Modular forms of non-integral weight and covering groups

Modular forms of integral weight $k \in \mathbb{Z}$ on a subgroup $\Gamma \subset SL_2(\mathbb{Z})$ of finite index are holomorphic functions $f: \mathcal{H} \to \mathbb{C}$ satisfying $\forall \alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ $\quad f|_k \alpha = f$ (more generally, $f|_k \alpha = \chi(\alpha) f$, for a suitable character of finite order $\chi: \Gamma \to \mathbb{C}^\times$) and having "moderate growth at ∞". Above, $(f|_k \alpha)(\tau) = J(\alpha,\tau)^{-k} f(\alpha(\tau))$, $J(\alpha,\tau) = c\tau + d$

Jacobi's theta function $\theta(\tau) = \sum_{n \in \mathbb{Z}} e^{\pi i n^2 \tau}$ satisfies

$$\forall \alpha \in \Gamma_\theta = \langle S, T^2 \rangle = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \,\middle|\, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \pmod 2 \right\}$$

$$\theta^2|_1 \alpha = \chi(\alpha)\theta^2, \quad \chi(T^2) = 1, \quad \chi(S) = -i \qquad \left(T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right).$$

($\Rightarrow \theta^2$ is a modular form of wt $= 1$ on Γ_θ, with character χ).

Question: in what sense is $\theta(\tau)$ a modular form of wt $= \frac{1}{2}$? What is the correct formulation of the relations

$$\forall \alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_\theta \quad \theta\left(\frac{a\tau + b}{c\tau + d}\right) = \varepsilon_\alpha \underbrace{(c\tau+d)^{1/2}}_{} f(\tau), \qquad \varepsilon_\alpha^8 = 1$$

the branch of $J(\alpha,\tau)^{1/2}$ on $\mathcal{H}$ such that $0 < \arg\left((c\tau+d)^{1/2}\right) < \frac{\pi}{2}$ (if $c > 0$)?

Answer: one needs to consider both branches of $J(\alpha,\tau)^{1/2}$:

Def. Let $G = SL_2(\mathbb{R})$. Define

$$G_2 := \{(\alpha, u) \mid \alpha \in G,\ u: \mathcal{H} \to \mathbb{C}^\times \text{ holomorphic},\ u(\tau)^2 = J(\alpha,\tau)\}$$

Prop. G_2 is a group with respect to the product $(\alpha_1, u_1)(\alpha_2, u_2) = (\alpha, u)$, where $\alpha = \alpha_1\alpha_2$, $u = (u_1 \circ \alpha_2) u_2$. The map $p: G_2 \to G$, $(\alpha, u) \mapsto \alpha$ is a surjective group homomorphism, $\operatorname{Ker}(p) = \{(I, \pm 1)\}$, there is a natural topology on G_2 for which p is a covering map (unramified).

Pf: $\mathcal{H}$ is simply connected $\Rightarrow$ p is surjective. The product makes sense and is associative, since $J(\alpha_1\alpha_2, \tau) = J(\alpha_1, \alpha_2(\tau)) J(\alpha_2, \tau)$.

Rmk: the centre

$$\begin{array}{ccc} Z(G_2) = \{(I, \pm 1), (-I, \pm i)\} & \simeq & \mathbb{Z}/4\mathbb{Z} \\ \downarrow p & & \downarrow p \text{ surjective} \\ Z(G) = \{I, -I\} & \simeq & \mathbb{Z}/2\mathbb{Z} \end{array}$$

$\Rightarrow$ the restriction of p to $p^{-1}(Z(G))$ does not split $\Rightarrow p$ does not split.

Modular forms of half-integral weight $\frac{m}{2}$ $(m \in \mathbb{Z})$ on $\Gamma \subset SL_2(\mathbb{Z})$ (with character $\chi : \tilde{\Gamma} = p^{-1}(\Gamma) \to \mathbb{C}^\times$) will satisfy

$\forall \tilde{\alpha} \in \tilde{\Gamma}$ with $p(\tilde{\alpha}) = \alpha \in \Gamma$ $\quad f(\alpha(\tau)) = \chi(\tilde{\alpha})\, u(\tau)^m f(\tau)$

$\tilde{\alpha} = (\alpha, u)$

Note: recall $\chi : \Gamma_\theta \to \{\pm 1, \pm i\}$, $\chi(T^2) = 1$, $\chi(S) = -i$.

Easy to see: $\Gamma := \operatorname{Ker}(\chi) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \;\middle|\; \begin{array}{l} a \equiv d \equiv 1 \pmod 4 \\ b \equiv c \equiv 0 \pmod 2 \end{array} \right\}$

$$= \left\langle T^2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, (ST^2S^{-1})^{-1} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \right\rangle.$$

As $\forall \alpha \in \Gamma_\theta$ $\quad (\theta^2|_1 \alpha)/\theta^2 = \chi(\alpha) \implies \forall \alpha \in \Gamma$ $\quad \tilde{\alpha} := \Big(\alpha, \underbrace{\frac{\theta(\alpha(\tau))}{\theta(\tau)}}_{u(\tau)}\Big) \in G_2$

and the map $\alpha \mapsto \tilde{\alpha}$ defines a splitting of p over Γ (s is a group homomorphism)

$$\begin{array}{c} p^{-1}(\Gamma) \\ s \uparrow\downarrow \\ \Gamma \end{array}$$

Exercise: if G is a connected Lie group, then its universal covering $\tilde{G} \xrightarrow{p} G$ has a natural structure of a topological group (in fact, a Lie group) and the projection p defines an exact sequence

$$1 \to \pi_1(G, e) \xrightarrow{i} \tilde{G} \xrightarrow{p} G \to 1 \qquad \text{such that } \operatorname{Im}(i) \subset Z(\tilde{G})$$

($\iff \tilde{G}$ is a central extension of G) $\implies$ $\pi_1(G,e)$ is abelian.

General construction: for every morphism of abelian groups $\lambda : \pi_1(G, e) \to A$ there is an induced central extension

$$\begin{tikzcd} 1 \arrow[r] & \pi_1(G,e) \arrow[r] \arrow[d, "\lambda"] & \tilde{G} \arrow[r, "p"] \arrow[d] & G \arrow[r] \arrow[d, equal] & 1 \\ 1 \arrow[r] & A \arrow[r] & \lambda_* \tilde{G} \arrow[r, "p_\lambda"] & G \arrow[r] & 1 \end{tikzcd}$$

Our case: $G = SL_2(\mathbb{R}) \supset K = SO(2)$. As $G/K \xrightarrow{\sim} \mathcal{H}$ is simply connected, $\mathbb{Z} = \pi_1(SO(2)) \xrightarrow{\sim} \pi_1(G)$. Taking $\lambda =$ the surjection $\mathbb{Z} \to \{\pm 1\}$, $n \mapsto (-1)^n$,

$\lambda_* \tilde{G} = G_2$

Relation between G_2 and the metaplectic group

Recall: the (big) metaplectic group $\widetilde{Mp}_2(\mathbb{R})$ is a central extension

$$1 \to \mathbb{C}_1^\times \to \widetilde{Mp}_2(\mathbb{R}) \xrightarrow{\tilde{p}} SL_2(\mathbb{R}) \to 1 \qquad \mathbb{C}_1^\times = U(1) = \{t \in \mathbb{C} \mid |t| = 1\}$$

such that $\forall \alpha \in SL_2(\mathbb{R})$ $\quad \tilde{p}^{-1}(\alpha) = \left\{ \begin{array}{l} \text{unitary linear maps } \tilde{\alpha} : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R}) \\ \text{such that } \tilde{\alpha}\,(P \;\; Q)\,\tilde{\alpha}^{-1} = (P \;\; Q)\,\alpha \end{array} \right\}$

$$\boxed{P = (2\pi i)x \,, \quad Q = -\frac{d}{dx}}$$

Why this sign convention? We want $\forall \tau \in \mathcal{H}$ $\quad \boxed{(P \;\; Q)\begin{pmatrix}\tau \\ 1\end{pmatrix} f_\tau = 0 \,, \quad f_\tau(x) = e^{\pi i \tau x^2} \quad (f_\tau(0) = 1)}$

Formulaire: $\left(\alpha = \begin{pmatrix} a & b \\ c & d\end{pmatrix} \in SL_2(\mathbb{R}) = G \,, \; \tilde{\alpha} \in \widetilde{Mp}_2(\mathbb{R}) \,, \; \tilde{p}(\tilde{\alpha}) = \alpha \right)$

(1) $\begin{pmatrix} a & b \\ c & d\end{pmatrix}\begin{pmatrix}\tau \\ 1\end{pmatrix} = \begin{pmatrix} \frac{a\tau+b}{c\tau+d} \\ 1 \end{pmatrix}(c\tau + d)$ $\qquad \boxed{\alpha\begin{pmatrix}\tau \\ 1\end{pmatrix} = \begin{pmatrix}\alpha(\tau) \\ 1\end{pmatrix} J(\alpha,\tau)}$

(2) $V_\tau := \{ f \in \mathcal{S}(\mathbb{R}) \mid \underbrace{(P \;\; Q)\begin{pmatrix}\tau \\ 1\end{pmatrix}}_{\tau P + Q} f = 0 \} = \mathbb{C}\cdot f_\tau \,, \qquad f_\tau(0) = 1$

(3) $f \in V_\tau \Rightarrow (P \;\; Q)\begin{pmatrix}\bar\tau \\ 1\end{pmatrix} f = (\bar\tau - \tau) Pf.$

(4) $\forall f \in \mathcal{S}(\mathbb{R}) \qquad \|\tilde{\alpha}(f)\|^2 = \|f\|^2$

(5) $\tilde{\alpha}\,(P \;\; Q)\,\tilde{\alpha}^{-1} = (P \;\; Q)\,\alpha$

Consequences: (a) $\forall \tau \in \mathcal{H} \qquad V_\tau \xrightarrow[\sim]{\tilde{\alpha}} V_{\alpha(\tau)}$, since $\forall f \in \mathcal{S}(\mathbb{R})$

$$(*) \quad \tilde{\alpha}\left((P\,Q)\begin{pmatrix}\tau \\ 1\end{pmatrix} f\right) = \Big(\underbrace{\tilde{\alpha}\,(P\,Q)\,\tilde{\alpha}^{-1}}_{(P\,Q)\alpha}\Big)\begin{pmatrix}\tau \\ 1\end{pmatrix}\tilde{\alpha}(f) = (P\,Q)\begin{pmatrix}\alpha(\tau) \\ 1\end{pmatrix}\tilde{\alpha}(f)\, J(\alpha,\tau)$$

$\Rightarrow \exists!\; c(\tilde{\alpha},\tau) \in \mathbb{C}^\times \quad \tilde{\alpha}(f_\tau) = c(\tilde{\alpha},\tau) f_{\alpha(\tau)} \,, \qquad c(t\tilde{\alpha},\tau) = t\,c(\tilde{\alpha},\tau) \quad \forall t \in \mathbb{C}_1^\times$

Key Proposition: $\quad |c(\tilde{\alpha},\tau)^2 J(\alpha,\tau)| = 1.$

Cor: Given $\alpha \in G$, there are two elements $\pm\tilde{\alpha} \in \tilde{p}^{-1}(\alpha)$ such that $c(\tilde{\alpha},\tau)^2 J(\alpha,\tau) = 1$.

Def: the (true) metaplectic group

$$Mp_2(\mathbb{R}) := \{ \tilde{\alpha} \in \widetilde{Mp}_2(\mathbb{R}) \mid \forall \tau \in \mathcal{H} \quad c(\tilde{\alpha},\tau)^2 J(\alpha,\tau) = 1 \} \subset \widetilde{Mp}_2(\mathbb{R})$$

is, indeed, a subgroup of $\widetilde{Mp}_2(\mathbb{R})$ (since both $c(\tilde{\alpha},\tau)$ and $J(\alpha,\tau)$ satisfy the 1-cocycle identity) and the sequence

$$1 \to \{\pm 1\} \to Mp_2(\mathbb{R}) \xrightarrow{\tilde{p}} SL_2(\mathbb{R}) \to 1 \quad \text{is exact.}$$

The map
$$\begin{tikzcd} Mp_2(\mathbb{R}) \arrow[r] \arrow[d, "\tilde{p}"] & G_2 \arrow[d, "p"] \\ G \arrow[r, equal] & G \end{tikzcd}$$
$\tilde{\alpha} \longmapsto (\alpha, c(\tilde{\alpha}, \tau)^{-1})$

is an <u>isomorphism $Mp_2(\mathbb{R}) \xrightarrow{\sim} G_2$</u> compatible with the projections $\tilde{p}, p$.

<u>Question</u>: where is the Fourier transform in this picture?

$\tilde{p}(\mathcal{F}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $\mathcal{F}(f_i) = f_i = e^{-\pi x^2} \Longrightarrow c(\mathcal{F}, i) = 1$.

However, $J\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, i\right) = i = e^{2\pi i/4} \Longrightarrow$ <u>$\pm e^{-2\pi i/8}\mathcal{F} \in Mp_2(\mathbb{R})$</u>

<u>Observe</u>: $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^4 = I$, but $(\pm e^{-2\pi i}\mathcal{F})^4 = -1 \in \mathrm{Ker}(\tilde{p})$
$\neq 1$

<u>Proof of Key Proposition</u>:

$$\underbrace{\tilde{\alpha}(P\ Q)\begin{pmatrix} \bar{\tau} \\ 1 \end{pmatrix} f_\tau}_{(\bar{\tau}-\tau)Pf_\tau} = \underbrace{(\tilde{\alpha}(P\ Q)\tilde{\alpha}^{-1})}_{(P\ Q)\alpha}\begin{pmatrix} \bar{\tau} \\ 1 \end{pmatrix} \underbrace{\tilde{\alpha}(f_\tau)}_{c(\tilde{\alpha},\tau) f_{\alpha(\tau)}} = \underbrace{(P\ Q)\begin{pmatrix} \overline{\alpha(\tau)} \\ 1 \end{pmatrix} c(\tilde{\alpha},\tau)\overline{J(\alpha,\tau)} f_{\alpha(\tau)}}_{c(\tilde{\alpha},\tau)\overline{J(\alpha,\tau)}(\overline{\alpha(\tau)}-\alpha(\tau))P f_{\alpha(\tau)}}$$

$\Longrightarrow$ <u>$\mathrm{Im}(\tau)\ \|Pf_\tau\| = |c(\tilde{\alpha},\tau)J(\alpha,\tau)|\ \mathrm{Im}(\alpha(\tau))\ \|Pf_{\alpha(\tau)}\|$</u>

$\|Pf_\tau\|^2 = (\text{const.})\int_{\mathbb{R}} x^2 e^{-2\pi\,\mathrm{Im}(\tau)x^2}dx = (\text{const.})\ \mathrm{Im}(\tau)^{-3/2}$

$\mathrm{Im}(\alpha(\tau)) / \mathrm{Im}(\tau) = |J(\alpha,\tau)|^{-2}$

$\Longrightarrow$ Prop.

<u>Rmk</u>. One can prove by abstract means, since

$|f_\tau|^2 = (f_{i\,\mathrm{Im}(\tau)})^2$ and $f_{i\,\mathrm{Im}(\tau)} = \begin{pmatrix} \mathrm{Im}(\tau)^{1/2} & 0 \\ 0 & \mathrm{Im}(\tau)^{-1/2} \end{pmatrix} f_i \cdot \mathrm{Im}(\tau)^{-1/4}$

Riemann's theta function

The theta series $\sum_{m\in\mathbb{Z}^n} e^{\pi i (Sm,m)}$ are very special cases of the Riemann theta function

$$\boxed{\Theta(T,z) = \sum_{m\in\mathbb{Z}^n} e^{\pi i (Tm,m) + 2\pi i (z,m)}} \qquad \begin{array}{l} z\in\mathbb{C}^n \qquad\qquad T>0 \\ T\in\mathcal{H}_n \quad (T={}^tT\in M_n(\mathbb{C})) \end{array}$$

$$\boxed{\text{and} \quad \Theta(T) = \Theta(T,0) = \sum_{m\in\mathbb{Z}^n} e^{\pi i (Tm,m)}}$$

Ex: If $T=\tau S$, $\tau\in\mathcal{H}=\mathcal{H}_1$ and $S={}^tS\in M_n(\mathbb{R})$, $S>0$, then
$\Theta(\tau S) = \Theta(S,\tau)$.

Note: $\alpha=\begin{pmatrix} A & B\\ C & D\end{pmatrix}\in G=Sp_{2n}(\mathbb{R})$ acts on $\mathcal{H}_n$ by $\underline{\alpha\begin{pmatrix} T\\ I_n\end{pmatrix} = \begin{pmatrix}\alpha(T)\\ I_n\end{pmatrix} J(\alpha,T)}$,
$\underline{J(\alpha,T) = CT+D}$, $\underline{\alpha(T) = (AT+B)(CT+D)^{-1}}$.

For $\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in SL_2(\mathbb{R})$, then $\begin{pmatrix} A & B\\ C & D\end{pmatrix} := \begin{pmatrix} aI_n & bS\\ cS^{-1} & dI_n\end{pmatrix}\in G$ and
$\begin{pmatrix} A & B\\ C & D\end{pmatrix}(\tau S) = \frac{a\tau+b}{c\tau+d}S$. In particular, if we know general transformation rules for $\Theta(T)$ under the action of a subgroup $\Gamma\subset Sp_{2n}(\mathbb{Z})$, then we obtain transformation rules for $\Theta(S,\tau)$ under the action of $\left\{\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in SL_2(\mathbb{Q}) \;\middle|\; \begin{pmatrix} aI_n & bS\\ cS^{-1} & dI_n\end{pmatrix}\in\Gamma\right\}$.

The Abstract method for studying $\Theta(T,z)$ and, more generally,

$$\boxed{\sum_{m\in\mathbb{Z}^n+a} e^{\pi i (Tm,m) + 2\pi i (z+b,m)}} \qquad (a,b\in\mathbb{R}^n),$$

relies on the action of the Heisenberg $\left\{\begin{matrix}\text{group}\\ \text{Lie algebra}\end{matrix}\right\}$ on $\left\{\begin{matrix} L^2(\mathbb{R}^n)\\ \mathcal{S}(\mathbb{R}^n)\end{matrix}\right\}$
($\Leftrightarrow$ transformation rules in the z-variable). Rigidity of this action $\Longrightarrow$ construction of the metaplectic groups

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{C}_1^\times & \longrightarrow & \widetilde{Mp}_{2n}(\mathbb{R}) & \longrightarrow & Sp_{2n}(\mathbb{R}) & \longrightarrow & 1 \\ & & \cup & & \cup & & \| & & \\ 1 & \longrightarrow & \{\pm 1\} & \longrightarrow & Mp_{2n}(\mathbb{R}) & \longrightarrow & Sp_{2n}(\mathbb{R}) & \longrightarrow & 1 \end{array}$$

acting on $\mathcal{S}(\mathbb{R}^n)$ and $L^2(\mathbb{R}^n)$.

Differentiation with respect do a or b $\Longrightarrow$ Θ-functions with pluriharmonic polynomials.

Good formalism: symmetric form of the Poisson formula

$\Longleftrightarrow$ symmetric form of the Heisenberg group

$\Longleftrightarrow$ writing the Heisenberg group using the exponential map from the Lie algebra.

Ex: (n=1) Heisenberg Lie algebra: $\mathbb{R}(2\pi i)\oplus\mathbb{R}\cdot P\oplus\mathbb{R}\cdot Q$

$P=(2\pi i)x$, $Q=-\frac{d}{dx}$, $[P,Q]=2\pi i$ central element

$\forall u,v\in\mathbb{R}$ $\quad (e^{uP}f)(x)=\underbrace{e^{2\pi i ux}f(x)}_{(e_u f)(x)}$, $\quad (e^{vQ}f)(x)=\underbrace{f(x-v)}_{(t_v f)(x)}$

Def: $\boxed{U\binom{u}{v}:=e^{uP+vQ}}$ $\quad$ Campbell–Hausdorff: $e^{uP}e^{vQ}=e^{uP+vQ+\underbrace{\frac{1}{2}[uP,vQ]}_{\pi i uv}}$

$\Longrightarrow$ $U\binom{u}{v}=e^{-\pi i uv}e_u t_v=e^{\pi i uv}t_v e_u$, $\quad \left(U\binom{u}{v}f\right)(x)=e^{-\pi i uv}e^{2\pi i ux}f(x-v)$

Fourier transform: $\mathcal{F}(uP+vQ)=\mathcal{F}(P\ Q)\binom{u}{v}=\underbrace{\mathcal{F}(P\ Q)\mathcal{F}^{-1}}_{(P\ Q)\begin{pmatrix}0&-1\\1&0\end{pmatrix}}\binom{u}{v}\mathcal{F}$

$\Longrightarrow$ $\mathcal{F}\,U\binom{u}{v}=U\left(\begin{pmatrix}0&-1\\1&0\end{pmatrix}\binom{u}{v}\right)\mathcal{F}$

The same calculation works for any $\begin{array}{c}\tilde{\alpha}\in\widetilde{Mp}_2(\mathbb{R})\\ \downarrow \quad \downarrow p\\ \alpha\in SL_2(\mathbb{R})\end{array}$

$\tilde{\alpha}(uP+vQ)=\tilde{\alpha}(P\ Q)\binom{u}{v}=\underbrace{\tilde{\alpha}(P\ Q)\tilde{\alpha}^{-1}}_{(P\ Q)\alpha}\binom{u}{v}\tilde{\alpha}=(P\ Q)\,\alpha\binom{u}{v}\tilde{\alpha}$

$\Longrightarrow$ $\boxed{\tilde{\alpha}\,U\binom{u}{v}\tilde{\alpha}^{-1}=U\left(\alpha\binom{u}{v}\right)}$ $\quad (u,v\in\mathbb{R},\ \tilde{\alpha}\in\widetilde{Mp}_2(\mathbb{R}))$

Action on Gaussians: $f_\tau(x)=e^{\pi i\tau x^2}$, $\quad f_{\tau,z}=e^{\pi i\tau x^2+2\pi i zx}$ $\quad (\tau\in\mathcal{H}, z\in\mathbb{C})$

symplectic form $\quad B\left(\binom{x_1}{x_2},\binom{y_1}{y_2}\right):=\begin{vmatrix}x_1&y_1\\x_2&y_2\end{vmatrix}$.

$\forall\alpha\in SL_2(\mathbb{R})$ $\quad \underbrace{B\left(\binom{u}{v},\binom{\tau}{1}\right)}_{\boxed{z:=u-v\tau}}=\underbrace{B\Big(\underbrace{\alpha\binom{u}{v}}_{\binom{u'}{v'}},\binom{\alpha(\tau)}{1}\Big)J(\alpha,\tau)}_{(z'=u'-v'\tau')\,J(\alpha,\tau)}$

$\Longrightarrow$ $z'=\frac{z}{c\tau+d}$ $\qquad \tau'=\alpha(\tau)=\frac{a\tau+b}{c\tau+d}$, $\quad J(\alpha,\tau)=c\tau+d$

$U\binom{u}{v}f_\tau=e^{-\pi i uv}e^{2\pi i ux}e^{\pi i\tau(x-v)^2}=e^{\pi i\tau x^2+2\pi i x\overbrace{(u-v\tau)}^{z}}e^{-\pi i vz}=f_{\tau,z}e^{-\pi i vz}$,

where $e^{-\pi i vz}=e^{\frac{\pi i(z-\bar{z})z}{\tau-\bar{\tau}}}=e^{\frac{\pi i z^2}{\tau-\bar{\tau}}}e^{-\frac{\pi|z|^2}{2\,\mathrm{Im}(\tau)}}$

let $\quad {}^\circ f_{\tau,z}:=e^{\frac{\pi i z^2}{\tau-\bar{\tau}}}f_{\tau,z}$ $\quad$ (we have already seen this function!)

$\Longrightarrow$ $\boxed{U\binom{u}{v}f_\tau={}^\circ f_{\tau,z}\ e^{-\pi|z|^2/2\,\mathrm{Im}(\tau)}\Big|_{\boxed{z=u-v\tau}}}$ $\quad$ Note: $|z|^2/\mathrm{Im}(\tau)=|z'|^2/\mathrm{Im}(\tau')$

This relation $\underline{U\binom{u}{v} f_\tau = {}^{\circ}f_{\tau,z}\; e^{-\pi|z|^2/2\operatorname{Im}(\tau)}}$, $\underline{z = u - v\tau}$

implies that one can deduce transformation rules for $f_{\tau,z}$ from those for f_τ: $\forall \tilde\alpha \in \widetilde{Mp_2}(\mathbb{R})$, $p(\tilde\alpha) = \alpha \in SL_2(\mathbb{R})$

$\underline{\tilde\alpha(f_\tau) = c(\tilde\alpha,\tau) f_{\alpha(\tau)}}$, $\underline{\tilde\alpha\, U\binom{u}{v} \tilde\alpha^{-1} = U(\alpha\binom{u}{v})}$

We also know that $(\tau', z') = \alpha(\tau,z) = \left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right)$ satisfy

$\underline{|z'|^2/\operatorname{Im}(\tau') = |z|^2/\operatorname{Im}(\tau)}$, $\underline{z' = u' - v'\tau'}$, $\underline{\alpha\binom{u}{v} = \binom{au+bv}{cu+dv} = \binom{u'}{v'}}$

$$\Longrightarrow \boxed{\alpha({}^{\circ}f_{\tau,z}) = c(\tilde\alpha,\tau)\, {}^{\circ}f_{\alpha(\tau,z)}}$$

Our earlier proof of this relation used canonical 1-cocycles for line bundles on $\mathbb{C}/(\mathbb{Z}\tau+\mathbb{Z})$ (τ was fixed, z was variable). Here we vary τ as well.

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n-dimensional formalism

Useful: write symplectic spaces in terms of duality (and Fourier transform)

Data: $L = X_{\mathbb{Z}} \subset \underbrace{X = X_{\mathbb{R}}}$

lattice $\qquad$ n-dimensional $\mathbb{R}$-vector space

$\Longrightarrow \quad L^* = X^*_{\mathbb{Z}} = \operatorname{Hom}_{\mathbb{Z}}(L,\mathbb{Z}) \subset X^* = X^*_{\mathbb{R}} = \operatorname{Hom}_{\mathbb{R}}(X,\mathbb{R})$

(e.g.: $L = \mathbb{Z}^n = \left\{\begin{pmatrix} m_1\\ \vdots \\ m_n\end{pmatrix} \middle| m_j \in \mathbb{Z}\right\}$, $L^* = (\mathbb{Z}^n)^* = \{(a_1,\dots,a_n) \mid a_j \in \mathbb{Z}\}$)

standard symplectic form B on $\begin{pmatrix} L^* \\ \oplus \\ L\end{pmatrix} \subset \begin{pmatrix} X^* \\ \oplus \\ X\end{pmatrix} = V$

$\underline{B\left(\binom{x^*}{x}, \binom{y^*}{y}\right) = (x^*, y) - (y^*, x)}$, $(\,,\,): L^* \times L \longrightarrow \mathbb{Z}$ duality pairing

Fix Lebesgue measures dx, dx^* so that $\int_{X/L} dx = 1 = \int_{X^*/L^*} dx^*$.

Fourier transform: $(\mathcal{F}f)(x^*) = \int_X e^{-2\pi i (x^*,x)} f(x)\, dx$ etc.

Heisenberg Lie algebra acting on $\mathcal{S}(X)$: contains

$P_{y^*} = (2\pi i)(y^*, \cdot)$ $\quad (y^* \in X^*)$, $Q_y = -\partial/\partial y$ $\quad (y \in X)$, $[P_{y^*}, Q_y] = (2\pi i)(y^*, y)$

$U\binom{y^*}{y} := e^{P_{y^*}+Q_y} = e^{P_{y^*}} e^{Q_y} e^{-\pi i (y^*,y)}$, $\boxed{(U\binom{y^*}{y} f)(x) = e^{-\pi i (y^*,y)} e^{2\pi i (y^*,x)} f(x-y)}$

$\boxed{U\binom{x^*}{x} U\binom{y^*}{y} = e^{\pi i B\left(\binom{x^*}{x}, \binom{y^*}{y}\right)}\, U\binom{x^*+y^*}{x+y}}$ $\Longrightarrow$ $G = Sp(V,B)$ acts on $\operatorname{Heis}(V)$ by group automorphisms

$\underline{\operatorname{Heis}(V) = \left\{t \cdot U\underbrace{\binom{x^*}{x}}_{v} \middle| t \in \mathbb{C}^\times_1, \binom{x^*}{x} \in V\right\}}$ $\qquad$ $\underline{g(t\cdot U(v)) = t \cdot U(gv)}$

Structure of $G = Sp(V,B)$: $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$,
$$\begin{tikzcd} X^* \arrow[r, "A"] \arrow[dr, "C" near end] & X^* \\ X \arrow[r, "D"'] \arrow[ur, "B" near end] & X \end{tikzcd}$$
belongs to G

$\Longleftrightarrow$ $A^*C = C^*A$, $B^*D = D^*B$, $D^*A - B^*C = I$ $\Longrightarrow$ $g^{-1} = \begin{pmatrix} D^* & -B^* \\ -C^* & A^* \end{pmatrix}$.

In particular: $P := G \cap \{C = 0\} = \left\{ \begin{pmatrix} I & B \\ 0 & I \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & (A^*)^{-1} \end{pmatrix} \,\middle|\, \begin{array}{l} B: X \to X^*,\ B^* = B \\ A \in GL(X^*) \end{array} \right\}$

Fact: G is generated by P and any fixed
$\begin{pmatrix} 0 & -S_0 \\ S_0^{-1} & 0 \end{pmatrix}$, where $S_0 : X \xrightarrow{\sim} X^*$, $S_0^* = S_0$.

Metaplectic groups:

Thm. (1) $\forall\, \alpha \in G = Sp(V,B)$ there exists a unitary linear map
$\tilde{\alpha} : \mathcal{S}(X) \to \mathcal{S}(X)$ such that $\forall \begin{pmatrix} y^* \\ y \end{pmatrix} \in \begin{matrix} X^* \\ \oplus \\ X \end{matrix} = V$
$$\tilde{\alpha}(P_{y^*} + Q_y)\tilde{\alpha}^{-1} = P_{y'^*} + Q_{y'}, \quad \text{where } \alpha\begin{pmatrix} y^* \\ y \end{pmatrix} = \begin{pmatrix} y'^* \\ y' \end{pmatrix}.$$
the map $\tilde{\alpha}$ is continuous and is unique up to $t \in \mathbb{C}_1^\times$.
($\Longrightarrow$ it extends to a unitary map $L^2(V) \to L^2(V)$).

(2) the collection of such maps forms a group (w.r.t. composition)
$$1 \to \mathbb{C}_1^\times \to \widetilde{Mp}(V,B) \xrightarrow{p} Sp(V,B) \to 1$$
$$\tilde{\alpha} \longmapsto \alpha$$

Explicit elements: (a) $\alpha = \begin{pmatrix} A & 0 \\ 0 & (A^*)^{-1} \end{pmatrix}$, $(\tilde{\alpha} f)(x) = |\det A|^{1/2} f(A^* x)$ $(A \in GL(X^*))$

(b) $\alpha = \begin{pmatrix} I & B \\ 0 & I \end{pmatrix}$, $B : X \to X^*$, $B^* = B$, $(\tilde{\alpha} f)(x) = e^{\pi i (Bx, x)} f(x)$

(c) $\begin{pmatrix} 0 & -S_0 \\ S_0^{-1} & 0 \end{pmatrix}$, $S_0 : X_{\mathbb{Z}} \xrightarrow{\sim} X^*_{\mathbb{Z}}$, $S_0^* = S_0$, $(\tilde{\alpha} f)(x) = (\mathcal{F} f)(S_0 x)$

Pf of Thm: the projections to $Sp(V,B)$ of the elements in (a)–(c) above generate $Sp(V,B)$
$\Longrightarrow$ p is surjective. Uniqueness of $\tilde{\alpha}$ up to $\mathbb{C}_1^\times$: as in the case $V = \mathbb{R}^2$.

Cor. $\forall\, \tilde{\alpha} \in \widetilde{Mp}(V,B)$ with $p(\tilde{\alpha}) = \alpha$, $\forall\, v \in V$ $\quad \tilde{\alpha}\, U(v)\, \tilde{\alpha}^{-1} = U(\alpha(v))$
$\left(v = \begin{pmatrix} y^* \\ y \end{pmatrix} \Longrightarrow U(v) = e^{P_{y^*} + Q_y} \right)$

Gaussians : $T \in \mathcal{H}_n \rightsquigarrow \boxed{f_T(x) = e^{\pi i (Tx,x)} \in \mathcal{S}(X)}$

$\forall y \in X_{\mathbb{C}}$ $\quad Q_y f_T = -\partial/\partial y\, f_T = -\pi i ((Tx,y)+(Ty,x)) f_T = -2\pi i (Ty,x) f_T = -P_{Ty} f_T$

$\Longrightarrow \underline{(P_{Ty} + Q_y) f_T = 0}$.

Exercise: $\underline{\{ f \in \mathcal{S}(V) \mid \forall y \in X_{\mathbb{C}} \quad (P_{Ty} + Q_y) f = 0 \} = \mathbb{C} \cdot f_T}$.

Prop. $\forall \tilde{\alpha} \in \widetilde{Mp}(V,B)$, $\tilde{\alpha} \xmapsto{p} \alpha \in Sp(V,B)$: (1) $\exists\, c(\tilde{\alpha},T) \in \mathbb{C}^\times$ ~~$\tilde{\alpha}(f_T) = c(\tilde{\alpha},$~~

$\underline{\tilde{\alpha}(f_T) = c(\tilde{\alpha},T) f_{\alpha(T)}}$

(2) $\underline{c(t\tilde{\alpha},T) = t\, c(\tilde{\alpha},T) \qquad \forall t \in \mathbb{C}_1^\times}$

(3) $|c(\tilde{\alpha},T)^2 \det(J(\alpha,T))| = 1$

Pf: (1) $\tilde{\alpha}(P_{Ty} + Q_y)\tilde{\alpha}^{-1} = P_{\alpha(T)y'} + Q_{y'}$, $\quad y' = J(\alpha,T)y$

$\Longrightarrow \quad \tilde{\alpha}(f_T) \in \mathbb{C} \cdot f_{\alpha(T)}$.

(2) Obvious. (3) As in the case $n=1$ (see above).

Cor: $Mp(V,B) := \{ \tilde{\alpha} \in \widetilde{Mp}(V,B) \mid c(\tilde{\alpha},T)^2 \det(J(\alpha,T)) = 1 \}$ is a subgroup of $\widetilde{Mp}(V,B)$ and $1 \to \{\pm 1\} \to Mp(V,B) \xrightarrow{p} Sp(V,B) \to 1$ is a central extension.

Transformation formulae for $\theta(T)$

Notation: (a) $e(a) = e^{2\pi i a}$

(b) tempered distribution $\delta_{X_{\mathbb{Z}}} : \mathcal{S}(X) \longrightarrow \mathbb{C}$, $f \longmapsto \sum_{m \in X_{\mathbb{Z}}} f(m)$

Fact: $Sp(V_{\mathbb{Z}}, B) = Sp(X^*_{\mathbb{Z}} \oplus X_{\mathbb{Z}}, B) \simeq Sp_{2n}(\mathbb{Z})$ is generated by

$$\left\{ \begin{pmatrix} A & 0 \\ 0 & (A^*)^{-1} \end{pmatrix} \,\middle|\, A \in GL(X^*_{\mathbb{Z}}) \right\}, \left\{ \begin{pmatrix} I & B \\ 0 & I \end{pmatrix} \,\middle|\, \begin{array}{c} B : X_{\mathbb{Z}} \to X^*_{\mathbb{Z}} \\ B^* = B \end{array} \right\}, \begin{pmatrix} 0 & -S_0 \\ S_0^{-1} & 0 \end{pmatrix} \quad \begin{array}{c} S_0 : X_{\mathbb{Z}} \xrightarrow{\sim} X^*_{\mathbb{Z}} \\ S_0^* = S_0 \end{array}$$

Note: for α^{ψ} or $\alpha = \begin{pmatrix} 0 & -S_0 \\ S_0^{-1} & 0 \end{pmatrix}$, the elements $\tilde{\alpha} \in Mp(V, B)$ defined above satisfy $\delta_{X_{\mathbb{Z}}} \tilde{\alpha} = \delta_{X_{\mathbb{Z}}}$.

On the other hand, if $\alpha = \begin{pmatrix} I & B \\ 0 & I \end{pmatrix}$, $B : X_{\mathbb{Z}} \to X^*_{\mathbb{Z}}$, $B^* = B$, then $m \longmapsto (Bm, m) \pmod 2$ is a linear map $X_{\mathbb{Z}} \longrightarrow \mathbb{Z}/2\mathbb{Z}$, hence equal to $m \longmapsto (\{B\}, m)$ for some $\{B\} \in X^*_{\mathbb{Z}}/2X^*_{\mathbb{Z}}$.

As
$$\langle \delta_{X_{\mathbb{Z}}}, \tilde{\alpha}(f) \rangle = \langle \delta_{X_{\mathbb{Z}}}, e^{\pi i (Bx, x)} f(x) \rangle$$
$$= \sum_{m \in X_{\mathbb{Z}}} (-1)^{(Bm, m)} f(m) = \sum_{m \in X_{\mathbb{Z}}} (-1)^{(\{B\}, m)} f(m)$$
$$= \left\langle \delta_{X_{\mathbb{Z}}}, U\begin{pmatrix} \{B\}/2 \\ 0 \end{pmatrix} f \right\rangle$$

$\boxed{\{B\} = \text{the diagonal terms of } B \pmod 2}$

Cor. $\exists$ group homomorphism $\lambda : Sp(V_{\mathbb{Z}}, B) \longrightarrow Heis(V_{\mathbb{Z}}/2V_{\mathbb{Z}})$ such that $\forall \tilde{\alpha} \in \widetilde{Mp}(V, B)$ with $p(\tilde{\alpha}) \in Sp(V_{\mathbb{Z}}, B)$

$$\langle \delta_{X_{\mathbb{Z}}}, \tilde{\alpha}(f) \rangle = \eta(\tilde{\alpha}) \langle \delta_{X_{\mathbb{Z}}}, U(\lambda(\tilde{\alpha})) f \rangle \qquad \eta(\tilde{\alpha}) \in \mathbb{C}^\times_1$$

More detailed analysis: if $q\left(\begin{pmatrix} x^* \\ x \end{pmatrix}\right) := (x^*, x)$

$$\Gamma_\theta = \left\{ \alpha \in Sp(V_{\mathbb{Z}}, B) \,\middle|\, \forall v = \begin{pmatrix} x^* \\ x \end{pmatrix} \in V_{\mathbb{Z}} \quad q(\alpha(v)) \equiv q(v) \pmod 2 \right\}$$

and $p(\tilde{\alpha}) = \alpha$, then $\langle \delta_{X_{\mathbb{Z}}}, \tilde{\alpha}(f) \rangle = \eta(\tilde{\alpha}) \langle \delta_{X_{\mathbb{Z}}}, f \rangle$

It remains to write various θ-functions as

$\langle \delta_{\mathbb{Z}}, (\text{certain operator})(f_T) \rangle$

Theta functions : $y \in X,\ y^* \in X^*$; $T: X_{\mathbb{C}} \xrightarrow{\sim} X^*_{\mathbb{C}}$, $T = T^*$, $\mathrm{Im}(T) > 0$

$$\langle \delta_{X_{\mathbb{Z}}}, U\binom{y^*}{y} f_T \rangle = e(-\tfrac{1}{2}(y^*, y)) \sum_{m \in X_{\mathbb{Z}}} f_T(m-y)\, e((y^*, m)) =$$

$$= \underbrace{e(\tfrac{1}{2}(y^*, y))}_{e(\frac{1}{2} q\binom{y^*}{y})} \underbrace{\sum_{m \in X_{\mathbb{Z}} - y} e(\tfrac{1}{2}(Tm, m) + (y^*, m))}_{\Theta\left[\begin{smallmatrix} y^* \\ y \end{smallmatrix}\right](T)} .$$

Notation : $q\binom{y^*}{y} := (y^*, y)$

We know : If $\tilde{\alpha} \in \widetilde{Mp}(V, B)$ and $p(\tilde{\alpha}) = \alpha \in \Gamma_\vartheta$, then

$$\langle \delta_{X_{\mathbb{Z}}} \tilde{\alpha}, \cdot \rangle = \eta(\tilde{\alpha}) \langle \delta_{X_{\mathbb{Z}}}, \cdot \rangle \quad \text{for some } \eta(\tilde{\alpha}) \in \mathbb{C}_1^\times$$

$$(\eta(t\tilde{\alpha}) = t^{-1}\eta(\tilde{\alpha}) \quad \forall t \in \mathbb{C}_1^\times)$$

$$\Downarrow$$

$$\langle \underbrace{\delta_{X_{\mathbb{Z}}} \tilde{\alpha}}_{\eta(\tilde{\alpha})\delta_{X_{\mathbb{Z}}}}, U\binom{y^*}{y} f_T \rangle = \langle \delta_{X_{\mathbb{Z}}}, \underbrace{\tilde{\alpha}\, U\binom{y^*}{y} \tilde{\alpha}^{-1}}_{U(\alpha\binom{y^*}{y})} \underbrace{\tilde{\alpha}(f_T)}_{c(\tilde{\alpha}, T) f_{\alpha(T)}} \rangle$$

$$\Downarrow$$

$$\eta(\tilde{\alpha})\, \underbrace{e(\tfrac{1}{2} q\binom{y^*}{y})}\, \Theta\left[\begin{smallmatrix} y^* \\ y \end{smallmatrix}\right](T) = c(\tilde{\alpha}, T)\, \underbrace{e(\tfrac{1}{2} q(\alpha\binom{y^*}{y}))}\, \Theta\left[\alpha\binom{y^*}{y}\right](\alpha(T))$$

~~since $\alpha \in \Gamma_\vartheta$~~

$$\Rightarrow \cancel{\eta(\tilde{\alpha})\,\Theta\left[\begin{smallmatrix} y^* \\ y \end{smallmatrix}\right](T) = c(\tilde{\alpha}, T)\,\Theta\left[\alpha\binom{y^*}{y}\right](\alpha(T))}$$

As in the case $n=1$, one can compute $\arg(c(\tilde{\alpha}, T))$ by letting T tend to infinity. The result again involves suitable quadratic Gauss sums.

From $\Theta(T)$ to $\Theta(T, z)$: as in the case $n=1$, one applies suitable $U\binom{y^*}{y}$. However, one has $z \in X^*_{\mathbb{C}}$, not $z \in X_{\mathbb{C}}$, so it is more logical to use notation $z^* \in X^*_{\mathbb{C}}$.

From f_T to $f_{T,z^*}(x) = e\left(\frac{1}{2}(Tx,x) + (z^*,x)\right)$

$z^* \in X^*_{\mathbb{C}}$
$T = T^*: X_{\mathbb{C}} \to X^*_{\mathbb{C}}$
$\mathrm{Im}(T) > 0$

$f_T(x) = e\left(\frac{1}{2}(Tx,x)\right)$ Write $z^* = y^* - Ty$, $\underbrace{y \in X, y^* \in X^*}_{\text{real}}$

Notation: $\kappa(z^*,T) := (z^*, (T-\overline{T})^{-1} z^*)$

$^\circ f(z^*,T) := f_{z^*,T}\, e\left(\frac{1}{2}\kappa(z^*,T)\right)$ section of the canonical 1-cocycle

$U\begin{pmatrix} y^* \\ y \end{pmatrix} f_T = {}^\circ f(z^*,T)\, e\left(-\frac{1}{2}(z^*, (T-\overline{T})^{-1} z^*)\right)$

$\alpha \in Sp(V,B)$: $\alpha\begin{pmatrix} y^* \\ y \end{pmatrix} = \begin{pmatrix} y'^* \\ y' \end{pmatrix}$, $T' = \alpha(T) = (AT+B)(CT+D)^{-1}$

$z'^* = y'^* - T'y'$, $\alpha(z^*,T) = \left(\underbrace{(J(\alpha,T)^*)^{-1} z^*}_{z'^*}, \underbrace{\alpha(T)}_{T'}\right)$

$(z'^*, (T'-\overline{T}')^{-1} \overline{z}'^*) = (z^*, (T-\overline{T})^{-1} \overline{z}^*)$

$\kappa(z'^*,T') - \kappa(z^*,T) = -(z^*, J(\alpha,T)^{-1} c_\alpha z^*)$

As in the case $n=1$: $\tilde{\alpha}\, U\begin{pmatrix} y^* \\ y \end{pmatrix} f_T = U\left(\alpha\begin{pmatrix} y^* \\ y \end{pmatrix}\right) \underbrace{\tilde{\alpha}(f_T)}_{c(\tilde{\alpha},T) f_{\alpha(T)}}$

$$\Rightarrow \boxed{\tilde{\alpha}\left({}^\circ f_{(z^*,T)}\right) = c(\tilde{\alpha},T)\, {}^\circ f_{\alpha(z^*,T)}}$$

theta functions: $q\begin{pmatrix} x^* \\ x \end{pmatrix} = (x^*,x)$

$$\langle \delta_{X_{\mathbb{Z}}}, U\begin{pmatrix} x^* \\ x \end{pmatrix} f_{z^*,T} \rangle = e\left(\frac{1}{2} q\begin{pmatrix} x^* \\ x \end{pmatrix}\right) \underbrace{\sum_{m \in X_{\mathbb{Z}} - x} e\left(\frac{1}{2}(Tm,m) + ((z^*+x^*),m)\right)}_{\theta\left[\begin{smallmatrix} x^* \\ x \end{smallmatrix}\right](T,z^*)}$$

$^\circ\theta\left[\begin{smallmatrix} x^* \\ x \end{smallmatrix}\right](T,z^*)$ – replace $f_{z^*,T}$ by $^\circ f_{z^*,T}$

Prop. If $\tilde{\alpha} \in \widetilde{Mp}(V,B)$, $p(\tilde{\alpha}) = \alpha \in \Gamma_\theta$, then

$$\eta(\tilde{\alpha})\, e\left(\frac{1}{2} q\begin{pmatrix} x^* \\ x \end{pmatrix}\right) {}^\circ\theta\left[\begin{smallmatrix} x^* \\ x \end{smallmatrix}\right](z^*,T) = c(\tilde{\alpha},T)\, e\left(\frac{1}{2} q\left(\alpha\begin{pmatrix} x^* \\ x \end{pmatrix}\right)\right) {}^\circ\theta\left[\alpha\begin{pmatrix} x^* \\ x \end{pmatrix}\right](\alpha(z^*,T))$$

Pf. $\underbrace{\langle \delta_{X_{\mathbb{Z}}}, \tilde{\alpha}\, U\begin{pmatrix} x^* \\ x \end{pmatrix} {}^\circ f_{z^*,T} \rangle}_{\eta(\tilde{\alpha}) \langle \delta_{X_{\mathbb{Z}}}, U\begin{pmatrix} x^* \\ x \end{pmatrix} {}^\circ f_{z^*,T} \rangle} = \langle \delta_{X_{\mathbb{Z}}}, U\left(\alpha\begin{pmatrix} x^* \\ x \end{pmatrix}\right) \underbrace{\tilde{\alpha}({}^\circ f_{z^*,T})}_{c(\tilde{\alpha},T)\, {}^\circ f_{\alpha(z^*,T)}} \rangle$

Theta functions with coefficients : given $\phi : X_{\mathbb{Z}} \longrightarrow X_{\mathbb{Z}}/NX_{\mathbb{Z}} \longrightarrow \mathbb{C}$

consider $\displaystyle \Theta(T, z^*, \phi) = \sum_{m \in X_{\mathbb{Z}}} \phi(m)\, e\left(\tfrac{1}{2}(Tm, m) + (z^*, m)\right) =$ (write $m = Nm' + r$)

$$= \sum_{r \in X_{\mathbb{Z}}/NX_{\mathbb{Z}}} \phi(-r) \underbrace{\sum_{m' \in X_{\mathbb{Z}}} e\left(\tfrac{1}{2}\left(TN^2\left(m' - \tfrac{r}{N}\right), m' - \tfrac{r}{N}\right) + \left(Nz^*, m' - \tfrac{r}{N}\right)\right)}_{\Theta\left[\begin{smallmatrix}0\\ r/N\end{smallmatrix}\right](N^2T,\, Nz^*) = \left\langle \delta_{X_{\mathbb{Z}}},\, U\left(\begin{smallmatrix}0\\ r/N\end{smallmatrix}\right) f_{Nz^*, N^2T}\right\rangle}$$

For simplicity: $z^* = 0$; then $\forall \tilde{\alpha} \in \widetilde{Mp}(V, B)$ s.t. $p(\tilde{\alpha}) = \alpha \in \Gamma_\theta$

$$\Theta\left[\begin{smallmatrix}0\\ r/N\end{smallmatrix}\right](N^2T) = \left\langle \delta_{X_{\mathbb{Z}}}\tilde{\alpha},\, U\left(\begin{smallmatrix}0\\ r/N\end{smallmatrix}\right) f_{N^2\tau}\right\rangle = \left\langle \delta_{X_{\mathbb{Z}}},\, \underbrace{\tilde{\alpha}\, U\left(\begin{smallmatrix}0\\ r/N\end{smallmatrix}\right)\tilde{\alpha}^{-1}}_{U\left(\alpha\left(\begin{smallmatrix}0\\ r/N\end{smallmatrix}\right)\right)}\, \underbrace{\tilde{\alpha}(f_{N^2\tau})}_{c(\tilde{\alpha}, N^2\tau) f_{\alpha(N^2\tau)}}\right\rangle$$

$$\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 \\ r/N \end{pmatrix} = \begin{pmatrix} br/N \\ dr/N \end{pmatrix}$$

Cor. If $\alpha \equiv \begin{pmatrix} * & 0 \\ * & I \end{pmatrix} \pmod N$, then $\alpha\begin{pmatrix} 0 \\ r/N \end{pmatrix} \equiv \begin{pmatrix} 0 \\ r/N \end{pmatrix} \pmod 1$, hence

$$\Theta\left[\begin{matrix} br/N \\ dr/N \end{matrix}\right](N^2T, Nz^*) = \Theta\left[\begin{matrix} 0 \\ r/N \end{matrix}\right](N^2T, Nz^*)$$

Rmk. In the discussion of $\Theta(T, z^*, \phi)$, one can decouple the discussion of the combinatorics of ϕ ($\Leftrightarrow$ of the action of $U\begin{pmatrix} r^*/N \\ r/N \end{pmatrix}$ for $r \in X_{\mathbb{Z}}$, $r^* \in X_{\mathbb{Z}}$) from what is happening to T and z^*.
This leads to a discussion of the action of the finite Heisenberg group attached to $V_{\mathbb{Z}}/NV_{\mathbb{Z}} = (X^*_{\mathbb{Z}} \oplus X_{\mathbb{Z}})/N$ on the set of functions $\{X_{\mathbb{Z}}/NX_{\mathbb{Z}} \longrightarrow \mathbb{C}\}$. A "passage to the limit with respect to N" gives rise to the Heisenberg group for $\hat{\mathbb{Q}}^{2n}$ and the metaplectic group $Mp_{2n}(\hat{\mathbb{Q}})$ ($\underbrace{\hat{\mathbb{Q}} = \hat{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}}_{\text{finite adèles}}$, $\hat{\mathbb{Z}} = \varprojlim_N \mathbb{Z}/N\mathbb{Z} = \prod_{p \text{ prime}} \mathbb{Z}_p$)

Together with $Mp_{2n}(\mathbb{R})$, this yield the adèlic metaplectic group $Mp_{2n}(\mathbb{A}_{\mathbb{Q}})$ over the adèles $\mathbb{A}_{\mathbb{Q}} = \mathbb{R} \times \hat{\mathbb{Q}}$ of $\mathbb{Q}$.
