Mathematical English (a brief summary)

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Arithmetic

Integers

0	zero	10	ten	20	twenty
1	one	11	eleven	30	thirty
2	two	12	twelve	40	forty
3	three	13	thirteen	50	fifty
4	four	14	fourteen	60	sixty
5	five	15	fifteen	70	seventy
6	six	16	sixteen	80	eighty
7	seven	17	seventeen	90	ninety
8	eight	18	eighteen	100	one hundred
9	nine	19	nineteen	1000	one thousand

−245	minus two hundred and forty-five
22 731	twenty-two thousand seven hundred and thirty-one
1 000 000	one million
56 000 000	fifty-six million
1 000 000 000	one billion [US usage, now universal]
7 000 000 000	seven billion [US usage, now universal]
1 000 000 000 000	one trillion [US usage, now universal]
3 000 000 000 000	three trillion [US usage, now universal]

Fractions [= Rational Numbers]

$\frac{1}{2}$	one half	$\frac{3}{8}$	three eighths
$\frac{1}{3}$	one third	$\frac{26}{9}$	twenty-six ninths
$\frac{1}{4}$	one quarter [= one fourth]	$-\frac{5}{34}$	minus five thirty-fourths
$\frac{1}{5}$	one fifth	$2\frac{3}{7}$	two and three sevenths
$-\frac{1}{17}$	minus one seventeenth		

Real Numbers

-0.067	minus nought point zero six seven
81.59	eighty-one point five nine
$-2.3 \cdot 10^6$	minus two point three times ten to the six
[$= -2\,300\,000$	minus two million three hundred thousand]
$4 \cdot 10^{-3}$	four times ten to the minus three
[$= 0.004 = 4/1000$	four thousandths]
π [$= 3.14159\ldots$]	pi [pronounced as ‘pie’]
e [$= 2.71828\ldots$]	e [base of the natural logarithm]

Complex Numbers

i	`i`
$3+4i$	`three plus four i`
$1-2i$	`one minus two i`
$\overline{1-2i}=1+2i$	`the complex conjugate of one minus two i equals one plus two i`

The real part and the imaginary part of $3+4i$ are equal, respectively, to 3 and 4.

Basic arithmetic operations

Addition:	$3+5=8$	`three plus five equals [= is equal to] eight`
Subtraction:	$3-5=-2$	`three minus five equals [= ...] minus two`
Multiplication:	$3\cdot 5=15$	`three times five equals [= ...] fifteen`
Division:	$3/5=0.6$	`three divided by five equals [= ...] zero point six`

$(2-3)\cdot 6+1=-5$	`two minus three in brackets times six plus one equals minus five`
$\frac{1-3}{2+4} = -1/3$	`one minus three over two plus four equals minus one third`
$4!\ [= 1\cdot 2\cdot 3\cdot 4]$	`four factorial`

Exponentiation, Roots

5^2	$[=5\cdot 5=25]$	`five squared`
5^3	$[=5\cdot 5\cdot 5=125]$	`five cubed`
5^4	$[=5\cdot 5\cdot 5\cdot 5=625]$	`five to the (power of) four`
5^{-1}	$[=1/5=0.2]$	`five to the minus one`
5^{-2}	$[=1/5^2=0.04]$	`five to the minus two`
$\sqrt{3}$	$[=1.73205\ldots]$	`the square root of three`
$\sqrt[3]{64}$	$[=4]$	`the cube root of sixty four`
$\sqrt[5]{32}$	$[=2]$	`the fifth root of thirty two`

In the complex domain the notation $\sqrt[n]{a}$ is ambiguous, since any non-zero complex number has n different n-th roots. For example, $\sqrt[4]{-4}$ has four possible values: $\pm 1 \pm i$ (with all possible combinations of signs).

$(1+2)^{2+2}$	`one plus two, all to the power of two plus two`
$e^{\pi i}=-1$	`e to the (power of) pi i equals minus one`

Divisibility

The multiples of a positive integer a are the numbers $a, 2a, 3a, 4a, \ldots$. If b is a multiple of a, we also say that a divides b, or that a is a divisor of b (notation: $a \mid b$). This is equivalent to $\frac{b}{a}$ being an integer.

Division with remainder

If a, b are arbitrary positive integers, we can divide b by a, in general, only with a remainder. For example, 7 lies between the following two consecutive multiples of 3:

$$2 \cdot 3 = 6 < 7 < 3 \cdot 3 = 9, \qquad 7 = 2 \cdot 3 + 1 \qquad \left(\iff \frac{7}{3} = 2 + \frac{1}{3} \right).$$

In general, if qa is the largest multiple of a which is less than or equal to b, then

$$b = qa + r, \qquad r = 0, 1, \ldots, a - 1.$$

The integer q (resp., r) is the *quotient* (resp., the *remainder*) of the division of b by a.

Euclid's algorithm

This algorithm computes the *greatest common divisor* (notation: $(a, b) = \gcd(a, b)$) of two positive integers a, b.

It proceeds by replacing the pair a, b (say, with $a \le b$) by r, a, where r is the remainder of the division of b by a. This procedure, which preserves the gcd, is repeated until we arrive at $r = 0$.

Example. Compute $\gcd(12, 44)$.

$$\begin{aligned} 44 &= 3 \cdot 12 + 8 \\ 12 &= 1 \cdot 8 + 4 \qquad \gcd(12, 44) = \gcd(8, 12) = \gcd(4, 8) = \gcd(0, 4) = 4. \\ 8 &= 2 \cdot 4 + 0 \end{aligned}$$

This calculation allows us to write the fraction $\frac{44}{12}$ in its lowest terms, and also as a continued fraction:

$$\frac{44}{12} = \frac{44/4}{12/4} = \frac{11}{3} = 3 + \cfrac{1}{1 + \cfrac{1}{2}}.$$

If $\gcd(a, b) = 1$, we say that a and b are *relatively prime*.

add additionner
algorithm algorithme
  Euclid's algorithm algorithme de division euclidienne
bracket parenthèse
  left bracket parenthèse à gauche
  right bracket parenthèse à droite
  curly bracket accolade
denominator denominateur

difference différence
divide diviser
divisibility divisibilité
divisor diviseur
exponent exposant
factorial factoriel
fraction fraction
  continued fraction fraction continue
gcd [= greatest common divisor] pgcd [= plus grand commun diviseur]
  lcm [= least common multiple] ppcm [= plus petit commun multiple]
infinity l'infini
iterate itérer
iteration itération
multiple multiple
multiply multiplier
number nombre
  even number nombre pair
  odd number nombre impair
numerator numerateur
pair couple
  pairwise deux à deux
power puissance
product produit
quotient quotient
ratio rapport; raison
rational rationnel(le)
  irrational irrationnel(le)
relatively prime premiers entre eux
remainder reste
root racine
sum somme
subtract soustraire

Algebra

Algebraic Expressions

$A = a^2$	capital a equals small a squared
$a = x + y$	a equals x plus y
$b = x - y$	b equals x minus y
$c = x \cdot y \cdot z$	c equals x times y times z
$c = xyz$	c equals x y z
$(x + y)z + xy$	x plus y in brackets times z plus x y
$x^2 + y^3 + z^5$	x squared plus y cubed plus z to the (power of) five
$x^n + y^n = z^n$	x to the n plus y to the n equals z to the n
$(x - y)^{3m}$	x minus y in brackets to the (power of) three m
	x minus y, all to the (power of) three m
$2^x 3^y$	two to the x times three to the y
$ax^2 + bx + c$	a x squared plus b x plus c
$\sqrt{x} + \sqrt[3]{y}$	the square root of x plus the cube root of y
$\sqrt[n]{x + y}$	the n-th root of x plus y
$\frac{a+b}{c-d}$	a plus b over c minus d
$\binom{n}{m}$	(the binomial coefficient) n over m

Indices

x_0	x zero; x nought
$x_1 + y_i$	x one plus y i
R_{ij}	(capital) R (subscript) i j; (capital) R lower i j
M^k_{ij}	(capital) M upper k lower i j;
	(capital) M superscript k subscript i j
$\sum_{i=0}^{n} a_i x^i$	sum of a i x to the i for i from nought [= zero] to n;
	sum over i (ranging) from zero to n of a i (times) x to the i
$\prod_{m=1}^{\infty} b_m$	product of b m for m from one to infinity;
	product over m (ranging) from one to infinity of b m
$\sum_{j=1}^{n} a_{ij} b_{jk}$	sum of a i j times b j k for j from one to n;
	sum over j (ranging) from one to n of a i j times b j k
$\sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}$	sum of n over i x to the i y to the n minus i for i
	from nought [= zero] to n

Matrices

column colonne
 column vector vecteur colonne
determinant déterminant
index (pl. indices) indice
matrix matrice
 matrix entry (pl. entries) coefficient d'une matrice
 $m \times n$ **matrix** [m **by** n **matrix**] matrice à m lignes et n colonnes
multi-index multiindice
row ligne
 row vector vecteur ligne
square carré
 square matrix matrice carrée

Inequalities

$x > y$	`x is greater than y`
$x \geq y$	`x is greater (than) or equal to y`
$x < y$	`x is smaller than y`
$x \leq y$	`x is smaller (than) or equal to y`
$x > 0$	`x is positive`
$x \geq 0$	`x is positive or zero; x is non-negative`
$x < 0$	`x is negative`
$x \leq 0$	`x is negative or zero`

The French terminology is different!

$x > y$	`x est strictement plus grand que y`
$x \geq y$	`x est supérieur ou égal à y`
$x < y$	`x est strictement plus petit que y`
$x \leq y$	`x est inférieur ou égal à y`
$x > 0$	`x est strictement positif`
$x \geq 0$	`x est positif ou nul`
$x < 0$	`x est strictement négatif`
$x \leq 0$	`x est négatif ou nul`

Polynomial equations

A polynomial equation of degree $n \geq 1$ with complex coefficients

$$f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n = 0 \qquad (a_0 \neq 0)$$

has n complex solutions (= roots), provided that they are counted with multiplicities.

For example, a quadratic equation

$$ax^2 + bx + c = 0 \qquad (a \neq 0)$$

can be solved by completing the square, *i.e.*, by rewriting the L.H.S. as

$$a(x + \text{constant})^2 + \text{ another constant.}$$

This leads to an equivalent equation

$$a\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a},$$

whose solutions are

$$x_{1,2} = \frac{-b \pm \sqrt{\Delta}}{2a},$$

where $\Delta = b^2 - 4ac$ $(= a^2(x_1 - x_2)^2)$ is the discriminant of the original equation. More precisely,

$$ax^2 + bx + c = a(x - x_1)(x - x_2).$$

If all coefficients a, b, c are real, then the sign of Δ plays a crucial rôle:

if $\Delta = 0$, then $x_1 = x_2$ $(= -b/2a)$ is a double root;
if $\Delta > 0$, then $x_1 \neq x_2$ are both real;
if $\Delta < 0$, then $x_1 = \overline{x_2}$ are complex conjugates of each other (and non-real).

coefficient coefficient
degree degré
discriminant discriminant
equation équation
L.H.S. [= left hand side] terme de gauche
  R.H.S. [= right hand side] terme de droite
polynomial *adj.* polynomial(e)
polynomial *n.* polynôme
provided that à condition que
root racine
  simple root racine simple
  double root racine double
  triple root racine triple
  multiple root racine multiple
  root of multiplicity m racine de multiplicité m

solution solution
solve résoudre

Congruences

Two integers a, b are *congruent* modulo a positive integer m if they have the same remainder when divided by m (equivalently, if their difference $a - b$ is a multiple of m).

$a \equiv b \pmod{m}$	`a is congruent to b modulo m`
$a \equiv b \ (m)$	

Some people use the following, slightly horrible, notation: $a = b\,[m]$.

Fermat's Little Theorem. *If p is a prime number and a is an integer, then $a^p \equiv a \pmod{p}$. In other words, $a^p - a$ is always divisible by p.*

Chinese Remainder Theorem. *If $m_1, \ldots, m_k$ are pairwise relatively prime integers, then the system of congruences*

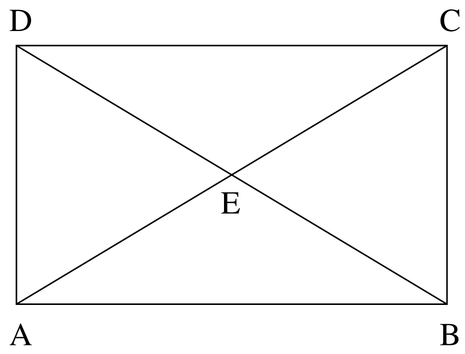
$$x \equiv a_1 \pmod{m_1} \qquad \cdots \qquad x \equiv a_k \pmod{m_k}$$

has a unique solution modulo $m_1 \cdots m_k$, for any integers $a_1, \ldots, a_k$.

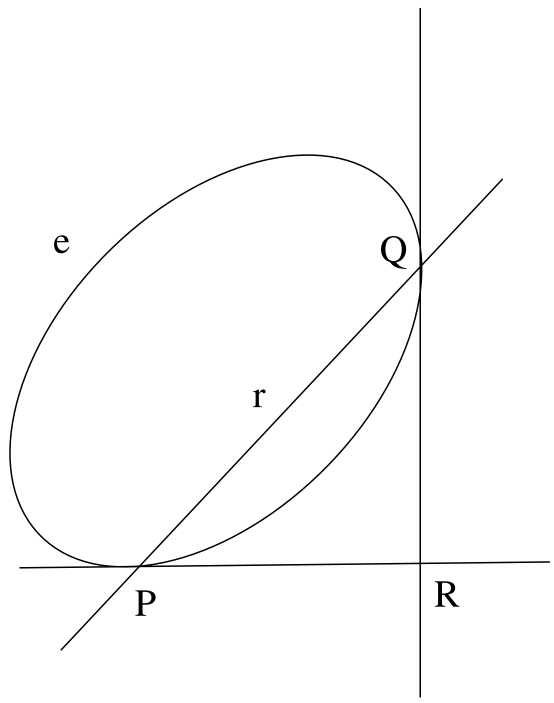
The definite article (and its absence)

measure theory	théorie de la mesure
number theory	théorie des nombres
Chapter one	le chapitre un
Equation (7)	l'équation (7)
Harnack's inequality	l'inégalité de Harnack
the Harnack inequality	
the Riemann hypothesis	l'hypothèse de Riemann
the Poincaré conjecture	la conjecture de Poincaré
Minkowski's theorem	le théorème de Minkowski
the Minkowski theorem	
the Dirac delta function	la fonction delta de Dirac
Dirac's delta function	
the delta function	la fonction delta

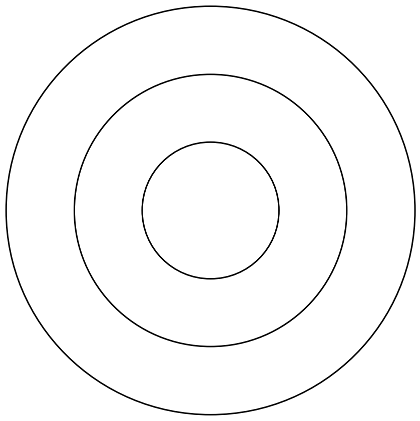
Geometry



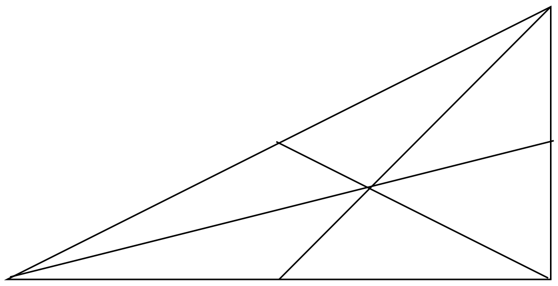
Let E be the intersection of the diagonals of the rectangle $ABCD$. The lines (AB) and (CD) are parallel to each other (and similarly for (BC) and (DA)). We can see on this picture several *acute angles:* $\angle EAD$, $\angle EAB$, $\angle EBA$, $\angle AED$, $\angle BEC \ldots$; *right angles:* $\angle ABC$, $\angle BCD$, $\angle CDA$, $\angle DAB$ and *obtuse angles:* $\angle AEB$, $\angle CED$.



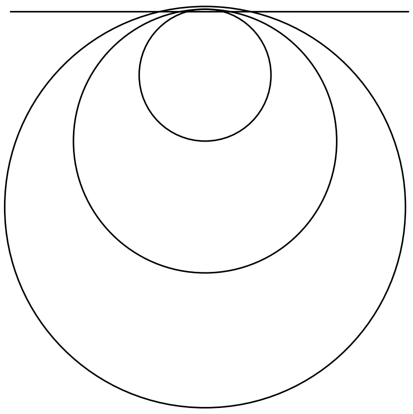
Let P and Q be two points lying on an ellipse e. Denote by R the intersection point of the respective tangent lines to e at P and Q. The line r passing through P and Q is called *the polar of the point R w.r.t. the ellipse e.*



Here we see three concentric circles with respective radii equal to 1, 2 and 3.



If we draw a line through each vertex of a given triangle and the midpoint of the opposite side, we obtain three lines which intersect at the barycentre (= the centre of gravity) of the triangle.



Above, three circles have a common tangent at their (unique) intersection point.

Euler's Formula

Let P be a convex polyhedron. Euler's formula asserts that

$$V - E + F = 2,$$

$$\begin{aligned} V &= \text{the number of vertices of } P, \\ E &= \text{the number of edges of } P, \\ F &= \text{the number of faces of } P. \end{aligned}$$

Exercise. *Use this formula to classify regular polyhedra (there are precisely five of them: tetrahedron, cube, octahedron, dodecahedron and icosahedron).*

For example, an icosahedron has 20 faces, 30 edges and 12 vertices. Each face is an isosceles triangle, each edge belongs to two faces and there are 5 faces meeting at each vertex. The midpoints of its faces form a dual regular polyhedron, in this case a dodecahedron, which has 12 faces (regular pentagons), 30 edges and 20 vertices (each of them belonging to 3 faces).

angle angle
  acute angle angle aigu
  obtuse angle angle obtus
  right angle angle droit
area aire
axis (pl. axes) axe
  coordinate axis axe de coordonnées
  horizontal axis axe horisontal
  vertical axis axe vertical
centre [US: center] centre
circle cercle
colinear (points) (points) alignés
conic (section) (section) conique
cone cône
convex convexe
cube cube
curve courbe
dimension dimension
distance distance
dodecahedron dodecaèdre
edge arête
ellipse ellipse
ellipsoid ellipsoïde
face face
hexagon hexagone
hyperbola hyperbole
hyperboloid hyperboloïde

one-sheet (two-sheet) hyperboloid hyperboloïde à une nappe (à deux nappes)
icosahedron icosaèdre
intersect intersecter
intersection intersection
lattice réseau
lettuce laitue
length longeur
line droite
midpoint of milieu de
octahedron octaèdre
orthogonal; perpendicular orthogonal(e); perpendiculaire
parabola parabole
parallel parallèl(e)
parallelogram parallélogramme
pass through passer par
pentagon pentagone
plane plan
point point
(regular) polygon polygone (régulier)
(regular) polyhedron (pl. polyhedra) polyèdre (régulier)
projection projection
central projection projection conique; projection centrale
orthogonal projection projection orthogonale
parallel projection projection parallèle
quadrilateral quadrilatère
radius (pl. radii) rayon
rectangle rectangle
rectangular rectangulaire
rotation rotation
side côté
slope pente
sphere sphère
square carré
square lattice réseau carré
surface surface
tangent to tangent(e) à
tangent line droite tangente
tangent hyper(plane) (hyper)plan tangent
tetrahedron tetraèdre
triangle triangle
equilateral triangle triangle équilatéral
isosceles triangle triangle isocèle
right-angled triangle triangle rectangle
vertex sommet

Linear Algebra

basis (pl. bases) base
  change of basis changement de base
bilinear form forme bilinéaire
coordinate coordonnée
(non-)degenerate (non) dégénéré(e)
dimension dimension
  codimension codimension
  finite dimension dimension finie
  infinite dimension dimension infinie
dual space espace dual
eigenvalue valeur propre
  eigenvector vecteur propre
(hyper)plane (hyper)plan
image image
isometry isométrie
kernel noyau
linear linéaire
  linear form forme linéaire
  linear map application linéaire
  linearly dependent liés; linéairement dépendants
  linearly independent libres; linéairement indépendants
multi-linear form forme multilinéaire
origin origine
orthogonal; perpendicular orthogonal(e); perpendiculaire
  orthogonal complement supplémentaire orthogonal
  orthogonal matrix matrice orthogonale
(orthogonal) projection projection (orthogonale)
quadratic form forme quadratique
reflection réflexion
represent représenter
rotation rotation
scalar scalaire
  scalar product produit scalaire
subspace sous-espace
(direct) sum somme (directe)
skew-symmetric anti-symétrique
symmetric symétrique
trilinear form forme trilinéaire
vector vecteur
  vector space espace vectoriel
  vector subspace sous-espace vectoriel
  vector space of dimension n espace vectoriel de dimension n

Mathematical arguments

Set theory

$x \in A$	`x is an element of A; x lies in A;` `x belongs to A; x is in A`
$x \notin A$	`x is not an element of A; x does not lie in A;` `x does not belong to A; x is not in A`
$x, y \in A$	`(both) x and y are elements of A; ...lie in A;` `...belong to A; ...are in A`
$x, y \notin A$	`(neither) x nor y is an element of A; ...lies in A;` `...belongs to A; ...is in A`
$\emptyset$	`the empty set (= set with no elements)`
$A = \emptyset$	`A is an empty set`
$A \neq \emptyset$	`A is non-empty`
$A \cup B$	`the union of (the sets) A and B; A union B`
$A \cap B$	`the intersection of (the sets) A and B; A intersection B`
$A \times B$	`the product of (the sets) A and B; A times B`
$A \cap B = \emptyset$	A `is disjoint from` B`; the intersection of A and B is empty`
$\{x \mid \ldots\}$	`the set of all x such that ...`
$\mathbf{C}$	`the set of all complex numbers`
$\mathbf{Q}$	`the set of all rational numbers`
$\mathbf{R}$	`the set of all real numbers`

$A \cup B$ contains those elements that belong to A or to B (or to both).
$A \cap B$ contains those elements that belong to both A and B.
$A \times B$ contains the ordered pairs (a, b), where a (resp., b) belongs to A (resp., to B).
$A^n = \underbrace{A \times \cdots \times A}_{n \text{ times}}$ contains all ordered n-tuples of elements of A.

belong to appartenir à
disjoint from disjoint de
element élément
empty vide
  non-empty non vide
intersection intersection
inverse l'inverse
  the inverse map to f l'application réciproque de f
  the inverse of f l'inverse de f
map application
  bijective map application bijective
  injective map application injective
  surjective map application surjective
pair couple

ordered pair couple ordonné
triple triplet
quadruple quadruplet
n**-tuple** n-uplet
relation relation
equivalence relation relation d'équivalence
set ensemble
finite set ensemble fini
infinite set ensemble infini
union réunion

Logic

$S \vee T$	`S or T`
$S \wedge T$	`S and T`
$S \implies T$	`S implies T; if S then T`
$S \iff T$	`S is equivalent to T; S iff T`
$\neg S$	`not S`
$\forall\, x \in A \ldots$	`for each [= for every] x in A ...`
$\exists\, x \in A \ldots$	`there exists [= there is] an x in A (such that) ...`
$\exists!\, x \in A \ldots$	`there exists [= there is] a unique x in A (such that) ...`
$\nexists\, x \in A \ldots$	`there is no x in A (such that)...`

$x > 0 \wedge y > 0 \Longrightarrow x + y > 0$	if both x and y are positive, so is $x+y$
$\nexists\, x \in \mathbf{Q} \quad x^2 = 2$	no rational number has a square equal to two
$\forall\, x \in \mathbf{R}\, \exists\, y \in \mathbf{Q} \quad \lvert x - y \rvert < 2/3$	for every real number x there exists a rational number y such that the absolute value of x minus y is smaller than two thirds

Exercise. Read out the following statements.

$$x \in A \cap B \iff (x \in A \wedge x \in B), \qquad x \in A \cup B \iff (x \in A \vee x \in B),$$
$$\forall\, x \in \mathbf{R} \quad x^2 \geq 0, \qquad \neg\exists\, x \in \mathbf{R} \quad x^2 < 0, \qquad \forall\, y \in \mathbf{C}\, \exists\, z \in \mathbf{C} \quad y = z^2.$$

Basic arguments

It follows from ... that ...

We deduce from ... that ...

Conversely, ... implies that ...

Equality (1) holds, by Proposition 2.

By definition, ...

The following statements are equivalent.

Thanks to ..., the properties ... and ... of ... are equivalent to each other.

... has the following properties.

Theorem 1 holds unconditionally.

This result is conditional on Axiom A.

... is an immediate consequence of Theorem 3.

Note that ... is well-defined, since ...

As ... satisfies ..., formula (1) can be simplified as follows.

We conclude (the argument) by combining inequalities (2) and (3).

(Let us) denote by X the set of all ...

Let X be the set of all ...

Recall that ..., by assumption.

It is enough to show that ...

We are reduced to proving that ...

The main idea is as follows.

We argue by contradiction. Assume that ... exists.

The formal argument proceeds in several steps.

Consider first the special case when ...

The assumptions ... and ... are independent (of each other), since ...

..., which proves the required claim.

We use induction on n to show that ...

On the other hand, ...

..., which means that ...

In other words, ...

argument argument
assume supposer
  assumption hypothèse
axiom axiome
case cas
  special case cas particulier
claim *v.* affirmer
  (the following) claim l'affirmation suivante; l'assertion suivante
concept notion
conclude conclure
  conclusion conclusion
condition condition
  a necessary and sufficient condition une condition nécessaire et suffisante
conjecture conjecture

consequence conséquence
consider considérer
contradict contredire
  contradiction contradiction
conversely réciproquement
corollary corollaire
deduce déduire
define définir
  well-defined bien défini(e)
  definition définition
equivalent équivalent(e)
establish établir
example exemple
exercise exercice
explain expliquer
  explanation explication
false faux, fausse
formal formel
hand main
  on one hand d'une part
  on the other hand d'autre part
iff [= **if and only if**] si et seulement si
imply impliquer, entraîner
induction on récurrence sur
lemma lemme
proof preuve; démonstration
property propriété
  satisfy property P satisfaire à la propriété P; vérifier la propriété P
proposition proposition
reasoning raisonnement
reduce to se ramener à
remark remarque(r)
required réquis(e)
result résultat
s.t. = such that
statement énoncé
t.f.a.e. = the following are equivalent
theorem théorème
true vrai
truth vérité
wlog = without loss of generality
word mot
  in other words autrement dit

Functions

Formulas/Formulae

$f(x)$	`f of x`
$g(x,y)$	`g of x (comma) y`
$h(2x,3y)$	`h of two x (comma) three y`
$\sin(x)$	`sine x`
$\cos(x)$	`cosine x`
$\tan(x)$	`tan x`
$\arcsin(x)$	`arc sine x`
$\arccos(x)$	`arc cosine x`
$\arctan(x)$	`arc tan x`
$\sinh(x)$	`hyperbolic sine x`
$\cosh(x)$	`hyperbolic cosine x`
$\tanh(x)$	`hyperbolic tan x`
$\sin(x^2)$	`sine of x squared`
$\sin(x)^2$	`sine squared of x; sine x, all squared`
$\frac{x+1}{\tan(y^4)}$	`x plus one, all over over tan of y to the four`
$3^{x-\cos(2x)}$	`three to the (power of) x minus cosine of two x`
$\exp(x^3+y^3)$	`exponential of x cubed plus y cubed`

Intervals

(a,b)	`open interval a b`
$[a,b]$	`closed interval a b`
$(a,b]$	`half open interval a b (open on the left, closed on the right)`
$[a,b)$	`half open interval a b (open on the right, closed on the left)`

The French notation is different!

$]a,b[$	`intervalle ouvert a b`
$[a,b]$	`intervalle fermé a b`
$]a,b]$	`intervalle demi ouvert a b (ouvert à gauche, fermé à droite)`
$[a,b[$	`intervalle demi ouvert a b (ouvert à droite, fermé à gauche)`

Exercise. Which of the two notations do you prefer, and why?

Derivatives

f'	`f dash; f prime; the first derivative of f`

f''	f double dash; f double prime; the second derivative of f
$f^{(3)}$	the third derivative of f
$f^{(n)}$	the n-th derivative of f
$\frac{dy}{dx}$	d y by d x; the derivative of y by x
$\frac{d^2y}{dx^2}$	the second derivative of y by x; d squared y by d x squared
$\frac{\partial f}{\partial x}$	the partial derivative of f by x (with respect to x); partial d f by d x
$\frac{\partial^2 f}{\partial x^2}$	the second partial derivative of f by x (with respect to x) partial d squared f by d x squared
∇f	nabla f; the gradient of f
Δf	delta f

Example. The (total) differential of a function $f(x, y, z)$ in three real variables is equal to

$$df = \frac{\partial f}{\partial x}\, dx + \frac{\partial f}{\partial y}\, dy + \frac{\partial f}{\partial z}\, dz.$$

The gradient of f is the vector whose components are the partial derivatives of f with respect to the three variables:

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right).$$

The Laplace operator Δ acts on f by taking the sum of the second partial derivatives with respect to the three variables:

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}.$$

The Jacobian matrix of a pair of functions $g(x, y)$, $h(x, y)$ in two real variables is the 2×2 matrix whose entries are the partial derivatives of g and h, respectively, with respect to the two variables:

$$\begin{pmatrix} \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \\ \frac{\partial h}{\partial x} & \frac{\partial h}{\partial y} \end{pmatrix}.$$

Integrals

$\int f(x)\, dx$	integral of f of x d x
$\int_a^b t^2\, dt$	integral from a to b of t squared d t
$\iint_S h(x, y)\, dx\, dy$	double integral over S of h of x y d x d y

Differential equations

An ordinary (resp., a partial) differential equation, abbreviated as ODE (resp., PDE), is an equation involving an unknown function f of one (resp., more than one) variable together with its derivatives (resp., partial derivatives). Its order is the maximal order of derivatives that appear in the equation. The equation is linear if f and its derivatives appear linearly; otherwise it is non-linear.

$f' + xf = 0$	first order linear ODE
$f'' + \sin(f) = 0$	second order non-linear ODE
$(x^2 + y)\frac{\partial f}{\partial x} - (x + y^2)\frac{\partial f}{\partial y} + 1 = 0$	first order linear PDE

The classical linear PDEs arising from physics involve the Laplace operator

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

$\Delta f = 0$	the Laplace equation
$\Delta f = \lambda f$	the Helmholtz equation
$\Delta g = \frac{\partial g}{\partial t}$	the heat equation
$\Delta g = \frac{\partial^2 g}{\partial t^2}$	the wave equation

Above, x, y, z are the standard coordinates on a suitable domain U in $\mathbf{R}^3$, t is the time variable, $f = f(x, y, z)$ and $g = g(x, y, z, t)$. In addition, the function f (resp., g) is required to satisfy suitable boundary conditions (resp., initial conditions) on the boundary of U (resp., for $t = 0$).

act *v.* agir
  action action
bound borne
  bounded borné(e)
  bounded above borné(e) supérieurement
  bounded below borné(e) inférieurement
  unbounded non borné(e)
comma virgule
concave function fonction concave
condition condition
  boundary condition condition au bord
  initial condition condition initiale
constant *n.* constante
constant *adj.* constant(e)
  constant function fonction constant(e)
  non-constant *adj.* non constant(e)

- **non-constant function** fonction non constante

continuous continu(e)
- **continuous function** fonction continue

convex function fonction convexe

decrease *n.* diminution

decrease *v.* décroître

decreasing function fonction décroissante
- **strictly decreasing function** fonction strictement décroissante

derivative dérivée
- **second derivative** dérivée seconde
- n**-th derivative** dérivée n-ième
- **partial derivative** dérivée partielle

differential *n.* différentielle
- **differential form** forme différentielle

differentiable function fonction dérivable
- **twice differentiable function** fonction deux fois dérivable
- n**-times continuously differentiable function** fonction n fois continument dérivable

domain domaine

equation équation
- **the heat equation** l'équation de la chaleur
- **the wave equation** l'équation des ondes

function fonction

graph graphe

increase *n.* croissance

increase *v.* croître

increasing function fonction croissante
- **strictly increasing function** fonction strictement croissante

integral intégrale

interval intervalle
- **closed interval** intervalle fermé
- **open interval** intervalle ouvert
- **half-open interval** intervalle demi ouvert

Jacobian matrix matrice jacobienne
- **Jacobian** le jacobien [= le déterminant de la matrice jacobienne]

linear linéaire
- **non-linear** non linéaire

maximum maximum
- **global maximum** maximum global
- **local maximum** maximum local

minimum minimum
- **global minimum** minimum global
- **local minimum** minimum local

monotone function fonction monotone
- **strictly monotone function** fonction strictement monotone

operator opérateur
  the Laplace operator opérateur de Laplace
ordinary ordinaire
order ordre
otherwise autrement
partial partiel(le)
  PDE [= **partial differential equation**] EDP
sign signe
value valeur
  complex-valued function fonction à valeurs complexes
  real-valued function fonction à valeurs réelles
variable variable
  complex variable variable complexe
  real variable variable réelle
  function in three variables fonction en trois variables
with respect to [= **w.r.t.**] par rapport à

This is all Greek to me

Small Greek letters used in mathematics

α	`alpha`	β	`beta`	γ	`gamma`	δ	`delta`
ϵ, ε	`epsilon`	ζ	`zeta`	η	`eta`	θ, ϑ	`theta`
ι	`iota`	κ	`kappa`	λ	`lambda`	μ	`mu`
ν	`nu`	ξ	`xi`	o	`omicron`	π, ϖ	`pi`
ρ, ϱ	`rho`	σ	`sigma`	τ	`tau`	υ	`upsilon`
ϕ, φ	`phi`	χ	`chi`	ψ	`psi`	ω	`omega`

Capital Greek letters used in mathematics

B	`Beta`	Γ	`Gamma`	Δ	`Delta`	Θ	`Theta`
Λ	`Lambda`	Ξ	`Xi`	Π	`Pi`	Σ	`Sigma`
Υ	`Upsilon`	Φ	`Phi`	Ψ	`Psi`	Ω	`Omega`

Sequences, Series

Convergence criteria

By definition, an infinite series of complex numbers $\sum_{n=1}^{\infty} a_n$ converges (to a complex number s) if the sequence of partial sums $s_n = a_1 + \cdots + a_n$ has a finite limit (equal to s); otherwise it diverges.

The simplest convergence criteria are based on the following two facts.

Fact 1. *If $\sum_{n=1}^{\infty} |a_n|$ is convergent, so is $\sum_{n=1}^{\infty} a_n$ (in this case we say that the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent).*

Fact 2. *If $0 \le a_n \le b_n$ for all sufficiently large n and if $\sum_{n=1}^{\infty} b_n$ converges, so does $\sum_{n=1}^{\infty} a_n$.*

Taking $b_n = r^n$ and using the fact that the geometric series $\sum_{n=1}^{\infty} r^n$ of ratio r is convergent iff $|r| < 1$, we deduce from Fact 2 the following statements.

The ratio test (d'Alembert). *If there exists $0 < r < 1$ such that, for all sufficiently large n, $|a_{n+1}| \le r\,|a_n|$, then $\sum_{n=1}^{\infty} a_n$ is (absolutely) convergent.*

The root test (Cauchy). *If there exists $0 < r < 1$ such that, for all sufficiently large n, $\sqrt[n]{|a_n|} \le r$, then $\sum_{n=1}^{\infty} a_n$ is (absolutely) convergent.*

What is the sum $1 + 2 + 3 + \cdots$ equal to?

At first glance, the answer is easy and not particularly interesting: as the partial sums

$$1, \quad 1+2=3, \quad 1+2+3=6, \quad 1+2+3+4=10, \quad \ldots$$

tend towards plus infinity, we have

$$1+2+3+\cdots = +\infty.$$

It turns out that something much more interesting is going on behind the scenes. In fact, there are several ways of "regularising" this divergent series and they all lead to the following surprising answer:

$$\text{(the regularised value of)} \;\; 1+2+3+\cdots = -\frac{1}{12}.$$

How is this possible? Let us pretend that the infinite sums

$$a = 1+2+3+4+\cdots$$
$$b = 1-2+3-4+\cdots$$
$$c = 1-1+1-1+\cdots$$

all make sense. What can we say about their values? Firstly, adding c to itself yields

$$\left.\begin{aligned} c &= 1-1+1-1+\cdots \\ c &= \phantom{1-{}} 1-1+1-\cdots \\ c+c &= 1+0+0+0+\cdots = 1 \end{aligned}\right\} \Longrightarrow c = \frac{1}{2}.$$

Secondly, computing $c^2 = c(1-1+1-1+\cdots) = c-c+c-c+\cdots$ by adding the infinitely many rows in the following table

$$\begin{aligned} c &= 1-1+1-1+\cdots \\ -c &= -1+1-1+\cdots \\ c &= \phantom{1-1+{}} 1-1+\cdots \\ -c &= -1+\cdots \\ &\vdots \qquad\qquad\qquad \ddots \end{aligned}$$

we obtain $b = c^2 = \frac{1}{4}$. Alternatively, adding b to itself gives

$$\left.\begin{aligned} b &= 1-2+3-4+\cdots \\ b &= \phantom{1-{}} 1-2+3-\cdots \\ b+b &= 1-1+1-1+\cdots = c \end{aligned}\right\} \Longrightarrow b = \frac{c}{2} = \frac{1}{4}.$$

Finally, we can relate a to b, by adding up the following two rows:

$$\left.\begin{aligned} a &= 1+2+3+4+\cdots \\ -4a &= \phantom{1+{}} -4 \phantom{{}+3} -8-\cdots \end{aligned}\right\} \Longrightarrow -3a = b = \frac{1}{4} \Longrightarrow a = -\frac{1}{12}.$$

Exercise. Using the same method, "compute" the sum

$$1^2 + 2^2 + 3^2 + 4^2 + \cdots.$$

$\lim_{x\to 1} f(x) = 2$ `the limit of f of x as x tends to one is equal to two`

approach approcher
close proche
  arbitrarily close to arbitrairement proche de
compare comparer
comparison comparaison
converge converger
convergence convergence
criterion (pl. criteria) critère
diverge diverger

divergence divergence
infinite infini(e)
infinity l'infini
  minus infinity moins l'infini
  plus infinity plus l'infini
large grand
  large enough assez grand
  sufficiently large suffisamment grand
limit limite
  tend to a limit admettre une limite
  tends to $\sqrt{2}$ tends vers $\sqrt{2}$
neighbo(u)rhood voisinage
sequence suite
  bounded sequence suite bornée
  convergent sequence suite convergente
  divergent sequence suite divergente
  unbounded sequence suite non bornée
series série
  absolutely convergent series série absolument convergente
  geometric series série géométrique
sum somme
  partial sum somme partielle

Prime Numbers

An integer $n > 1$ is a *prime (number)* if it cannot be written as a product of two integers $a, b > 1$. If, on the contrary, $n = ab$ for integers $a, b > 1$, we say that n is a *composite number*. The list of primes begins as follows:

$$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61 \ldots$$

Note the presence of several "twin primes" (pairs of primes of the form p, $p + 2$) in this sequence:

$$11, 13 \qquad 17, 19 \qquad 29, 31 \qquad 41, 43 \qquad 59, 61$$

Two fundamental properties of primes – with proofs – were already contained in Euclid's Elements:

Proposition 1. *There are infinitely many primes.*

Proposition 2. *Every integer $n \geq 1$ can be written* **in a unique way** *(up to reordering of the factors) as a product of primes.*

Recall the proof of Proposition 1: given any finite set of primes $p_1, \ldots, p_j$, we must show that there is a prime p different from each p_i. Set $M = p_1 \cdots p_j$; the integer $N = M + 1 \geq 2$ is divisible by at least one prime p (namely, the smallest divisor of N greater than 1). If p was equal to p_i for some $i = 1, \ldots, j$, then it would divide both N and $M = p_i(M/p_i)$, hence also $N - M = 1$, which is impossible. This contradiction implies that $p \neq p_1, \ldots, p_j$, concluding the proof.

The beauty of this argument lies in the fact that we do not need to know in advance any single prime, since the proof works even for $j = 0$: in this case $N = 2$ (as the empty product M is equal to 1, by definition) and $p = 2$.

It is easy to adapt this proof in order to show that there are infinitely many primes of the form $4n + 3$ (resp., $6n + 5$). It is slightly more difficult, but still elementary, to do the same for the primes of the form $4n + 1$ (resp., $6n + 1$).

Questions About Prime Numbers

Q1. *Given a large integer n (say, with several hundred decimal digits), is it possible to decide whether or not n is a prime?*

Yes, there are algorithms for "primality testing" which are reasonably fast both in theory (the Agrawal-Kayal-Saxena test) and in practice (the Miller-Rabin test).

Q2. *Is it possible to find concrete large primes?*

Searching for huge prime numbers usually involves numbers of special form, such as the Mersenne numbers $M_n = 2^n - 1$ (if M_n is a prime, n is necessarily also a prime). The point is that there is a simple test (the Lucas-Lehmer criterion) for deciding whether M_n is a prime or not.

In practice, if we wish to generate a prime with several hundred decimal digits, it is computationally feasible to pick a number randomly and then apply a primality testing algorithm to numbers in its vicinity (having first eliminated those which are divisible by small primes).

Q3. *Given a large integer n, is it possible to make explicit the factorisation of n into a product of primes?* *[For example,* $999\,999 = 3^3 \cdot 7 \cdot 11 \cdot 13 \cdot 37$*.]*

At present, no (unless n has special form). It is an open question whether a fast "prime factorisation" algorithm exists (such an algorithm is known for a hypothetical quantum computer).

Q4. *Are there infinitely many primes of special form?*

According to Dirichlet's theorem on primes in arithmetic progressions, there are infinitely many primes of the form $an + b$, for fixed integers $a, b \geq 1$ without a common factor.

It is unknown whether there are infinitely many primes of the form $n^2 + 1$ (or, more generally, of the form $f(n)$, where $f(n)$ is a polynomial of degree $\deg(f) > 1$).

Similarly, it is unknown whether there are infinitely many primes of the form $2^n - 1$ (the Mersenne primes) or $2^n + 1$ (the Fermat primes).

Q5. *Is there anything interesting about primes that one can actually prove?*

Green and Tao have recently shown that there are arbitrarily long arithmetic progressions consisting entirely of primes.

digit chiffre
prime number nombre premier
  twin primes nombres premiers jumeaux
progression progression
  arithmetic progression progression arithmétique
  geometric progression progression géométrique

Probability and Randomness

Probability theory attempts to describe in quantitative terms various random events.

For example, if we roll a die, we expect each of the six possible outcomes to occur with the same probability, namely $\frac{1}{6}$ (this should be true for a fair die; professional gamblers would prefer to use loaded dice, instead).

The following basic rules are easy to remember. Assume that an event A (resp., B) occurs with probability p (resp., q).

Rule 1. *If A and B are independent, then the probability of both A and B occurring is equal to pq.*

For example, if we roll the die twice in a row, the probability that we get twice 6 points is equal to $\frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36}$.

Rule 2. *If A and B are mutually exclusive (= they can never occur together), then the probability that A or B occurs is equal to $p + q$.*

For example, if we roll the die once, the probability that we get 5 or 6 points is equal to $\frac{1}{6} + \frac{1}{6} = \frac{1}{3}$.

It turns out that human intuition is not very good at estimating probabilities. Here are three classical examples.

Example 1. *The winner of a regular TV show can win a car hidden behind one of three doors. The winner makes a preliminary choice of one of the doors (the "first door"). The show moderator then opens one of the remaining two doors behind which there is no car (the "second door"). Should the winner open the initially chosen first door, or the remaining "third door"?*

Example 2. *The probability that two randomly chosen people have birthday on the same day of the year is equal to $\frac{1}{365}$ (we disregard the occasional existence of February 29). Given $n \geq 2$ randomly chosen people, what is the probability P_n that at least two of them have birthday on the same day of the year? What is the smallest value of n for which $P_n > \frac{1}{2}$?*

Example 3. *100 letters should have been put into 100 addressed envelopes, but the letters got mixed up and were put into the envelopes completely randomly. What is the probability that no (resp., exactly one) letter is in the correct envelope?*

See the next page for answers.

coin pièce (de monnaie)
toss [= flip] a coin lancer une pièce
die (pl. dice) dé
fair [= unbiased] die dé non pipé
biased [= loaded] die dé pipé
roll [= throw] a die lancer un dé
heads face
probability probabilité
random aléatoire
randomly chosen choisi(e) par hasard
tails pile
with respect to [= w.r.t.] par rapport à

Answer to Example 1. The third door. The probability that the car is behind the first (resp., the second) door is equal to $\frac{1}{3}$ (resp., zero); the probability that it is behind the third one is, therefore, equal to $1 - \frac{1}{3} - 0 = \frac{2}{3}$.

Answer to Example 2. Say, we have n people with respective birthdays on the days $D_1, \ldots, D_n$. We compute $1 - P_n$, namely, the probability that all the days D_i are distinct. There are 365 possibilities for each D_i. Given D_1, only 364 possible values of D_2 are distinct from D_1. Given distinct D_1, D_2, only 363 possible values of D_3 are distinct from D_1, D_2. Similarly, given distinct $D_1, \ldots, D_{n-1}$, only $365-(n-1)$ values of D_n are distinct from $D_1, \ldots, D_{n-1}$. As a result,

$$1 - P_n = \frac{364}{365} \cdot \frac{363}{365} \cdots \frac{365-(n-1)}{365},$$
$$P_n = 1 - \left(1 - \frac{1}{365}\right)\left(1 - \frac{2}{365}\right)\cdots\left(1 - \frac{n-1}{365}\right).$$

One computes that $P_{22} = 0.476\ldots$ and $P_{23} = 0.507\ldots$.

In other words, it is more likely than not that a group of 23 randomly chosen people will contain two people who share the same birthday!

Answer to Example 3. Assume that there are N letters and N envelopes (with $N \geq 10$). The probability p_m that there will be exactly m letters in the correct envelopes is equal to

$$p_m = \frac{1}{m!}\left(\frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots \pm \frac{1}{(N-m)!}\right)$$

(where $m! = 1 \cdot 2 \cdots m$ and $0! = 1$, as usual). For small values of m (with respect to N), p_m is very close to the infinite sum

$$q_m = \frac{1}{m!}\left(\frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots\right) = \frac{1}{e \cdot m!} = \frac{1^m}{m!}\, e^{-1},$$

which is the probability occurring in the Poisson distribution, and which **does not depend on the (large) number of envelopes**.

In particular, both p_0 and p_1 are very close to $q_0 = q_1 = \frac{1}{e} = 0.368\ldots$, which implies that the probability that there will be at most one letter in the correct envelope is greater than 73% !

depend on dépendre de
(to be) independent of (d'être) indépendant de
correspondence correspondance
transcendental transcendant