}^*, t + t') = \varphi(z,t)\varphi(z',t'). \qquad \square$$

Such a function φ is an exponential with respect to t, and is completely determined by its restriction to $V(1)\times\mathbb{R}$ and this restriction is $U(q+1)$-invariant: for $z \in V(1) \simeq \mathbb{C}^{q+1}$, $t \in \mathbb{R}$,

$$\varphi(z,t) = e^{\lambda t}\Phi(\|z\|^2),$$

where Φ is a continuous function $[0, \infty[$, and $\lambda \in \mathbb{C}$. ∎

b) *The topological space* Ξ

Before introducing the spherical dual of the Olshanski spherical pair (G, K), we define

$$\Xi = \{\xi = (\alpha, \gamma) \mid \alpha = (\alpha_j),\ \alpha_j \geq 0, \sum_{j=1}^{\infty} \alpha_j < \infty,\ \gamma \geq 0\},$$

and consider on Ξ the following topology. To a continuous function ψ on $[0, \infty[$ one associates the function L_ψ on Ξ given by

$$L_\psi(\xi) = \gamma\psi(0) + \sum_{j=1}^{\infty} \alpha_j \psi(\alpha_j) \quad \big(\xi = (\alpha, \gamma)\big).$$

The topology on Ξ is the initial topology with respect to the functions L_ψ. Observe that $\psi \mapsto L_\psi(\xi)$ is a positive measure whose support is bounded. Hence Ξ is embeded in the set $\mathcal{M}([0, \infty[)$ of bounded positive measures on $[0, \infty[$. The topology on Ξ is induced by the weak topology on $\mathcal{M}([0, \infty[)$. The subset of the $\xi = (\alpha, \gamma)$ for which only finitely many α_j are non zero, and $\gamma = 0$ is dense in Ξ. Furthermore, the set Ξ is closed in $\mathcal{M}([0, \infty[)$ ([Rabaoui,2008], Theorem 4.3).

Lemma 6.3. *For $C > 0$, the set*

$$\Xi_C = \{\xi = (\alpha, \gamma) \mid \sum_{j=1}^{\infty} \alpha_j + \gamma \leq C\}$$

is compact.

Proof. Since, for $\psi \equiv 1$, $L_\psi(\xi) = \sum \alpha_j + \gamma$, the set Ξ_C is closed. Seen as a subset of $\mathcal{M}([0, \infty[)$, Ξ_C is a set of measures with supports in $[0, C]$, and total measure $\leq C$, therefore relatively compact. ∎

The following Pólya type function will play an important role in this section:

$$\Phi(\xi; x) = e^{-\gamma x} \prod_{j=1}^{\infty} \frac{1}{1 + \alpha_j x} \quad \big(\xi = (\alpha, \gamma)\big).$$

The function Φ is continuous on $\Xi \times [0, \infty[$. We will study its Taylor expansion at 0. For that we will define an algebra morphism $f \mapsto \tilde{f}$ from the algebra Λ of symmetric functions into the space $\mathcal{C}(\Xi)$ of continuous functions on Ξ. The Newton power sums p_m generate the algebra Λ, hence this morphism is well defined as soon as their images $\tilde{p}_m$ are given: put

$$\tilde{p}_1(\xi) = \gamma + \sum_{j=1}^{\infty} \alpha_j,$$

and, for $m \geq 2$,

$$\tilde{p}_m(\xi) = \sum_{j=1}^{\infty} \alpha_j^m.$$

Proposition 6.4. (i) *For* $\xi \in \Xi$, $x \geq 0$,

$$\Phi(\xi; x) = \sum_{m=1}^{\infty} \tilde{h}_m(\xi)(-x)^m.$$

(ii) *More generally, for* $x = (x_1, \ldots, x_n)$ *with* $x_j \geq 0$,

$$\prod_{j=1}^{n} \Phi(\xi; x_j) = \sum_{\mathbf{m}} \tilde{s}_{\mathbf{m}}(\xi) s_{\mathbf{m}}(-x).$$

Statement (ii) can be written: for $y \in Herm(n, \mathbb{C})$, semi-positive definite,

$$\det \Phi(\xi; y) = \sum_{\mathbf{m}} \tilde{s}_{\mathbf{m}}(\xi) \chi_{\mathbf{m}}(-y).$$

Proof. Let us compute the logarithmic derivative of $\Phi(\xi, x)$ with respect to x:

$$\begin{aligned}
-\frac{d}{dx} \log \Phi(\xi; x) &= \gamma + \sum_{j=1}^{\infty} \frac{\alpha_j}{1 + \alpha_j x} \\
&= \gamma + \sum_{j=1}^{\infty} \alpha_j \Big(\sum_{m=0}^{\infty} (-1)^m \alpha_j^m x^m \Big) \\
&= \gamma + \sum_{m=0}^{\infty} (-1)^m \Big(\sum_{j=1}^{\infty} \alpha_j^{m+1} \Big) x^m \\
&= \sum_{m=0}^{\infty} \tilde{p}_{m+1}(\xi)(-x)^m.
\end{aligned}$$

It means that, as a function of ξ, $-\frac{d}{dx} \log \Phi(\xi, x)$ is the image, by the morphism $f \mapsto \tilde{f}$, of the function

$$\sum_{m=0}^{\infty} p_{m+1}(z)(-x)^m.$$

One observes that

$$\sum_{m=0}^{\infty} p_{m+1}(z)(-x)^m = -\frac{d}{dx} \log H(z, -x),$$

with

$$H(z,-x)=\prod_i \frac{1}{1+z_i x}=\sum_{m=1}^{\infty} h_m(z)(-x)^m.$$

By using the morphism property of the map $f\mapsto \tilde f$, one obtains

$$\Phi(\xi;x)=\sum_{m=1}^{\infty}\tilde h_m(\xi)(-x)^m.$$

Formula (ii) follows from (i) and the identity:

$$\prod_{j=1}^{n} H(z,-x_j)=\sum_{\mathbf{m}} s_{\mathbf{m}}(z)s_{\mathbf{m}}(-x) \qquad \square.$$

c) *Asymptotics of spherical functions*

As we saw at the end of Section 3 the spherical dual Σ_n for the Gelfand pair $\big(G(n),K(n)\big)$ can be described as $\Sigma_n^1\cup\Sigma_n^2$, with

$$\begin{aligned}\Sigma_n^1&=\{(\lambda,\mathbf{m})\mid \lambda\in\mathbb{R}^*,\ \mathbf{m}\text{ is a partition},\ \ell(\mathbf{m})\le n\},\\ \Sigma_n^2&=\{\rho\in\mathbb{R}^n\mid \rho_1\ge\cdots\ge\rho_n\ge 0\}.\end{aligned}$$

For $\omega=(\lambda,\xi)\in\Omega:=\mathbb{R}\times\Xi$, $(z,t)\in H$, define

$$\varphi(\omega;z,t)=e^{i\lambda t}e^{-\frac12|\lambda|\|z\|^2}\det\Phi(\xi;zz^*),$$

where Φ is the Pólya type function we have introduced.

For every n define the map

$$\mathcal{T}_n:\Sigma_n\to\Omega,\ \sigma\mapsto\omega=(\lambda,\xi)=(\lambda,\alpha,\gamma),$$

with, if $\sigma=(\lambda,\mathbf{m})\in\Sigma_n^1$,

$$\alpha_j=\frac{1}{n^2}|\lambda|m_j\ (1\le j\le n),\ \alpha_j=0\ (j>n),\ \gamma=0,$$

and, if $\sigma=\rho\in\Sigma_n^2$,

$$\lambda=0,\ \alpha_j=\frac{1}{n^2}\rho_j\ (1\le j\le n),\ \alpha_j=0\ (j>n),\ \gamma=0.$$

Theorem 6.5. *Let $(\sigma^{(n)})$ be a sequence with $\sigma^{(n)}\in\Sigma_n$. Assume that*

$$\lim_{n\to\infty}\mathcal{T}_n(\sigma^{(n)})=\omega,$$

for the topology of Ω. *Then*

$$\lim_{n\to\infty} \varphi_n(\sigma^{(n)}; z, t) = \varphi(\omega; z, t),$$

uniformly on compact sets in H.

Before giving the proof we need some preliminaries. Let $f^* \in \Lambda^*$ be a shifted symmetric function (see [Okounkov-Olshanski,1998a] for the definition), $\ell = \deg f^*$, and f the homogeneous part of degree ℓ of f^*. For $\sigma \in \Sigma_n$, define $Q(f^*, \sigma)$ as follows:

$$\text{if } \sigma = (\lambda, \mathbf{m}) \in \Sigma_n^1, \text{ then } Q(f^*, \sigma) = |\lambda|^\ell f^*(\mathbf{m}),$$
$$\text{if } \sigma = \rho \in \Sigma_n^2, \text{ then } Q(f^*, \sigma) = f(\rho).$$

With the notation introduced at the end of Section 3, $a_{\mathbf{k}}(\sigma) = Q(s^*_{\mathbf{k}}, \sigma)$.

Proposition 6.6. *Let* $(\sigma^{(n)})$ *be a sequence with* $\sigma^{(n)} \in \Sigma_n$. *Assume that*

$$\lim_{n\to\infty} \mathcal{T}_n(\sigma^{(n)}) = \omega = (\lambda, \xi),$$

for the topology of Ω. *Then, for every* $f^* \in \Lambda^*$ *of degree* ℓ,

$$\lim_{n\to\infty} \frac{1}{n^{2\ell}} Q(f^*, \sigma^{(n)}) = \tilde{f}(\xi).$$

Proof. By the definition of $\mathcal{T}_n$ and the topology of Ξ, for every continuous function ψ on $[0, \infty[$,

$$\lim_{n\to\infty} \sum_{j=1}^n \frac{\mu_j^{(n)}}{n^2} \psi\Big(\frac{\mu_j^{(n)}}{n^2}\Big) = \sum_{j=1}^\infty \alpha_j \psi(\alpha_j) + \gamma\psi(0),$$

with

$$\mu_j = |\lambda| m_j, \text{ if } \sigma = (\lambda, \mathbf{m}) \in \Sigma_n^1,$$
$$\mu_j = \rho_j, \text{ if } \sigma = \rho \in \Sigma_n^2,$$
$$\omega = (\lambda, \alpha, \gamma).$$

The shifted power functions p^*_ℓ

$$p^*_\ell(x) = \sum_i (x_i - i)^\ell - (-i)^\ell,$$

generate the algebra Λ^*. Hence it suffices to prove Proposition 6.6 in case of $f^* = p^*_\ell$. For $\ell = 1$,

$$p^*_1(x) = \sum_i x_i, \ \tilde{p}_1(\alpha, \gamma) = \sum_{j=1}^\infty \alpha_j + \gamma.$$

By taking $\psi \equiv 1$, one gets from the assumption

$$\lim_{n\to\infty} \frac{1}{n^2} \sum_{j=1}^{n} \mu_j^{(n)} = \sum_{j+1}^{\infty} \alpha_j + \gamma,$$

or

$$\lim_{n\infty} \frac{1}{n^2} Q(p_1^*, \sigma^{(n)}) = \tilde{p}_1(\alpha, \gamma).$$

Assume $\ell \geq 2$, and expand p_ℓ^*:

$$p_\ell^*(x) = \sum_i (x-i)^\ell - (-i)^\ell = \sum_i \Big(\sum_{k=1}^{\ell} \binom{\ell}{k} x_i^k (-i)^{\ell-k} \Big).$$

Hence, if $\sigma = (\lambda, \mathbf{m}) \in \Sigma_n^1$,

$$\frac{1}{n^{2\ell}} Q(p_\ell^*, \sigma) = \sum_{j=1}^{n} \Big(\frac{|\lambda| m_j}{n^2} \Big)^\ell + \sum_{k=1}^{\ell-1} \binom{\ell}{k} \sum_{j=1}^{n} \Big(\frac{|\lambda| m_j}{n^2} \Big)^k \Big(\frac{-j}{n^2} \Big)^{\ell-k},$$

and, if $\sigma = \rho \in \Sigma^2$,

$$\frac{1}{n^{2\ell}} Q(p_\ell^*, \sigma) = \sum_{j=1}^{n} \Big(\frac{\rho_j}{n^2} \Big)^\ell.$$

By taking $\psi(s) = s^{k-1}$ $(k \geq 2)$ one gets

$$\lim_{n\to\infty} \sum_{j=1}^{n} \Big(\frac{\mu_j^{(n)}}{n^2} \Big)^k = \tilde{p}_\ell(\alpha, \gamma) = \sum_{j=1}^{\infty} \alpha_j{}^k.$$

It follows that, for $k < \ell$,

$$\Big| \sum_{j=1}^{n} \Big(\frac{\mu_j^{(n)}}{n^2} \Big)^k \Big(\frac{-j}{n^2} \Big)^{\ell-k} \Big| \leq \frac{1}{n} \sum_{j=1}^{n} \Big(\frac{\mu_j^{(n)}}{n^2} \Big)^k = \mathcal{O}\Big(\frac{1}{n} \Big),$$

and finally

$$\lim_{n\to\infty} \frac{1}{n^{2\ell}} Q(p_\ell^*, \sigma^{(n)}) = \tilde{p}_\ell(\alpha, \gamma). \qquad \square$$

Proof of Theorem 6.5. Recall that, for $\sigma \in \Sigma_n$,

$$\varphi_n(\sigma; z, t) = e^{i\lambda t} e^{-\frac{1}{2}|\lambda| \|z\|^2} \sum_{\mathbf{k}} (-1)^{|\mathbf{k}|} \frac{1}{(n)_{\mathbf{k}}} \frac{1}{(n+q)_{\mathbf{k}}} a_{\mathbf{k}}(\sigma) \chi_{\mathbf{k}}(zz^*),$$

and that $a_{\mathbf{k}}(\sigma) = Q(s^*_{\mathbf{k}}, \sigma)$. By Proposition 6.6,

$$\lim_{n\to\infty} \frac{1}{n^{2|\mathbf{k}|}} a_{\mathbf{k}}(\sigma^{(n)}) = \tilde{s}_{\mathbf{k}}(\alpha, \gamma).$$

Since $(n)_{\mathbf{k}}(n+q)_{\mathbf{k}} \sim n^{2|\mathbf{k}|}$ $(n \to \infty)$, it follows, by Lemma 3.4, that

$$\begin{aligned}\lim_{n\to\infty} \sum_{\mathbf{k}} (-1)^{|\mathbf{k}|} \frac{1}{(n)_{\mathbf{k}}} \frac{1}{(n+q)_{\mathbf{k}}} a_{\mathbf{k}}(\sigma^{(n)}) \chi_{\mathbf{k}}(zz^*) \\ = \sum_{\mathbf{k}} (-1)^{|\mathbf{k}|} \tilde{s}_{\mathbf{k}}(\alpha, \gamma) \chi_{\mathbf{k}}(zz^*) = \det \Phi(\alpha, \gamma; zz^*),\end{aligned}$$

by Proposition 6.4. We have proven

$$\lim_{n\to\infty} \varphi_n(\sigma^{(n)}; z, t) = \varphi(\omega; z, t). \qquad \square$$

By using the same method it is possible to study asymptotics of the spherical functions as n and p go to infinity with $\lim \frac{p}{n} = c$ $(1 \leq c < \infty)$.

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