Theta functions as functions of z

Fix a lattice $L \subset \mathbb{C}$

Def. A theta-function w.r.t. L is a $\overbrace{\text{holomorphic}}^{\text{non-zero}}$ function $F: \mathbb{C} \to \mathbb{C}$ such that $\forall u \in L \ \exists\, a(u), b(u) \in \mathbb{C} \ \forall z \in \mathbb{C}$ $F(z+u) = e^{a(u)z+b(u)} F(z)$.

Fundamental example: $L = \mathbb{Z}\tau + \mathbb{Z}$, $\tau \in \mathcal{H}$

$$\boxed{\theta(z,\tau) = \sum_{n\in\mathbb{Z}} e^{\pi i n^2 \tau + 2\pi i n z} = \sum_{n \in \mathbb{Z}} e\left(\frac{n^2\tau}{2} + nz\right)} \qquad e(x) = e^{2\pi i x}$$

Properties: $\theta(z+1,\tau) = \theta(z,\tau)$, $\theta(z+\tau,\tau) = e^{-(\pi i \tau + 2\pi i z)}\theta(z,\tau)$ $\quad(*)$

$\overset{\text{induction}}{\Longrightarrow}$ $\forall u,v \in \mathbb{Z}$ $\quad \theta(z+u\tau+v,\tau) = e^{-\pi i u^2\tau - 2\pi i u z}\theta(z,\tau)$

Abstract formulation: for $u,v \in \mathbb{R}$, define operators A_u, B_v on $\{f: \mathbb{C} \to \mathbb{C}\}$ by
$$\boxed{(A_u f)(z) := e^{\pi i u^2 \tau + 2\pi i u z} f(z+u\tau), \quad (B_v f)(z) := f(z+v)}$$

Commutation rules: $A_u A_{u'} = A_{u+u'}$, $B_v B_{v'} = B_{v+v'}$, $B_v A_u = e^{2\pi i u v} A_u B_v$

Group generated by $\{A_u\}$ and $\{B_v\}$: the real Heisenberg group Heis

$$\text{Heis} = \{\lambda A_u B_v = U_{(\lambda,a,b)} \mid \lambda \in \mathbb{C}^\times,\ |\lambda| = 1,\ u,v \in \mathbb{R}\} \qquad (\lambda \in U(1))$$

with $(\lambda A_u B_v)(\lambda' A_{u'} B_{v'}) = \lambda\lambda' A_{u+u'} B_{v+v'} e^{2\pi i u v}$

$U_{(\lambda,u,v)} U_{(\lambda',u',v')} = U_{(\lambda\lambda' e^{2\pi i u v},\, u+u',\, v+v')}$

$$\boxed{\begin{array}{l}\text{Action of Heis on } L^2(\mathbb{R}):\\ (U_{\lambda,u,v}F)(x) = \lambda e^{2\pi i u x} F(x+v)\end{array}}$$

Central extension $\quad 1 \to U(1) \to \text{Heis} \to \mathbb{R}\times\mathbb{R} \to 0$

$$U_{(\lambda,u,v)} \mapsto (u,v)$$

Functional eqns $(*)$ of $\theta(z,\tau)$ $\iff$ $\theta(z,\tau)$ is invariant under the subgroup $\{U_{(1,\mathbb{Z},\mathbb{Z})}\}$ of Heis

$$\boxed{\begin{array}{l}\text{Note: } \theta(z,\tau) = \sum_{n\in\mathbb{Z}} A_n(\mathbb{1})\\ \mathbb{1}(z) = 1 \quad \forall z \in \mathbb{C}\end{array}}$$

Applying Heis to θ:
$$\boxed{\theta\begin{bmatrix}u\\v\end{bmatrix}(z,\tau) := B_v A_u \theta = \sum_{n\in\mathbb{Z}} e^{\pi i (n+u)^2\tau + 2\pi i (n+u)(z+v)} \quad (u,v \in \mathbb{R})}$$

Properties: $\underbrace{B_1\left(\theta\begin{bmatrix}u\\v\end{bmatrix}\right)}_{\theta\begin{bmatrix}u\\v\end{bmatrix}(z+1,\tau)} = B_v e^{2\pi i u} A_u \underbrace{B_1\theta}_{\theta} = e^{2\pi i u}\theta\begin{bmatrix}u\\v\end{bmatrix}(z,\tau)$

$A_1\left(\theta\begin{bmatrix}u\\v\end{bmatrix}\right) = e^{-2\pi i v} B_v A_u \underbrace{A_1\theta}_{\theta} \implies \theta\begin{bmatrix}u\\v\end{bmatrix}(z+\tau,\tau) = e^{-(\pi i\tau + 2\pi i z)} e^{-2\pi i v}\theta\begin{bmatrix}u\\v\end{bmatrix}(z,\tau)$

General principle: (a) transformation rules for θ-fns in variable z are given by Heis

(b) — " — in variable τ are given by $SL_2(\mathbb{R})$ (rather, its 2-fold covering)

General background

$L \subset \mathbb{C}$ lattice

Goal: write elliptic functions $f \in \mathcal{M}(\mathbb{C}/L)$ as quotients of "nice" holomorphic functions $\mathbb{C} \to \mathbb{C}$

Prop. (1) Each $f \in \mathcal{M}(\mathbb{C}) \setminus \{0\}$ can be written as $f = F_1/F_2$, $F_1, F_2 \in \mathcal{O}(\mathbb{C})$ with no common zeroes.

(2) If $f = G_1/G_2$ for another pair $G_1, G_2 \in \mathcal{O}(\mathbb{C})$ with no common zeroes, then $\exists\ H : \mathbb{C} \to \mathbb{C}^\times$ holomorphic such that $F_k H = G_k$ $(k=1,2)$

(H is of the form $H = e^{\tilde{H}}$, $\tilde{H} : \mathbb{C} \to \mathbb{C}$ holomorphic, since $\pi_1(\mathbb{C}) = \{1\}$)

Pf: (1) theory of Weierstrass products $\Rightarrow \exists\ F_1 \in \mathcal{O}(\mathbb{C})$ with the same zeroes as f (including multiplicities); then $F_2 := F_1/f \in \mathcal{O}(\mathbb{C})$ and $f = F_1/F_2$.

(2) $\operatorname{div}(F_k) = \operatorname{div}(G_k)$ $(k=1,2)$ $\Longrightarrow$ $G_1/F_1 = G_2/F_2$ is holomorphic with no zeroes

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Apply Prop. to $0 \neq f \in \mathcal{M}(\mathbb{C}/L)$: $\forall u \in L$ $\dfrac{F_1(z+u)}{F_2(z+u)} = \dfrac{F_1(z)}{F_2(z)} \overset{(2)}{\Longrightarrow} F_k(z+u) = e_u(z) F_k(z)$ $(k=1,2)$

$e_u : \mathbb{C} \to \mathbb{C}^\times$ holomorphic

Compatibility w.r.t. $(z+u)+v = z+(u+v)$: $\boxed{e_{u+v}(z) = e_u(z+v)\, e_v(z) \quad \text{1-cocycle } (u,v \in L)}$ (*)

non-uniqueness of F_k $\Longleftrightarrow$ gauge transformation $\tilde{F}_k = HF_k$ $(k=1,2)$
$H : \mathbb{C} \to \mathbb{C}^\times$ holomorphic

$\tilde{e}_u(z) = e_u(z) \dfrac{H(z+u)}{H(z)}$ cohomologous 1-cocycles (equivalent)

Geometry: given $\{e_u : \mathbb{C} \to \mathbb{C}^\times$ holomorphic$\}_{u \in L}$ satisfying the 1-cocycle rule (*)

L acts on

$$\begin{array}{ccccc} \mathbb{C} \times \mathbb{C} \ni (z,t) & & (z,t) & \longmapsto & (z+u,\ e_u(z)t) \\ \downarrow p_1 \quad\ \downarrow & \text{by} & \downarrow & & \downarrow \\ \mathbb{C} \ni z & & z & \longmapsto & z+u \end{array}$$

Quotient by L: $\mathcal{L} = (\mathbb{C} \times \mathbb{C})/L \xrightarrow{\overline{p_1}} \mathbb{C}/L$ is a holomorphic line bundle (vector bundle of $rk = 1$)

Sections of $\overline{p_1}$: $\left\{\begin{matrix}\text{holomorphic} \\ \text{meromorphic}\end{matrix}\right\}$ sections of $\overline{p_1}$ are $\left\{\begin{matrix}\text{hol.} \\ \text{merom.}\end{matrix}\right\}$ $F : \mathbb{C} \to \mathbb{C}$ such that $\forall u \in L$ $F(z+u) = e_u(z) F(z)$

Space of such sections: $\left\{\begin{matrix}\Gamma(\mathbb{C}/L, \mathcal{O}(\mathcal{L})) \\ \Gamma(\mathbb{C}/L, \mathcal{M}(\mathcal{L}))\end{matrix}\right\}$

Cohomologous 1-cocycles $\longleftrightarrow$ isomorphic line bundles:

$\mathcal{L} \xrightarrow{\sim} \tilde{\mathcal{L}}$, $\Gamma(\mathbb{C}/L, \mathcal{O}(\mathcal{L})) \xrightarrow{\sim} \Gamma(\mathbb{C}/L, \mathcal{O}(\tilde{\mathcal{L}}))$

$(z,t) \longmapsto (z, H(z)t)$ $\qquad F \longmapsto HF$

Fundamental result on line bundles on $\mathbb{C}/L$:

every 1-cocycle $\{e_u(z)\}$ as above is cohomologous to a 1-cocycle of the form $\{e^{a(u)z+b(u)}\}_{u\in L}$ (cf. [Mu AV], ch. 1 – works for $\mathbb{C}^n$/lattice)

From now on: consider only $\{e_u(z)=e^{a(u)z+b(u)}\}_{u\in L}$ $(a(u), b(u)\in\mathbb{C})$.

Equivalence between two such cocycles is given by a gauge transformation by $H(z)=e^{Az^2+Bz}$ $(A,B\in\mathbb{C})$.

Prop (1) 1-cocycle identity for $e_u(z) \iff \forall u,v\in L$ $a(u+v)=a(u)+a(v)$

$$b(u+v)\equiv a(u)v+b(u)+b(v) \pmod{2\pi i\mathbb{Z}}$$
$$\equiv a(v)u+b(v)+b(u)$$

(2) $\forall u,v\in\mathbb{Z}$ $\begin{vmatrix} a(u) & u \\ a(v) & v\end{vmatrix}\in 2\pi i\mathbb{Z}$

(3) let $b(u)=: c(u)+\frac{1}{2}ua(u)$, $(u\in L)$; then

$\forall u,v\in L$ $\quad c(u+v)-c(u)-c(v)\equiv\frac{1}{2}\begin{vmatrix} a(u) & u \\ a(v) & v\end{vmatrix} \pmod{2\pi i\mathbb{Z}} \equiv \pi i E(u,v) \in \pi i\mathbb{Z} \pmod{2\pi i\mathbb{Z}}$

(4) $E: L\times L\longrightarrow\mathbb{Z}$, $\quad E(u,v):=\frac{1}{2\pi i}\begin{vmatrix} u & a(u) \\ v & a(v)\end{vmatrix}$ is $\mathbb{Z}$-bilinear and skew-symmetric.

Pf: (1) by definition; (2) subtract the two lines in (1); (3) use linearity of $u\mapsto a(u)$ (4) linearity of $u\mapsto a(u)$ and (2).

Rmk: $E: H_1(\mathbb{C}/L,\mathbb{Z})\times H_1(\mathbb{C}/L,\mathbb{Z})\longrightarrow\mathbb{Z}$ is the 1st Chern class $c_1(\mathcal{L})\in H^2(\mathbb{C}/L,\mathbb{Z})$ of the line bundle $\mathcal{L}$ defined by $\{e_u(z)\}$

Zeroes of sections of $\mathcal{L}$: if $0\neq F\in\Gamma(\mathbb{C}/L,\mathcal{M}(\mathcal{L}))$, then

$$\sum_{x\in\mathbb{C}}\mathrm{ord}_x(F)(x) \quad\text{satisfies}\quad \forall u\in L\ \forall x\in\mathbb{C}\ \ \mathrm{ord}_{x+u}(F)=\mathrm{ord}_x(F)$$

It makes sense, therefore, to consider only the finite sum

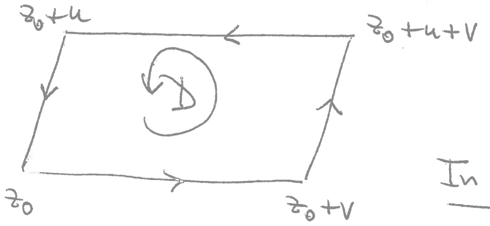
$$\boxed{\mathrm{div}(F):=\sum_{x\in\mathbb{C}/L}\mathrm{ord}_x(F)(x)\in\mathrm{Div}(\mathbb{C}/L)}$$

Prop. If $0\neq F\in\Gamma(\mathbb{C}/L,\mathcal{M}(\mathcal{L}))$, where $\mathcal{L}$ is given by $\{e_u(z)=e^{a(u)z+b(u)}\}$, then, for any $u,v\in L$ such that $\mathrm{Im}(u/v)>0$,

(1) $$\boxed{\sum_{x\in\mathbb{C}/(\mathbb{Z}u+\mathbb{Z}v)}\mathrm{ord}_x(F)=E(u,v)}$$

(2) $$\boxed{\sum_{x\in\mathbb{C}/(\mathbb{Z}u+\mathbb{Z}v)}\mathrm{ord}_x(F)x\equiv\frac{u+v}{2}E(u,v)+\frac{1}{2\pi i}\begin{vmatrix} c(u) & u \\ c(v) & v\end{vmatrix} \pmod{L}}$$

Pf: fix generic $z_0 \in \mathbb{C}$; need to compute: (1) $\frac{1}{2\pi i}\int_{\partial D} \frac{F'}{F}(z)\,dz$



(2) $\frac{1}{2\pi i}\int_{\partial D} z\frac{F'}{F}(z)\,dz \pmod L$

In (1): $\frac{1}{2\pi i}\Big[\underbrace{\int_{z_0}^{z_0+v} -(a(u)z+b(v))'\,dz}_{-v\,a(u)} + \underbrace{\int_{z_0}^{z_0+u} (a(v)z+b(v))'\,dz}_{u\,a(v)}\Big]$

In (2): exercise.

Canonical normalisation of 1-cocycles $\{e^{a(u)z+b(u)}\}$:

Observe: (1) $a: L \longrightarrow \mathbb{C}$ $\mathbb{Z}$-linear $\Longrightarrow$ extends uniquely to $a: L\otimes\mathbb{R} = \mathbb{C} \longrightarrow \mathbb{C}$ $\mathbb{R}$-linear

$\Longrightarrow$ $a(w) = a_1\bar{w} + a_2 w$ $\forall w\in\mathbb{C}$ ($a_1, a_2 \in \mathbb{C}$ unique)

(2) $\mathrm{Re}(c): L \longrightarrow \mathbb{R}$ is $\mathbb{Z}$-linear $\Longrightarrow$ extends uniquely to $\mathrm{Re}(c): L\otimes\mathbb{R}=\mathbb{C}\longrightarrow\mathbb{R}$ $\mathbb{R}$-linear

$\Longrightarrow$ $\mathrm{Re}(c)(w) = \mathrm{Re}(bw)$ $\forall w\in\mathbb{C}$ ($b\in\mathbb{C}$ unique)

(3) Gauge transformation by $e^{-a_2z^2/2 - bz}$ gives an equivalent 1-cocycle in which $a_2 = 0 = b$.

Conclusion: Each equivalence class of 1-cocycles $\{e^{a(u)z+b(u)}\}$ contains a canonical representative for which

$a: L\otimes\mathbb{R} = \mathbb{C}\longrightarrow\mathbb{C}$ is $\mathbb{C}$-antilinear

and $c: L \longrightarrow i\mathbb{R}$.

Explicitly: (a) $H(w,z) := \frac{a(w)z}{\pi}: \mathbb{C}\times\mathbb{C}\longrightarrow\mathbb{C}$ is hermitian (antilinear in w, linear in z)

(b) $E = \mathrm{Im}(H): L\times L \longrightarrow \mathbb{Z}$ (skew-symmetric)

(c) $\alpha(u) := e^{c(u)}$ $(u\in L)$ satisfy $|\alpha(u)| = 1$,

$\alpha(u+v) = \alpha(u)\,\alpha(v)\,(-1)^{E(u,v)}$

(d) $e_u(z) = e^{\pi H(z,u) + \frac{\pi}{2}H(u,u)}\,\alpha(u)$

Rmk. the same classification holds for $\mathbb{C}^n/L$, $L\subset\mathbb{C}^n$ lattice (see [Mu, AV], ch. 1)

1-cocycles trivial on $\mathbb{Z}\omega_2 \subset L$ (another, very useful, normalisation) satisfied by $\theta(z,\tau)$

Fix: positive basis $L=\mathbb{Z}\omega_1+\mathbb{Z}\omega_2$, $\mathrm{Im}\left(\frac{\omega_1}{\omega_2}\right)>0$

Trivialisation of $\{e_u\}$ along $\mathbb{Z}\omega_2$: apply gauge transformation by

$e^{-P(z)}$, where $P(z)=Az^2+Bz$, $P(z+\omega_2)-P(z)=a(\omega_2)z+b(\omega_2)$

$\Longrightarrow$ get equivalent 1-cocycle for which $e_{\omega_2}(z)=1$ ($\Longrightarrow \forall n\in\mathbb{Z}\ e_{n\omega_2}(z)=1$)

Conditions on $e_{\omega_1}(z)=e^{a(\omega_1)z+b(\omega_1)}$: $b(\omega_1)=\frac{\omega_1 a(\omega_1)}{2}+c(\omega_1)$

$$\mathbb{Z}\ni\boxed{m:=E(\omega_1,\omega_2)}=\frac{1}{2\pi i}\begin{vmatrix}\omega_1 & a(\omega_1)\\ \omega_2 & \underbrace{a(\omega_2)}_{=0}\end{vmatrix}\Longrightarrow a(\omega_1)=-2\pi i m/\omega_2$$

$$\underbrace{e_{\omega_1}(z)=e^{-2\pi i\left(m\left(z+\frac{\omega_1}{2}\right)+c\right)/\omega_2}}_{\text{depends only on } c \ (\mathrm{mod}\ \mathbb{Z}\omega_2).}, \quad -2\pi i c=\omega_2 c(\omega_1)$$

Gauge transformations preserving $e_{\omega_2}(z)=1$:

by e^{Bz}, $B\omega_2\in 2\pi i\mathbb{Z}\Longrightarrow e_{\omega_1}(z)$ changes by $e^{B\omega_1}$,

$B\omega_1\in\frac{2\pi i\omega_1}{\omega_2}\mathbb{Z}\Longrightarrow$ c changes by $-\frac{\omega_2}{2\pi i}B\omega_1\in\omega_1\mathbb{Z}$.

Conclusion: thm. Equivalence classes of $\{\underbrace{e^{a(u)z+b(u)}}_{e_u(z)}\}$ are classified by pairs $(m,c)\in\mathbb{Z}\times(\mathbb{C}/L)$, $\boxed{m=E(\omega_1,\omega_2)}$

$$\updownarrow$$

$$\left(e_{\omega_2}(z)=1,\ e_{\omega_1}(z)=e^{-2\pi i\left(m\left(z+\frac{\omega_1}{2}\right)+c\right)/\omega_2}\right)$$

Ex: normalised lattice $\omega_1=\tau\in\mathcal{H}$, $\omega_2=1$

$\theta(z,\tau)$ corresponds to $e_1(z)=1$, $e_\tau(z)=e^{-2\pi i\left(z+\frac{\tau}{2}\right)}$ $\Longleftrightarrow$ $m=1$, $c=0$

comparison of the two normalisations for this class of 1-cocycles:

if $H(z,w)=\frac{z\bar{w}}{\mathrm{Im}(\tau)}$, then the gauge transformation

by $e^{-\pi z^2/2\mathrm{Im}(\tau)}$ transforms the canonical 1-cocycle

$$e_u^{can}(z)=\alpha(u)\,e^{\pi H(z,u)+\frac{\pi}{2}H(u,u)} \qquad \underset{a,b\in\mathbb{Z}}{u=a\tau+b},\ \alpha(u)=(-1)^{ab}$$

to the above 1-cocycle $e_1(z)=1$, $e_\tau(z)=e^{-2\pi i\left(z+\frac{\tau}{2}\right)}$

Dimension of $\Gamma(\mathbb{C}/L, \mathcal{O}(\mathcal{L}))$

Thm. If $\mathcal{L}$ is given by $\{e^{a(u)z+b(u)}\}$ and if $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$, $\mathrm{Im}\left(\frac{\omega_1}{\omega_2}\right) > 0$

then
$$\dim_{\mathbb{C}} \Gamma(\mathbb{C}/L, \mathcal{O}(\mathcal{L})) = \begin{cases} 0 & \text{if } E(\omega_1,\omega_2) < 0 \\ E(\omega_1,\omega_2) & \text{if } E(\omega_1,\omega_2) > 0. \end{cases}$$

$$\left(E(u,v) = \frac{1}{2\pi i} \begin{vmatrix} u & a(u) \\ v & a(v) \end{vmatrix} \right).$$

Pf: $\forall\, 0 \neq F \in \Gamma(\mathbb{C}/L, \mathcal{M}(\mathcal{L}))$ $\quad \sum_{x \in \mathbb{C}/L} \mathrm{ord}_x(F) = \underbrace{E(\omega_1,\omega_2)}_{m}$

If $m<0 \Rightarrow F \notin \mathcal{O}(\mathbb{C})$.

If $m>0$: can assume $\omega_1 = \tau \in \mathcal{H}$, $\omega_2 = 1$, $e_1(z) = 1$, $e_\tau(z) = e^{-2\pi i(m(z+\frac{\tau}{2})+c)}$

If $F \in \Gamma(\mathbb{C}/L, \mathcal{O}(\mathcal{L}))$, then $F(z+1) = F(z) \Rightarrow F(z) = \sum_{n\in\mathbb{Z}} a_n e^{2\pi i n z}$

$F(z+\tau) = e_\tau(z)F(z) \Leftrightarrow \forall n \in \mathbb{Z}$
$$a_n e^{2\pi i n\tau} = e^{-2\pi i(m\tau/2+c)} a_{n+m}$$

$\Leftrightarrow \forall k \in \mathbb{Z}$ $\quad a_{\ell+mk} = a_\ell\, e^{2\pi i\left(\left(\frac{k^2\tau}{2} + k\ell\tau\right)+kc\right)}$ $\quad$ (and such F lies in $\Gamma(\mathbb{C}/L, \mathcal{O}(\mathcal{L}))$)

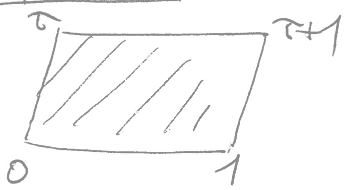
$$F(z) = \sum_{\ell=1}^{m} \underbrace{a_\ell}_{\text{arbitrary}} \sum_{k\in\mathbb{Z}} e^{\pi i k^2\tau + 2\pi i k(\ell\tau + c)}.$$

Basic θ-function $\theta(z,\tau)$: $\quad e_{a\tau+b}(z) = e^{-2\pi i\left(az + \frac{a^2\tau}{2}\right)}$ $\quad (a,b \in \mathbb{Z})$

$L = \mathbb{Z}\tau + \mathbb{Z}$

$$E(a\tau+b, c\tau+d) = \frac{1}{2\pi i}\begin{vmatrix} a\tau+b & -2\pi i a \\ c\tau+d & -2\pi i c \end{vmatrix} = ad-bc$$

$\Rightarrow E(\tau,1) = 1$ $\Rightarrow$ θ has a unique (simple) zero in the fundamental domain



$a(m\tau+n) = -2\pi i m$

$b(m\tau+n) = -\pi i m^2\tau = \frac{(m\tau+n)a(m\tau+n)}{2} + c(m\tau+n)$

$\Rightarrow c(m\tau+n) = \pi i m n \Rightarrow c(\tau) = c(1) = 0$

So: the unique zero of $\theta(z,\tau)$ is at $\underbrace{\frac{\tau+1}{2}E(\tau,1) + \frac{1}{2\pi i}\begin{vmatrix} c(\tau) & \tau \\ c(1) & 1 \end{vmatrix}}_{\frac{\tau+1}{2}} \pmod L$

Cor: $\theta\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right](z,\tau) = e^{\pi i u^2\tau + 2\pi i u(z+v)}\,\theta(z+u\tau+v,\tau)$ has a unique

$\forall u,v \in \mathbb{R}$ $\quad$ (simple) zero on $\mathbb{C}/L$, at $\left(\frac{1}{2}-u\right)\tau + \left(\frac{1}{2}-v\right) \pmod L$

$= B_v A_u \theta$

Projective embeddings $\mathbb{C}/L \longrightarrow \mathbb{P}^{N^2-1}(\mathbb{C})$ via θ-functions of level $N \geq 2$

Recall: $(A_u f)(z) = e^{2\pi i(\frac{u^2}{2}\tau + uz)} f(z+u\tau)$, $\quad (B_v f)(z) = f(z+v) \quad (u,v\in\mathbb{R})$

Heisenberg group: $\mathrm{Heis} = \{ U_{(\lambda,u,v)} = \lambda A_u B_v \mid \lambda \in U(1),\ u,v\in\mathbb{R}\}$

$A_u A_{u'} = A_{u+u'}$, $\quad B_v B_{v'} = B_{v+v'}$, $\quad B_v A_u = e^{2\pi i uv} A_u B_v \qquad (*)$

$\theta(z,\tau) = \sum_{n\in\mathbb{Z}} A_n(\mathbb{1})$, $\quad \theta\begin{bmatrix}u\\v\end{bmatrix} := B_v A_u \theta = \sum_{n\in\mathbb{Z}} e^{\pi i(n+u)^2\tau + 2\pi i(n+u)(z+v)}$

$\forall z\in\mathbb{C}\ \ \mathbb{1}(z) = z$

$$\boxed{\begin{aligned}\theta\begin{bmatrix}u\\v\end{bmatrix}(z+1,\tau) &= e^{2\pi i u}\,\theta\begin{bmatrix}u\\v\end{bmatrix}(z,\tau)\\ \theta\begin{bmatrix}u\\v\end{bmatrix}(z+\tau,\tau) &= e^{-2\pi i(z+\tau/2)}e^{-2\pi i v}\,\theta\begin{bmatrix}u\\v\end{bmatrix}(z,\tau)\end{aligned}}$$

We know: $\mathbb{C}\theta = \{ f\in\mathcal{O}(\mathbb{C}) \mid$ invariant by the action of $A_u, B_v\ \forall u,v\in\mathbb{Z}\}$
(calculation with Fourier coefficients of $f = \sum a_n e^{2\pi i n z}$)

Prop. For every integer $N\geq 1$,

$V_N := \{ f\in\mathcal{O}(\mathbb{C}) \mid$ invariant by the action of $A_u, B_v\ \forall u,v\in N\mathbb{Z}\} = \bigoplus_{u,v\in\frac{1}{N}\mathbb{Z}/\mathbb{Z}} \theta\begin{bmatrix}u\\v\end{bmatrix}$

$(\dim_{\mathbb{C}} = N^2)$

Pf: $\supseteq$ clear from $(*)$

$\subseteq$ $B_N f = f \iff f(z) = \sum_{n\in\frac{1}{N}\mathbb{Z}} c_n e^{2\pi i n z} = \sum_{n\in\frac{1}{N}\mathbb{Z}} c'_n \underbrace{e^{\pi i n^2\tau + 2\pi i n z}}_{A_n(\mathbb{1})}$

$A_N f = f \iff c'_n = c'_m$ if $n - m\in N\mathbb{Z}$ $\Big\} \Rightarrow V_N$ is of $\dim = N^2$.

linear independence of characters $n \mapsto e^{2\pi i u n}$ $(n\in\mathbb{Z})$ for $u = \frac{1}{N}, \frac{2}{N}, \dots, \frac{N}{N}$

$\Rightarrow \theta\begin{bmatrix}u\\v\end{bmatrix}$ $(u,v\in\frac{1}{N}\mathbb{Z}/\mathbb{Z})$ are linearly independent.

Rmk: If $u-u', v-v'\in\mathbb{Z} \Rightarrow \theta\begin{bmatrix}u'\\v'\end{bmatrix} = (\mathrm{const})\,\theta\begin{bmatrix}u\\v\end{bmatrix}$

~~Operator: $(D_N f)(z) = f(Nz)$~~

Finite Heisenberg group: $\mathrm{Heis}_N \subset \mathrm{Heis}$: generated by A_u, B_v $(u,v\in\frac{1}{N}\mathbb{Z})$

$\mathrm{Heis}_N = \{\lambda A_u B_v \mid Nu, Nv\in\mathbb{Z},\ \lambda^{N^2} = 1\}$

Prop. Heis_N acts on V_N.

Pf: $u,v\in\frac{1}{N}\mathbb{Z}$, $u',v'\in N\mathbb{Z}$
$\Rightarrow A_u, B_v$ commute with $A_{u'}, B_{v'}$
$\Rightarrow$ preserve V_N.

Explicit formulas:
$\forall u,v,u',v'\in\frac{1}{N}\mathbb{Z}$

(1) $\theta\begin{bmatrix}u\\v\end{bmatrix} = B_v A_u\theta$ | (2) $B_{v'}\theta\begin{bmatrix}u\\v\end{bmatrix} = \theta\begin{bmatrix}u\\v+v'\end{bmatrix}$ | (3) $A_{u'}\theta\begin{bmatrix}u\\v\end{bmatrix} = e^{-2\pi i u'v}\,\theta\begin{bmatrix}u+u'\\v\end{bmatrix}$

(4) $\forall m,n\in\mathbb{Z}\quad \theta\begin{bmatrix}u+m\\v+n\end{bmatrix} = e^{2\pi i n u}\,\theta\begin{bmatrix}u\\v\end{bmatrix}$

Thm. The action of Heis_N on V_N is irreducible. ("rigidity of the system $\theta\begin{bmatrix}u\\v\end{bmatrix}$, $u,v\in\frac{1}{N}\mathbb{Z}/\mathbb{Z}$")

Prop. $\forall u, v \in \frac{1}{N}\mathbb{Z}$, $F(z) := \theta\left[\begin{smallmatrix}u\\v\end{smallmatrix}\right](Nz,\tau)$ satisfies
$$F(z+1) = F(z), \quad F(z+\tau) = e^{-2\pi i\left(\frac{N^2}{2}\tau + Nz\right)} F(z)$$

Pf: $\theta\left[\begin{smallmatrix}u\\v\end{smallmatrix}\right]$ is invariant under A_N, B_N.

Cor. Fix representatives of $\frac{1}{N}\mathbb{Z}/\mathbb{Z}$ in $\frac{1}{N}\mathbb{Z}$. ($N \geq 2$) Then the map
$$z \stackrel{\varphi_N'}{\longmapsto} \left(\theta\left[\begin{smallmatrix}u\\v\end{smallmatrix}\right](Nz,\tau)\right)_{u,v\in\frac{1}{N}\mathbb{Z}/\mathbb{Z}} \in \mathbb{C}^{N^2}$$
$z \in \mathbb{C}$

defines a map $\varphi_N : \mathbb{C}/\underbrace{L}_{\mathbb{Z}\tau+\mathbb{Z}} \longrightarrow \mathbb{P}^{N^2-1}(\mathbb{C})$.

Pf: $\forall w \in L = \mathbb{Z}\tau + \mathbb{Z}$, $\forall z \in \mathbb{C}$: $\varphi_N'(z+w) \stackrel{\text{Prop.}}{=} \begin{pmatrix}\text{non-zero scalar}\\ \text{depending on } z, w\end{pmatrix} \varphi_N'(z)$

$$\left\{\text{zeroes of } \theta\left[\begin{smallmatrix}u\\v\end{smallmatrix}\right](Nz)\right\} = \frac{(1-u)\tau + (1-v)}{2N} + \frac{1}{N}L \implies \forall z \in \mathbb{C} \quad \varphi_N'(z) \neq (0,\dots,0).$$

Rmk: in fact, $\varphi_N : \mathbb{C}/L \longrightarrow \mathbb{P}^{N^2-1}(\mathbb{C})$ is an embedding (it is injective on points and on tangent vectors). Its image is an algebraic curve in $\mathbb{P}^{N^2-1}(\mathbb{C})$.

The case $N=2$

$\theta_{ab} := \theta_{ab}(0)$

Classical notation: for $a, b \in \{0,1\}$, $\theta_{ab}(z) = \theta_{ab}(z,\tau) := \theta\left[\begin{smallmatrix}a/2\\b/2\end{smallmatrix}\right](z,\tau)$

$$\theta_{ab}(z,\tau) = \sum_{n\in\mathbb{Z}} e^{\pi i(n+\frac{a}{2})^2\tau + 2\pi i(n+\frac{a}{2})(z+\frac{b}{2})} = \theta_{a0}(z+\tfrac{b}{2},\tau) = e^{\pi i a(z+\frac{b}{2}) + \frac{\pi i a\tau}{4}}\,\theta\left(z+\tfrac{a\tau+b}{2},\tau\right)$$

$\boxed{\theta_{00} = \theta}$ (classical $\theta_{11} = -$ our θ_{11})

Facts: (1) $\theta_{ab}(z+1) = (-1)^a\theta_{ab}(z)$, $\theta_{ab}(z+\tau) = (-1)^b e^{-2\pi i(z+\frac{\tau}{2})}\theta_{ab}(z)$ (and is characterised by this, up to a scalar multiple).

(2) $\theta_{ab}(-z) = \theta_{ab}(z)\cdot\begin{cases}-1 & a=b=1\\ 1 & \text{if not}\end{cases}$

(3) $\theta_{ab}(z)$ has simple zeroes at $\left(\frac{a+1}{2}\tau + \frac{b+1}{2}\right) + \mathbb{Z}\tau + \mathbb{Z}$

Quadratic relations between $\theta_{ab}(z)$

Note: each $F \in \{\theta_{ab}(z)^2\}$ satisfies
$$F(z+1) = F(z), \quad F(z+\tau) = e^{-2\pi i(2(z+\frac{\tau}{2}))}F(z)$$

We know: the space of $F \in \mathcal{O}(\mathbb{C})$ satisfying these relations has $\dim = 2$.

Cor. there exist two independent linear relations between $\theta_{00}(z)^2$, $\theta_{01}(z)^2$, $\theta_{10}(z)^2$, $\theta_{11}(z)^2$.

First quadratic relation

$$\operatorname{div}(\theta_{ab}(z)) = \frac{(a+1)\tau}{2} + \frac{b+1}{2} + \mathbb{Z}\tau + \mathbb{Z}$$

$\exists\, A, B \in \mathbb{C}$ $\quad \theta_{00}(z)^2 = A\,\theta_{01}(z)^2 + B\,\theta_{10}(z)^2$

$z = \frac{1}{2}$: $\quad \theta_{00}(\tfrac{1}{2})^2 = A\,\theta_{01}(\tfrac{1}{2})^2 \qquad z = \frac{\tau}{2}$: $\quad \theta_{00}(\tfrac{\tau}{2})^2 = B\,\theta_{10}(\tfrac{\tau}{2})^2$

$$\theta_{ab} = \theta_{ab}(0) = e^{\pi i a(b/2 + \tau/4)}\,\theta_{00}\left(\frac{a\tau+b}{2}\right), \quad \theta_{ab}(z) = \theta_{a0}(z + \tfrac{b}{2})$$

$$\theta_{00}(\tfrac{1}{2}) = \theta_{01}, \quad \theta_{00}(\tfrac{\tau}{2}) = \theta_{10}\, e^{-\pi i\tau/4}; \quad \theta_{10}(\tfrac{\tau}{2}) = e^{3\pi i\tau/4}\,\theta_{00}(\tau) = e^{-\pi i\tau/4}\,\theta_{00}$$

$$\theta_{01}(\tfrac{1}{2}) = \theta_{00}(1) = \theta_{00} \implies A = \left(\frac{\theta_{01}}{\theta_{00}}\right)^2, \quad B = \left(\frac{\theta_{10}}{\theta_{00}}\right)^2$$

Second quadratic relation

$$\theta_{01}(z)^2 = C\,\theta_{11}(z)^2 + D\,\theta_{10}(z)^2 \qquad = E\,\theta_{11}(z)^2 + F\,\theta_{00}(z)^2$$

$z = \frac{1}{2}$: $\quad \theta_{01}(\tfrac{1}{2})^2 = C\,\theta_{11}(\tfrac{1}{2})^2, \qquad z = 0$: $\quad \theta_{01}^2 = D\,\theta_{10}^2 = F\,\theta_{00}^2$

$$\theta_{11}(\tfrac{1}{2}) = \theta_{10}(1) = -\theta_{10} \implies C = \frac{\theta_{00}^2}{\theta_{10}^2}, \quad D = \frac{\theta_{01}^2}{\theta_{10}^2}, \quad F = \left(\frac{\theta_{01}}{\theta_{00}}\right)^2$$

$$\boxed{\left(\frac{\theta_{11}(z)\,\theta_{00}}{\theta_{01}(z)\,\theta_{10}}\right)^2 + \left(\frac{\theta_{10}(z)\,\theta_{01}}{\theta_{01}(z)\,\theta_{10}}\right)^2 = 1} \qquad (*)$$

$z = \frac{\tau+1}{2}$: $\quad \theta_{01}(\tfrac{\tau+1}{2})^2 = E\,\theta_{11}(\tfrac{\tau+1}{2})^2, \quad \theta_{01}(\tfrac{\tau+1}{2}) = \theta_{00}(\tfrac{\tau}{2}+1) = \theta_{00}(\tfrac{\tau}{2}) = \theta_{10}\,e^{-\pi i\tau/4}$

$$\theta_{11}(\tfrac{\tau+1}{2}) = e^{\pi i(\frac{\tau}{2}+1) + \frac{\pi i\tau}{4}}\,\theta(\tau+1) = -e^{-\pi i\tau/4}\,\theta_{00} \implies E = (\theta_{10}/\theta_{00})^2$$

$$\boxed{\left(\frac{\theta_{11}(z)\,\theta_{10}}{\theta_{01}(z)\,\theta_{00}}\right)^2 + \left(\frac{\theta_{00}(z)\,\theta_{01}}{\theta_{01}(z)\,\theta_{00}}\right)^2 = 1} \qquad (**)$$

Notation: $\quad (*) \quad x + (1-x) = 1$

$\qquad (**) \quad k^2 x + (1 - k^2 x) = 1$

$$x = \left(\frac{\theta_{11}(z)\,\theta_{00}}{\theta_{01}(z)\,\theta_{10}}\right)^2, \quad 1-x = \left(\frac{\theta_{10}(z)\,\theta_{01}}{\theta_{01}(z)\,\theta_{10}}\right)^2, \quad 1-k^2x = \left(\frac{\theta_{00}(z)\,\theta_{01}}{\theta_{01}(z)\,\theta_{00}}\right)^2$$

$$k = \left(\frac{\theta_{10}}{\theta_{00}}\right)^2, \quad k' = \left(\frac{\theta_{01}}{\theta_{00}}\right)^2$$

Prop: $\theta_{01}^4 + \theta_{10}^4 = \theta_{00}^4 \quad (\iff k^2 + k'^2 = 1) \qquad (q = e^{2\pi i\tau})$

Pf: Take $z = 0$ in the first quadratic equation.

$$\theta_{00} = \sum_{n\in\mathbb{Z}} q^{n^2/2}, \quad \theta_{01} = \sum_{n\in\mathbb{Z}} (-1)^n q^{n^2/2}, \quad \theta_{10} = \sum_{n\in\mathbb{Z}} q^{(n+1/2)^2/2}$$

Parameterisation of $E_\lambda : y^2 = x(1-x)(1-\lambda x)$, $\lambda = k^2$

$$x = \left(\frac{\theta_{11}(z)\theta_{00}}{\theta_{01}(z)\theta_{10}}\right)^2, \quad 1-x = \left(\frac{\theta_{10}(z)\theta_{01}}{\theta_{01}(z)\theta_{10}}\right)^2, \quad 1-k^2x = \left(\frac{\theta_{00}(z)\theta_{01}}{\theta_{01}(z)\theta_{00}}\right)^2, \quad k = \left(\frac{\theta_{10}}{\theta_{00}}\right)^2$$

$$y = -\frac{\theta_{00}(z)\theta_{10}(z)\theta_{11}(z)}{\theta_{01}(z)^3}\,\frac{\theta_{01}^2}{\theta_{10}^2}$$

Question: relate dz on $\mathbb{C}/(\mathbb{Z}\tau+\mathbb{Z})$ to $\frac{dx}{y}$ on E_λ.

Need to compute dx/dz

Prop. $\begin{vmatrix} \theta_{11}'(z) & \theta_{01}'(z) \\ \theta_{11}(z) & \theta_{01}(z) \end{vmatrix} =: F(z)$ satisfies $F(z+1) = -F(z)$
$F(z+\tau) = e^{-2\pi i(2(z+\frac{\tau}{2}))}F(z)$

$\Longrightarrow$ space of solutions has dim $=2$, contains $\underbrace{\theta_{11}(z)\theta_{01}(z)}_{\text{odd}}, \underbrace{\theta_{00}(z)\theta_{10}(z)}_{\text{even}}$

F even $\overset{z\to 0}{\Longrightarrow}$ $F(z) = \frac{\theta_{11}'\theta_{01}}{\theta_{00}\theta_{10}}\theta_{00}(z)\theta_{10}(z)$ $(*)$

Cor. $$\frac{dx}{dz} = 2\frac{\theta_{00}^2}{\theta_{10}^2}\,\frac{\theta_{11}'\theta_{01}}{\theta_{00}\theta_{10}}\,\frac{\theta_{11}(z)\theta_{00}(z)\theta_{10}(z)}{\theta_{01}(z)^3} = -2y\frac{\theta_{11}'\theta_{00}}{\theta_{10}\theta_{01}}$$

Prop. $\theta_{11}' = -\pi\theta_{00}\theta_{01}\theta_{10}$

Cor. $\frac{dx}{dz} = 2\pi y\theta_{00}^2$, $\boxed{\frac{dx}{y} = 2\pi\theta_{00}^2\,dz}$

Pf: $\left(\frac{d}{dz}\right)^2\Big|_{z=0}$ of the formula gives $\frac{\theta_{11}'''}{\theta_{11}'} = \frac{\theta_{01}''}{\theta_{01}} + \frac{\theta_{10}''}{\theta_{10}} + \frac{\theta_{00}''}{\theta_{00}}$

$\left(\theta_{ab}^{(n)} := \left(\frac{d}{dz}\right)^n\theta_{ab}(z)\Big|_{z=0}\right)$. Heat equation: $\underbrace{\left(\left(\frac{\partial}{\partial z}\right)^2 - 4\pi i\frac{\partial}{\partial\tau}\right)}\theta_{ab}(z,\tau) = 0$

Pf: kills each term $q^{(n+a/2)^2/2}t^{n+a/2}$
$(q = e^{2\pi i\tau},\ t = e^{2\pi iz})$

Hence $\frac{\partial}{\partial\tau}\log\left(\frac{\theta_{11}'}{\theta_{01}\theta_{10}\theta_{00}}\right) = 0$

$\Longrightarrow \exists\, c\in\mathbb{C}$ $\theta_{11}' = c\,\theta_{01}\theta_{10}\theta_{00}$. If $q = e^{2\pi i\tau}\to 0$, then $\theta_{00}\sim\theta_{01}\sim 1$
$\theta_{10}\sim 2q^{1/8}$, $\theta_{11}'\sim -2\pi q^{1/8}$ $\Longrightarrow$ result $(c = -\pi)$.

Final formulas: $\boxed{\frac{dx}{y} = 2\pi\theta_{00}^2\,dz}$

z	0	$\frac{1}{2}$	$\frac{\tau+1}{2}$	$\frac{\tau}{2}$
x	0	1	$1/\lambda$	∞
y	0	0	0	∞

For $0 < \lambda = k^2 < 1$:

$$2K(k) = \underbrace{\int_0^1\frac{dx}{y}}_{\frac{\omega_2}{2}} = \int_0^{1/2}2\pi\theta_{00}^2\,dz = \pi\theta_{00}^2$$

$$\frac{\omega_1}{2i} = \frac{\tau\omega_2}{2i} = \frac{\pi\tau}{i}\theta_{00}^2 = \int_1^{1/\lambda}\frac{dx}{y}$$

Jacobi's triple product formula

Note: $F(z) := \prod_{n=1}^{\infty} (1+q^{n-\frac{1}{2}}t)(1+q^{n-\frac{1}{2}}t^{-1})$ $\quad (t = e^{2\pi i z},\ q = e^{2\pi i \tau})$

satisfies $F(z+1) = F(z)$, $F(z+\tau) = e^{-2\pi i (z+\frac{\tau}{2})} F(z)$, $\mathrm{div}(F) = \frac{\tau+1}{2} + \mathbb{Z}\tau + \mathbb{Z}$

$\longrightarrow$ $\theta(z,\tau) = \theta_{00}(z,\tau) = c(\tau) F(t)$, $\quad c(\tau) \to 1$ if $q \to 0$

$$\Rightarrow \quad \theta_{01}(z,\tau) = c(\tau) \prod_{n=1}^{\infty} (1-q^{n-\frac{1}{2}}t)(1-q^{n-\frac{1}{2}}t^{-1})$$

$$\theta_{10}(z,\tau) = c(\tau)\,(t^{1/2}+t^{-1/2})\, q^{1/8} \prod_{n=1}^{\infty} (1+q^n t)(1+q^n t^{-1})$$

$$\theta_{11}(z,\tau) = c(\tau)\ i\,(t^{1/2}-t^{-1/2})\, q^{1/8} \prod_{n=1}^{\infty} (1-q^n t)(1-q^n t^{-1})$$

let $z \to 0$: $\quad \theta_{00} = c(\tau) \prod_{n=1}^{\infty} (1+q^{n-\frac{1}{2}})^2, \quad \theta_{01} = c(\tau) \prod_{n=1}^{\infty} (1-q^{n-\frac{1}{2}})^2$

$$\theta_{10} = 2c(\tau)\, q^{1/8} \prod_{n=1}^{\infty} (1+q^n)^2, \quad \theta_{11}' = -2\pi\, c(\tau)\, q^{1/8} \prod_{n=1}^{\infty} (1-q^n)^2$$

$$\theta_{11}' = -\pi\, \theta_{00}\theta_{01}\theta_{10} \Longrightarrow c(\tau)^2 = \prod_{n=1}^{\infty} (1-q^n)^2 \Longrightarrow \boxed{c(\tau) = \prod_{n=1}^{\infty} (1-q^n) =: P(q)}$$

$P(q) = q^{-1/24}\eta(\tau)$

Jacobi's triple product formula:

$$\theta(z,\tau) = \sum_{n\in\mathbb{Z}} q^{n^2/2} t^n = \prod_{n=1}^{\infty} (1-q^n)(1+q^{n-\frac{1}{2}}t)(1+q^{n-\frac{1}{2}}t^{-1})$$

$$\Longleftrightarrow \quad \theta_{01}(z,\tau) = \sum_{n\in\mathbb{Z}} (-1)^n q^{n^2/2} t^n = \prod_{n=1}^{\infty} (1-q^n)(1-q^{n-\frac{1}{2}}t)(1-q^{n-\frac{1}{2}}t^{-1})$$

Special cases: (a) $t=1$: $\quad \theta_{01}(\tau) = \sum_{n\in\mathbb{Z}} (-1)^n q^{n^2/2} = \prod_{n=1}^{\infty} (1-q^n)(1-q^{n-\frac{1}{2}})^2$

$$\theta(2\tau+1) = \theta_{01}(2\tau) = \sum_{n\in\mathbb{Z}} (-1)^n q^{n^2} = \prod_{n=1}^{\infty} (1-q^{2n})(1-q^{2n-1})^2 = \prod_{n=1}^{\infty} \frac{(1-q^n)^2}{1-q^{2n}} = \frac{P(q)^2}{P(q^2)} = \frac{\eta(\tau)^2}{\eta(2\tau)}$$

(b) $q = u^3$, $t = u^{1/2}$: then replace u by q

$$\sum_{n\in\mathbb{Z}} (-1)^n q^{\frac{3n^2+n}{2}} = \prod_{n=1}^{\infty} (1-q^{3n})(1-q^{3n-1})(1-q^{3n-2})$$

Euler's pentagonal number formula $\quad = \prod_{n=1}^{\infty} (1-q^n) = P(q)$

Cor. $\eta(\tau) = q^{1/24} P(q) = \sum_{m\in\mathbb{Z}} (-1)^m q^{(6m+1)^2/24} = \sum_{\substack{n=1 \\ (n,12)=1}}^{\infty} \chi(n)\, q^{n^2/24}$

$$\chi(n) = \begin{cases} 1, & n \equiv \pm 1 \pmod{12} \\ -1, & n \equiv \pm 5 \pmod{12} \\ 0, & (n,12) \neq 1 \end{cases}$$

Note: $\chi(n) = \left(\frac{3}{n}\right)$ (Jacobi symbol)

Exercise. Deduce from Jacobi's triple product formula

$$\theta(z,\tau) = \prod_{n=1}^{\infty} (1-q^n)(1+q^{n-\frac{1}{2}}t)(1+q^{n-\frac{1}{2}}t^{-1}) \qquad (q = e^{2\pi i\tau},\ t = e^{2\pi i z})$$

product formulas for $\theta(z+\frac{a\tau+b}{2},\tau)$ $(a,b\in\{0,1\})$.

Express the values $\theta(\frac{a\tau+b}{2},\tau)$ $(a,b\in\{0,1\})$ in terms of the function $\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n) = q^{1/24}P(q)$

Solution.

$$\theta(z+\tfrac{1}{2},\tau) \overset{t:=-t}{=} \prod_{n=1}^{\infty}(1-q^n)(1-q^{n-\frac{1}{2}}t)(1-q^{n-\frac{1}{2}}t^{-1})$$

$$\theta(z+\tfrac{\tau}{2},\tau) \overset{t:=q^{1/2}t}{=} \prod_{n=1}^{\infty}(1-q^n)(1+q^n t)(1+q^{n-1}t)$$

$$\theta(z+\tfrac{\tau+1}{2},\tau) \overset{t:=-q^{1/2}t}{=} \prod_{n=1}^{\infty}(1-q^n)(1-q^n t)(1-q^{n-1}t^{-1})$$

$z=0$:

$$\theta(0,\tau) = \prod_{n=1}^{\infty}(1-q^n)(1+q^{n-\frac{1}{2}})^2 = \prod_{m=1}^{\infty}\frac{(1-e^{\pi i(\tau+1)m})^2}{1-q^m} = \frac{\eta(\frac{\tau+1}{2})^2}{\eta(\tau)}$$

$$\theta(\tfrac{1}{2},\tau) = \prod_{n=1}^{\infty}(1-q^n)(1-q^{n-\frac{1}{2}})^2 = \prod_{n=1}^{\infty}\frac{(1-q^{n/2})^2}{1-q^n} = \frac{\eta(\tau/2)^2}{\eta(\tau)}$$

$$\theta(\tfrac{\tau}{2},\tau) = 2\prod_{n=1}^{\infty}(1-q^n)(1+q^n)^2 = 2\prod_{n=1}^{\infty}\frac{(1-q^{2n})^2}{1-q^n} = 2\,\frac{\eta(2\tau)^2}{\eta(\tau)}$$

$$\theta(\tfrac{\tau+1}{2},\tau) = 0, \quad \theta'(\tfrac{\tau+1}{2},\tau) = 2\pi i\prod_{n=1}^{\infty}(1-q^n)^3 = (2\pi i)\, q^{-1/8}\,\eta(\tau)^3$$

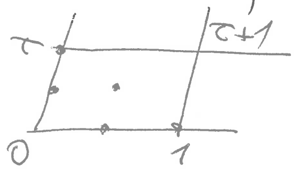
Rmk. Above,

$$\theta(0,\tau) = \theta(\tau) = \theta_{00}(\tau)$$

$$\theta(\tfrac{1}{2},\tau) = \theta_{01}(\tau)$$

$$\theta(\tfrac{\tau}{2},\tau) = q^{-1/8}\,\theta_{10}(\tau)$$

θ and $\wp(z)$



$L = \mathbb{Z}\tau + \mathbb{Z}$, $\omega_1 = \tau$, $\omega_2 = 1$, $\omega_3 = \tau + 1$, $e_j = \wp\left(\frac{\omega_j}{2}\right)$

$$\operatorname{div}\left(\wp(z) - e_j\right) = 2\left(\frac{\omega_j}{2}\right) - 2(0) = \operatorname{div}\left(\frac{\theta_{ab}(z)}{\theta_{11}(z)}\right)^2,$$

where

j	1	2	3
a	0	1	0
b	1	0	0

$\wp(z) \sim z^{-2}$ as $z \to 0$

$$\Longrightarrow \boxed{\wp(z) - e_j = \left(\frac{\theta_{ab}(z)\,\theta_{11}'}{\theta_{11}(z)\,\theta_{ab}}\right)^2}$$

Cor 1: $$\boxed{\wp'(z) = -2\,\frac{\theta_{00}(z)\theta_{01}(z)\theta_{10}(z)}{\theta_{11}(z)^3}\;\frac{(\theta_{11}')^3}{\theta_{00}\theta_{01}\theta_{10}}}$$

Cor 2: $$e_1 - e_2 = \wp\left(\frac{\tau}{2}\right) - \wp\left(\frac{1}{2}\right) = -\left(\frac{\theta_{00}\,\theta_{11}'}{\theta_{10}\,\theta_{01}}\right)^2 = -(\pi\theta_{00}^2)^2$$

$$e_1 - e_3 = -\left(\wp\left(\frac{\tau+1}{2}\right) - \wp\left(\frac{\tau}{2}\right)\right) = -\left(\frac{\theta_{10}\,\theta_{11}'}{\theta_{00}\theta_{01}}\right)^2 = -(\pi\theta_{10}^2)^2$$

$$e_2 - e_3 = \wp\left(\frac{1}{2}\right) - \wp\left(\frac{\tau+1}{2}\right) = \left(\frac{\theta_{01}\,\theta_{11}'}{\theta_{10}\,\theta_{00}}\right)^2 = (\pi\theta_{01}^2)^2.$$

$$\Longrightarrow \boxed{\frac{e_1 - e_3}{e_1 - e_2} = \left(\frac{\theta_{10}}{\theta_{00}}\right)^4 \;\Big|\; = k^2}$$

Up to a renumbering of $\{e_j\}$, this is the function $\lambda(\tau)$.

So: $\lambda(\tau) = \left(\frac{\theta_{10}}{\theta_{00}}\right)^4$

Cor 3: $$\Delta(\mathbb{Z}\tau + \mathbb{Z}) = 16\prod_{j<k}(e_j - e_k)^2 = 2^4\left(\frac{(\theta_{11}')^3}{\theta_{00}\theta_{01}\theta_{10}}\right)^4 = 2^4\pi^{4}(\theta_{11}')^8 = (2\pi)^{12}\,q\prod_{n=1}^{\infty}(1-q^n)^{24}$$

(Jacobi's formula)

Rmk: $$\underbrace{-\theta_{11}'/2\pi q^{1/8}}_{\prod_{n=1}^{\infty}(1-q^n)^3} = \sum_{n=0}^{\infty}(-1)^n(2n+1)\,q^{n(n+1)/2}$$

This can also be seen directly from Jacobi's triple product formula: take $t = q^{1/2}x$, divide by $1-x$ and let $x \to 0$.

Product formulas for $\lambda(\tau) = \dfrac{\wp\left(\frac{\tau}{2}\right) - \wp\left(\frac{\tau+1}{2}\right)}{\wp\left(\frac{\tau}{2}\right) - \wp\left(\frac{1}{2}\right)} = \dfrac{e_1 - e_3}{e_1 - e_2}$:

$$1 - \lambda(\tau) = \left(\frac{\theta_{01}}{\theta_{00}}\right)^4 = \prod_{n=1}^{\infty}\left(\frac{1 - q^{n-\frac{1}{2}}}{1 + q^{n-\frac{1}{2}}}\right)^8 = \left(\frac{\eta(\tau/2)}{\eta\left(\frac{\tau+1}{2}\right)}\right)^8$$

$$\lambda(\tau) = \left(\frac{\theta_{10}}{\theta_{00}}\right)^4 = 2^4 q^{1/2}\prod_{n=1}^{\infty}\left(\frac{1 + q^n}{1 + q^{n-\frac{1}{2}}}\right)^8 = 2^4\left(\frac{\eta(2\tau)}{\eta\left(\frac{\tau+1}{2}\right)}\right)^8$$

Complex tori V/L and complex abelian varieties

Data: $V \simeq \mathbb{C}^n$, $L \subset V$ lattice $(L \simeq \mathbb{Z}^{2n})$

As in the case $n=1$, meromorphic functions on V/L can be written as F_1/F_2, where $F_1, F_2 \in \mathcal{O}(V)$ have divisors with no common component (Poincaré for $n=2$; Cousin in general)

$\Longrightarrow$ $F = F_j$ satisfies $F(z+u) = e^{\alpha_u(z)} F(z)$ $(z \in V, u \in L)$

$\alpha_u \in \mathcal{O}(V)$

Again, $\exists$ gauge transformation $F(z) \mapsto F(z)g(z)$ that replaces $e^{\alpha_u(z)}$ by an equivalent 1-cocycle of the form

$e^{\langle A(u), z\rangle + B(u)}$, where $A(u) \in \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R}) = V_{\mathbb{R}}^*$, $B(u) \in \mathbb{C}$

Furthermore, there is an additional (unique) gauge transformation with $g(z) = e^{\langle Q(z), z\rangle + \langle R, z\rangle}$, $Q = Q^* : V_{\mathbb{R}} \longrightarrow V_{\mathbb{R}}^*$, $R \in V_{\mathbb{R}}^*$, after which the 1-cocycle will be as in the case $n=1$:

$\alpha(u)\, e^{\pi H(z,u) + \frac{\pi}{2} H(u,u)}$ $(u \in L)$, where $H : V \times V \longrightarrow \mathbb{C}$ is a hermitian form whose imaginary part $E = \mathrm{Im}(H)$ satisfies $E(L \times L) \subseteq \mathbb{Z}$, and $\alpha(u+v) = \alpha(u)\alpha(v)(-1)^{E(u,v)}$.

Facts: (1) $\exists$ non-degenerate hermitian form $H : V \times V \longrightarrow \mathbb{C}$ s.t. $\mathrm{Im}(H)(L \times L) \subseteq \mathbb{Z}$

$\Updownarrow$

$\exists$ $L' \subset L$ of finite index, $\exists$ basis of V over $\mathbb{C}$ such that $L' =$ the rows of $\begin{pmatrix} T \\ I_n \end{pmatrix}$, $\underbrace{T = {}^tT \in M_n(\mathbb{C})}_{\text{symmetric}}$ and $\underbrace{\mathrm{Im}(T) \in GL_n(\mathbb{R})}_{\text{invertible}}$

(2) $\exists$ positive definite hermitian form $H : V \times V \longrightarrow \mathbb{C}$, $\mathrm{Im}(H)(L \times L) \subseteq \mathbb{Z}$

$\Updownarrow$

T in (2) lies in $\mathcal{H}_n$ $(\Longleftrightarrow \mathrm{Im}(T) > 0$, $T = {}^tT)$

$\Updownarrow$

$\mathrm{tr.deg}_{\mathbb{C}}\, \mathcal{M}(V/L) = n$ $\Longleftrightarrow$ V/L is a projective algebraic variety