Theta functions as functions of τ

Goal: understand transformation rules (symmetries) of Jacobi's θ-function $\theta(z,\tau) = \sum_{n\in\mathbb{Z}} e^{\pi i n^2\tau + 2\pi i n z} = \sum_{n\in\mathbb{Z}} e(\tfrac{1}{2}n^2\tau + nz)$ and its generalisations.

Qualitative analysis ($\tau\in\mathbb{H}$ fixed, $z\in\mathbb{C}$ variable):

$\theta(z+1,\tau) = \theta(z,\tau)$, $\theta(z+\tau) = e(-z-\tfrac{\tau}{2})\theta(z,\tau)$ ($\theta(\cdot,\tau)$ is a θ-function for $\mathbb{Z}\tau+\mathbb{Z}$)

$\mathrm{div}(\theta) = \frac{1+\tau}{2} + (\mathbb{Z}\tau+\mathbb{Z}) \Longrightarrow \theta(z,\tau) = e(P_\tau(z))\sigma(z-\frac{1+\tau}{2}, \mathbb{Z}\tau+\mathbb{Z})$, $P_\tau\in\mathbb{C}[z]$, $\deg(P_\tau)\le 2$

symmetries of the oriented lattice $L_\tau = \mathbb{Z}\tau+\mathbb{Z}$: $\alpha = \begin{pmatrix} a & b\\ c & d\end{pmatrix}\in SL_2(\mathbb{Z})$

$L_\tau = \mathbb{Z}(a\tau+b) + \mathbb{Z}(c\tau+d) = (c\tau+d)\left(\mathbb{Z}\frac{a\tau+b}{c\tau+d} + \mathbb{Z}\right) = (c\tau+d)L_{\tau'}$, $\tau' = \frac{a\tau+b}{c\tau+d} = \alpha(\tau)$

$\sigma(\lambda z, \lambda L) = \lambda^{-1}\sigma(z,L)$

$(c\tau+d)^{-1}\frac{1+\tau}{2} - \frac{1+\tau'}{2} = (c\tau+d)^{-1}\frac{1}{2}((1-(a+c))\tau + (1-(b+d)))$

$\Rightarrow$ if $a+c\equiv b+d\equiv 1 \pmod 2$ (which is equivalent to $2|ac$ and $2|bd$),

then $\theta\left(\frac{z}{c\tau+d}, \frac{a\tau+b}{c\tau+d}\right)$ is a θ-function for $\mathbb{Z}\frac{a\tau+b}{c\tau+d}+\mathbb{Z} = L_{\tau'}$

with divisor $\underbrace{\frac{1+\tau'}{2} + L_{\tau'}}_{\text{(invariant under } z\mapsto -z)}$

$$\Rightarrow \boxed{\theta\left(\frac{z}{c\tau+d}, \frac{a\tau+b}{c\tau+d}\right) = A(\alpha,\tau)\, e(B(\alpha,\tau)z^2)\,\theta(z,\tau) \quad \forall \alpha = \begin{pmatrix} a & b\\ c & d\end{pmatrix}\in SL_2(\mathbb{Z}),\ 2|ac,\ 2|bd}$$

Symmetry of $\theta(z,\tau)$ under $\tau\mapsto -\frac{1}{\tau} = \begin{pmatrix} 0 & -1\\ 1 & 0\end{pmatrix}(\tau)$

Key point: relation to Fourier transform. Notation: $e_\lambda(x) = e^{2\pi i\lambda x} = e(\lambda x)$

Def. For $f\in L^1(\mathbb{R})$

$$(\mathcal{F}f)(y) = \hat{f}(y) := \int_{\mathbb{R}} e^{-2\pi i xy} f(x)\,dx = \int_{\mathbb{R}} e_{-y} f\,dx$$

$$(\mathcal{F}^\vee f)(y) = f^\vee(y) := \int_{\mathbb{R}} e^{2\pi i xy} f(x)\,dx = \int_{\mathbb{R}} e_y f\,dx$$

Schwartz functions: $\mathcal{S}(\mathbb{R}) = \{f\in C^\infty(\mathbb{R}) \mid \forall l,m\ \sup_{x\in\mathbb{R}} |x^l (\frac{d}{dx})^m f(x)| < \infty\}$

topology of $\mathcal{S}(\mathbb{R})$: given by norms $p_{l,m}(f) = \sup_{|\alpha|\le l,\ |\beta|\le m,\ x\in\mathbb{R}} |x^\alpha (\frac{d}{dx})^\beta f(x)|$

($\Leftrightarrow$ by the metric $d(f,g) = \sum_{l,m\ge 0} 2^{-(l+m)} \frac{p_{l,m}(f-g)}{1+p_{l,m}(f-g)}$)

Tempered distributions: $\mathcal{S}(\mathbb{R})' = \{$continuous linear maps $\mathcal{S}(\mathbb{R})\to\mathbb{C}\}$

Ex: $\delta_{\mathbb{Z}} = \sum_{m\in\mathbb{Z}} \delta_m : f\mapsto \sum_{n\in\mathbb{Z}} f(n) = \langle\delta_{\mathbb{Z}}, f\rangle \Longrightarrow \boxed{\theta(z,\tau) = \langle\delta_{\mathbb{Z}}, f_{z,\tau}\rangle,\quad f_{z,\tau}(x) = e^{\pi i\tau x^2 + 2\pi i x z}\in\mathcal{S}(\mathbb{R})}$

$\begin{pmatrix}\tau\in\mathbb{H}\\ z\in\mathbb{C}\end{pmatrix}$

Basic rules: (1) $f \in \mathcal{S}(\mathbb{R}) \Longrightarrow \mathcal{F}f, \mathcal{F}^\vee f \in \mathcal{S}(\mathbb{R})$ and $\mathcal{F}, \mathcal{F}^\vee : \mathcal{S}(\mathbb{R}) \circlearrowleft$ are continuous

(2) $f, g \in \mathcal{S}(\mathbb{R}) \Longrightarrow \int_X f(x)\overline{\mathcal{F}^\vee g(x)}\,dx = \int_{X\times X} f(x)\overline{g(y)}\,e(-xy)\,dx\,dy = \int_X (\mathcal{F}f)(y)\overline{g(y)}\,dy$

(3) $\mathcal{F}^\vee\mathcal{F} = id$, $\mathcal{F}\mathcal{F}^\vee = id$ on $\mathcal{S}(\mathbb{R})$ $\quad (\Longleftrightarrow \mathcal{F}\mathcal{F} = r_{-1})$

Fact: $\mathcal{S}(\mathbb{R}) \subset L^2(\mathbb{R})$ is a continuous embedding with dense image

$\overset{(1)-(3)}{\Longrightarrow}$ $\mathcal{F}, \mathcal{F}^\vee$ extend canonically to unitary operators $\mathcal{F}, \mathcal{F}^\vee = \mathcal{F}^{-1} : L^2(\mathbb{R}) \circlearrowleft$

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(4) $\frac{d}{dx}\mathcal{F} = \mathcal{F}(-2\pi i x)$, $\quad (2\pi i x)\mathcal{F} = \mathcal{F}\frac{d}{dx}$ $\quad$ (on $\mathcal{S}(\mathbb{R})$)

(5) Translations: $(t_y f)(x) := f(x-y)$ $\quad t_y\mathcal{F} = \mathcal{F}e_y$, $\quad \mathcal{F}t_y = e_{-y}\mathcal{F}$

(6) Homotheties: $(r_\lambda f)(x) := f(\lambda^{-1}x)$ $(\lambda \in \mathbb{R}^\times)$ $\quad \mathcal{F}r_\lambda = |\lambda|\, r_{\lambda^{-1}}\mathcal{F}$, $\quad r_\lambda\mathcal{F} = |\lambda|^{-1}\mathcal{F}r_\lambda$

Poisson summation formula: $f \in \mathcal{S}(\mathbb{R})$, $\quad F(x) := \sum_{n\in\mathbb{Z}} f(x+n)$

$F : \mathbb{R}/\mathbb{Z} \to \mathbb{C}$, $\quad F \in C^\infty(\mathbb{R}/\mathbb{Z})$

$\Longrightarrow$ Fourier series $\sum_{n\in\mathbb{Z}} \hat{F}(n)e_n(x)$ $\quad (\hat{F}(n) = \int_{\mathbb{R}/\mathbb{Z}} e_{-n}F\,dx)$ converges to $F(x)$ $\forall x \in \mathbb{R}$

$$\hat{F}(n) = \int_0^1 F(x)e(-nx)\,dx = \int_0^1 \sum_{m\in\mathbb{Z}} f(x+m)e(-nx)\,dx = \sum_{m\in\mathbb{Z}} \int_m^{m+1} f(y)\underbrace{e(-n(y-m))}_{e(-ny),\ \text{since } mn\in\mathbb{Z}}\,dy =$$

$$= \int_{\mathbb{R}} f(y)e(-ny)\,dy = (\mathcal{F}f)(n)$$

$$\Longrightarrow \boxed{\forall x \in \mathbb{Z} \quad \sum_{n\in\mathbb{Z}} f(x+n) = \sum_{n\in\mathbb{Z}} (\mathcal{F}f)(n)\,e(nx) \qquad (f\in\mathcal{S}(\mathbb{R}))}$$

Special case $x = 0$:

$(*)$ $\quad \boxed{\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} (\mathcal{F}f)(n)}$ $\quad$ (applied to $t_{-x}f$, this gives the previous formula)

Reformulation: $\boxed{\delta_{\mathbb{Z}} = \delta_{\mathbb{Z}}\mathcal{F}}$ $\in \mathcal{S}(\mathbb{R})'$

Exercise: Show that $(*)$ holds if $f \in C^2(\mathbb{R})$ and if $\int_{\mathbb{R}} f$, $\int_{\mathbb{R}} |f''|$ exist.

Poisson summation formula for $\boxed{f_{z,\tau}(x) = e\left(\frac{\tau x^2}{2} + zx\right)}$ and $\boxed{f_\tau(x) = e\left(\frac{\tau x^2}{2}\right)}$.

Key point: $\boxed{\mathcal{F}(e^{-\pi x^2}) = e^{-\pi x^2}}$ $\quad (= f_i)$

Pf: $\left(\frac{d}{dx} + 2\pi x\right)e^{-\pi x^2} = 0 \Longrightarrow 0 = \mathcal{F}\left(\frac{d}{dx} + 2\pi x\right)e^{-\pi x^2} = i\left(2\pi x + \frac{d}{dx}\right)\mathcal{F}e^{-\pi x^2}$

$\Longrightarrow \mathcal{F}e^{-\pi x^2} = c\,e^{-\pi x^2}$, $\quad c = \int_{\mathbb{R}} e^{-\pi x^2}dx > 0$

$\mathcal{F}\mathcal{F} = r_{-1} \Longrightarrow c^2 = 1$

$\Longrightarrow c = 1.$

Cor 1. $\forall\, t>0$ $\quad \mathcal{F} e^{-\pi t x^2} = \mathcal{F}\, r_{-1/\sqrt{t}}\, e^{-\pi x^2} = t^{-1/2}\, r_{t^{1/2}}\, \mathcal{F} e^{-\pi x^2} = t^{-1/2} e^{-\pi x^2/t}$

Cor 2. $\forall\, \tau\in\mathbb{H}$ $\quad \mathcal{F} e^{\pi i \tau x^2} = \underbrace{\left(\frac{\tau}{i}\right)^{-1/2}}_{\text{branch equal to } 1 \text{ at } \tau = i} e^{-\pi i x^2/\tau}$

Pf: this holds if $\tau \in \mathbb{H}\cap i\mathbb{R}$, by Cor. 1. For fixed $x\in\mathbb{R}$, both sides are holomorphic functions of $\tau \Longrightarrow$ equality for all $\tau\in\mathbb{H}$.

Direct proof: $f_\tau(x) = e^{\pi i \tau x^2} = e\left(\frac{1}{2}\tau x^2\right)$ $\quad$ ($\tau\in\mathbb{H}$ fixed, $x\in\mathbb{R}$)

satisfies $\left(\frac{d}{dx} - 2\pi i \tau x\right) f_\tau = 0 \Longrightarrow 0 = \mathcal{F}\left(\frac{d}{dx} - 2\pi i\tau x\right) f_\tau = \left(2\pi i x + \tau\frac{d}{dx}\right)\mathcal{F} f_\tau$

$\Longrightarrow \mathcal{F} f_\tau \in \left\{ f\in\mathcal{S}(\mathbb{R}) \;\middle|\; \left(\frac{d}{dx} + \frac{2\pi i}{\tau} x\right) f = 0\right\} = \mathbb{C}\cdot f_{-1/\tau}$

$\mathcal{F} f_\tau = c(\tau) f_{-1/\tau}$. For $\lambda>0$, $r_{\lambda^{-1}} f_\tau = f_{\lambda^2\tau} \Longrightarrow c(\lambda^2\tau) = \lambda^{-1} c(\tau)$

$\mathcal{F}\mathcal{F} = r_{-1} \Longrightarrow c(\tau)c(-1/\tau) = 1 \Longrightarrow \left.\begin{array}{l} c(i)^2 = 1 \\ c(i)>0 \end{array}\right\} \Longrightarrow c(i) = 1 \quad \Longrightarrow c(i\lambda^2) = \lambda^{-1}$.

$\tau \longmapsto c(\tau)$ is holomorphic $\Longrightarrow c(\tau) = \left(\frac{\tau}{i}\right)^{-1/2}$, $c(i)=1$.

Cor: $\forall\,\tau\in\mathbb{H}$ $\forall\, x\in\mathbb{R}$ $\quad \sum_{n\in\mathbb{Z}} e^{\pi i \tau (n+x)^2} = \left(\frac{\tau}{i}\right)^{-1/2} \sum_{n\in\mathbb{Z}} e^{2\pi i n x}\, e^{-\pi i n^2/\tau}$

$x=0$: $\quad \theta(\tau) = \sum_{n\in\mathbb{Z}} e^{\pi i \tau n^2} = \left(\frac{\tau}{i}\right)^{-1/2} \underbrace{\sum_{n\in\mathbb{Z}} e^{-\pi i n^2/\tau}}_{\theta(-1/\tau)}$

Rk: limit values for $\mathrm{Im}(\tau)\longrightarrow 0+$:

If $a\in\mathbb{R}$, then $f_a(x) = e^{\pi i a x^2} \notin \mathcal{S}(\mathbb{R})$, but $\left|\left(\frac{d}{dx}\right)^k f_a(x)\right| \le (\text{const.})\,|x|^k$ on $\mathbb{R}$

$\Longrightarrow$ multiplication by f_a is a continuous map $\mathcal{S}(\mathbb{R}) \longrightarrow \mathcal{S}(\mathbb{R})$

$\Longrightarrow$ defines $[f_a]\in\mathcal{S}(\mathbb{R})'$ given by $\langle [f_a], g\rangle = \int_{\mathbb{R}} f_a g\, dx$.

(if $a=0$, then $f_0 = 1$ and $[f_0]\mathcal{F} = \delta_0$)

If $a\in\mathbb{R}^\times$, then an easy limit argument (exercise!) shows that $[f_a]\mathcal{F} = c(a)[f_{-1/a}]$, where $c(a)$ is obtained by continuity from the branch $c(\tau) = \left(\frac{\tau}{i}\right)^{-1/2}$, $c(i)=1$, $\tau\in\mathbb{H}$:

$$\boxed{c(a) = |a|^{-1/2} e^{(2\pi i/8)\,\mathrm{sgn}(a)}}$$

Exercise (Fresnel integrals): $\int_{\mathbb{R}} e^{\pm\pi i x^2} dx := \lim_{A,B\to+\infty} \int_{-A}^{B} e^{\pm\pi i x^2} = e^{\pm(2\pi i/8)}$

Reformulation: $\left(\theta^2|_1\begin{pmatrix}0&-1\\1&0\end{pmatrix}\right)(\tau) := \tau^{-1}\theta^2(-1/\tau) = -i\,\theta^2(\tau)$

As $\underbrace{\theta(\tau+2)=\theta(\tau)}_{\theta|_1\begin{pmatrix}1&2\\0&1\end{pmatrix}}$, $\begin{pmatrix}1&2\\0&1\end{pmatrix}, \begin{pmatrix}0&-1\\1&0\end{pmatrix}$ generate $\Gamma_\theta = \left\{\begin{pmatrix}a&b\\c&d\end{pmatrix}\in SL_2(\mathbb{Z}) \mid 2|ac,\ 2|bd\right\}$

$\Longrightarrow \forall \alpha\in\Gamma_\theta \quad (\theta^2|_1\alpha)/\theta^2 \in \{\pm1, \pm i\}$

$\Rightarrow \forall\ \alpha=\begin{pmatrix}a&b\\c&d\end{pmatrix}\in\Gamma_\theta \qquad \theta\left(\frac{a\tau+b}{c\tau+d}\right) = \varepsilon_\alpha \underbrace{(c\tau+d)^{1/2}}_{\text{branch with}\ 0<\arg(c\tau+d)^{1/2}<\frac{\pi}{2}} \theta(\tau) \qquad (\tau\in\mathbb{H})$

$c>0 \qquad \varepsilon_\alpha^8 = 1$

Fourier transform of $f_{z,\tau}(x) = e^{\pi i\tau x^2+2\pi i x z}$ $\quad (\tau\in\mathbb{H},\ z\in\mathbb{C})$

formally: $f_{z,\tau} = f_\tau e_z \Longrightarrow \mathcal{F}f_{z,\tau} = \mathcal{F}e_z f_\tau = t_z\mathcal{F}f_\tau = c(\tau)\,t_z f_{-1/\tau}$

$t_z f_{-1/\tau}(x) = f_{-1/\tau}(x-z) = e^{-\pi i(x-z)^2/\tau} = e^{-\pi i z^2/\tau}\underbrace{e^{-\pi i x^2/\tau+2\pi i x z/\tau}}_{f_{z/\tau,-1/\tau}}$

problem: $e_z(x) = e^{2\pi i x z} \notin \mathcal{S}(\mathbb{R})$ if $z\notin\mathbb{R}$

Exercise: justify the above calculation by analytic continuation.

Alternative calculation: write $z = u - v\tau$, $u,v\in\mathbb{R}$

$f_{z,\tau}(x) = f_{u-v\tau,\tau}(x) = e\left(\tfrac12\tau x^2 + (u-v\tau)x\right) = e\left(\tfrac12\tau(x-v)^2+ux\right)e\left(-\tfrac12\tau v^2\right)$

$= \underbrace{e\left(\tfrac12\tau(x-v)^2+u(x-v)\right)}_{t_v e_u f_\tau} e\left(-\tfrac12\tau v^2+uv\right)$

$\Rightarrow (\mathcal{F}f_{z,\tau})(x) = e\left(-\tfrac12\tau v^2+uv\right)\underbrace{\left(e_{-v}t_u\underbrace{\mathcal{F}f_\tau}_{c(\tau)f_{-1/\tau}}\right)(x)}_{c(\tau)\,e(-vx)\,e\left(-\frac12(x-u)^2/\tau\right)} = c(\tau)\underbrace{e\left(-\frac{x^2}{2\tau}+\frac{x}{\tau}(u-v\tau)\right)}_{f_{z/\tau,-1/\tau}}\underbrace{e\left(-\frac{\tau v^2}{2}+uv-\frac{u^2}{2\tau}\right)}_{-z^2/2\tau}$

Poisson formula for $f_{z,\tau}$:

$$\theta(z,\tau) = \sum_{n\in\mathbb{Z}} e^{\pi i n^2\tau+2\pi i n z} = \left(\frac{\tau}{i}\right)^{-1/2} e^{-\pi i z^2/\tau}\,\theta\left(\frac{z}{\tau}, -\frac{1}{\tau}\right)$$

$\theta(z,\tau+2) = \theta(z,\tau)$

What is the general formula for $\theta(\alpha(z,\tau)) = \theta\left(\frac{z}{c\tau+d}, \frac{a\tau+b}{c\tau+d}\right)$ if $\alpha=\begin{pmatrix}a&b\\c&d\end{pmatrix}\in\Gamma_\theta = \left\langle\begin{pmatrix}1&2\\0&1\end{pmatrix}, \begin{pmatrix}0&-1\\1&0\end{pmatrix}\right\rangle$? Could the exponential factor $\frac{z^2}{\tau}$ be replaced by $\frac{z^2}{c\tau+d}$? No! that would not be compatible under $\alpha\beta=\alpha\circ\beta$.

Invariance under $\frac{a\tau+b}{c\tau+d} = \frac{-a\tau-b}{-c\tau-d} \Longrightarrow$ need to take $\boxed{\frac{cz^2}{c\tau+d}}$

Need to check: Prop. Let $\alpha = \begin{pmatrix} a_\alpha & b_\alpha \\ c_\alpha & d_\alpha \end{pmatrix}$, $\beta = \begin{pmatrix} a_\beta & b_\beta \\ c_\beta & d_\beta \end{pmatrix} \in SL_2(\mathbb{R})$, $(z,\tau) \in \mathbb{C} \times \mathbb{H}$.

(1) the formula $\alpha(z,\tau) = \left(\frac{z}{c_\alpha \tau + d_\alpha}, \frac{a_\alpha \tau + b_\alpha}{c_\alpha \tau + d_\alpha} \right)$ defines a (left) action of $SL_2(\mathbb{R})$ on $\mathbb{C} \times \mathbb{H}$.

(i.e. $\alpha(\beta(z,\tau)) = (\alpha\beta)(z,\tau)$.

(2) the function $F_\alpha(z,\tau) = \frac{cz^2}{c\tau+d}$ satisfies

$F_{\alpha\beta}(z,\tau) = F_\alpha(\beta(z,\tau)) + F_\beta(z,\tau))$

Pf (1) It follows from $\alpha \begin{pmatrix} \tau \\ 1 \end{pmatrix} = \begin{pmatrix} a\tau + b \\ c\tau + d \end{pmatrix} = \begin{pmatrix} \alpha(\tau) \\ 1 \end{pmatrix} (c\tau + d)$ that $J_\alpha(\tau) := c\tau + d$ satisfies $J_{\alpha\beta}(\tau) = J_\alpha(\beta\tau) J_\beta(\tau) \implies (1)$.

(2) $$\frac{F_{\alpha\beta}(z,\tau) - F_\beta(z,\tau)}{\cancel{z}} = \left(\frac{c_{\alpha\beta}}{\underbrace{c_{\alpha\beta}\tau + d_{\alpha\beta}}_{(c_\alpha \beta(\tau) + d_\alpha)(c_\beta \tau + d_\beta)}} - \frac{c_\beta}{c_\beta \tau + d_\beta} \right) z^2 = \frac{\overbrace{(c_{\alpha\beta} d_\beta - d_{\alpha\beta} c_\beta)}^{c_\alpha} z^2}{(c_\beta \tau + d_\beta)^2 (c_\alpha \beta(\tau) + d_\alpha)} = F_\alpha(\beta(z,\tau))$$

Cor. For $k, m \in \mathbb{Z}$, $F : \mathbb{C} \times \mathbb{H} \longrightarrow \mathbb{C}$, $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$, define

$$(F|_{k,m}\alpha)(z,\tau) := e\left(-\frac{m}{2} \frac{cz^2}{c\tau+d} \right) (c\tau+d)^{-k} F\left(\frac{z}{c\tau+d}, \frac{a\tau+b}{c\tau+d} \right).$$

then $F|_{k,m}(\alpha\beta) = (F|_{k,m}\alpha)|_{k,m}\beta$. (transformation rule for "Jacobi forms")

$$\boxed{\text{Cor. } \forall \alpha \in \Gamma_\theta \quad \theta^2|_{1,2}(z,\tau) / \theta^2(z,\tau) \in \{\pm 1, \pm i\}}$$

Later: conceptual approach $\Rightarrow$ no need to guess a formula, it falls out of the general formalism.

Alternative proof of (2) above: the function $g(z,\tau) := \frac{z^2}{\tau - \bar{\tau}}$ satisfies

$$g(\alpha(z,\tau)) = \frac{z^2}{(c\tau+d)^2} \frac{|c\tau+d|^2}{\tau - \bar{\tau}} = \frac{z^2}{\tau - \bar{\tau}} \left(\frac{c\bar{\tau}+d}{c\tau+d} \right)$$

$$g(\alpha(z,\tau)) - g(z,\tau) = \frac{-cz^2}{c\tau+d} = -F_\alpha(z,\tau)$$

Question: what is the value of $\varepsilon_\alpha := \frac{(\theta|_{1/2,1}\alpha)(z,\tau)}{\theta(z,\tau)} \in \mu_8$ equal to?

$(\alpha \in \Gamma_\theta,\ c > 0)$ $$(F|_{1/2,1}\alpha)(z,\tau) = e\left(\frac{-cz^2}{2(c\tau+d)} \right) \underbrace{\left((c\tau+d)^{1/2}\right)^{-1}}_{\text{branch with } 0 < \mathrm{Arg} < \frac{\pi}{2} \text{ on } \mathbb{H}} F\left(\frac{z}{c\tau+d}, \frac{a\tau+b}{c\tau+d} \right)$$

We know: $\alpha = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \implies \varepsilon_\alpha = 1$

$\alpha = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \implies \varepsilon_\alpha = e^{-2\pi i/8}$

From the functional equation for $\theta(\tau)$ to that of $\theta(z,\tau)$ (2^{nd} proof)

Recall: for $\mathbb{Z}\tau+\mathbb{Z}$, the 1-cocycle (trivialised on $\mathbb{Z}\subset\mathbb{Z}\tau+\mathbb{Z}$)

$$e^{triv}_{m\tau+n}(z) = e^{-\pi i m^2\tau - 2\pi i m z}$$

defines a bundle $\mathcal{L}_{triv}$ such that

$$\Gamma(\mathbb{C}/(\mathbb{Z}\tau+\mathbb{Z}), \mathcal{L}_{triv}) = \{ f\in\mathcal{O}(\mathbb{C}) \mid \forall m,n\in\mathbb{Z} \;\; f(z+m\tau+n) = e^{triv}(m\tau+n)\, f(z)\} = \mathbb{C}\cdot\theta(z,\tau)$$

the corresponding canonical 1-cocycle is given by

$$e^{can}_u(z) = \alpha(u)\, e^{\pi H(z,u) + \frac{\pi}{2}H(u,u)}, \quad H_\tau(z,w) = \frac{z\bar{w}}{\operatorname{Im}(\tau)}, \quad \alpha(m\tau+n) = (-1)^{mn}$$

The gauge transformation between the two:

$$e^{can}_u(z) = e^{triv}_u(z)\,\frac{F(z+u)}{F(z)}, \quad F(z) = e^{\pi z^2/2\operatorname{Im}(\tau)} = e\left(\frac{1}{2}\frac{z^2}{\tau-\bar\tau}\right) = e^{\pi H(z,\bar z)/2}$$

$\Gamma(\mathcal{L}_{triv}) \xrightarrow{\sim} \Gamma(\mathcal{L}_{can})$ is given by $f\longmapsto fF$.

$$\Longrightarrow \Gamma(\mathbb{C}/(\mathbb{Z}\tau+\mathbb{Z}), \mathcal{L}_{can}) = \mathbb{C}\cdot {}^{\circ}\theta(z,\tau), \quad {}^{\circ}\theta(z,\tau) = e\left(\frac{1}{2}\frac{z^2}{\tau-\bar\tau}\right)\theta(z,\tau)$$

Notation: $\boxed{\kappa(z,\tau) := \dfrac{z^2}{\tau-\bar\tau}}$

Change of basis of $\mathbb{Z}\tau+\mathbb{Z}$: $g = \begin{pmatrix} a & b\\ c & d\end{pmatrix}\in SL_2(\mathbb{Z})$, $g\begin{pmatrix}\tau\\ 1\end{pmatrix} = \begin{pmatrix} g(\tau)\\ 1\end{pmatrix}(c\tau+d)$

$g(\tau) = \dfrac{a\tau+b}{c\tau+d}$. Define $g(z,\tau) := \left(\dfrac{z}{c\tau+d}, \dfrac{a\tau+b}{c\tau+d}\right)$; $c\tau+d = J(g,\tau)$

Note: $\mathbb{Z}\tau+\mathbb{Z} = \mathbb{Z}(a\tau+b)+\mathbb{Z}(c\tau+d) = (\mathbb{Z}g(\tau)+\mathbb{Z})(c\tau+d)$, hence

$$\Gamma(\mathbb{C}/(\mathbb{Z}\tau+\mathbb{Z}), \mathcal{L}_{can}) \xrightarrow{\sim} \Gamma(\mathbb{C}/(\mathbb{Z}g(\tau)+\mathbb{Z}), \mathcal{L}),$$
$$f(z) \longmapsto f\left(\frac{z}{c\tau+d}\right)$$

where $\mathcal{L}$ is given by the 1-cocycle

$$e^{\mathcal{L}}_u(z) = \alpha((c\tau+d)u)\, e^{\underbrace{(\pi H_\tau(z,u) + \frac{\pi}{2}H_\tau(u,\bar u))|c\tau+d|^2}_{\pi H_{g(\tau)}(z,u) + \frac{\pi}{2}H_{g(\tau)}(u,\bar u)}}, \quad u\in \underbrace{(c\tau+d)^{-1}(\mathbb{Z}\tau+\mathbb{Z})}_{\mathbb{Z}g(\tau)+\mathbb{Z}}$$

(since $\operatorname{Im}(g(\tau)) = \operatorname{Im}(\tau)/|c\tau+d|^2$)

Summary: if $g\in SL_2(\mathbb{Z})$ preserves the function $\alpha(m\tau+n) = (-1)^{mn}$

$\underbrace{\text{(i.e., if } \forall m,n \;\; m\tau+n \equiv m(a\tau+b)+n(c\tau+d) \pmod 2)}_{\Longleftrightarrow\; g = \begin{pmatrix} a & b\\ c & d\end{pmatrix}\in\Gamma_\theta}$, then

$$\boxed{e^{\mathcal{L}}_u(z) = e^{can}_u(z) \text{ on } \mathbb{Z}g(\tau)+\mathbb{Z}} \Longrightarrow \boxed{{}^{\circ}\theta(g(z,\tau)) = A(g,\tau)\,{}^{\circ}\theta(z,\tau)}$$

To compute $A(g,\tau)$: let $z=0$

$$\Longrightarrow \boxed{A(g,\tau) = \varepsilon_g\,(c\tau+d)^{1/2}, \quad \varepsilon_g^8 = 1}$$

Applications of $\sum\limits_{n\in\mathbb{Z}} e^{\pi i\tau(n+x)^2} = \left(\frac{\tau}{i}\right)^{-1/2}\sum\limits_{n\in\mathbb{Z}} e^{-\pi i n^2/\tau + 2\pi i n x}$

(I) Apply $\frac{1}{2\pi i}\frac{d}{dx}$:
$$\sum_{n\in\mathbb{Z}} (n+x)\, e^{\pi i\tau(n+x)^2} = \tau^{-1}\left(\frac{\tau}{i}\right)^{-1/2}\sum_{n\in\mathbb{Z}} n\, e^{-\pi i n^2/\tau + 2\pi i n x}$$

(II) Apply $\frac{1}{2\pi i}\frac{d}{dx}$ once again and take $x=0$:
$$\sum_{n\in\mathbb{Z}}\left(n^2+\frac{1}{2\pi i\tau}\right) e^{\pi i\tau n^2} = \tau^{-2}\left(\frac{\tau}{i}\right)^{-1/2}\sum_{n\in\mathbb{Z}} n^2 e^{-\pi n^2/\tau}$$

Want: find $f(\tau)$ such that $F(\tau) := \underbrace{\sum\limits_{n\in\mathbb{Z}} (n^2 - f(\tau))e^{\pi i\tau n^2}}_{\left(\sum_n n^2 e^{\pi i\tau n^2}\right) - f(\tau)\theta(\tau)}$ has "nice" transformation properties, such as

(*) $F(\tau) = \tau^{-2}\left(\frac{\tau}{i}\right)^{-1/2} F(-1/\tau)$. But this is equivalent to
$$0 = \text{RHS} - \text{LHS} = -\tau^{-2}\left(\frac{\tau}{i}\right)^{-1/2} f(-1/\tau)\,\theta(-1/\tau) + \frac{1}{2\pi i\tau}\theta(\tau) + f(\tau)\theta(\tau)$$
$$\iff \quad -\tau^{-2} f(-1/\tau) + \frac{1}{2\pi i\tau} + f(\tau) = 0 \quad \iff \quad \left(f - f|_2\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\right)(\tau) = \frac{i}{2\pi\tau}$$

one solution: $f(\tau) = \frac{i}{4\pi\tau} \implies F(\tau) = \sum\limits_{n\in\mathbb{Z}}\left(n^2 - \frac{i}{4\pi\tau}\right)e^{\pi i n^2\tau}$ satisfies (*).

problem: $F(\tau+2) \neq F(\tau)$!! ($f|_2\begin{pmatrix}1 & 2\\ 0 & 1\end{pmatrix}$ is unrelated to f)

remedy: try $f(\tau)$ that depends only on $\mathrm{Im}(\tau)$:

$\forall \alpha = \begin{pmatrix}a & b\\ c & d\end{pmatrix} \in SL_2(\mathbb{R})$ $\quad \mathrm{Im}(\alpha(\tau)) = \mathrm{Im}\left(\frac{a\tau+b}{c\tau+d}\right) = \frac{\mathrm{Im}(\tau)}{|c\tau+d|^2}$

$$\implies \left(\underbrace{\frac{1}{\mathrm{Im}}}_{\frac{2i}{\tau-\bar\tau}}\Big|_2\alpha\right)(\tau) = (c\tau+d)^{-2}\frac{1}{\mathrm{Im}(\alpha(\tau))} = \frac{\cancel{(c\tau+d)}(c\bar\tau+d)}{\mathrm{Im}(\tau)(c\tau+d)}$$

$$\implies \frac{1}{\tau-\bar\tau}\Big|_2\alpha - \frac{1}{\tau-\bar\tau} = \frac{1}{\tau-\bar\tau}\left(\frac{c\bar\tau+d}{c\tau+d} - 1\right) = \frac{-c}{c\tau+d}$$

$\alpha = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$: $\quad f(\tau) = \frac{1}{4\pi\,\mathrm{Im}(\tau)} = \frac{i}{2\pi(\tau-\bar\tau)}$ works

$$\implies F(\tau) := \sum_{n\in\mathbb{Z}}\left(n^2 - \frac{1}{4\pi\,\mathrm{Im}(\tau)}\right)e^{\pi i\tau n^2} \;\; ~~satisfies~~ \boxed{= \underbrace{\frac{1}{\pi i}\left(\frac{d}{d\tau} + \frac{1}{4i\,\mathrm{Im}(\tau)}\right)}_{\text{why this operator?}}\theta(\tau)}$$

satisfies
$$\boxed{\begin{aligned} F(\tau+2) &= F(\tau)\\ F(-1/\tau) &= \tau^{-2}\left(\tfrac{\tau}{i}\right)^{1/2}F(\tau)\end{aligned}} \implies \begin{aligned}&\forall\alpha = \begin{pmatrix}a & b\\ c & d\end{pmatrix}\in\Gamma_\theta\\ &F\left(\frac{a\tau+b}{c\tau+d}\right) = \varepsilon_\alpha (c\tau+d)^{-2}(c\tau+d)^{1/2}F(\tau)\end{aligned}$$

$\varepsilon_\alpha^8 = 1$, $\quad 0 < \mathrm{Arg}\,(c\tau+d)^{1/2} < \frac{\pi}{2}$ (if $c>0$)

(III) θ-functions of higher level : fix $\phi : \mathbb{Z}/N\mathbb{Z} \longrightarrow \mathbb{C}$

let $\theta_\phi(\tau) := \sum_{n\in\mathbb{Z}} \phi(n) e^{\pi i n^2 \tau} = \sum_{a=0}^{N-1} \phi(a) \underbrace{\sum_{b\in\mathbb{Z}} e^{\pi i (b+a/N)^2 N^2 \tau}}_{\left(\frac{N^2\tau}{i}\right)^{-1/2} \sum_{n\in\mathbb{Z}} e^{-\pi i n^2/N^2\tau + 2\pi i n a/N}}$

$(n = a+Nb)$

$$= N^{-1}\left(\frac{\tau}{i}\right)^{-1/2} \sum_{n\in\mathbb{Z}} \underbrace{\left(\sum_{a\in\mathbb{Z}/N\mathbb{Z}} \phi(a) e^{2\pi i n a/N} \right)}_{\widehat{\phi}(n) \text{ (discrete Fourier transform)}} e^{-\pi i n^2/N^2\tau}$$

$$\Longrightarrow \quad \boxed{\theta_\phi(\tau) = N^{-1}\left(\frac{\tau}{i}\right)^{-1/2} \theta_{\widehat{\phi}}(-1/N^2\tau)}$$

Similarly : $\theta^*_\phi(\tau) := \sum_{n\in\mathbb{Z}} n\phi(n) e^{\pi i n^2\tau} = N \sum_{a=0}^{M-1} \phi(a) \sum_{b\in\mathbb{Z}} \left(b+\frac{a}{N}\right) e^{\pi i (b+\frac{a}{N})^2 N^2 \tau}$

$$= N \sum_{a=0}^{M-1} \phi(a) (N^2\tau)^{-1} \left(\frac{N^2\tau}{i}\right)^{-1/2} \sum_{n\in\mathbb{Z}} n e^{-\pi i n^2/N^2\tau + 2\pi i n a/N} = N^{-2}\tau^{-1}\left(\frac{\tau}{i}\right)^{-1} \theta^*_{\widehat{\phi}}(-1/N^2\tau)$$

(IV) Limit values for $\tau \longrightarrow i\infty$, $\frac{a}{c} \in \mathbb{Q}$:

$$\lim_{t\to+\infty} \theta_\phi(it) = \phi(0).$$

Fix $r = \frac{a}{c} \in \mathbb{Q}$, $a, c \in \mathbb{Z}$, $c \neq 0$, $(a,c) = 1$. then $F_r(n) = e^{\pi i n^2 r}$ depends only on $\begin{cases} n \pmod{2|c|} & \text{if } 2\nmid ac \\ n \pmod{|c|} & \text{if } 2\mid ac \end{cases}$. Assume : $2c \mid N$, $\phi : \mathbb{Z}/N\mathbb{Z} \longrightarrow \mathbb{C}$

For $t>0$: $\theta_\phi(r+it) = \sum_{n\in\mathbb{Z}} (\phi(n) \underbrace{e^{\pi i n^2 r}}_{F_r(n)}) e^{-\pi n^2 t} = \theta_{\phi F_r}(it) =$

$$= N^{-1} t^{-1/2} \theta_{\widehat{\phi F_r}}(i/N^2 t) \quad \Longrightarrow \quad \lim_{t\to 0+} t^{1/2}\theta_\phi(r+it) = N^{-1}\,\widehat{\phi F_r}(0) = N^{-1} \sum_{x\in\mathbb{Z}/N\mathbb{Z}} \phi(x) e^{\pi i r x^2}$$

Special cases : (1) $\phi = 1$, $\theta_\phi = \theta$

$$\lim_{t\to 0+} t^{1/2} \theta\left(\frac{a}{c}+it\right) = \frac{1}{2|c|} \sum_{x\in\mathbb{Z}/2|c|\mathbb{Z}} e^{\pi i a x^2/c}$$

(2) $r=0$ $\quad \lim_{t\to 0+} t^{1/2}\theta_\phi(it) = N^{-1}\sum_{x\in\mathbb{Z}/N\mathbb{Z}} \phi(x)$, $\quad \lim_{t\to 0+} t^{1/2}\theta(it) = 1$

Reciprocity formula : if $r = \frac{a}{c} \neq 0$, $-\left(\frac{a}{c}+it\right)^{-1} = -\frac{c}{a} + it\left(\frac{c}{a}\right)^2 + O(t^2)$ $(t \to 0+)$

$$\lim_{t\to 0+} t^{1/2}\theta\left(\frac{a}{c}+it\right) = \lim_{t\to 0+} t^{1/2}\left(\frac{a}{ci}+t\right)^{-1/2}\theta\left(-\left(\frac{a}{c}+it\right)^{-1}\right) = \left|\frac{a}{c}\right| \left(\frac{a}{ci}\right)^{-1/2} \lim_{t\to 0+} t^{1/2}\theta\left(-\frac{c}{a}+it\right)$$

$$\boxed{\frac{1}{2|c|^{1/2}} \sum_{x\in\mathbb{Z}/2|c|\mathbb{Z}} e^{\pi i a x^2/c} = \underbrace{\left(\frac{\mathrm{sgn}(ac)}{i}\right)^{-1/2}}_{e^{(2\pi i/8)\,\mathrm{sgn}(ac)}} \frac{1}{2|a|^{1/2}} \sum_{x\in\mathbb{Z}/2|a|\mathbb{Z}} e^{-\pi i c x^2/a}}$$

Exercise: Apply the same argument to θ_ϕ instead of θ.

(V) the value of ε_α in $\theta\left(\frac{a\tau+b}{c\tau+d}\right) = \varepsilon_\alpha (c\tau+d)^{1/2}\theta(\tau)$, $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_\theta$

branch of $(c\tau+d)^{1/2}$: $0 < \mathrm{Arg} < \frac{\pi}{2}$ if $c>0$ (say, if $c>0$)

Take $\tau = i\lambda$, $\lambda \to +\infty$: $\frac{a\tau+b}{c\tau+d} - \frac{a}{c} = -\frac{1}{c(c\tau+d)} = \frac{it}{c^2} + O(t^2)$

$(t = \lambda^{-1} \to 0+)$

$$\Longrightarrow \underbrace{\lim_{t\to 0+} t^{1/2}\theta\left(\frac{ait^{-1}+b}{cit^{-1}+d}\right)}_{|c|\cdot \lim_{t\to 0+} t^{1/2}\theta\left(\frac{a}{c}+it\right) = \frac{1}{2}\sum_{x\in\mathbb{Z}/2|c|\mathbb{Z}} e^{\pi i a x^2/c}} = \varepsilon_\alpha (ci)^{1/2} \lim_{\lambda\to+\infty}\theta(i\lambda) = \varepsilon_\alpha (ci)^{1/2}$$

$i^{1/2} = e^{2\pi i/8}$

$$\Longrightarrow \text{(if } c>0\text{)} \quad \varepsilon_\alpha = (e^{-2\pi i/8})\, \underbrace{\frac{1}{2c^{1/2}} \sum_{x\in\mathbb{Z}/2c\mathbb{Z}} e^{\pi i a x^2/c}}_{\frac{1}{c^{1/2}}\sum_{x\in\mathbb{Z}/c\mathbb{Z}} e^{\pi i a x^2/c}} \quad (\text{since } 2|ac)$$

If $c=0 \Rightarrow \pm\alpha = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}^k$ $(k\in\mathbb{Z}) \Rightarrow \varepsilon_\alpha = 1$.

Computing quadratic Gauss sums $S(a,c) := \frac{1}{2}\sum_{x\in\mathbb{Z}/2|c|\mathbb{Z}} e\left(\frac{1}{2}ax^2/c\right)$:
$(a,c\in\mathbb{Z}\setminus\{0\},\ (a,c)=1)$

(1) $$S(a,c) = \frac{1}{2}\sum_{x\in\mathbb{Z}/|c|\mathbb{Z}} e\left(\tfrac{1}{2}ax^2/c\right) + \underbrace{e\left(\tfrac{1}{2}a(x+c)^2/c\right)}_{e\left(\frac{1}{2}ax^2/c\right)(-1)^{ac}} = \begin{cases} 0, & 2\nmid ac \\ \sum_{x\in\mathbb{Z}/|c|\mathbb{Z}} e\left(\frac{1}{2}ax^2/c\right), & 2|ac \end{cases}$$

(2) We know: $$\frac{S(a,c)}{|c|^{1/2}} = \left(e^{2\pi i/8}\right)^{\mathrm{sgn}(ac)} \frac{S(-c,a)}{|a|^{1/2}}.$$

(3) If $2\nmid c$, $2|a$ $\Longrightarrow$ $S(a,c) = G\left(\frac{a}{2},c\right)$, $\quad G(b,c) = \sum_{x\in\mathbb{Z}/|c|\mathbb{Z}} e(bx^2/c)$

(4) $G(b,c)$ depends only on $b \pmod{|c|}$

(5) $u\in(\mathbb{Z}/|c|\mathbb{Z})^* \Longrightarrow G(bu^2,c) = G(b,c)$

(6) $c = c_1c_2$, $(c_1,c_2)=1$ $\quad \frac{1}{c_1c_2} = \frac{u_1}{c_1} + \frac{u_2}{c_2}$, $\quad u_1c_2 \equiv 1 \pmod{|c_1|}$, $u_2c_1 \equiv 1 \pmod{|c_2|}$

$$\Longrightarrow G(b,c_1c_2) = \sum_{x_k\in\mathbb{Z}/|c_k|\mathbb{Z}} e\left(\frac{bu_1x_1^2}{c_1} + \frac{bu_2x_2^2}{c_2}\right) = G(bu_1,c_1)G(bu_2,c_2) \overset{(5)}{=} G(bc_2,c_1)G(bc_1,c_2)$$

$x \equiv x_k \pmod{|c_k|}$ $(k=1,2)$

(7) $2\nmid c = c_1c_2^2$: $\quad (x_1 + c_1c_2x_2)^2 \equiv x_1^2 + 2c_1c_2x_1x_2 \pmod{c_1c_2^2}$

$$\sum_{x_2\in\mathbb{Z}/|c_2|\mathbb{Z}} e\left(\frac{2x_1x_2}{c_2}\right) = \begin{cases} |c_2|, & c_2|2x_1 \ (\Leftrightarrow c_2|x_1) \\ 0, & c_2\nmid 2x_1 \ (\Leftrightarrow c_2\nmid x_1) \end{cases}$$

$$\Longrightarrow G(b,c_1c_2^2) = |c_2| \sum_{y\in\mathbb{Z}/|c_1|\mathbb{Z}} e\left(\frac{by^2}{c_1}\right) = |c_2|\, G(b,c_1)$$

$(x_1 = c_2y)$

(8) $\underline{p\neq 2 \text{ prime},\ p\nmid b}$: $\mathbb{F}_p^\times = \mathbb{F}_p^{\times 2} \sqcup u\mathbb{F}_p^{\times 2}$, any u such that $\left(\frac{u}{p}\right)=-1$

$$\mathbb{F}_p^\times/\mathbb{F}_p^{\times 2} \xrightarrow{\sim} \{\pm 1\}$$ legendre's symbol

$$a \pmod p \longmapsto \left(\frac{a}{p}\right)$$

$$G(b,p) = 1 + 2\sum_{y\in b\mathbb{F}_p^{\times 2}} e(y/p) \quad , \text{ but } 1+\sum_{y\in\mathbb{F}_p^\times} e(y/p) = 0$$

$$\Rightarrow \quad G(b,p) + G(ub,p) = 0 \quad \Rightarrow \quad \underline{G(b,p) = \left(\frac{b}{p}\right) G(1,p)}$$

(9) $\underline{\text{Jacobi symbol}}$: $(2b,c)=1$, $c>0$: $c = \prod p_k^{r_k}$, $p_k \neq 2$ primes

$$\left(\frac{b}{c}\right) := \prod \left(\frac{b}{p_k}\right)^{r_k} \qquad \left(\frac{b_1 b_2}{c}\right) = \left(\frac{b_1}{c}\right)\left(\frac{b_2}{c}\right)$$

(10) $p\neq 2$ prime , $p\nmid b$, $r\geq 0$:

$$G(b,p^{2r}) = p^r G(b,1) = p^r = \left(\frac{p^{2r}}{p}\right) G(b,1)$$
$$G(b,p^{2r+1}) = p^r G(b,p) = p^r \left(\frac{b}{p}\right) G(1,p)$$

(11) $\underline{c = c_1 c_2,\ (c_1,c_2)=1,\ (2b,c)=1}$

$$\underline{G(b,c_1c_2) = \left(\frac{c_1}{c_2}\right)\left(\frac{c_2}{c_1}\right) G(b,c_1)\, G(b,c_2)}$$

~~(enough to check for $c_2 = p^r$, $p\neq 2$)~~

$\Uparrow$ (6)

(10.5) $\underline{(2b_1b_2,c)=1}$: $\quad G(b_1b_2,c) = \left(\frac{b_2}{c}\right) G(b_1,c)$

$\underline{\text{Pf}}$: $p_1,\dots,p_k \neq 2$ distinct primes, $c = p_1\cdots p_k$, $(b,c)=1$

$$G(b,c) = G(b,p_1\cdots p_k) = \underbrace{G(bp_k, \overbrace{p_1\cdots p_{k-1}}^{c/p_k})}_{G(bp_kp_{k-1},\, p_1\cdots p_{k-2})\, G(bc/p_{k-1},\, p_{k-1})} G(bp_1\cdots p_{k-1}, p_k) = \dots = \prod_{j=1}^{k} \underbrace{G(bc/p_j, p_j)}_{\left(\frac{bc/p_j}{p_j}\right) G(1,p_j)}$$

$$\Rightarrow \quad G(b_1b_2,c)/G(b_1,c) = \prod_j \left(\frac{b_2}{p_j}\right) = \left(\frac{b_2}{c}\right) \longrightarrow \text{OK if } c = p_1\cdots p_k.$$

the general case then follows from (7).

(12) $(2\nmid c)$

$$G(1,c) = S(2,c) = \left|\frac{c}{2}\right|^{1/2} e^{(2\pi i/8)\operatorname{sgn}(c)} S(-c,2)$$

$$S(-c,2) = \frac{1}{2}\sum_{x\in\mathbb{Z}/4\mathbb{Z}} e^{-\pi i c x^2/2} = \frac{2}{2}\Big(1 + \underbrace{e^{-\pi i c/2}}_{i^{-c}}\Big) = \sqrt{2}\cdot\begin{cases} e^{2\pi i/8}, & c\equiv -1 \pmod 4 \\ e^{-2\pi i/8}, & c\equiv 1 \pmod 4 \end{cases}$$

$$c>0: \quad \frac{G(1,c)}{c^{1/2}} = \begin{cases} i, & c\equiv -1 \pmod 4 \\ 1, & c\equiv 1 \pmod 4 \end{cases} = \frac{1+i^{-c}}{1+i^{-1}}$$

$$c<0: \quad \frac{G(1,c)}{|c|^{1/2}} = \begin{cases} 1, & c\equiv -1 \pmod 4 \\ -i, & c\equiv 1 \pmod 4 \end{cases}$$

$\underline{\text{Note}}$: $S(-c,-2) = S(c,2) = 1+i^c \Rightarrow G(-1,c) = S(-2,c) = \left|\frac{c}{2}\right|^{1/2}\left(e^{-2\pi i/8}\right)^{\operatorname{sgn}(c)} S(-c,-2)$

$$= \overline{G(1,c)}$$

(13) Quadratic reciprocity law : $p \neq q$ primes $\neq 2$

- $\underbrace{G(1,pq)}_{\frac{1+i^{-pq}}{1+i^{-1}}\sqrt{pq}} = \left(\frac{p}{q}\right)\left(\frac{q}{p}\right) \underbrace{G(1,p)\,G(1,q)}_{\left(\frac{1+i^{-p}}{1+i^{-1}}\sqrt{p}\right)\left(\frac{1+i^{-q}}{1+i^{-1}}\sqrt{q}\right)}$ $\Longrightarrow \left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = \begin{cases} -1, & p \equiv q \equiv -1 \pmod 4 \\ 1, & \text{otherwise} \end{cases}$

- $\underbrace{G(-1,p)}_{\left(\frac{-1}{p}\right)G(1,p)} = \overline{G(1,p)}$ $\Longrightarrow \left(\frac{-1}{p}\right) = \frac{\overline{G(1,p)}}{G(1,p)} = \begin{cases} -1, & p \equiv -1 \pmod 4 \\ 1, & p \equiv 1 \pmod 4 \end{cases}$

(idem for $\left(\frac{-1}{c}\right)$)

- $\underbrace{G(2,p)\left(= \left(\frac{2}{p}\right) G(1,p)\right)}_{\left(\frac{2}{p}\right) \frac{p^{1/2}}{\sqrt{2}} e^{2\pi i/8} S(-p,2)} = S(4,p) = \frac{p^{1/2}}{2} e^{2\pi i/8} S(-p,4)$ $\Longrightarrow \left(\frac{2}{p}\right) = \frac{S(-p,4)}{\sqrt{2}\, S(-p,2)}$

---

(14) General formula for $G(b,c)$: $(2b,c)=1,\ c>0$

$$G(b,c) = \left(\frac{b}{c}\right) G(1,c) = \left(\frac{b}{c}\right) c^{1/2} \frac{1+i^{-c}}{1+i^{-1}} \qquad (c>0)$$

$$G(b,c) = G(-b,-c) = \left(\frac{-b}{-c}\right) G(1,-c) = \left(\frac{-b}{-c}\right)(-c)^{1/2} \frac{1+i^{c}}{1+i^{-1}} \qquad (c<0)$$

Cor 1 : for $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_\theta$ such that $c>0$ and $2 \nmid c$ ($\Rightarrow 2 \mid a$),

$$\varepsilon_\alpha = e^{-2\pi i/8} c^{-1/2} G\left(\tfrac{a}{2}, c\right) = e^{-2\pi i/8} \left(\frac{a/2}{c}\right) \frac{1+i^{-c}}{1+i^{-1}}$$

($\Rightarrow \varepsilon_\alpha^2 = -i\left(\frac{-1}{c}\right)$) ?

$$= \left(\frac{a/2}{c}\right) \cdot \begin{cases} e^{-2\pi i/8}, & c \equiv 1 \pmod 4 \\ e^{2\pi i/8}, & c \equiv -1 \pmod 4 \end{cases}$$

Cor 2 : for $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_\theta$ such that $2 \mid c$ ($\Rightarrow 2 \nmid a$) and $c>0$:

$$\varepsilon_\alpha = e^{-2\pi i/8} c^{-1/2} S(a,c) = \left(e^{2\pi i/8}\right)^{(\mathrm{sgn}(a)-1)} \underbrace{\frac{S(-c,a)}{|a|^{1/2}}}_{G(-c/2,a)/|a|^{1/2}}$$

$$G(-c/2,a)/|a|^{1/2} = \left(\frac{-c/2}{a}\right) \frac{G(1,a)}{|a|^{1/2}} \quad \text{if } a>0$$

$$= \left(\frac{c/2}{-a}\right) \frac{G(1,-a)}{|a|^{1/2}} \quad \text{if } a<0$$

$$= \left(\frac{(-c/2)\,\mathrm{sgn}(a)}{a}\right) \underbrace{\frac{G(1,|a|)}{|a|^{1/2}}}_{\frac{1+i^{-|a|}}{1+i^{-1}}} \cdot \left(e^{2\pi i/8}\right)^{\mathrm{sgn}(a)-1}$$

$$\Longrightarrow \varepsilon_\alpha^2 = \begin{cases} 1, & a>0,\ a \equiv 1\ [4] \\ -1, & a>0,\ a \equiv -1\ [4] \\ -1, & a<0,\ a \equiv -1\ [4] \\ 1 & a<0,\ a \equiv 1\ [4] \end{cases}$$

?

$\theta(\tau)$ and $\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}$ (Riemann)

Recall: $\Gamma(s)=\int_0^\infty e^{-t}t^s\frac{dt}{t}$ $\quad(\mathrm{Re}(s)>0)$ $\qquad (t^s=e^{s\ln(t)})$

Mellin transform: $\boxed{(Mf)(s):=\int_0^\infty f(t)\,t^s\frac{dt}{t}}$ $\qquad t=au \Rightarrow \frac{dt}{t}=\frac{du}{u}\quad (a>0)$

$$\int_0^\infty e^{-au}u^s\frac{du}{u}=a^{-s}\Gamma(s)$$

$$Z(s):=\pi^{-s}\Gamma(s)\zeta(2s)=\sum_{n=1}^\infty\int_0^\infty e^{-\pi n^2 t}t^s\frac{dt}{t}=\int_0^\infty\underbrace{\Big(\sum_{n=1}^\infty e^{-\pi n^2 t}\Big)}_{(\theta(it)-1)/2\ \in\mathcal{S}(\mathbb{R})}t^s\frac{dt}{t}\qquad (\mathrm{Re}(s)>\tfrac{1}{2})$$

$$\underline{\theta(it)=t^{-1/2}\theta(\tfrac{i}{t})}\quad\left\{\begin{array}{ll}\sim t^{-1/2} & \text{as } t\to 0+\\ =1+O(e^{-ct}) & \text{as } t\to+\infty\end{array}\right\}$$

$$Z(s)=\pi^{-s/2}\Gamma(\tfrac{s}{2})\zeta(s)=\underbrace{\int_0^1\Big(\frac{\theta(it)-1}{2}\Big)t^{s/2}\frac{dt}{t}}_{I_0(s)}+\underbrace{\int_1^\infty\Big(\frac{\theta(it)-1}{2}\Big)t^{s/2}\frac{dt}{t}}_{I_\infty(s)\ \text{holomorphic in } s\in\mathbb{C}}\qquad (\mathrm{Re}(s)>1)$$

transform I_0 to I_∞ by $t=\frac{1}{u}$: $\quad \frac{dt}{t}=-\frac{du}{u}$, $\quad \theta(it)=u^{1/2}\theta(iu)$

$$I_0(s)=\int_1^\infty\Big(\frac{u^{1/2}\theta(iu)-1}{2}\Big)u^{-s/2}\frac{du}{u}\qquad (\mathrm{Re}(s)>1)$$

$$Z(s)=\underbrace{\int_1^\infty\underbrace{\Big(\frac{\theta(it)-1}{2}\Big)}_{\in\mathcal{S}(\mathbb{R})}\big(t^{s/2}+t^{(1-s)/2}\big)\frac{dt}{t}}_{\substack{\text{holomorphic in } s\in\mathbb{C},\\ \text{invariant under } s\leftrightarrow 1-s}}+\underbrace{\int_1^\infty\Big(\frac{t^{(1-s)/2}-t^{-s/2}}{2}\Big)\frac{dt}{t}}_{\frac{1}{s-1}-\frac{1}{s}}\qquad (\mathrm{Re}(s)>1)$$

$\Rightarrow$ Thm. $\underline{Z(s)=\pi^{-s/2}\Gamma(\tfrac{s}{2})\zeta(s)}$ has meromorphic continuation to $\mathbb{C}$, which has simple poles at $s=0,1$ with residues

$\mathrm{Res}_{s=1}=1$, $\mathrm{Res}_{s=0}=-1$, and is holomorphic in $\mathbb{C}\setminus\{0,1\}$.

In addition, $Z(1-s)=Z(s)$.

Cor. $\lim_{s\to 0}\frac{s\Gamma(s)}{\Gamma(s+1)}=\Gamma(1)=1 \Longrightarrow 2\zeta(0)=\mathrm{Res}_{s=0}Z(s)\Rightarrow \zeta(0)=-\frac{1}{2}$.

(Recall: $\Gamma(s)$ has meromorphic continuation to $\mathbb{C}$ satisfying $\Gamma(s+1)=s\Gamma(s)$. It is holomorphic in $\mathbb{C}\setminus\{0,-1,-2,\dots\}$, with simple poles at each $s=-n$ $(n\in\mathbb{N})$, $\mathrm{Res}_{s=-n}\Gamma(s)=\frac{(-1)^n}{n!}$)

Cor. For $\mathrm{Re}(s)>1$, $\zeta(s)\neq 0$

For $\mathrm{Re}(s)<0$, $\left\{\begin{array}{l}\zeta(s)\neq 0 \quad \text{if } s\neq -2,-4,-6,\dots\\ \zeta(s) \text{ has simple zeros at each } s=-2,-4,-6,\dots\end{array}\right.$

θ and η

$$\eta(\tau) = q^{1/24} P(q) \quad , \quad P(q) = \prod_{n=1}^{\infty} (1-q^n) \qquad (q = e^{2\pi i \tau} \, , \, q^{1/N} = e^{2\pi i \tau / N})$$

Euler's pentagonal number formula: $P(q) = \sum_{n\in\mathbb{Z}} (-1)^n q^{\frac{3n^2+n}{2}}$

$$\Longrightarrow \quad \eta(24\tau) = \sum_{n\in\mathbb{Z}} (-1)^n q^{1+12(3n^2+n)} = \sum_{n\in\mathbb{Z}} (-1)^n q^{(6n+1)^2} = \frac{1}{2}\sum_{m\in\mathbb{Z}} \phi(m) q^{m^2}$$

$$\phi(m) = \begin{cases} 0, & (m,12)\neq 1 \\ 1, & m \equiv \pm 1 \, [12] \\ -1, & m \equiv \pm 5 \, [12] \end{cases} \qquad (\phi : \mathbb{Z} \longrightarrow \mathbb{Z}/12\mathbb{Z} \longrightarrow \{0, \pm 1\})$$

$$2\eta(\tau) = \sum_{n\in\mathbb{Z}} \phi(n) e^{\pi i n^2 \tau / 12} \quad , \quad 2\eta(12\tau) = \sum_{n\in\mathbb{Z}} \phi(n) e^{\pi i n^2 \tau} = \theta_\phi(\tau)$$

Poisson formula $\Longrightarrow \quad \theta_\phi(\tau) = 12^{-1} \left(\frac{\tau}{i}\right)^{-1/2} \theta_{\hat{\phi}}(-1/12^2\tau)$

$$\left(\Longleftrightarrow \quad 2\eta(\tau) = 12^{-1/2} \left(\frac{\tau}{i}\right)^{-1/2} \theta_{\hat{\phi}}(-1/12\tau) \right)$$

$$\hat{\phi}(n) = \zeta_{12}^{n} + \zeta_{12}^{-n} - (\zeta_{12}^{5n} + \zeta_{12}^{-5n}) \qquad (\zeta_{12} = e^{2\pi i/12} \, , \quad \zeta_{12}^5 = -\zeta_{12}^{-1})$$

$$= (1-(-1)^n)(\zeta_{12}^{n} + \zeta_{12}^{-n}) = \hat{\phi}(-n) = \begin{cases} 0 & \text{if } 2 \mid n \text{ or } 3 \mid n \\ \underbrace{2(\zeta_{12} + \zeta_{12}^{-1})}_{12^{1/2}} \phi(n) & \text{if } (n,12) = 1 \end{cases}$$

$$\Longrightarrow \quad \hat{\phi} = 12^{1/2} \phi$$

Conclusion: $\boxed{\eta(\tau) = \left(\frac{\tau}{i}\right)^{-1/2} \frac{\theta_\phi(-1/12\tau)}{2} = \left(\frac{\tau}{i}\right)^{-1/2} \eta\left(-\frac{1}{\tau}\right)}$