Functions $f: \mathbb{C}\setminus\mathbb{R} \longrightarrow V$ vs functions $F: \underbrace{\{L \subset \mathbb{C}\}}_{\text{lattice}} \longrightarrow V$

<u>Goal</u>: formalise the relation $G_k(\tau) \longleftrightarrow G_k(\omega_1,\omega_2) = G_k(\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2)$

<u>Recall</u>: lattice $L \subset \mathbb{C}$: $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$, $\underbrace{\omega_1, \omega_2 \text{ linearly indep. over } \mathbb{R}}_{\iff \frac{\omega_1}{\omega_2} \notin \mathbb{R}}$

<u>Def</u>: the $\mathbb{Z}$-basis ω_1, ω_2 of L is <u>positive</u> if <u>$\mathrm{Im}(\omega_1/\omega_2) > 0$</u> ($\iff \omega_1/\omega_2 \in \mathcal{H}$).

<u>Change of basis</u>: $L = \mathbb{Z}\omega_1' + \mathbb{Z}\omega_2' \iff \exists\, \alpha = \begin{pmatrix} a & b \\ c & d\end{pmatrix} \in GL_2(\mathbb{Z})$ $\begin{pmatrix} a & b \\ c & d\end{pmatrix}\begin{pmatrix}\omega_1 \\ \omega_2\end{pmatrix} = \begin{pmatrix}\omega_1' \\ \omega_2'\end{pmatrix}$

$GL_2(\mathbb{Z}) = \{\alpha \in M_2(\mathbb{Z}) \mid \det(\alpha) \in \mathbb{Z}^\times = \{\pm 1\}\}$

$\cup$

$GL_2(\mathbb{Z}) \cap GL_2(\mathbb{R})^+ = SL_2(\mathbb{Z})$ preserves positivity

<u>Formulas</u>: $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 = (\mathbb{Z}\frac{\omega_1}{\omega_2} + \mathbb{Z})\omega_2 = (\mathbb{Z}\tau + \mathbb{Z})\omega_2$, $\tau = \frac{\omega_1}{\omega_2}$

$= \mathbb{Z}\omega_1' + \mathbb{Z}\omega_2' = (\mathbb{Z}\tau' + \mathbb{Z})\omega_2'$, $\boxed{\tau' = \frac{\omega_1'}{\omega_2'} = \frac{a\tau+b}{c\tau+d}}$

$\mathrm{Im}(\tau') = \frac{\det(\alpha)\,\mathrm{Im}(\tau)}{|c\tau+d|^2}$ $\det(\alpha) = ad - bc = \pm 1$

<u>Thm</u>. Let V be a complex vector space, let $\chi: \mathbb{C}^\times \longrightarrow \mathbb{C}^\times$ be a group homomorphism. There are natural bijections between the following spaces of functions: $\left(\alpha = \begin{pmatrix} a & b \\ c & d\end{pmatrix}\right)$

(1) $\{F: \{\text{lattices } L \subset \mathbb{C}\} \to V \mid \forall t \in \mathbb{C}^\times \quad F(tL) = \chi(t)^{-1}F(L)\}$

(2) $\{f: \mathbb{C}\setminus\mathbb{R} \longrightarrow V \mid \forall \alpha \in GL_2(\mathbb{Z}) \quad f\left(\frac{a\tau+b}{c\tau+d}\right) = \chi(c\tau+d)f(\tau)\}$

(2') $\{f: \mathcal{H} \longrightarrow V \mid \forall \alpha \in SL_2(\mathbb{Z}) \quad$ —— " —— $\}$

<u>Note</u>: if $\chi(-1) = -1$, then $F(L) = F(-L) = -F(L) \implies F(L) = 0$.

<u>Formulas</u>: $f(\tau) = F(\mathbb{Z}\tau + \mathbb{Z})$, $F(\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2) = \chi(\omega_2)^{-1}f\left(\frac{\omega_1}{\omega_2}\right)$

<u>Pf</u>: <u>(1) $\Rightarrow$ (2)</u>: define f by ↑ ; then $\forall \alpha = \begin{pmatrix} a & b \\ c & d\end{pmatrix} \in GL_2(\mathbb{Z})$ $\forall \tau \in \mathbb{C}\setminus\mathbb{R}$

$f\left(\frac{a\tau+b}{c\tau+d}\right) = F\left(\mathbb{Z}\frac{a\tau+b}{c\tau+d} + \mathbb{Z}\right) = \chi(c\tau+d)F(\underbrace{\mathbb{Z}(a\tau+b) + \mathbb{Z}(c\tau+d)}_{\mathbb{Z}\tau+\mathbb{Z}})$

<u>(2) $\Rightarrow$ (2')</u>: restriction to $\tau \in \mathcal{H}$

[<u>(2') $\Rightarrow$ (2)</u>: for $\mathrm{Im}(\tau) < 0$ define $f(\tau) := f(-\tau)$]

<u>(2') $\Rightarrow$ (1)</u>: define $F(\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2) = \chi(\omega_2)^{-1}f\left(\frac{\omega_1}{\omega_2}\right)$, for a fixed positive basis ω_1, ω_2. Another positive basis: $\begin{pmatrix}\omega_1' \\ \omega_2'\end{pmatrix} = \underbrace{\begin{pmatrix} a & b \\ c & d\end{pmatrix}}_{\alpha \in SL_2(\mathbb{Z})}\begin{pmatrix}\omega_1 \\ \omega_2\end{pmatrix}$

then $\tau := \frac{\omega_1}{\omega_2}$, $\tau' := \frac{\omega_1'}{\omega_2'} \in \mathcal{H}$, $\frac{a\tau+b}{c\tau+d} = \tau'$, $\left(\frac{\omega_2'}{\omega_2} = c\tau+d\right)$

$\implies \chi(\omega_2')^{-1}f(\tau') \overset{(2')}{=} \underbrace{\chi(\omega_2')^{-1}\chi(c\tau+d)}_{\chi(\omega_2)^{-1}}f(\tau)$, so F is well-defined.

Exercise: $\{$continuous group homomorphisms $\mathbb{R}^\times \longrightarrow \mathbb{C}^\times\} =$
$= \{ t \longmapsto t^k |t|^s \mid k \in \mathbb{Z},\ s \in \mathbb{C} \}$

Reformulation: for $\chi : \mathbb{C}^\times \longrightarrow \mathbb{C}^\times$ as in Thm, define
for $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{R})^+$ and $f : \mathcal{H} \longrightarrow V$ (or $f : \mathbb{C} \setminus \mathbb{R} \longrightarrow V$)

$$(f|_\chi \alpha)(\tau) := \chi(\det(\alpha)^{1/2})\, \chi(c\tau+d)^{-1} f\Big(\underbrace{\frac{a\tau+b}{c\tau+d}}_{\alpha(\tau)}\Big) \qquad \left(= \chi(\det(\alpha))^{1/2}\, \chi(J(\alpha,\tau))^{-1} f(\alpha(\tau)) \right)$$

Properties: (1) $t > 0 \Longrightarrow f|_\chi \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix} = f$

(2) this is a right action of $GL_2(\mathbb{R})^+$ on $\{ f : \mathcal{H} \longrightarrow V \}$:
$((f|_\chi)\alpha)|_\chi \alpha' = f|_\chi \alpha\alpha'$ (since $J(\alpha\alpha', \tau) = J(\alpha, \alpha'(\tau))\, J(\alpha', \tau)$)

Conditions in Thm (2'): $\forall \alpha \in SL_2(\mathbb{Z})$ $f|_\chi \alpha = f$

Special notation: for $\chi = \chi_{k,s} : t \longmapsto t^k |t|^s$ ($k \in \mathbb{Z}, s \in \mathbb{C}$)
(resp $\chi = \chi_k : t \longmapsto t^k$ ($k \in \mathbb{Z}$)) write
$f|_{k,s}$ (resp. $f|_k$) instead of $f|\chi$.

Ex: $$G_{k,s}(\tau) := \sum_{m,n \in \mathbb{Z}}{}' \frac{1}{(m\tau+n)^k |m\tau+n|^s} \qquad (\tau \in \mathbb{C} \setminus \mathbb{R})$$
corresponds to $$F(L) = \sum_{0 \neq u \in L}{}' \frac{1}{u^k |u|^s} \qquad (k + \mathrm{Re}(s) > 2)$$

Modification: getting rid of s !!

$\forall \alpha \in GL_2(\mathbb{R})^+$
$\forall \tau \in \mathcal{H}$
$$\boxed{\mathrm{Im}(\alpha(\tau))^{s/2} = \left(\frac{\det(\alpha)^{1/2}}{|c\tau+d|} \right)^s \mathrm{Im}(\tau)^{s/2}}$$

If $\tilde{f}(\tau) = \mathrm{Im}(\tau)^{s/2} f(\tau)$ $\forall \tau \in \mathcal{H}$, then
$\forall \alpha \in GL_2(\mathbb{R})^+$
$$(\tilde{f}|_k \alpha)(\tau) = \left(\frac{\det(\alpha)^{1/2}}{c\tau+d} \right)^k \tilde{f}(\alpha(\tau)) = \underbrace{\left(\frac{\det(\alpha)^{1/2}}{c\tau+d} \right)^k \left(\frac{\det(\alpha)^{1/2}}{|c\tau+d|} \right)^s f(\alpha(\tau))}_{(f|_{k,s}\alpha)(\tau)} \mathrm{Im}(\tau)^{s}$$

Cor: $\forall \alpha \in SL_2(\mathbb{Z})$ $\tilde{f}|_k \alpha = \tilde{f}$
$\Updownarrow$
$\forall \alpha \in SL_2(\mathbb{Z})$ $f|_{k,s} = f$

Ex: $f = G_{k,s}$, $$\boxed{\tilde{f}(\tau) = \sum_{m,n \in \mathbb{Z}}{}' \frac{\mathrm{Im}(\tau)^{s/2}}{(m\tau+n)^k |m\tau+n|^s}} \qquad (\tau \in \mathcal{H})$$

$\forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ $\forall \tau \in \mathcal{H}$ $\tilde{f}\left(\frac{a\tau+b}{c\tau+d} \right) = (c\tau+d)^k \tilde{f}(\tau)$

Back to $E_2(\tau) = 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n) q^n \qquad (q = e^{2\pi i \tau})$

Recall : $\forall \tau \in \mathcal{H}$ $\forall \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ $\quad (E_2|\gamma)(\tau) = (c\tau+d)^{-2} E_2\left(\frac{a\tau+b}{c\tau+d}\right)$

$$= E_2(\tau) + \underbrace{\frac{12c}{2\pi i(c\tau+d)}}_{\frac{12c}{2\pi i\, J(\gamma,\tau)}}$$

Question : is there an elementary correction term CT such that $(E_2 + CT)|_2 \gamma = E_2 + CT \quad \forall \gamma \in SL_2(\mathbb{Z})$?

Try something involving $\mathrm{Im}(\tau) = y(\tau)$ $\qquad (\tau = x + iy)$

$\forall g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$ $\qquad y(g(\tau)) = \frac{y(\tau)}{J(g,\tau)\, J(g,\bar\tau)}$, $\quad y(\tau) = \frac{\tau - \bar\tau}{2i}$

$J(g,\tau) - J(g,\bar\tau) = c(\tau - \bar\tau) = 2ic\, y(\tau)$

$$\Longrightarrow \frac{1}{y(g(\tau))} \frac{1}{J(g,\tau)^2} = \frac{J(g,\bar\tau)}{y(\tau)\, J(g,\tau)} = \frac{J(g,\tau) - 2icy(\tau)}{J(g,\tau)\, y(\tau)} = \frac{1}{y(\tau)} - \frac{2ic}{J(g,\tau)}$$

$$\Longrightarrow \quad \left(\frac{1}{y}\Big|_2 g\right)(\tau) - \frac{1}{y}(\tau) = -\frac{2ic}{J(g,\tau)}$$

Cor : $\quad \left(E_2(\tau) - \frac{3}{\pi} \cdot \frac{1}{\mathrm{Im}(\tau)}\right)\Big|_2 \gamma = E_2(\tau) - \frac{3}{\pi} \cdot \frac{1}{\mathrm{Im}(\tau)} \quad \forall \gamma \in SL_2(\mathbb{Z})$

$\boxed{\text{Functions} \quad f: \mathcal{H} \longrightarrow V \quad \text{and functions} \quad \tilde{f}: SL_2(\mathbb{R}) \longrightarrow V}$

Group – theoretical description of $\mathcal{H}$: $G = SL_2(\mathbb{R})$ acts transitively on $\mathcal{H}$, the stabiliser of i (= the base point) is

$$G_i = \{g \in G \mid g(i) = i\} = \left\{\begin{pmatrix} a & -c \\ c & a \end{pmatrix} \middle| \begin{matrix} a, c \in \mathbb{R} \\ a^2+c^2=1 \end{matrix}\right\} = \left\{\underbrace{\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}}_{h_\theta} \middle| \theta \in \mathbb{R}\right\}$$
$$= SO(2)$$

Automorphy factor : $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G, \ \tau \in \mathcal{H}$

$\boxed{J(g, \tau) = c\tau + d}$ $\qquad (\Longrightarrow \boxed{J(h_\theta, i) = e^{i\theta}})$

1-cocycle identity : $J(g_1 g_2, \tau) = J(g_1, g_2(\tau))\, J(g_2, \tau)$

($\Longrightarrow$ the same identity for $\chi(J(g,\tau))$, if $\chi: \mathbb{C}^\times \longrightarrow \mathbb{C}^\times$ is a group hom.)

Interesting functions : $f: \mathcal{H} \longrightarrow V$ (V = $\mathbb{C}$-vector space)

such that $\forall \gamma \in \Gamma \quad f(\gamma(\tau)) = \chi(J(\gamma, \tau))\, f(\tau) \qquad (\Gamma = SL_2(\mathbb{Z}))$

Abstract situation :

Data : • G group acting transitively on a set X

• base point $x_0 \in X$ ($\Longrightarrow$ stabiliser $G_{x_0} = \{g \in G \mid g(x_0) = x_0\}$)

($\Longrightarrow$ identification $\boxed{G/G_{x_0} \xrightarrow{\sim} X, \quad gG_{x_0} \longmapsto g(x_0)}$)

• V set

• $\rho: G \times X \longrightarrow \mathrm{Aut}(V)$ automorphy factor :

$\forall g \in G \ \forall x \in X \quad \boxed{\rho(g_1 g_2, x) = \rho(g_1, g_2 x)\, \rho(g_2, x)}$ (1-cocycle)

$$(\Longrightarrow \quad \begin{aligned} \rho(\cdot, x_0): G_{x_0} &\longrightarrow \mathrm{Aut}(V) \\ h &\longmapsto \rho(h, x_0) \end{aligned} \quad \text{is a group homomorphism} \ (h(x_0) = x_0))$$

Thm. $\exists$ Bijection

$$\left\{\text{maps } f: \underset{G/G_{x_0}}{X} \longrightarrow V\right\} \longleftrightarrow \left\{\begin{matrix}\text{maps } \tilde{f}: G \longrightarrow V \text{ such that} \\ \forall g \in G \ \forall h \in G_{x_0} \quad \tilde{f}(gh) = \rho(h, x_0)^{-1}(\tilde{f}(g))\end{matrix}\right\}$$

$$\tilde{f}(g) = \rho(g, x_0)^{-1}(f(gx_0)) \qquad (*)$$

Pf : Given f, define $\tilde{f}$ by $(*)$ $\Longrightarrow$ $\forall h \in G_{x_0}$

$$\tilde{f}(gh) = \rho(gh, x_0)^{-1}(f(gh(x_0))) = (\rho(g, \underbrace{h(x_0)}_{x_0})\, \rho(h, x_0))^{-1} f(gx_0) = \rho(h, x_0)^{-1}(\tilde{f}(g)).$$

Given $\tilde{f}$, define f by $f(gx_0) = \rho(g, x_0)(\tilde{f}(g))$. If $g'(x_0) = g(x_0)$, then $g' = gh$, $h \in G_{x_0}$ and $\rho(g', x_0)(\tilde{f}(g')) = \rho(g, \underbrace{hx_0}_{x_0})\, \underbrace{\rho(h, x_0)(\tilde{f}(gh))}_{\tilde{f}(g)} \Longrightarrow f$ is well-defined.

Additional data: • $\Gamma \subset G$ subgroup

then: it is equivalent: $(x = g x_0)$

$$(\forall \gamma\in\Gamma)(\forall g\in G)\quad \tilde f(\gamma g)=\tilde f(g) \implies (\forall\gamma\in\Gamma)(\forall x\in X)\quad f(\gamma(x)) = f(\gamma g(x_0)) =$$

$$= \underbrace{\rho(\gamma g, x_0)}_{\rho(\gamma,x)\rho(g,x_0)}(\tilde f(\gamma g)) = \rho(\gamma,x)\underbrace{\rho(g,x_0)(\tilde f(g))}_{f(x)}$$

$$\underbrace{\tilde f(\gamma g)=\rho(\gamma g,x_0)^{-1}(\underbrace{f(\gamma g(x_0))}_{\rho(\gamma, g(x_0))(f(gx_0))})}_{\rho(g,x_0)^{-1}(f(gx_0)) = \tilde f(g).} \qquad \Longleftarrow$$

Summary: bijection $\qquad (\tilde f(g) = \rho(g,x_0)^{-1}(f(gx_0)))$

$$\{f : X = G/G_{x_0} \longrightarrow V \mid \forall\gamma\in\Gamma\ \forall x\in X\quad f(\gamma(x)) = \rho(\gamma,x)f(x)\}$$

$$\updownarrow\wr$$

$$\{\tilde f : G \longrightarrow V \mid \forall g\in G\ \forall h\in G_{x_0}\ \forall\gamma\in\Gamma\quad \tilde f(\gamma g h) = \rho(h,x_0)^{-1}(\tilde f(g))\}$$

(so $\tilde f$ factors through $\Gamma\backslash G \longrightarrow V$, while f through G/G_{x_0})

Our case: $\forall k\in\mathbb Z$ bijections $\qquad$ ($V = \mathbb R$-vector space, $\Gamma\subset SL_2(\mathbb R)$ subgroup)

(1) $\{f : \mathcal H \longrightarrow V\} \xleftrightarrow{\sim} \{\tilde f : SL_2(\mathbb R)\longrightarrow V \mid \forall h_\theta\in SO(2)\ \forall g\in SL_2(\mathbb R)\quad \tilde f(g h_\theta) = e^{-ik\theta}\tilde f(g)\}$

$$\tilde f(g) = \underbrace{J(g,i)^{-k}}_{(ci+d)^{-k}} f(\underbrace{g(i)}_{\frac{ai+b}{ci+d}}), \quad g=\begin{pmatrix}a&b\\c&d\end{pmatrix} \qquad h_\theta = \begin{pmatrix}\cos(\theta) & -\sin(\theta)\\ \sin(\theta) & \cos(\theta)\end{pmatrix}$$

$$f(\tau) = f(x+iy) = f(g_\tau(i)) = J(g_\tau,i)^k\,\tilde f(g_\tau) = \mathrm{Im}(\tau)^{-k/2}\,\tilde f(g_\tau)$$

$$g_\tau = \begin{pmatrix}1&x\\0&1\end{pmatrix}\begin{pmatrix}y^{1/2}&0\\0&y^{-1/2}\end{pmatrix} = \begin{pmatrix}y^{1/2}&xy^{-1/2}\\0&y^{-1/2}\end{pmatrix}, \quad J(g_\tau,i) = y^{-1/2}$$

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(2) It is equivalent:

$$\left(\begin{array}{l}\forall\ \gamma = \begin{pmatrix}a&b\\c&d\end{pmatrix}\in\Gamma \qquad f(\gamma(\tau)) = (c\tau+d)^k f(\tau)\\ \forall\tau\in\mathcal H\end{array}\right)$$

$$\Updownarrow$$

$$\forall\gamma\in\Gamma\ \forall g\in SL_2(\mathbb R)\qquad \tilde f(\gamma g) = \tilde f(g)$$

Summary: interesting functions: $\tilde f : SL_2(\mathbb R)\longrightarrow V$

$$\forall\gamma\in\Gamma\ \forall g\in SL_2(\mathbb R)\ \forall\theta\in\mathbb R \qquad \tilde f(\gamma g h_\theta) = e^{-ik\theta}\tilde f(g)$$

Geometric interpretation of automorphy factors

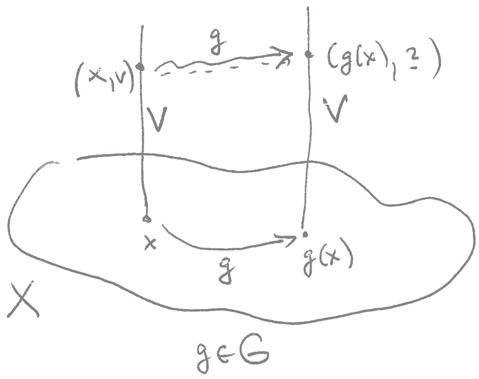
Data
- X, V sets
- G group acting on X (on the left)

$\Longrightarrow$ product

$$\begin{tikzcd} X\times V \arrow[d, "p"] & (x,v) \arrow[d, mapsto, "p"] \\ G \circlearrowright X & x \end{tikzcd}$$

projection

Goal: lift the G-action on X to a G-action on $X\times V$



$$\begin{tikzcd} (x,v) \arrow[rrr, dashed, mapsto] \arrow[d, mapsto] & X\times V \arrow[r, dashed, "g"] \arrow[d, "p"] & X\times V \arrow[d, "p"] & (gx, \rho(g,x)(v)) \arrow[d, mapsto] \\ x \arrow[rrr, mapsto] & X \arrow[r, "g"] & X & gx \end{tikzcd}$$

$$\rho : G\times X \longrightarrow \mathrm{Aut}(V)$$

Compatibility condition required: $\forall g,h\in G \quad \forall x\in X \quad \forall v\in V$

$$\begin{tikzcd} (x,v) \arrow[r, mapsto, "h"] \arrow[rr, bend right, "gh"'] & (hx, \rho(h,x)(v)) \arrow[r, mapsto, "g"] & (g(hx), \rho(g,hx)\rho(h,x)(v)) \\ & & ((gh)x, \rho(gh,x)(v)) \arrow[u, equal] \end{tikzcd}$$

(*) $\boxed{\rho(gh,x)=\rho(g,hx)\rho(h,x)}$ 1-cocycle identity ($\overset{h=e}{\Longrightarrow}$ $\rho(e,x)=id_X$)

Summary: lifts of the G-action from X to $X\times V$ (compatible via p) are given by $g(x,v)=(gx, \rho(g,x)(v))$, where $\rho : G\times X\longrightarrow \mathrm{Aut}(V)$ satisfies (*).

Sections of p: maps

$$\begin{tikzcd} X\times V \arrow[d, "p"] \\ X \arrow[u, bend left, "s"] \end{tikzcd}$$

such that $p\circ s = id_X$: $s(x)=(x, f(x))$, $f : X\longrightarrow V$ map

Equivariant sections: for fixed $\gamma\in G$, s satisfies

$\gamma\circ s = s\circ\gamma$

"s is γ-equivariant"

$$\begin{tikzcd} X\times V \arrow[r, "\gamma"] & X\times V \\ X \arrow[u, "s"] \arrow[r, "\gamma"] & X \arrow[u, "s"'] \end{tikzcd} \iff \forall x\in X \quad f(\gamma x)=\rho(\gamma,x)(f(x))$$

So: for a subgroup $\Gamma\subset G$,

$$\{f : X\longrightarrow V \mid \forall\gamma\in\Gamma \;\; f(\gamma x)=\rho(\gamma,x)(f(x))\} \overset{\sim}{\longleftrightarrow} \left\{\Gamma\text{-equivariant sections } s \text{ of } \begin{tikzcd} X\times V \arrow[d, "p"] \\ X \end{tikzcd}\right\}$$

$$s(x)=(x, f(x))$$

G-actions on $\{f: X \longrightarrow V\}$ vs $\{\tilde f : G \longrightarrow V\}$

left multiplication $\quad L(g): G \longrightarrow G$
$$g' \longmapsto gg'$$

induces standard right action G on $\{\tilde f : G \longrightarrow V\}$:
$(\tilde f | g)(g') = \tilde f (gg')$.

This can be transported to a right action of G on $\{f: X = G/G_{x_0} \longrightarrow V\}$ via the automorphy factor $\rho : G \times X \longrightarrow \mathrm{Aut}(V)$:

$$(f |_\rho g)(g' x_0) = \rho(g', x_0)\big(\underbrace{(\tilde f | g)(g')}_{\tilde f(gg') = \underbrace{\rho(gg', x_0)^{-1}}_{(\rho(g, g'x_0)\rho(g', x_0))^{-1}} f(gg'x_0)}\big)$$

$$= \rho(g, g'x_0)^{-1} f(gg'x_0), \quad \text{i.e.}$$

$$\boxed{(f|_\rho g)(x) = \rho(g,x)^{-1} f(gx)} \qquad (*)$$

Geometric interpretation :

$$\{f : G/G_{x_0} = X \longrightarrow V\} \overset{\sim}{\longleftrightarrow} \left\{\text{sections } s \text{ of } \begin{array}{c} X \times V \\ {\scriptstyle s}\uparrow \downarrow {\scriptstyle p} \\ X \end{array}\right\}$$
$$s(x) = (x, f(x))$$

$g \in G$ acts on $X \times V$ by $g(x,v) = (gx, \rho(g,x)v)$

$\Rightarrow$ right action of G on $\{\text{sections } s \text{ of } X \times V \xrightarrow{p} X\}$ by

$$(s|g)(x) = g^{-1}(s(gx))$$
$$= g^{-1}(gx, f(gx)) = (x, \underbrace{\rho(g,x)^{-1} f(gx)}_{(f|_\rho g)(x)})$$

Summary : the natural right actions of G on $\{\tilde f : G \longrightarrow V\}$ and on $\{\text{sections of } X \times V \xrightarrow{p} X\}$ (with G action on $X \times V$ via ρ)

correspond to the ρ-twisted right action $(*)$ of G on $\{f : X \longrightarrow V\}$

Cor: $\{\Gamma\text{-equivariant sections of } p\} \overset{\sim}{\longleftrightarrow} \{\tilde f : \Gamma \backslash G \longrightarrow V\}$
$$\overset{\sim}{\longleftrightarrow} \{f : X \longrightarrow V \mid \forall \gamma \in \Gamma \;\; f|_\rho \gamma = f\}$$
$(\Gamma \subset G$ subgroup$)$

Γ-equivariant sections (continued):

a Γ-equivariant section s of
$$\begin{tikzcd} X\times V \arrow[d, "p"] \\ X \arrow[u, bend left, "s"] \end{tikzcd}$$
defines a section

$\bar s$ of the quotient
$$\boxed{\begin{tikzcd} \Gamma\backslash(X\times V) \arrow[d, "\bar p"] \\ \Gamma\backslash X \arrow[u, bend left, "\bar s"] \end{tikzcd} \qquad \bar s(\Gamma x) = \Gamma s(x)}$$

(and vice versa). Under good circumstances, $\bar p$ is a bundle with fibre V.

Summary:

$$\{ f: X \longrightarrow V \mid \forall \gamma\in\Gamma \ f(\gamma x) = \underbrace{\rho(\gamma, x)}_{\text{defines the lift of the action of } \Gamma \text{ from } X \text{ to } X\times V}(f(x))\} \longleftrightarrow \left\{ \text{sections } \bar s \text{ of } \begin{tikzcd} \Gamma\backslash(X\times V) \arrow[d, "\bar p"] \\ \Gamma\backslash X \end{tikzcd} \right\}$$

$$\bar s(\Gamma x) = \Gamma(x, f(x))$$

Ex: $X = W\setminus\{0\}$, $V, W =$ K-vector spaces, $\dim W = n+1$, $t\in\Gamma = K^\times$ acting on X by multiplication by t and on $X\times V$ by $(x, v)\longmapsto(tx, t^{k}v)$ for fixed $k\in\mathbb{Z}$. Sections of
$$\begin{tikzcd} \Gamma\backslash(X\times V) =: \arrow[d] & L_k(V) \arrow[d] \\ \Gamma\backslash X & =: \mathbb{P}(W) \end{tikzcd}$$
, correspond to maps $f: W\setminus\{0\} \longrightarrow V$ homogeneous of degree k: $f(tx) = t^k f(x)$

If $V = K$: $\quad L_k(K) = \mathcal{O}(k) = \mathcal{O}(1)^{\otimes k}$

the tautological bundle
$$\begin{tikzcd} E \arrow[d] \\ \mathbb{P}(W) \end{tikzcd} = \{(\ell, w) \mid \ell\subset W \text{ vector subspace of } \dim = 1,\ w\in\ell\}$$
is isomorphic to $L_{-1}(K)$.

Reformulation: general principle: G group acting on the left on sets X, Y $\Rightarrow$ G acts on $\mathrm{Hom}(X, Y) := \{\text{maps } X \xrightarrow{f} Y\}$ by

$(g*f)(x) := g(f(g^{-1}x))$ $\quad (g*f = g\circ f\circ g^{-1})$ – left action

$(f*g)(x) := g^{-1}(f(gx))$ $\quad (f*g = g^{-1}\circ f\circ g)$ – right action

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For a subgroup $\Gamma\subset G$, the set of Γ-invariants
$\mathrm{Hom}(X, Y)^\Gamma := \{f\in \mathrm{Hom}(X, Y) \mid \forall\gamma\in\Gamma \ \ \gamma * f = f\}$ is the set of Γ-equivariant maps: $f(\gamma x) = \gamma f(x) \ \ \forall\gamma\in\Gamma \ \forall x\in X$

Above: Γ-equivariant sections of $p: X\times V \longrightarrow X$ are just Γ-invariant elements of $\{\text{sections of } p\} \subset \mathrm{Hom}(X, X\times V)$

__Gauge tranformations__ : "change of coordinates on the fibres of p"

given by a map $\underline{a : X \longrightarrow \mathrm{Aut}(V)}$

Transform the action $g = g_{old} : (x,v) \longmapsto (gx, \rho(g,x)(v))$ to

$$\begin{tikzcd}
(x,\cdot) \arrow[r, "g_{new}"] & (gx,\cdot) \\
(x,v) \arrow[u, "a(x)"] \arrow[r, "g_{old}"] & (gx,\cdot) \arrow[u, "a(gx)"']
\end{tikzcd}$$

$$\boxed{\rho_{new}(g,x) = a(gx)\rho(g,x)\,a(x)^{-1}}$$ (1-cocycle cohomologous to ρ)

__Old section__ : $s(x) = s_{old}(x) = (x, f(x))$ new section: $s_{new}(x) = (x, a(x)f(x))$

s_{old} is γ_{old} – equivariant $\Longleftrightarrow$ s_{new} is γ_{new} – equivariant.

$(\gamma_{old} \circ s_{old} = s_{old} \circ \gamma)$

$$\begin{tikzcd}
X \times V \arrow[d, "p"] \\
X
\end{tikzcd}$$

is a trivial bundle over X with fibre V

__G – equivariant bundles over X__ (with fibre V)

In a suitable category of spaces ($\mathcal{C}^\infty$ – manifolds, complex manifolds, ...)

map $\begin{tikzcd} E \arrow[d, "p"] \\ X \end{tikzcd}$, open covering $X = \bigcup\limits_\alpha U_\alpha$, trivialisations

$$\begin{tikzcd}
E|_{U_\alpha} \arrow[r, "t_\alpha", "\sim"'] \arrow[d, "p"'] & U_\alpha \times V \arrow[dl, "p"] \\
U_\alpha
\end{tikzcd}$$

such that over each $U_\alpha \cap U_\beta$ $t_\alpha|_{U_\alpha \cap U_\beta}$, $t_\beta|_{U_\alpha \cap U_\beta}$ are

compatible as follows: $t_\beta \circ t_\alpha^{-1}(x,v) = (x, u_{\alpha\beta}(x)(v))$ $\forall x \in U_\alpha \cap U_\beta$

$u_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow \mathrm{Aut}(V)$ being again $\mathcal{C}^\infty$

or holomorphic or ...

Such a bundle is __G – equivariant__ if G is a group acting on E and X, compatibly via p.

__Ex__: $X = \mathbb{P}^1(\mathbb{C}) = \mathbb{P}(\mathbb{C}^2) = \{\ell \subset \mathbb{C}^2 \mid \text{subspace}, \dim(\ell) = 1\}$, $G = Gl_2(\mathbb{C})$

$V = \mathbb{C}$, E = the tautological bundle over $\mathbb{P}^1(\mathbb{C})$

$$\begin{tikzcd}
E \arrow[d, "p"'] \\
\mathbb{P}^1(\mathbb{C})
\end{tikzcd}$$

$= \{(\ell, v) \mid v \in \mathbb{C}^2, \ell \subset \mathbb{C}^2 \text{ vector subspace}, \dim(\ell) = 1, v \in \ell\}$

$p(\ell, v) = \ell$

Trivialisation over $U = \mathbb{P}^1(\mathbb{C}) \setminus \{\infty\} \cong \mathbb{C} \ni t$ corresponds to $\mathbb{C}\binom{t}{1} \in \mathbb{P}^1(\mathbb{C})$

via $E|_U \xrightarrow{\sim} U \times \mathbb{C}$ given by the section

$$\begin{tikzcd}
E|_U \arrow[d, "p"', shift right] & \ni & \left(\mathbb{C}\binom{t}{1}, \binom{t}{1}\right) \\
U \arrow[u, "s"', shift right] & \ni & \mathbb{C}\binom{t}{1} \arrow[u]
\end{tikzcd}$$

$$\left(\mathbb{C}\binom{t}{1}, \lambda\binom{t}{1}\right) \longmapsto \left(\mathbb{C}\binom{t}{1}, \lambda\right)$$

the Canonical (diagonal) action of $G = GL_2(\mathbb{C})$ on $\mathbb{P}^1(\mathbb{C}) \times \mathbb{C}^2$ preserves the subset E

$\Longrightarrow$ makes $\begin{array}{c} E \\ \downarrow \\ \mathbb{P}^1(\mathbb{C}) \end{array}$ a $GL_2(\mathbb{C})$- ~~bundle~~ equivariant bundle.

the above trivialisation over $U = \mathbb{P}^1(\mathbb{C}) \setminus \{\infty\} \simeq \mathbb{C}$ is then given by the usual formula

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} t \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{at+b}{ct+d} \\ 1 \end{pmatrix} (ct+d)$$

$\Longrightarrow$ given by the 1-cocycle $\rho\Big(\underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{g}, t\Big) = ct+d = J(g, t)$

(over $U \cap g^{-1}(U)$).

Previous discussion: the restriction of $E|_{\mathcal{H}}$ over $\mathcal{H}$ is an $SL_2(\mathbb{R})$- equiv. bundle over $\mathcal{H}$ ($V = \mathbb{C}$)

It is trivialised as above, and its Γ- equivariant sections correspond to $f : \mathcal{H} \longrightarrow \mathbb{C}$ such that $(\Gamma \subset SL_2(\mathbb{R}))$

$$\forall\ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \qquad f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)\, f(\tau).$$

If we want to replace $(c\tau+d)$ by $(c\tau+d)^k$, we need to consider Γ- equivariant sections of $\begin{array}{c} E^{\otimes k} \\ \downarrow \\ \mathcal{H} \end{array}$.

Various incarnations of $\mathcal{H}$ and its symmetry group $SL_2(\mathbb{R})$

Unbounded: complex: $\mathcal{H} = \{\tau \in \mathbb{C} \mid \mathrm{Im}(\tau) > 0\}$ cplx upper half plane

real: $H^2 = \{x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 \mid x_2 > 0\}$ real hyperbolic space of dim 2

Bounded: complex: $D = \{z \in \mathbb{C} \mid |z| < 1\}$ (cplx) unit disc

real: $\{y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in \mathbb{R}^2 \mid \|y\|^2 = y_1^2 + y_2^2 < 1\}$ real unit disc

Cayley map:

$\mathcal{H}$

$$\tau \longmapsto z = \frac{\tau - i}{\tau + i} = g(\tau), \qquad g = \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$$

Symmetry groups: $\boxed{\begin{array}{ccc} SL_2(\mathbb{R})/SO(2) & \xrightarrow{\sim} & \mathcal{H} \\ h & \longmapsto & h(i) \end{array}}$

$$g \begin{pmatrix} a & b \\ c & d \end{pmatrix} g^{-1} = \begin{pmatrix} u & \bar{v} \\ v & \bar{u} \end{pmatrix}, \quad u = \frac{a+d}{2} + \frac{i(b-c)}{2}, \; v = \frac{a-d}{2} + \frac{i(b+c)}{2}, \quad |u|^2 - |v|^2 = ad - bc$$

$$g\, SL_2(\mathbb{R})\, g^{-1} = \left\{ \begin{pmatrix} u & \bar{v} \\ v & \bar{u} \end{pmatrix} \;\middle|\; u, v \in \mathbb{C},\ |u|^2 - |v|^2 = 1 \right\} = SU(1,1)$$

$$g\, SO(2)\, g^{-1} = \left\{ \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{-i\alpha} \end{pmatrix} \;\middle|\; \alpha \in \mathbb{R} \right\} \simeq U(1).$$

$$g(i) = 0 \implies \boxed{\begin{array}{ccc} SU(1,1)/U(1) & \xrightarrow{\sim} & D \\ h & \longmapsto & h(0) \end{array}}$$

(Bi)linear algebra versions:

Symplectic form $\omega: \mathbb{C}^2 \times \mathbb{C}^2 \longrightarrow \mathbb{C}$, $\quad \omega\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right) = \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}$

$$\mathcal{H} \xrightarrow{\sim} \left\{ \mathbb{C}x \;\middle|\; \frac{\omega(x, \bar{x})}{i} > 0 \right\} \subset \mathbb{P}^1(\mathbb{C}) = \{1\text{-dim } \mathbb{C}\text{-subspaces of } \mathbb{C}^2\}$$

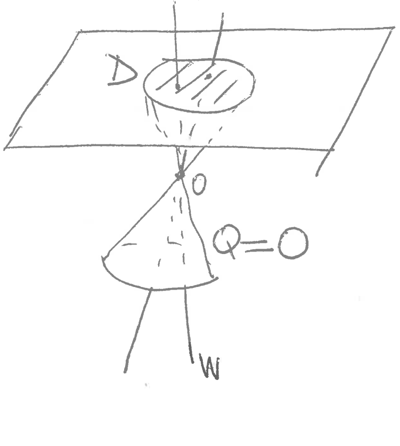
$$\tau \longmapsto \mathbb{C}\begin{pmatrix} \tau \\ 1 \end{pmatrix} \qquad = Gr_1(\mathbb{C}^2)$$

Quadratic form $Q(Y_0, Y_1, Y_2) = -Y_0^2 + Y_1^2 + Y_2^2$ on $\mathbb{R}^{2,1}$

$$D \xrightarrow{\sim} \underbrace{\{\mathbb{R}Y \mid Q(Y) < 0\}}_{Gr_1^-(\mathbb{R}^{2,1}) \,=\, \{\text{negative lines in } \mathbb{R}^{2,1}\}} \subset \{1\text{-dim } \mathbb{R}\text{-subspaces } W \text{ of } \mathbb{R}^{2,1}\} = Gr_1(\mathbb{R}^{2,1})$$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \longmapsto \mathbb{R}\begin{pmatrix} 1 \\ y_1 \\ y_2 \end{pmatrix}$$

$$W \cap \{Y_0 = 1\} \longleftarrow W$$



$Y_0 = 1$

Symmetry group

$SO(2,1)^+$

Higher-dimensional versions of $G/K \xrightarrow{\sim} X$

Algebraic: $X = \{$positive definite real quadratic forms of $\det = 1\}$, $G = SL_n(\mathbb{R})$, $K = SO(n)$

Complex: unbounded: Siegel upper half-space

$$X = \mathcal{H}_n = \{ \underset{\text{symmetric}}{T \in M_n(\mathbb{C})} \mid \underbrace{\operatorname{Im}(T) > 0}_{\text{positive definite}} \}, \quad G = Sp_{2n}(\mathbb{R}), \quad K = U(n)$$

bounded: $X = \{ z = \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} \in \mathbb{C}^n \mid {}^t\bar{z} z = |z_1|^2 + \dots + |z_n|^2 < 1 \}$ n-dim cplx unit ball

$G = SU(n,1)$, $K = S(U(n) \times U$

matrix generalisation: $X = \{ Z \in M_{p,q}(\mathbb{C}) \mid I_q - {}^t\bar{Z} Z > 0 \}$ $(p, q \geq 1)$

$$\Big\downarrow \wr$$

$Z \mapsto$ span of the columns of $\begin{pmatrix} Z \\ I_q \end{pmatrix}$

$Gr_q^-(\mathbb{C}^{p,q}) = \{ q$-dim $\mathbb{C}$-subspaces W of $\mathbb{C}^{p,q}$ on which $|w_1|^2 + \dots + |w_p|^2 - |w_{p+1}|^2 - \dots - |w_{p+q}|^2$ is negative definite $\}$

$G = SU(p,q)$, $K = S(U(p) \times U(q)) := (U(p) \times U(q)) \cap SL_{p+q}$

symmetry with respect to $p \leftrightarrow q$: $W \mapsto W^\perp$

Real: unbounded: n-dim real hyperbolic space

$$X = H^n = \{ x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n \mid x_n > 0 \}$$

$\Big\downarrow \wr$ generalised Cayley map

bounded: $\{ y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \in \mathbb{R}^n \mid y_1^2 + \dots + y_n^2 < 1 \}$ n-dim real unit ball

$G = SO(n,1)^+$, $K = SO(2)$

matrix generalisation: $(p, q \geq 1)$

$$X = \{ Y \in M_{p,q}(\mathbb{R}) \mid I_q - {}^tY Y > 0 \} \xrightarrow{\sim} Gr_q^-(\mathbb{R}^{p,q})$$

$$Y \longmapsto \left(\text{span of the columns of } \begin{pmatrix} Y \\ I_q \end{pmatrix}\right) = W$$

$G = SO(p,q)^+$, $K = SO(p) \times SO(q)$

(note: $SO(p,q)^+ = SO(p,q)$ if $p, q \geq 2$)

again: symmetry w.r.t. $p \leftrightarrow q$: $W \mapsto W^\perp$

Accidental isomorphisms in small dimensions:

$(A_1$ outer$) = (A_1$ inner$) = (C_1) = (B_1)$

$SU(1,1) \simeq SL_2(\mathbb{R}) = Sp_2(\mathbb{R}) \simeq Spin(2,1)$

$$\left(\begin{array}{ccc} GL_2(\mathbb{R}) & \xrightarrow{\sim} & GSpin(2,1) \\ PGL_2(\mathbb{R}) & \xrightarrow{\sim} & SO(2,1) \\ \cup & & \cup \\ SL_2(\mathbb{R})/\{\pm I\} & \xrightarrow{\sim} & SO(2,1)^+ \end{array} \right)$$

$SU(2) \simeq SL_1(\mathbb{H}) \simeq Spin(3)$

$$\left(\begin{array}{ccc} \mathbb{H}^\times & \xrightarrow{\sim} & GSpin(3) \\ \mathbb{H}^\times/\mathbb{R}^\times & \xrightarrow{\sim} & SO(3) \\ \uparrow\wr & & \\ SL_1(\mathbb{H})/\{\pm 1\} & & \end{array} \right)$$

$(A_1 \times A_1 = D_2)$ $SL_2(\mathbb{C})/\{\pm I\} = PGL_2(\mathbb{C}) \xrightarrow{\sim} SO(3,1)^+$

$(SL_2(\mathbb{R}) \times SL_2(\mathbb{R}))/\{\pm(I,I)\} \xrightarrow{\sim} SO(2,2)$, $(SL_1(\mathbb{H}) \times SL_1(\mathbb{H}))/\{\pm(1,1)\} \xrightarrow{\sim} SO(4)$