Rankin – Cohen brackets

$sl(2)$ – modules with lowest weight

$sl(2) = \langle X, Y, H \rangle \qquad [X,Y] = H, \quad [H,X] = 2X, \quad [H,Y] = -2Y$

$Z(U(sl(2))) = \mathbb{C}[\Omega], \quad \Omega = XY + YX + \frac{H^2}{2} = 2XY + \frac{(H-1)^2-1}{2}$

$[\Omega, X] = [\Omega, Y] = [\Omega, H] = 0 \qquad = 2YX + \frac{(H+1)^2-1}{2}$

Lowest weight modules: $(\lambda, \mu \in \mathbb{C})$

$M(\lambda) = \bigoplus_{n\geq 0} \mathbb{C}u_n, \quad u_n = X^n u_0, \quad \underbrace{Yu_0 = 0}, \quad Hu_0 = \lambda u_0$

lowest weight vector, of weight λ

$\Rightarrow \; Hu_n = (\lambda+2n)u_n,$

Action of Ω: $\Omega u_0 = \left(2XY + \frac{(H-1)^2-1}{2}\right) u_0 = \frac{(\lambda-1)^2-1}{2} u_0 \Rightarrow \boxed{\Omega = \frac{(\lambda-1)^2-1}{2} \cdot I}$

$n \to n-1$

Action of Y: $\Omega u_n = \left(2XY + \frac{(\lambda+2n-1)^2-1}{2}\right) u_n \Rightarrow \boxed{Yu_n = n(\lambda+n-1)u_{n-1}}$

Lowest weight vectors in $M(\lambda) \otimes M(\mu)$:

$M(\lambda) = \bigoplus_{m\geq 0} \mathbb{C}\underbrace{X^m u_0}_{u_m}, \quad Hu_0 = \lambda u_0, \; Yu_0 = 0$

$M(\mu) = \bigoplus_{n\geq 0} \mathbb{C}\underbrace{X^n v_0}_{v_n}, \quad Hv_0 = \mu v_0, \; Yv_0 = 0$

Action on $M \otimes N$: $X(u\otimes v) = (Xu)\otimes v + u \otimes (Xv)$

So: (0) $u_0 \otimes v_0 \xmapsto{Y} 0$

(1) $u_0 \otimes v_1 \xmapsto{Y} u_0 \otimes Yv_1 = u_0 \otimes \mu v_0$

$u_1 \otimes v_0 \xmapsto{Y} \lambda u_0 \otimes v_0$

$\Rightarrow Y(\lambda u_0 \otimes v_1 - u_0 \otimes \mu v_1) = 0$

(2) $Y(\lambda(\lambda+1)u_0 \otimes v_2 - 2(\lambda+1)(\mu+1)u_1 \otimes v_1 + \mu(\mu+1)u_2 \otimes v_0) = 0$

Prop. $(M(\lambda) \otimes M(\mu))^{Y=0} \supseteq \bigoplus_{n \geq 0} \mathbb{C}w_n$ (equality if $\lambda+\mu, \lambda, \mu \notin \mathbb{Z}_{\leq 0}$) $\quad (a)_k = a(a+1)\cdots(a+k-1)$

$$w_n = \sum_{\substack{k+l=n \\ k,l\geq 0}} (-1)^k \binom{n}{k} (\lambda+k)_l (\mu+l)_k \, u_k \otimes v_l .$$

$(Yw_n = 0, \quad Hw_n = \lambda+\mu+2n)$

PR. Explicit calculation.

Cor. $M(\lambda) \otimes M(\mu) \simeq \bigoplus_{n\geq 0} M(\lambda+\mu+2n)$ (if $\lambda, \mu \notin \mathbb{Z}_{\leq 0}$, $\lambda+\mu \notin$)

Rmk: If $\lambda = -n \in \mathbb{Z}_{\leq 0}$, then $\bigoplus_{k>n} \mathbb{C}u_k \subset M(-n)$ is a submodule, since $Yu_{n+1} = 0 \Rightarrow$ the quotient $M(-n)/\bigoplus_{k>n} \mathbb{C}u_k =: L(-n)$ is the $(n+1)$-dimensional representation of $sl(2)$ on homogeneous polynomials $F(X,Y)$ of degree n.

<u>Our situation</u>: $\underbrace{\langle X_+, X_-, H_0 \rangle}_{\mathfrak{sl}(2)}$ acting on $\{F : G = SL_2(\mathbb{R}) \to \mathbb{C}\}$

$$X_\pm = e^{\mp 2i\theta}\left(y(\partial_y \pm i\partial_x) \pm \frac{i}{2}\partial_\theta\right)$$

$$H_0 = i\partial_\theta$$

Under weight λ lift $\{f : \mathcal{H} \to \mathbb{C}\} \longrightarrow \{\tilde{f} : G \to \mathbb{C}$, $\tilde{f}(g h_\theta) = e^{-i\lambda\theta}\tilde{f}(g)\}$

$$\tilde{f}(g) = f(x+iy)\, y^{\lambda/2} e^{-i\lambda\theta}$$

$$\left(x+iy = g(i),\quad g = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix} h_\theta\right)$$

$X_+ \tilde{f} = \widetilde{R_\lambda f}$ (lift of wt $\lambda+2$)

$X_- \tilde{f} = \widetilde{L_\lambda f}$ (— " — $\lambda-2$)

$$R_\lambda = 2i\partial_\tau + \frac{\lambda}{y}$$

$$L_\lambda = -2iy^2\partial_{\bar{\tau}}$$

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f is holomorphic $\iff$ $X_-\tilde{f} = 0$ $\iff$ $\tilde{f}$ is a lowest weight vector, of weight λ.

<u>Notation</u>: $(k \geq 1)$

$R_\lambda^k := R_{\lambda+2(k-1)} \circ \cdots \circ R_{\lambda+2} \circ R_\lambda$ (increases weight by $2k$)

$L_\lambda^k := L_{\lambda-2(k-1)} \circ \cdots \circ L_{\lambda-2} \circ L_\lambda$ (decreases wt by $2k$)

<u>Previous discussion implies</u>: for fixed λ, μ, define (for $f, g \in \mathcal{C}^\infty(\mathcal{H})$)

$$[f,g]'_n := \sum_{\substack{k+l=n \\ k,l \geq 0}} (-1)^k \binom{n}{k} (\lambda+k)_l (\mu+l)_k (R_\lambda^k f)(R_\mu^l g).$$

(1) If $\alpha \in GL_2(\mathbb{R})^+$ and if $f|_\lambda \alpha = f$, $g|_\mu \alpha = g$, then

$$[f,g]'_n \big|_{\lambda+\mu+2n} \alpha = [f,g]'_n$$

(2) If f, g are holomorphic, so is $[f,g]'_n$.

<u>Cor</u>: $[f,g]_n := (-4\pi)^{-n}[f,g]'_n$ is equal to

$$\sum_{\substack{k+l=n \\ k,l \geq 0}} (-1)^k \binom{n}{k} (\lambda+k)_l (\mu+l)_k (D^k f)(D^l g), \quad \left(D = \frac{1}{2\pi i}\frac{d}{d\tau} = q\frac{d}{dq}\right)$$

(the non-holomorphic terms $y^{(a+b)-n}(D^a f)(D^b g)$ cancel each other if $a+b < n$)

Rmk: More precisely, $\forall \alpha \in GL_2(\mathbb{R})^+$

$$[f|_\lambda \alpha, g|_\mu \alpha]_n = [f,g]_n \big|_{\lambda+\mu+2n} \alpha$$

Terminology: $[f,g]_n$ is the n-th Rankin – Cohen bracket of f, g.
(up to the factor $(2\pi i)^{-n}$)

Application to Rankin products: $f=\sum_{n\ge 1} a_n q^n$, $g=\sum_{n\ge 0} b_n q^n$

Prop. If $f\in S_k(SL_2(\mathbb{Z}))$, $g\in M_{l_1}(SL_2(\mathbb{Z}))$, $k=l_1+l_2+2m$, where $l_2>2$ is even, and $m\ge 0$, then

$$(f,[g,E_{l_2}]_m)_{SL_2(\mathbb{Z})} = \frac{\Gamma(k-1)\,\Gamma(l_2+m)}{(4\pi)^{k-1}\,\Gamma(l_2)} \sum_{n=1}^{\infty} \frac{a_n \overline{b_n}}{n^{k-m-1}}$$

Pf: For each $\alpha=\begin{pmatrix}a&b\\c&d\end{pmatrix}\in SL_2(\mathbb{Z})$ and $m\ge 0$,

$$(\partial_\tau^m g)(\alpha(\tau)) = \sum_{l=0}^{m} \binom{m}{l}\frac{\Gamma(l_1+m)}{\Gamma(l_1+l)}\, c^{m-l}(c\tau+d)^{l_1+m+l} \underbrace{(\partial_\tau^l g)(\tau)}_{(2\pi i)^l \sum_{n\ge 0} n^l b_n q^n}$$

$\Gamma = SL_2(\mathbb{Z})$

let $G_n(\tau) := \sum_{\alpha=\begin{pmatrix}a&b\\c&d\end{pmatrix}\in\Gamma_\infty\backslash\Gamma} \underbrace{(c\tau+d)^{-k}\, e^{2\pi i n\left(\frac{a\tau+b}{c\tau+d}\right)}}_{q^n|_k\alpha}$; then

$$(2\pi i)^m \sum_{n\ge 0} n^m b_n G_n(\tau) = \sum_{\alpha\in\Gamma_\infty\backslash\Gamma} (c\tau+d)^{-k}\, \partial_\tau^m(\alpha(\tau)) =$$

$$= \sum_{l=0}^{m} \binom{m}{l}\frac{\Gamma(l_1+m)}{\Gamma(l_1+l)}(\partial_\tau^l g)(\tau) \underbrace{\sum_{\alpha\in\Gamma_\infty\backslash\Gamma} c^{m-l}(c\tau+d)^{l-m-l_2}}_{(-1)^{m-l}\,\partial_\tau^{m-l} E_{l_2}}$$

$$= (2\pi i)^m \frac{\Gamma(l_2)}{\Gamma(l_2+m)}\,[g,E_{l_2}]_m$$

As $(f,G_n)_{SL_2(\mathbb{Z})} = \frac{\Gamma(k-1)}{(4\pi n)^{k-1}}\, a_n \quad \forall n\ge 1$, Prop. follows.

Special case: $g = E_{l_1}$, $f \in S_k(SL_2(\mathbb{Z}))$ normalised Eigenform

$k = l_1 + l_2 + 2m$ $\qquad$ $l_1, l_2 > 2$ even, $l_1 \neq l_2$

$$(f, [E_{l_1}, E_{l_2}]_m)_{SL_2(\mathbb{Z})} = (-1)^{l_2/2} \frac{2l_1}{B_{l_1}} \frac{2l_2}{B_{l_2}} \frac{\Gamma(k-1)}{2^{k-1}\Gamma(k-m-1)} \cdot \Lambda(f, k-m-1)\Lambda(f, l_2+m)$$

(if $l_1 > l_2$, interchange $E_{l_1} \leftrightarrow E_{l_2}$)

$$\boxed{\begin{aligned}\Lambda(f,s) &= (2\pi)^{-s}\Gamma(s)L(f,s)\\ &= (-1)^k \Lambda(f, k-s)\end{aligned}}$$

Rmk: $[E_{l_1}, E_{l_2}]_m$ has rational Fourier coefficients

$$\Longrightarrow \frac{(f, [E_{l_1}, E_{l_2}]_m)}{(f,f)} \in K_f = \mathbb{Q}(a_1, a_2, \ldots)$$

Exercise: For $f = \Delta_{16} = \Delta E_4 \in S_{16}(SL_2(\mathbb{Z}))$, check that:

(a) $\dfrac{\Lambda(f,a)\Lambda(f,b)}{(f,f)}$ is an explicit rational number whenever ($8 < a, b < 16$ and $a \not\equiv b \pmod 2$)

(b) $\Lambda(f, a+2)/\Lambda(f,a)$ is an explicit rational number if $0 < a < 14$.
