Functional equation of $\theta(\tau) = \sum_{n\in\mathbb{Z}} e^{\pi i n^2 \tau}$ – reformulation of the proof.

__Thm__: $\forall\, \alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_\theta := \left\{ \alpha \in SL_2(\mathbb{Z}) \;\middle|\; \alpha \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \pmod 2 \right\}$

$$\left(\theta\left(\frac{a\tau+b}{c\tau+d}\right) / \theta(\tau)\right)^2 (c\tau+d)^{-1} \in \{\pm 1, \pm i\}$$

__Summary of the proof above__: (1) enough to prove for generators of Γ_θ (by the 1-cocycle identity for $J(\alpha,\tau) = c\tau + d$).

(2) $T^2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ generate Γ_θ.

(3) Thm is trivial for $\alpha = T^2$.

(4) $f_\tau(x) = e^{\pi i \tau x^2} \in \mathcal{S}(\mathbb{R})$ is the unique solution (up to a mult. constant) of $(\tau P + Q) f = 0$, $P = -(2\pi i)x$, $Q = \frac{d}{dx}$

(5) the Fourier transform $\mathcal{F}$ satisfies

$$\mathcal{F} \circ (P\ \ Q) \circ \mathcal{F}^{-1} = (Q\ \ -P) = (P\ \ Q)\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = (P\ \ Q)S$$

$$\Rightarrow \underbrace{\mathcal{F} \circ (P\ \ Q)\begin{pmatrix} \tau \\ 1 \end{pmatrix}}_{\tau P + Q} \circ \mathcal{F}^{-1} = (P\ \ Q)\begin{pmatrix} -1/\tau \\ 1 \end{pmatrix}\tau = \tau\left(-\tfrac{1}{\tau}P + Q\right) \quad \Big| \quad \underline{S\begin{pmatrix} \tau \\ 1 \end{pmatrix} = \begin{pmatrix} -1/\tau \\ 1 \end{pmatrix}\tau}$$

$$\Rightarrow (-\tau^{-1}P + Q)\mathcal{F} f_\tau = 0 \xRightarrow{(4)} \underline{\mathcal{F} f_\tau = c(\tau) f_{-1/\tau}}$$

(6) $c(\tau)^2 = \frac{i}{\tau}$

(7) Poisson summation formula $\xRightarrow{(5)}$ $\underbrace{\sum_{n\in\mathbb{Z}} f_\tau(n)}_{\theta(\tau)} = \sum_{n\in\mathbb{Z}} (\mathcal{F} f_\tau)(n) = c(\tau) \underbrace{\sum_{n\in\mathbb{Z}} f_{-1/\tau}(n)}_{\theta(-1/\tau)}$

$$\Rightarrow \left(\theta(-1/\tau)/\theta(\tau)\right)^2 = c(\tau)^{-2} = \frac{\tau}{i} \Rightarrow \text{Thm for } \alpha = S.$$

__Goal__: Reformulate in more abstract terms.

In particular, generalise (5) to __arbitrary__ matrices

Unitary Integral operators generalising the Fourier transform

Ex. The operators $P = -(2\pi i)x$, $Q = \frac{d}{dx}$ acting on $\mathcal{S}(\mathbb{R})$ satisfy

$$[P,Q] = PQ - QP = (2\pi i), \quad \mathcal{F} Q \mathcal{F}^{-1} = -P, \quad \mathcal{F} P \mathcal{F}^{-1} = Q$$

$$(\mathcal{F}f)(x^*) = \int_{\mathbb{R}} e^{-2\pi i x^* x} f(x)\,dx \qquad \text{Fourier transform}$$

Heisenberg Lie algebra: $\mathbb{R}P \oplus \mathbb{R}Q \oplus \mathbb{R}\cdot(2\pi i)$

elements $y_1 Q + y_2 P \longleftrightarrow y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in \mathbb{R}^2$

$$\mathcal{F}(y_1 \mathbf{P} + y_2 \mathbf{Q})\mathcal{F}^{-1} = -y_2 \mathbf{P} + y_1 \mathbf{Q} \qquad \begin{pmatrix} -y_2 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

Question: given $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{R})$, is there an operator $U = U(g)$ such that

$$\boxed{\begin{array}{l} UPU^{-1} = aP + cQ \\ UQU^{-1} = bP + dQ \end{array}} \;? \quad \left(\Leftrightarrow U(y_1 \mathbf{P} + y_2 \mathbf{Q})U^{-1} = y_1' \mathbf{P} + y_2' \mathbf{Q}\right)$$

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

Necessary condition: $[aP + cQ, bP + dQ] = (ad - bc)[P,Q]$

$$\Longrightarrow \quad ad - bc = 1, \quad g \in SL_2(\mathbb{R})$$

Ex: (1) $g = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$: $(U(g)f)(x) = |a|^{1/2} f(ax)$ (unitary on $L^2(\mathbb{R})$)

(2) $g = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$: $(U(g)f)(x) = e^{\pi i b x^2} f(x)$ (— " —)

(3) $g = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$: $U(g) =$ Fourier transform $\mathcal{F}$

All of them preserve the Schwartz space $\mathcal{S}(\mathbb{R})$.

Thm: (1) $\forall\, g \in SL_2(\mathbb{R})$ there exists an invertible linear map $U(g): \mathcal{S}(\mathbb{R}) \longrightarrow \mathcal{S}(\mathbb{R})$, such that $U(g)\,(y_1 P + y_2 Q)\,U(g)^{-1} = y_1' P + y_2' Q$ $\quad \forall y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in \mathbb{R}^2$, where $y' = \begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = g\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$, and $\int_{\mathbb{R}} |U(g)f|^2 = \int_{\mathbb{R}} |f|^2 \quad \forall f \in \mathcal{S}(\mathbb{R})$.

(2) Such a linear map is unique up to multiplication by $t \in \mathbb{C}^\times$, $|t| = 1$ ($t \in U(1)$).

(3) Each $U(g)$ is continuous and extends (uniquely) to a unitary operator on $L^2(\mathbb{R})$.

Cor: The set of such $U(g)$ (for all $g \in SL_2(\mathbb{R})$) is a group $\widetilde{Mp}_2(\mathbb{R})$ under composition and there is an exact sequence

$$1 \longrightarrow \underbrace{U(1)}_{\{t \in \mathbb{C}^\times \mid |t| = 1\}} \longrightarrow \widetilde{Mp}_2(\mathbb{R}) \longrightarrow SL_2(\mathbb{R}) \longrightarrow 1 \qquad \text{(central extension)}$$

$$U(g) \longmapsto g$$

<u>Pf.</u> (1) Given $g=\begin{pmatrix} a & b \\ c & d\end{pmatrix} \in SL_2(\mathbb{R})$, write

$$g = \begin{cases} \begin{pmatrix} a & 0 \\ 0 & a^{-1}\end{pmatrix}\begin{pmatrix} 1 & ba^{-1} \\ 0 & 1\end{pmatrix} = g_1 g_2 & \text{if } c=0 \\ \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1\end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0\end{pmatrix}\begin{pmatrix} c & 0 \\ 0 & c^{-1}\end{pmatrix}\begin{pmatrix} 1 & c^{-1}d \\ 0 & 1\end{pmatrix} = g_1g_2g_3g_4 & \text{if } c\neq 0 \end{cases}$$

and let $U(g) := U(g_1)\cdots U(g_k)$, where $U(g_j)$ are defined as above.

(2) Given two such $U(g)$, $\tilde{U}(g)$, the linear map $V := U(g)^{-1}\tilde{U}(g)$, $V: \mathcal{S}(\mathbb{R}) \longrightarrow \mathcal{S}(\mathbb{R})$ satisfies $x\circ V = V\circ x$, $\frac{d}{dx}\circ V = V \circ \frac{d}{dx}$.

<u>Lemma</u>. If a linear map $V: \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n)$ satisfies $\forall j=1,\dots,n$ $\underline{x_j\circ V = V\circ x_j}$, then $V=$ multiplication by some $g\in \mathcal{C}^\infty(\mathbb{R}^n)$.

<u>Pf.</u> Fix $a\in\mathbb{R}^n$. Every $\phi\in\mathcal{S}(\mathbb{R}^n)$ can be written as

$$\phi = \phi(a) + \sum_{j=1}^n (x_j - a_j)\phi_j, \quad \phi_j \in \mathcal{S}(\mathbb{R}^n) \qquad \text{(exercise!)}$$

In particular, if $\phi(a)=0$, then $V(\phi) = \sum_{j=1}^n (x_j-a_j)V(\phi_j) \Rightarrow \underline{(V(\phi))(a)=0}$.

Fix $\psi\in\mathcal{S}(\mathbb{R}^n)$ such that $\forall x\in\mathbb{R}^n$ $\psi(x)\neq 0$.

If $\phi\in\mathcal{S}(\mathbb{R}^n)$ and $\phi(a)\neq 0$, then $\tilde{\phi}(x) := \phi(x) - \frac{\psi(x)\phi(a)}{\psi(a)}$ satisfies $\tilde\phi\in\mathcal{S}(\mathbb{R}^n)$ and $\tilde\phi(a)=0 \Longrightarrow (V\tilde\phi)(a)=0$

$\Longrightarrow (V\phi)(a) = \frac{(V\psi)(a)}{\psi(a)}$, and so $\forall\phi\in\mathcal{S}(\mathbb{R}^n)$ $V\phi = g\phi$, $g=\frac{V\psi}{\psi}\in\mathcal{C}^\infty(\mathbb{R}^n)$.

<u>Cor.</u> If, in addition, $\forall j=1,\dots,n$ $\frac{\partial}{\partial x_j}\circ V = V\circ\frac{\partial}{\partial x_j}$, then $V=$ multiplication by constant.

$\boxed{\Longrightarrow \text{(2) of Thm}}$

---

(3) True for $U(g)$ constructed in (1) $\xRightarrow{(2)}$ true for all possible $\tilde U(g) = t\cdot U(g)$, $t\in\mathbb{C}^\times$, $|t|=1$

<u>Corresponding action of the Lie algebra</u> <u>$\mathfrak{sl}(2,\mathbb{R}) = \mathrm{Lie}\, SL_2(\mathbb{R})$</u>:

$\mathfrak{sl}(2,\mathbb{R}) = \mathbb{R}\cdot X \oplus \mathbb{R}\cdot Y \oplus \mathbb{R}\cdot H$, $X=\begin{pmatrix} 0 & 1 \\ 0 & 0\end{pmatrix}$, $Y=\begin{pmatrix} 0 & 0 \\ 1 & 0\end{pmatrix}$, $H=\begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}$

$e^{tX} = \begin{pmatrix} 1 & t \\ 0 & 1\end{pmatrix}$, $e^{tY} = \begin{pmatrix} 1 & 0 \\ t & 1\end{pmatrix}$, $e^{tH} = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t}\end{pmatrix}$; $\mathbb{R}(-X+Y) = \mathbb{R}\cdot\begin{pmatrix} 0 & -1 \\ 1 & 0\end{pmatrix} = \mathrm{Lie}(SO(2))$

$e^{t(-X+Y)} = \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t)\end{pmatrix} = h_t$

$$\boxed{[X,Y]=H, \quad [H,X]=2X, \quad [H,Y]=-2Y}$$

$$(X*f)(x) = \frac{d}{dt}\Big(\underbrace{U\begin{pmatrix} 1 & t \\ 0 & 1\end{pmatrix}f)(x)}_{e^{\pi i t x^2}f(x)}\Big|_{t=0} = \pi i x^2 f(x) = \left(\frac{1}{2}\frac{P^2}{2\pi i}f\right)(x)$$

$$(H*f)(x) = \frac{d}{dt}\Big(\underbrace{U\begin{pmatrix} e^t & 0 \\ 0 & e^{-t}\end{pmatrix}f)(x)}_{e^{t/2}f(e^t x)}\Big|_{t=0} = \frac{1}{2}f(x) + x f'(x) = \left(\left(\frac{1}{2} + \frac{PQ}{2\pi i}\right)f\right)(x)$$

As $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ satisfies $J \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} J^{-1} = \begin{pmatrix} 1 & 0 \\ -b & 1 \end{pmatrix}$, we can take

$$U\left(\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}\right) = \mathcal{F} \circ U\left(\begin{pmatrix} 1 & -t \\ 0 & 1 \end{pmatrix}\right) \circ \mathcal{F}^{-1}$$

$$(Y \cancel{\ast} f)(x) = \frac{d}{dt}\left(U\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} f\right)(x)\Big|_{t=0} = (\mathcal{F} \circ (-X) \circ \mathcal{F}^{-1}) f\ (x)$$

$$= \frac{1}{2} \frac{\mathcal{F} \circ (-P^2) \circ \mathcal{F}^{-1}}{2\pi i} = \frac{1}{2} \cdot \frac{-Q^2}{2\pi i} = \frac{1}{2} \cdot \left(-\frac{1}{2\pi i}\left(\frac{d}{dx}\right)^2\right) f$$

Six-dimensional group of symmetries of $\mathcal{S}(\mathbb{R})$ and $L^2(\mathbb{R})$:

Its Lie algebra is $\underbrace{\mathbb{R} \cdot (2\pi i) \oplus \mathbb{R} \cdot \underbrace{(2\pi i x)}_{P} \oplus \mathbb{R} \cdot \underbrace{\frac{d}{dx}}_{Q}}_{\text{Heisenberg Lie algebra}} \oplus \underbrace{\mathbb{R} \cdot \pi i x^2 \oplus \mathbb{R} \cdot \pi i \left(\frac{d}{dx}\right)^2 \oplus \mathbb{R}\left(\frac{1}{2} + x\frac{d}{dx}\right)}_{\mathfrak{sl}(2,\mathbb{R})}$

the group is (essentially) a semi-direct product of $\widetilde{Mp_2}(\mathbb{R})$ (the metaplectic group) and the Heisenberg group generated by operators

$$(e^{tP} f)(x) = e^{-2\pi i t x} f(x), \quad (e^{tQ} f)(x) = f(x+t) \qquad (t \in \mathbb{R})$$

Functional equations of $\theta(z, \tau)$ and its variants are explained by this group of operators (and its variants)

Example: the function $f_i(x) = e^{-\pi x^2}$ satisfies

$$(Q + iP) f_i = 0 \implies (-X+Y) f_i = -\frac{i}{2} f \implies \underbrace{e^{t(-X+Y)} f_i}_{\text{"}U\underbrace{\begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}}_{h_t} f_i\text{"}} = e^{-it/2} f_i$$

this shows that $\{e^{t(-X+Y)} \mid t \in \mathbb{R}\} \subset \widetilde{Mp_2}(\mathbb{R})$ is a 2-fold covering of $SO(2) = \{h_t\}$, $SO(2) \subset SL_2(\mathbb{R})$

Moreover, $\forall \tau = u + vi \in \mathcal{H}$

$$g_\tau = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\begin{pmatrix} v^{1/2} & 0 \\ 0 & v^{-1/2} \end{pmatrix} : i \longmapsto \tau \quad \text{and}$$

$$U(g_\tau) := U\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} U\begin{pmatrix} v^{1/2} & 0 \\ 0 & v^{-1/2} \end{pmatrix} : f \longmapsto e^{\pi i u x^2} v^{1/4} f(v^{1/2} x)$$

sends f_i to $(U(g_\tau) f_i)(x) = v^{1/4} e^{\pi i u x^2} e^{-\pi v x^2}$

$$= v^{1/4} \underbrace{e^{\pi i \tau x^2}}_{f_\tau(x)} = v^{1/4} f_\tau(x)$$

Therefore: $\operatorname{Im}(\tau)^{1/4} \theta(\tau) = \operatorname{Im}(\tau)^{1/4} \sum_{n \in \mathbb{Z}} f_\tau(n) = \sum_{n \in \mathbb{Z}} (U(g_\tau) f_i)(n)$

$$= \langle \delta_{\mathbb{Z}}, U(g_\tau) f_i \rangle \quad \text{matrix element of the representation } U$$

$$\langle \delta_{\mathbb{Z}}, F \rangle = \sum_{n \in \mathbb{Z}} F(n) \qquad (F \in \mathcal{S}(\mathbb{R}))$$

$\widetilde{Mp_2(\mathbb{R})}$ and $\theta(\tau) = \sum_{n\in\mathbb{Z}} f_\tau(n) = \langle \delta_{\mathbb{Z}}, f_\tau \rangle$

$f_\tau(x) = e^{\pi i \tau x^2}$

Recall: $1 \longrightarrow \mathbb{C}_1^\times \longrightarrow \underbrace{\widetilde{Mp_2(\mathbb{R})}}_{\text{acts on } \mathcal{S}(\mathbb{R})} \xrightarrow{p} SL_2(\mathbb{R}) \longrightarrow 1$ ($\mathbb{C}_1^\times$ central)

$|t|=1$, $t \in \mathbb{C}_1^\times$ acts by multiplication by t

Key property: $\forall \tilde\alpha \in \widetilde{Mp_2(\mathbb{R})}$ with $p(\tilde\alpha) = \alpha \in SL_2(\mathbb{R})$

$(P, Q)\alpha = \tilde\alpha (P, Q) \tilde\alpha^{-1}$ $P = -(2\pi i)x$, $Q = \frac{d}{dx}$

$\mathbb{C} f_\tau = \{ f \in \mathcal{S}(\mathbb{R}) \mid (P, Q)\begin{pmatrix}\tau\\1\end{pmatrix} f = 0 \}$

$\alpha\begin{pmatrix}\tau\\1\end{pmatrix} = \begin{pmatrix}\alpha(\tau)\\1\end{pmatrix} J(\alpha,\tau)$

$\Rightarrow$ $\tilde\alpha(f_\tau) = c(\tilde\alpha,\tau) f_{\alpha(\tau)}$, $c(\tilde\alpha,\tau) \in \mathbb{C}^\times$ | $c(t\tilde\alpha,\tau) = t\,c(\tilde\alpha,\tau)$

Prop: $c(\tilde\alpha,\tau)^2 J(\alpha,\tau) \in \mathbb{C}_1^\times = \{t \in \mathbb{C} \mid |t| = 1\}$

Pf. True for $\alpha = \begin{pmatrix}1 & b\\0 & 1\end{pmatrix}, \begin{pmatrix}a & 0\\0 & a^{-1}\end{pmatrix}, S$ (by explicit formulas)

$\Rightarrow$ true for all α.

Def: $Mp_2(\mathbb{R}) := \{ \tilde\alpha \in \widetilde{Mp_2(\mathbb{R})} \mid c(\tilde\alpha,\tau)^2 J(\alpha,\tau) = 1 \}$

this is a subgroup of $\widetilde{Mp_2(\mathbb{R})}$ and there is an exact sequence

$1 \longrightarrow \{\pm 1\} \longrightarrow Mp_2(\mathbb{R}) \longrightarrow SL_2(\mathbb{R}) \longrightarrow 1$ (non-split)

(we have multiplied earlier $U(g)$ by suitable $u(g) \in \mathbb{C}_1^\times$ so that new $\tilde U(g) = U(g)u(g)$ satisfy $\tilde U(gh) = \pm \tilde U(g)\tilde U(h)$)

Prop. the tempered distribution $\delta_{\mathbb{Z}} : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathbb{C}$

$f \longmapsto \sum_{n\in\mathbb{Z}} f(n)$

satisfies $\forall \tilde\alpha \in \widetilde{Mp_2(\mathbb{R})}$ s.t. $p(\tilde\alpha) = \alpha \in \Gamma_\theta$ $\delta_{\mathbb{Z}}\tilde\alpha = \underset{\mathbb{C}_1^\times}{\eta(\tilde\alpha)}\,\delta_{\mathbb{Z}}$.

Pf. True for $\alpha = T^2$ (trivially), for $\alpha = S$ by Poisson $\Rightarrow$ for all $\alpha \in \Gamma_\theta$.

Moreover: if $\tilde\alpha \in Mp_2(\mathbb{R})$ and $p(\tilde\alpha) = \alpha \in \Gamma_\theta$, then

$\eta(\tilde\alpha)^2 \in \{\pm 1, \pm i\}$ depends only on α.

Abstract proof of the functional equation of $\theta(\tau)$

$\forall \tilde{\alpha} \in Mp_2(\mathbb{R})$, $\tilde{\alpha} \xmapsto{p} \alpha \in \Gamma_\theta$

$$\theta(\alpha(\tau)) = \langle \delta_{\mathbb{Z}}, f_{\alpha(\tau)} \rangle = c(\tilde{\alpha},\tau)^{-1} \underbrace{\langle \delta_{\mathbb{Z}}, \tilde{\alpha}(f_\tau) \rangle}_{\langle \delta_{\mathbb{Z}}\tilde{\alpha}, f_\tau \rangle = \eta(\tilde{\alpha}) \underbrace{\langle \delta_{\mathbb{Z}}, f_\tau \rangle}_{\theta(\tau)}}$$

$$\theta(\alpha(\tau)) = \eta(\tilde{\alpha})\, c(\tilde{\alpha},\tau)^{-1}\, \theta(\tau)$$

$$(c(\tilde{\alpha},\tau)^{-1})^2 = J(\alpha,\tau)$$

Exercise : $\forall n \geq 0$, $f_{\tau,n} := (\bar{\tau} P + Q)^n f_\tau = f_\tau(x)\, He_n(\sqrt{4\pi \operatorname{Im}(\tau)}\, x)$

$\left(He_n(x) = (-1)^n e^{x^2/2} \left(\frac{d}{dx}\right)^n e^{-x^2/2} = 2^{-n/2} H_n(x/\sqrt{2}) \right.$ Hermite polynomial$\left.\right)$

satisfies

$\forall \tilde{\alpha} \in \widetilde{Mp_2}(\mathbb{R})$ $\quad \tilde{\alpha}(f_{\tau,n}) = (c\bar{\tau}+d)^n\, c(\tilde{\alpha},\tau)\, f_{\alpha(\tau),n}$

$\Longrightarrow$ functional equation for $\langle \delta_{\mathbb{Z}}, f_{\tau,n} \rangle = \sum_{m\in\mathbb{Z}} e^{\pi i m^2 \tau}\, He_m(\sqrt{4\pi y}\, n)$

$\tau = x+iy$

Rmk : for $\tau = i$, the functions $f_{i,n}(x)$ (and the operators $(\pm i P + Q)$) appear in quantum mechanics, as eigenfunctions of the Hamiltonian operator for the harmonic oscillator.

The action of $\widetilde{Mp_2}(\mathbb{R})$ on $L^2(\mathbb{R})$ is often called the oscillator representation, or the Segal – Shale – Weil representation, or just the Weil representation.

Hermite polynomials

$P = 2\pi i x$, $Q = -\frac{d}{dx}$

$\mathfrak{F}P\mathfrak{F}^{-1} = Q$, $\mathfrak{F}Q\mathfrak{F}^{-1} = -P$

$P \longleftrightarrow \begin{pmatrix}1\\0\end{pmatrix}$, $Q \longleftrightarrow \begin{pmatrix}0\\1\end{pmatrix}$

$\mathfrak{F}(aP+bQ)\mathfrak{F}^{-1} = -bP+aQ$

$\begin{pmatrix}a\\b\end{pmatrix} \longmapsto \underbrace{\begin{pmatrix}0&-1\\1&0\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix}}_{\begin{pmatrix}-b\\a\end{pmatrix}}$

$f_\tau(x) = e^{\pi i \tau x^2}$

$(\tau P + Q)(f_\tau) = 0$ $\qquad \tau P + Q \longleftrightarrow \begin{pmatrix}\tau\\1\end{pmatrix}$

$\overset{\tau P+Q}{\longleftarrow} \quad \overset{\bar\tau P+Q}{\longrightarrow}$

$[\bar\tau P + Q, \tau P + Q] = (\bar\tau - \tau)[P,Q] = 2\pi i(\bar\tau - \tau) = 4\pi \operatorname{Im}(\tau)$

$\mathfrak{sl}(2)$: $X_\tau = a_1(\bar\tau P+Q)^2/2$, $Y_\tau = a_2(\tau P+Q)^2/2$, $H_\tau = a_0((\bar\tau P+Q)(\tau P+Q) + (\tau P+Q)(\bar\tau P+Q))/2$

$[X_\tau, Y_\tau] = \frac{a_1 a_2}{a_0}(2\pi i)(\bar\tau-\tau)H_\tau$, $[H_\tau, X_\tau] = 2a_0(2\pi i)(\bar\tau-\tau)X_\tau$, $[H_\tau, Y_\tau] = -2a_0(2\pi i)(\bar\tau - \tau)Y_\tau$

We want: $a_0 = a_1 = a_2 = \frac{1}{2\pi i(\bar\tau - \tau)} = \frac{1}{4\pi \operatorname{Im}(\tau)}$

$[H_\tau, \bar\tau P+Q] = \bar\tau P + Q$, $[H_\tau, \tau P+Q] = -(\tau P+Q)$

$\boxed{H_\tau = a_0(\bar\tau P+Q)(\tau P+Q) + \frac{1}{2}}$

Goal: for each $n \in \mathbb{N}$, compute $f_{\tau,n} := (\bar\tau P+Q)^n f_\tau$ and $\mathfrak{F} f_{\tau,n}$.

$(\tau P+Q)(f_\tau) = 0 \implies f_\tau^{-1}\circ(\tau P+Q)\circ f_\tau = Q$, $f_{\bar\tau}^{-1}\circ(\bar\tau P+Q)\circ f_{\bar\tau} = Q$

$\bar\tau P + Q = f_{\bar\tau}\circ Q \circ f_{\bar\tau}^{-1} \implies f_{\tau,n} = f_{\bar\tau}\cdot Q^n(f_{\bar\tau}^{-1} f_\tau) = f_{\bar\tau} Q^n(f_{\tau-\bar\tau})$, $f_\tau^{-1} f_{\tau,n} = f_{\tau-\bar\tau}^{-1} Q^n(f_{\tau-\bar\tau})$

We know: $\{f \in \mathcal{S}(\mathbb{R}) \mid (\tau P+Q)(f)\} = \mathbb{C} f_\tau$, $H_\tau(f_\tau) = \frac{1}{2} f_\tau$

representation theory of $\mathfrak{sl}(2) \implies$
(a) $H_\tau(f_{\tau,n}) = (n+\frac{1}{2}) f_{\tau,n}$
(b) $(\tau P+Q)^n = f_\tau \cdot (\text{constant} \neq 0)$
(c) $\{f \in \mathcal{S}(\mathbb{R}) \mid (\tau P+Q)^{n+1} f = 0$, $H_\tau(f) = (n+\frac{1}{2})f\} = \mathbb{C} f_{\tau,n}$.

change of τ: if $\alpha = \begin{pmatrix}a&b\\c&d\end{pmatrix} \in SL_2(\mathbb{R})$ and $\alpha(\tau) = \frac{a\tau+b}{c\tau+d}$, assume that $F_\alpha : \mathcal{S}(\mathbb{R}) \longrightarrow \mathcal{S}(\mathbb{R})$ satisfies

$F\circ P \circ F^{-1} = aP + cQ \longleftrightarrow \begin{pmatrix}a\\c\end{pmatrix} = \alpha\begin{pmatrix}1\\0\end{pmatrix}$

$F\circ Q \circ F^{-1} = bP + dQ \longleftrightarrow \begin{pmatrix}b\\d\end{pmatrix} = \alpha\begin{pmatrix}0\\1\end{pmatrix}$

then $F\circ(\tau P+Q)\circ F^{-1} = (a\tau+b)P + (c\tau+d)Q = (c\tau+d)(\alpha(\tau)P+Q)$

$\updownarrow \begin{pmatrix}\tau\\1\end{pmatrix}$ $\qquad \updownarrow \begin{pmatrix}a\tau+b\\c\tau+d\end{pmatrix} = \alpha\begin{pmatrix}\tau\\1\end{pmatrix}$

$F\circ(\bar\tau P+Q)\circ F^{-1} = (c\bar\tau+d)(\alpha(\bar\tau)P+Q)$

$\implies F(f_\tau) = c(F,\tau) f_{\alpha(\tau)}$, $F(f_{\tau,n}) = (F\circ(\bar\tau P+Q)\circ F^{-1})^n c(F,\tau) f_{\alpha(\tau)}$
$= (c\bar\tau+d)^n c(F,\tau) f_{\alpha(\tau),n}$

Rmk: $(c\bar\tau+d)^n = (c\tau+d)^{-n}|c\tau+d|^{2n} = (c\tau+d)^{-n}\left(\frac{\operatorname{Im}(\tau)}{\operatorname{Im}(\alpha(\tau))}\right)^n$.

Hermite polynomials: $He_n(x) = (-1)^n e^{x^2/2}\left(\frac{d}{dx}\right)^n e^{-x^2/2} = \left(x - \frac{d}{dx}\right)^n(1) = 2^{-n/2} H_n\left(\frac{x}{\sqrt 2}\right) = x^n + \cdots$

$H_n(x) = (-1)^n e^{x^2}\left(\frac{d}{dx}\right)^n e^{-x^2} = \left(2x - \frac{d}{dx}\right)^n(1) = (2x)^n + \cdots$

$\implies \underline{f_{\tau,n}(x) = f_\tau(x)\, He_n(\sqrt{4\pi\operatorname{Im}(\tau)}\, x)}$

$\underline{f_{\tau,2}(x) = e^{\pi i \tau x^2}(4\pi \operatorname{Im}(\tau)x^2 - 1)}$

n	$He_n(y)$
0	1
1	y
2	y^2-1
3	y^3-3y

More general theta functions

We have proved (by two methods): transformation rules for

$$\vartheta(\tau) = \sum_{n\in\mathbb{Z}} e^{\pi i n^2\tau}, \quad \vartheta(\tau,z) = \sum_{n\in\mathbb{Z}} e^{\pi i n^2\tau + 2\pi i n z}:$$

$$\forall \alpha = \begin{pmatrix} a & b \\ c & d\end{pmatrix} \in \Gamma_\vartheta = \left\{ \alpha\in SL_2(\mathbb{Z}) \mid \alpha \equiv \begin{pmatrix}1&0\\0&1\end{pmatrix}, \begin{pmatrix}0&1\\1&0\end{pmatrix} \pmod 2\right\}$$

$$\vartheta\left(\frac{a\tau+b}{c\tau+d}\right) = \varepsilon_\alpha (c\tau+d)^{1/2}\vartheta(\tau), \quad \varepsilon_\alpha^8 = 1, \quad 0<\arg(c\tau+d)^{1/2}<\frac{\pi}{2} \ (\text{if } c>0)$$

$$\vartheta\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right) = \varepsilon_\alpha (c\tau+d)^{1/2} e^{\frac{\pi i c z^2}{c\tau+d}}\vartheta(\tau,z).$$

Goal: generalise this to more general ϑ-functions, such as

$$\vartheta_\phi(\tau) = \sum_{n\in\mathbb{Z}} \phi(n) e^{\pi i n^2\tau} \quad (\phi: \mathbb{Z}/N\mathbb{Z}\to\mathbb{C}), \quad \vartheta_\phi^*(\tau) = \sum_{n\in\mathbb{Z}}\phi(n) n e^{\pi i n^2\tau}$$

$$\vartheta_\phi(S,\tau) = \sum_{n\in\mathbb{Z}^k}\phi(n) e^{\pi i S[n]\tau}, \quad S = {}^tS \in M_k(\mathbb{Q}) \text{ positive definite}, \quad S[x] = {}^tx\,Sx \ (x\in\mathbb{R}^k)$$

$$\vartheta_\phi(S,P,\tau) = \sum_{n\in\mathbb{Z}^k}\phi(n)P(n)e^{\pi i S[n]\tau} \quad \text{for certain polynomials } P(x)$$

(we know that $\vartheta(\tau)^{10} = (\text{Eisenstein series}) + (\text{const.})\sum_{n_1,n_2\in\mathbb{Z}}(n_1+in_2)^4 e^{\pi i\tau(n_1^2+n_2^2)}$ (Liouville))

We first do it by an explicit approach, and then from a more conceptual point of view.

Example: we know that $\sum_{m\in\mathbb{Z}} e^{-\pi t(m+x)^2} = t^{-1/2}\sum_{m\in\mathbb{Z}} e^{-\pi m^2/t + 2\pi i m x}$

$$\Rightarrow \sum_{m,n\in\mathbb{Z}} e^{-\pi t((m+x)^2+(n+y)^2)} = t^{-1}\sum_{m,n\in\mathbb{Z}} e^{\pi(m^2+n^2)/t + 2\pi i(mx+ny)}$$

Apply $\left(\frac{1}{2\pi}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\right)^k$ and take $x=y=0$:

$$(-t)^k\sum_{m,n\in\mathbb{Z}}(m+ni)^k e^{-\pi t(m^2+n^2)} = t^{-1} i^k\sum_{m,n\in\mathbb{Z}}(m+ni)^k e^{-\pi(m^2+n^2)/t}$$

$\Rightarrow$ the function $F_k(\tau) := \sum_{m,n\in\mathbb{Z}}(m+ni)^k e^{\pi i\tau(m^2+n^2)}$ $(k\geq 0)$ $\quad (4\nmid k \Rightarrow F_k = 0)$

satisfies $F_k\left(-\frac{1}{\tau}\right) = \underbrace{i^k}_{=1 \text{ if } 4\mid k}\tau^{k+1}F_k(\tau)$, $\quad F_k(\tau+2) = F_k(\tau)$

$$\Rightarrow \forall\alpha = \begin{pmatrix} a&b\\c&d\end{pmatrix}\in\Gamma_\vartheta \quad F_k\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^{k+1}F_k(\tau)$$

The Poisson formula – a symmetric version

Recall: the Poisson summation formula

(A) $$\boxed{\sum_{m\in\mathbb{Z}} f(m) = \sum_{n\in\mathbb{Z}} (\mathcal{F}f)(n)}$$ $f\in\mathcal{S}(\mathbb{R})$, $(\mathcal{F}f)(y) = \int_{\mathbb{R}} e^{-2\pi i xy} f(x)\,dx$

was proved by writing down the Fourier series of the $\mathbb{Z}$-periodic function $F(x) := \sum_{m\in\mathbb{Z}} f(x+m)$:

$F(x) = \sum_{n\in\mathbb{Z}} \hat{F}(n) e^{2\pi i n x}$, $\hat{F}(n) = \int_{\mathbb{R}/\mathbb{Z}} F(x) e^{-2\pi i n x}\,dx = \int_{\mathbb{R}} f(x) e^{-2\pi i n x}\,dx = (\mathcal{F}f)(n)$,

$a = x \Downarrow$

(B) $$\boxed{\sum_{m\in\mathbb{Z}} f(m+a) = \sum_{n\in\mathbb{Z}} (\mathcal{F}f)(n) e^{2\pi i a n}}$$

Note: (A) $\Leftrightarrow$ [(B) for $a=0$], and [(A) for $t_{-a}f$] $\Leftrightarrow$ (B)), where $(t_b f)(x) = f(x-b)$ (translation). The operator $(e_a f)(x) = e^{2\pi i a x} f(x)$ on the R.H.S. of (2) is of a different nature.

Properties of e_a, t_b acting on $L^2(\mathbb{R})$: (1) e_a and t_b are unitary,
$(a, b\in\mathbb{R})$ (2) $e_a t_b = e^{2\pi i ab} t_b e_a$ $\qquad (e_a t_b f)(x) = e^{2\pi i a x} f(x-b)$

$\Longrightarrow$ the operators $\lambda e_a t_b$ $(\lambda\in\mathbb{C}^\times, |\lambda|=1, a,b\in\mathbb{R})$ define a unitary action of the Heisenberg group on $L^2(\mathbb{R})$

(3) $\mathcal{F} e_a = t_a \mathcal{F}$, $\mathcal{F} t_b = e_{-b}\mathcal{F}$ $\Longrightarrow$ $\mathcal{F}(e_a t_b) = e^{2\pi i ab} e_{-b} t_a$

Applying (A) to $e_a t_b f$:

(C) $$\boxed{\sum_{m\in\mathbb{Z}} e^{2\pi i m a} f(m-b) = e^{2\pi i ab} \sum_{n\in\mathbb{Z}} e^{-2\pi i n b} (\mathcal{F}f)(n-a)} \qquad (a, b\in\mathbb{R})$$

Symmetric version: distribute $e^{2\pi i ab}$ equally between the two sides

$e^{-\pi i ab} e_a t_b = e^{\pi i ab} t_b e_a$, $\quad \mathcal{F}(e^{-\pi i ab} e_a t_b) = e^{\pi i ab} e_{-b} t_a \mathcal{F}$

Define: $U\binom{a}{b} := e^{-\pi i ab} e_a t_b = e^{\pi i ab} t_b e_a \Longrightarrow \mathcal{F}\, U\binom{a}{b} = U\binom{-b}{a}\mathcal{F}$ (*)

Explanation: $t_b = e^{bQ}$, $e_a = e^{aP}$, $P = (2\pi i)x$, $Q = -\frac{d}{dx}$, $[P, Q] = 2\pi i$

Campbell – Hausdorff formula: $e^A e^B = e^C$, $C = A+B+\frac{1}{2}[A,B]+\frac{1}{12}[A,[A,B]] + \frac{1}{12}[[A,B],B] + \cdots$

$\Longrightarrow e_a t_b = e^{aP} e^{bQ} = e^{aP+bQ+\pi i ab} \Longrightarrow \boxed{U\binom{a}{b} = e^{aP+bQ}}$

$\mathcal{F}(P\ Q) = (P\ Q)\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\mathcal{F} \Longrightarrow \mathcal{F}\underbrace{(P\ Q)\binom{a}{b}}_{aP+bQ} = \underbrace{(P\ Q)\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\binom{a}{b}}_{-bP+aQ}\mathcal{F} \xRightarrow{\exp}$ (*)

$\boxed{\text{The Lie algebra of the Heisenberg group} = \mathbb{R}\cdot 1 \oplus \mathbb{R}\cdot P \oplus \mathbb{R}\cdot Q}$

Theta functions attached to positive definite quadratic forms
(a direct approach)

Goal: investigate $\boxed{\theta(S,\tau) := \sum_{m\in\mathbb{Z}^n} e^{\pi i S[m]\tau}}$ $(\tau\in\mathcal{H})$, $\theta(I_n,\tau)=\theta(\tau)^n$

where $S={}^tS\in M_n(\mathbb{Q})$ is positive definite (notation: $S>0$)

and $S[x] = {}^tx\,Sx = (x, Sx) = (Sx, x)$, where $x=\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix}\in\mathbb{R}^n$ and

$(x,y)={}^txy=(y,x)$ is the standard scaler product on $\mathbb{R}^n$.

Fourier transform on $\mathbb{R}^n$: $(\mathcal{F}f)(y) := \int_{\mathbb{R}^n} e^{-2\pi i(x,y)} f(x)\,dx$, $f,g\in\mathcal{S}(\mathbb{R}^n)$ Schwartz space

inverse $(\mathcal{F}^\vee g)(x) = \int_{\mathbb{R}^n} e^{2\pi i(x,y)} g(y)\,dy$

Formulas: (1) translations $(t_bf)(x) := f(x-b)$, exponential factors

$(e_af)(x) := e^{2\pi i(a,x)}f(x)$ $(a,b\in\mathbb{R}^n)$, $e_at_b = e^{2\pi i(a,b)}t_b\,e_a$

(2) $\mathcal{F}e_a = t_a\mathcal{F}$, $\mathcal{F}t_b = e_{-b}\mathcal{F}$

(3) Linear change of variables: $A\in GL_n(\mathbb{R})$, $(r_Af)(x) := f(A^{-1}x)$

dual map $({}^tAx, y) = (x, Ay)$, $\mathcal{F}r_A = |\det A|\, r_{{}^tA^{-1}}\mathcal{F}$

(4) Derivatives: $(\partial_yf)(x) := \frac{d}{dh}f(x+hy)\Big|_{h=0}$ $(y\in\mathbb{R}^n)$

$t_b = e^{-\partial_b}$, $\partial_y r_A = r_A\,\partial_{A^{-1}y}$, $t_yr_A = r_A\,t_{A^{-1}y}$

$\mathcal{F}\partial_b = 2\pi i(b,\cdot)\mathcal{F}$, $\partial_b\mathcal{F} = \mathcal{F}(-2\pi i)(b,\cdot)$

---

the Poisson formula for e_at_bf: $(f\in\mathcal{S}(\mathbb{R}^n),\ a,b\in\mathbb{R}^n)$

$$\sum_{m\in\mathbb{Z}^n} e^{2\pi i(m,a)}f(m-b) = e^{2\pi i(a,b)}\sum_{m\in\mathbb{Z}^n} e^{-2\pi i(m,b)}(\mathcal{F}f)(m-a) \qquad (*)$$

Standard Gaussian function: $e^{-\pi(x,x)} = e^{-\pi(x_1^2+\cdots+x_n^2)}\in\mathcal{S}(\mathbb{R}^n)$

We know: $\boxed{\mathcal{F}(e^{-\pi(x,x)}) = e^{-\pi(x,x)}}$ (since true for $n=1$).

General Gaussian function on $\mathbb{R}^n$: $\boxed{\tilde{f}_S(x) := e^{-\pi(Sx,x)},\quad S={}^tS\in M_n(\mathbb{R}),\ S>0 \text{ (S positive definite)}}$

Next step: apply $(*)$ to $\tilde{f}_S$ and then repeatedly differentiate with respect to a and b.

Prop. the Fourier transform of $\tilde f_S(x) := e^{-\pi(Sx,x)} \in \mathcal{S}(\mathbb{R}^n)$ $\quad (S={}^tS\in M_n(\mathbb{R}),\ S>0)$ is equal to $\boxed{\mathcal{F}\tilde f_S = (\det S)^{-1/2}\,\tilde f_{S^{-1}}}$.

Pf. Lemma $\exists A\in GL_n(\mathbb{R})$ / $A={}^tA$ $\quad S={}^tAA$ (since $\exists U\in O(n)$ ${}^tUSU=U^{-1}SU=D$, $D=\begin{pmatrix}\lambda_1 & & 0\\ & \ddots & \\ 0 & & \lambda_n\end{pmatrix}$, $\lambda_j>0$; take $A:=D^{1/2}U^{-1}$, $D^{1/2}=\begin{pmatrix}\lambda_1^{1/2} & & 0\\ & \ddots & \\ 0 & & \lambda_n^{1/2}\end{pmatrix}$).

Then $\tilde f_S(x) = e^{-\pi({}^tAAx,x)} = e^{-\pi(Ax,Ax)} = \tilde f_I(Ax)$, $\quad \tilde f_I = e^{-\pi(x,x)}$ standard Gaussian

$\tilde f_S = r_{A^{-1}}\tilde f_I$, $\quad \mathcal{F}\tilde f_I=\tilde f_I \implies \mathcal{F}\tilde f_S = \mathcal{F}r_{A^{-1}}\tilde f_I = |\det A|^{-1}\, r_{{}^tA}\,\mathcal{F}\tilde f_I =$

$= |\det S|^{-1/2}\, r_{({}^tA)}\tilde f_I = |\det S|^{-1/2}\tilde f_{S^{-1}}$.

Cor. $\forall v>0$ $\quad \mathcal{F}\tilde f_{vS} = v^{-n/2}(\det S)^{-1/2}\,\tilde f_{v^{-1}S^{-1}}$.

Apply $(*)$ to $f=\tilde f_{vS}$: $\quad \forall a,b\in\mathbb{R}^n$, $\forall v>0$

$$\sum_{m\in\mathbb{Z}^n} e^{2\pi i(m,a)}\, e^{-\pi v(S(m-b),\,m-b)} = v^{-n/2}(\det S)^{-1/2}\, e^{2\pi i(a,b)} \sum_{m\in\mathbb{Z}^n} e^{-2\pi i(m,b)}\, e^{-\pi v^{-1}(S^{-1}(m-a),\,m-a)}$$

which implies that $\forall \tau\in\mathcal{H}$

$$\sum_{m\in\mathbb{Z}^n} e^{2\pi i(m,a)}\, e^{\pi i\tau(S(m-b),\,m-b)} = \underbrace{(\tau/i)^{-n/2}}_{\text{the branch }=1\text{ at }\tau=i}(\det S)^{-1/2} e^{2\pi i(a,b)} \sum_{m\in\mathbb{Z}^n} e^{-2\pi i(m,b)}\, e^{-\pi i(S^{-1}(m-a),\,m-a)\tau^{-1}}$$

(Pf: both sides are holomorphic for $\tau\in\mathcal{H}$, and the equality holds if $\tau=iv$, $v>0$)

Special case $a=b=0$:

$$\theta(S,\tau) := \sum_{m\in\mathbb{Z}^n} e^{\pi i\tau(Sm,m)} = (\tau/i)^{-n/2}(\det S)^{-1/2}\,\theta(S^{-1},-\tau^{-1})$$
$$\theta(S,\tau) = \theta_S(\tau)$$

Special case $a=0$: replace b by $-b$ and denote the new b by x:

$$\sum_{m\in\mathbb{Z}^n} e^{\pi i\tau(S(m+x),\,m+x)} = (\tau/i)^{-n/2}(\det S)^{-1/2}\sum_{m\in\mathbb{Z}^n} e^{-\pi i(S^{-1}m,m)\tau^{-1} + 2\pi i(m,x)}$$
$(x\in\mathbb{R}^n)$

Rmk: if $A\in GL_n(\mathbb{Z})$, then $\mathbb{Z}^n \xrightarrow{A} \mathbb{Z}^n$, $m\longmapsto Am$ is bijective, hence

$\theta(S,\tau) = \theta({}^tASA,\tau)$, since $(S(Am),Am) = ({}^tASAm,m)$.

Arithmetically interesting cases

(a) $\underline{S \text{ integral}}$: $\boxed{S = {}^tS \in M_n(\mathbb{Z})} \Longrightarrow \forall x \in \mathbb{Z}^n \ \overbrace{(Sx,x)}^{S[x]} \in \mathbb{Z} \Longrightarrow \underline{\theta_S(\tau+2) = \theta_S(\tau)}$

(b) $\underline{S \text{ even}}$: $\boxed{\forall x \in \mathbb{Z}^n \quad (Sx,x) \in 2\mathbb{Z}}$ $\quad (\Longrightarrow S \in M_n(\mathbb{Z}))$

$$\Longrightarrow \underline{\theta_S(\tau+1) = \theta_S(\tau)}$$

(c) $\underline{S \text{ integral unimodular}}$: $S = {}^tS \in M_n(\mathbb{Z})$, $\underset{S>0}{S \in GL_n(\mathbb{Z})}$ $\left(\Longrightarrow S \in SL_n(\mathbb{Z}),\ \text{since } S>0\right)$

In this case $\underline{\theta_S(\tau+2) = \theta_S(\tau)}$ (cf (a)) and

$\underline{\theta(S^{-1},\tau) = \theta({}^tS S^{-1} S, \tau) = \theta(S,\tau)}$. We have proved:

$$\boxed{\theta_S(\tau) = (\tau/i)^{-n/2} \det(S)^{-1/2}\, \theta_{S^{-1}}(-1/\tau) = \underbrace{\left(\frac{\tau}{i}\right)^{-n/2}}_{\boxed{\text{branch equal to } 1 \text{ at } \tau = i}} \theta_S(-1/\tau)}$$

(d) $\underline{S \text{ even unimodular}}$: $\boxed{\underset{S>0}{S \in SL_n(\mathbb{Z})},\ S = {}^tS,\ \forall x \in \mathbb{Z}^n \ S[x] \in 2\mathbb{Z}.}$

In this case $\underbrace{\theta_S(\tau+1) = \theta_S(\tau)}_{\text{action of} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = T}$, $\underbrace{\theta_S(\tau) = (\tau/i)^{-n/2} \theta_S(-1/\tau)}_{\boxed{\text{action of } S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}}$.

$\underline{\text{Prop}}$. If S is even unimodular and positive definite, then $8 | n$.

$\underline{\text{Pf}}$. The matrices $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ (abuse of notation – sorry)

satisfy $TS = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$, $STS = \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}$, $TSTS = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$, $STSTS = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$

$$(ST)^3 = STSTST = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \cancel{(\ldots)}, \quad (TS)^3 = TSTSTS = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I$$

$$\theta_S(\tau) = \theta_S(TSTSTS(\tau)) = \theta_S(STSTS(\tau)) = \left(\frac{TSTS(\tau)}{i}\right)^{n/2} \theta_S(TSTS(\tau))$$

$$\theta_S(TSTS(\tau)) = \theta_S(STS(\tau)) = \left(\frac{TS(\tau)}{i}\right)^{n/2} \theta_S(TS(\tau))$$

$$\theta_S(TS(\tau)) = \theta_S(S(\tau)) = \left(\frac{\tau}{i}\right)^{n/2} \theta_S(\tau)$$

$$\Longrightarrow 1 = \left(\Big(\underbrace{\frac{TSTS(\tau)}{i}}_{\frac{1}{(1-\tau)i}}\Big)^{1/2} \Big(\underbrace{\frac{TS(\tau)}{i}}_{\frac{\tau-1}{i\tau}}\Big)^{1/2} \left(\frac{\tau}{i}\right)^{1/2} \right)^n$$

$\left(\frac{\tau}{i}\right)^{1/2}$ = the branch equal to 1 at $\tau = i$, $\boxed{\text{so } -\frac{\pi}{4} < \arg < \frac{\pi}{4} \quad (\tau \in \mathcal{H})}$

$\underline{\text{take } \tau = i}$:

$$1 = \underbrace{\left(e^{-2\pi i/16} \cdot 2^{-1/4} \cdot e^{-2\pi i/16} \cdot 2^{1/4} \right)^n}_{\left(e^{-2\pi i/8}\right)^n} \Longrightarrow 8 | n.$$

$\underline{\text{Rmk}}$: For $n = 8$ there is only one such S (up to change of coordinates $x = Ay$, $A \in SL_n(\mathbb{Z})$): the matrix of S is the Cartan matrix of the root system with the Dynkin diagram E_8 :

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Theta functions with harmonic polynomials

Data: $S = {}^tS \in M_n(\mathbb{R})$, $S > 0$. Write $\tilde{f}_S(x) := e^{-\pi(Sx,x)} \in \mathcal{S}(\mathbb{R}^n)$

Extend the standard scalar product $(x,y) = {}^t x y$ on $\mathbb{R}^n$ to a $\mathbb{C}$-bilinear product $(x,y) = {}^t x y$ on $\mathbb{C}^n$.

Goal: compute the Fourier transform of $P(x)\tilde{f}_S(x)$, for certain polynomials $P(x) \in \mathbb{C}[x_1, \dots, x_n]$, and apply the Poisson formula to the function $P(x)\tilde{f}_S(x)$. In fact, this can be done by differentiating the Poisson formula for $\tilde{f}_S(x)$, as in the example of $\sum_{n_1, n_2 \in \mathbb{Z}} (n_1 + i n_2)^k e^{\pi i \tau (n_1^2 + n_2^2)}$.

Notation: for $a \in \mathbb{C}^n$, let $(a,\cdot)$ be the (linear) function $\mathbb{C}^n \longrightarrow \mathbb{C}$, $x \longmapsto (a,x)$

Differentiating $\tilde{f}_S(x) = e^{-\pi(Sx,x)}$: in the direction $a \in \mathbb{C}^n$

$$\partial_a \tilde{f}_S = -2\pi (Sa,\cdot)\tilde{f}_S \Rightarrow \partial_a^2 \tilde{f}_S = -2\pi(Sa,a) + (2\pi)^2 (Sa,\cdot)^2 \tilde{f}_S$$

Prop. If $a \in \mathbb{C}^n$ is an isotropic vector for S (i.e., if $S[a] = (Sa,a) = 0$), then $\forall k \geq 0$ $\left(-\frac{1}{2\pi}\partial_a\right)^k \tilde{f}_S = (Sa,\cdot)^k \tilde{f}_S$. (for $k=0,1$ the assumption $S[a]=0$ is unnecessary)

Pf: $k=1,2$ – see above, then easy induction.

Cor. If $a \in \mathbb{C}^n$ and $(Sa,a) = 0$, then $\forall k \geq 0$

$$\mathcal{F}\left((Sa,\cdot)^k \tilde{f}_S\right) = (-i)^k (a,\cdot)^k (\det S)^{-1/2} \tilde{f}_{S^{-1}}$$

Pf: $\mathcal{F}\partial_a = 2\pi i (a,\cdot)\mathcal{F}$ and $\mathcal{F}\tilde{f}_S = (\det S)^{-1/2}\tilde{f}_{S^{-1}}$

Cor. For any homogeneous polynomial $P \in \mathbb{C}[x_1, \dots, x_n]$ of $\deg(P) = k$ which is
- arbitrary if $k = 0,1$
- a linear combination of polynomials $(Sa,\cdot)^k$, where $S[a]=0$, if $k \geq 2$

$$\boxed{\mathcal{F}(P(x)\tilde{f}_S(x)) = (-i)^k P(S^{-1}x)(\det S)^{-1/2}\tilde{f}_{S^{-1}}(x)}$$

(note: $\tilde{f}_{S^{-1}}(x) = \tilde{f}_S(S^{-1}x)$)

Pf: Above, if $P(x) = (Sa,x)^k$, then $(a,x) = (a, SS^{-1}x) = (Sa, S^{-1}x) = P(S^{-1}x)$

Question: How can one characterise the polynomials $P(x)$ above?

Notation: $\mathrm{Pol} := \mathbb{C}[x_1, \dots, x_n] = \bigoplus_{k\geq 0} \underbrace{\mathrm{Pol}_k}_{\text{homogeneous polynomials, } \deg = k}$

Mysterious spaces to determine:

$$M_k = M_k(S) = \begin{cases} \mathrm{Pol}_k & \text{if } k=0,1 \\ \{\mathbb{C}\text{-linear combinations of } (Sa,\cdot)^k, \text{ where } a\in\mathbb{C}^n, (Sa,a)=0\} & k\geq 2 \end{cases}$$

Ex: $n=1$: $\quad M_k = 0 \quad \forall k \geq 2$

$n=2$: $\quad S = I, \quad S[x] = x_1^2 + x_2^2 \Rightarrow M_k = \mathbb{C}\cdot(x_1+ix_2)^k \oplus \mathbb{C}\cdot(x_1-ix_2)^k \quad (k\geq 2)$

Note: For $n=1,2$ and $S=I$, $\quad M_k = \underbrace{\{f\in \mathrm{Pol}_k \mid \Delta f = 0\}}_{\text{harmonic polynomials}}$

(since $\Delta = \left(\frac{\partial}{\partial x_1}\right)^2 + \dots + \left(\frac{\partial}{\partial x_n}\right)^2 = 4 \frac{\partial}{\partial z}\frac{\partial}{\partial \bar z}$ if $n=2$, $z = x_1+ix_2$, $\bar z = x_1 - ix_2$)

Need to investigate: harmonic polynomials w.r.t. S

Recall: if (X,g) is a Riemannian mfd with metric $g = g_{ij}\,dx^i dx^j$, then the Laplace–Beltrami operator

$\Delta =$

is well-defined and is invariant under isometries of (X,g).

Our case: $\quad X = \mathbb{R}^n, \quad g = \sum S_{jk}\, dx_j\, dx_k$

$$\Rightarrow \quad \Delta_S := \sum_{j,k=1}^n (S^{-1})_{jk} \frac{\partial}{\partial x_j}\frac{\partial}{\partial x_k} \quad \text{is invariant under}$$

$O(S) = \{U \in GL_n(\mathbb{R}) \mid {}^tUSU = S\}$.

Def. $\quad H_k = H_k(S) := \{f\in \mathrm{Pol}_k \mid \Delta_S = 0\}$ (harmonic polynomials)

$H = \bigoplus_{k\geq 0} H_k \subset \mathrm{Pol}$

Prop. $\quad M_k \subseteq H_k$.

Pf. OK if $k=0,1$. If $k\geq 2$, $a \in \mathbb{C}$ and $(Sa,a)=0$, then

$$\frac{\Delta_S\,(Sa,\cdot)^k}{k(k-1)} = \underbrace{\left(\sum_{j,l} S^{-1}_{jl}\,(Sa)_j\,(Sa)_l\right)}_{(S^{-1}Sa,\,Sa) = (a,Sa) = 0.}(Sa,\cdot)^{k-2}$$

Goal: Describe the structure of Pol in terms of H and show that $M_k = H_k$ $\forall k$.

Exercise: If $n=3$, $S=I$ and $k \geq 0$, then

$$H_k(S = \underbrace{x_1^2+x_2^2+x_3^2}_{r^2}) = \left\{ \text{linear combinations of } r^{2k+1}\left(\frac{\partial}{\partial x_1}\right)^{j_1}\left(\frac{\partial}{\partial x_2}\right)^{j_2}\left(\frac{\partial}{\partial x_3}\right)^{j_3}\frac{1}{r},\ j_1+j_2+j_3=k \right\}$$

(Maxwell). Does this generalise?

Two maps: $\mathrm{Pol}_k \underset{\Delta_S}{\overset{S}{\rightleftarrows}} \mathrm{Pol}_{k+2}$. Are they related?

Prop. There are natural <u>positive definite</u> hermitian products $\langle\,,\,\rangle_k$ on each Pol_k satisfying $\boxed{\langle \Delta_S f, g\rangle_k = \langle f, Sg\rangle_{k+2}.}$

Pf. Writing $S = {}^tAA$ and $Ax = y$, use y as new coordinates

$\Longrightarrow$ can assume $S = I_n$, $S[x] = x_1^2+\dots+x_n^2$, $\Delta_S = \Delta$.

The products $\langle f, g\rangle_k := (f(\partial)\bar{g})(0)$ work: if $x^\alpha = x_1^{\alpha_1}\cdots x_n^{\alpha_n}$,

then $$\langle x^\alpha, x^\beta\rangle_k = \begin{cases} 0, & \alpha \neq \beta \\ \alpha_1!\cdots\alpha_n!, & \alpha = \beta \end{cases} \qquad (|\alpha| = \alpha_1+\dots+\alpha_n = k)$$

Note: in fact, (for $S = I_n$) $\left\langle \frac{\partial}{\partial x_j} f, g\right\rangle = \langle f, x_j g\rangle \Longrightarrow \langle \partial_a f, g\rangle = \langle f, (a,\cdot) g\rangle$ $\forall a \in \mathbb{C}^n$

Thm. (1) $\forall k \geq 0$ $\underline{\mathrm{Pol}_{k+2} = H_{k+2} \oplus S\,\mathrm{Pol}_k}$ and $\underline{H_{k+2}^\perp = S\,\mathrm{Pol}_k}$.

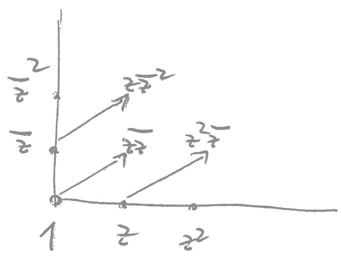
(2) $\mathrm{Pol} = \bigoplus_{j=0}^{\infty} S^j H$ (i.e., each $f \in \mathrm{Pol}$ can be written uniquely as $f = \sum_{j\geq 0} S^j f_j$, $f_j \in H$)

(3) $\forall k \geq 0$ $M_k = H_k$.

Ex: $\underline{n=1}$: $H = \mathbb{C}\cdot 1 \oplus \mathbb{C}\cdot x$, $S = x^2$, $\mathbb{C}[x] = \bigoplus_{j\geq 0}\left(\mathbb{C}\cdot 1\cdot (x^2)^j \oplus \mathbb{C}\cdot x\cdot (x^2)^j\right)$

$\underline{n=2}$: $z = x_1 + ix_2$, $\bar{z} = x_1 - ix_2$, $\mathrm{Pol} = \mathbb{C}[z, \bar{z}]$, $\underline{S = z\bar{z}}$

$\underline{H = \mathbb{C}\cdot 1 \oplus \bigoplus_{k\geq 1}(\mathbb{C}\cdot z^k \oplus \mathbb{C}\cdot \bar{z}^k)}$



Cor. $\dim H_k(\mathbb{R}^n) = \underbrace{\dim \mathrm{Pol}_k(\mathbb{R}^n)}_{\binom{n+k-1}{n-1}} - \dim \mathrm{Pol}_{k-2}(\mathbb{R}^n)$

$n=3$: $\binom{k+2}{2} - \binom{k}{2} = 2k+1$; $n=4$: $\binom{k+3}{3} - \binom{k+1}{3} = (k+1)^2$

$\Longrightarrow$ $\dim H_{2k}(\mathbb{R}^4) = (\dim H_k(\mathbb{R}^3))^2$. Exercise: Why is this true?

Pf of Thm: (1) $\langle \Delta_S f, g\rangle = \langle f, Sg\rangle \Longrightarrow \underbrace{\mathrm{Ker}(\Delta)}_{H} \perp \mathrm{Im}(S)$

If $\Delta_S Sg = 0 \Longrightarrow \langle Sg, Sg\rangle = 0 \overset{\text{positivity}}{\underset{\text{of } \langle,\rangle}{\Longrightarrow}} Sg = 0$. therefore $\mathrm{Ker}(\Delta) \cap \mathrm{Im}(S) = 0$

If $f \in \mathrm{Im}(S)^\perp \Longrightarrow \Delta_S f \in \mathrm{Pol}^\perp = \{0\}$. This proves

$\mathrm{Ker}(\Delta) = \mathrm{Im}(S)^\perp \Longrightarrow H = \mathrm{Ker}(\Delta) \oplus \mathrm{Im}(S)$.

(2) Follows immediately from $\mathrm{Pol} = H \oplus S\,\mathrm{Pol}$.

(3) We know that $M_k \subseteq H_k \Longrightarrow M_k^\perp \supseteq H_k^\perp = S\,\mathrm{Pol}_{k-2}$.

Enough to show: $M_k^\perp \subseteq S\,\mathrm{Pol}_{k-2}$. We can assume $S = I_n$.

Assume $\overset{k\geq 2}{\ } f \in \mathrm{Pol}_k$ and $\forall a \in \mathbb{C}^n$ such that $(a,a)=0$ $\langle f, (a,\cdot)^k\rangle = 0$

We know: $\langle f, (a,\cdot)^k\rangle = \langle \partial_a^k f, 1\rangle = \underbrace{\partial_a^k f}_{\text{constant polynomial}}$

So: $\forall a \in \mathbb{C}^n$ such that $S[a] = a_1^2 + \cdots + a_n^2 = 0$,

$\partial_a^k f = 0 \Longrightarrow f(a) = 0$. Nullstellensatz $\Longrightarrow S \mid f$.

Cor. If $P \in \mathbb{C}[x_1, \dots, x_n]$ is homogeneous of $\deg = k$ and $\Delta_S P = 0$, then

$\mathcal{F}(P(x)\tilde{f}_S(x)) = (-i)^k (\det S)^{-1/2} P(S^{-1}x)\tilde{f}_{S^{-1}}(x)$

Cor. $\forall \tau \in \mathcal{H}$ $\quad \boxed{\mathcal{F}(P(x)e^{\pi i \tau S[x]}) = (-i)^k (\det S)^{-1/2} \underbrace{(\tau/i)^{-k-n/2}}_{=1 \text{ at } \tau=1} P(S^{-1}x) e^{-\pi i S^{-1}[x]\tau^{-1}}}$

$(P \in H_k(S))$

Pf. Both sides are holomorphic functions of $\tau \in \mathcal{H}$ and the statement holds if $\tau = iv$ $(v>0)$, by the previous Corollary applied to vS.

Cor. $\Theta(S, P, \tau) := \sum_{m \in \mathbb{Z}^n} P(m) e^{\pi i \tau S[m]}$ satisfies

$$\Theta(S, P, \tau) = (-i)^k (\det S)^{-1/2} (\tau/i)^{-k-n/2}\, \Theta(S^{-1}, P\circ S^{-1}, -\tfrac{1}{\tau})$$

Pf. Apply Poisson's summation formula.

The Epstein zeta-function: given $S = {}^tS \in M_n(\mathbb{R})$ positive definite and $P \in H_k(S)$ homogeneous harmonic polynomial ($\Delta_S P = 0$) of degree $k \geq 0$, the series

$$\zeta(S, P, s) := \sum_{0 \neq m \in \mathbb{Z}^n} \frac{P(m)}{(S[m])^s} \qquad (S[m] = (Sm, m))$$

is absolutely convergent if $\mathrm{Re}(s) > k + \frac{n}{2}$.

Mellin transform: $\int_0^\infty v^s e^{-av} \frac{dv}{v} = \frac{\Gamma(s)}{a^s} \qquad (a, \mathrm{Re}(s) > 0)$

$$\Rightarrow \boxed{\pi^{-s}\Gamma(s)\zeta(S, P, s) = \int_0^\infty v^s (\Theta(S, P, iv) - P(0)) \frac{dv}{v} = Z(S, P, s)}$$

(Note: $P(0) = 0$ if $k > 0$). Ex: $n = 1$, $S = 1$, $k = 0$, $P = 1$: $\zeta(S, P, s) = 2\zeta(2s)$

Exercise: Show that $Z(S, P, s)$ has meromorphic continuation to $\mathbb{C}$, with only possible poles being simple poles at $s = 0$ and $s = n/2$ when $k = 0$, with

$$\mathrm{Res}_{s=0}\, Z(S, P, s) = -P(0).$$

Furthermore, $Z(S^{-1}, P \circ S^{-1}, k + \frac{n}{2} - s) = (-1)^k (\det S)^{1/2} Z(S, P, s)$.

Exercise: State and prove a similar statement for

$$\zeta(S, P, s, a, b) := \sum_{m \in \mathbb{Z}^n} \frac{P(m+a)}{(S[m+a])^s} e^{2\pi i (m, b)} \qquad (a, b \in \mathbb{R}^n)$$

Ex: Let $\tau = u + iv \in \mathcal{H}$, and consider $S_\tau(x, y) := \frac{|x\tau + y|^2}{\mathrm{Im}(\tau)}$ $(x, y \in \mathbb{R})$
$P = 1$ $(k = 0, n = 2)$. $(\Rightarrow \det(S_\tau) = 1)$

the corresponding series

$$(\mathrm{Re}(s) > 1) \qquad \zeta(S_\tau, 1, s) = \sum_{m, n \in \mathbb{Z}}{}' \frac{\mathrm{Im}(\tau)^s}{|m\tau + n|^{2s}} =: F_s(\tau) = E(\tau, s)$$

non-holomorphic Eisenstein series

has meromorphic continuation to $\mathbb{C}$, with simple poles at $s = 0$ and 1; it satisfies

$$E(\tau, s) = E(\tau, 1-s)$$

and $\mathrm{Res}_{s=0}\, E(\tau, s) = -1$, $\mathrm{Res}_{s=1}\, E(\tau, s) = 1$
