Introduction to Modular Forms — Questions and exercises 1
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(1) **(A combinatorial proof of Euler's pentagonal number formula)** Interpret in terms of suitable partitions the coefficients of

$$\prod_{n=1}^{\infty}(1+q^n) = (1+q)(1+q^2)(1+q^3)\cdots = 1+\sum_{n=1}^{\infty} p^*(n)q^n.$$

Idem for the coefficients of

$$\prod_{n=1}^{\infty}(1-q^n) = (1-q)(1-q^2)(1-q^3)\cdots = 1+\sum_{n=1}^{\infty} c(n)q^n.$$

Find a combinatorial proof of the fact that $c(n) = (-1)^m$ if $n = (3m^2 \pm m)/2$ (and $c(n) = 0$ otherwise).

(2) **(Spaces of Eisenstein series)** Let $k > 2$ and $N \geq 1$ be integers. Denote by $\mathrm{Eis}_{k,N}$ the complex vector space of holomorphic Eisenstein series of weight k and level N. Its elements are the functions

$$G_k(\tau,\phi) = \sum_{m,n\in\mathbf{Z}}{}' \frac{\phi(m,n)}{(m\tau+n)^k} \qquad (\tau \in \mathbf{C}\setminus\mathbf{R}),$$

where $\phi : (\mathbf{Z}/N\mathbf{Z})^2 \longrightarrow \mathbf{C}$ is a function.

Express $G_k(\tau,\phi)$ in terms of the Weierstrass function $\wp(z; L_\tau)$ of the lattice $L_\tau = \mathbf{Z}\tau + \mathbf{Z}$. Write it as a power series in $q_N = e^{2\pi i\tau/N}$.

Show that there is a natural left action of $\alpha \in GL_2(\mathbf{Z}/N\mathbf{Z})$ on $\mathrm{Eis}_{k,N}$ given by $\alpha * G_k(\cdot,\phi) = G_k(\cdot, \alpha*\phi)$ (where $(\alpha*\phi)(m,n) = \phi((m,n)\alpha)$) and a natural right action of $GL_2(\mathbf{Q})$ on $\mathrm{Eis}_k = \bigcup_{N\geq 1}\mathrm{Eis}_{k,N}$ given by $(f|_k g)(\tau) = |\det(g)|^{k/2}(c\tau + d)^{-k} f(g(\tau))$. Are these two actions related in any way? Is there an action of a bigger group on Eis_k that incorporates both of these actions? How can one recover the subspace $\mathrm{Eis}_{k,N}$ from Eis_k in terms of these actions?

Show that all of the above makes sense for $k = 2$ if one considers only functions ϕ satisfying $\sum_{m,n\in\mathbf{Z}/N\mathbf{Z}} \phi(m,n) = 0$.

More generally, consider non-holomorphic Eisenstein series

$$G_{k,s}(\tau,\phi) = (\mathrm{Im}(\tau))^s \sum_{m,n\in\mathbf{Z}}{}' \frac{\phi(m,n)}{(m\tau+n)^k|m\tau+n|^{2s}},$$

where $s \in \mathbf{C}$ and $k + 2\,\mathrm{Re}(s) > 2$. Can one obtain another Eisenstein series $G_{k',s'}(\tau,\phi)$ from $G_{k,s}(\tau,\phi)$ by applying a suitable differential (or integral) operator?

In particular, can one relate the function

$$\sum_{m,n\in\mathbf{Z}}{}' \frac{\phi(m,n)}{(m\tau+n)^\alpha(m\overline{\tau}+n)^\beta} \qquad (\alpha,\beta\in\mathbf{Z},\ \alpha+\beta>2)$$

to $G_{\alpha+\beta}(\tau,\phi)$? Can this function be written in terms of q_N and $\mathrm{Im}(\tau)$?

(3) Show that the subgroup $\langle S, T^2\rangle \subset SL_2(\mathbf{Z})$ generated by the matrices $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $T^2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ is equal to the theta group

$$\Gamma_\theta = \left\{ \gamma \in SL_2(\mathbf{Z}) \;\middle|\; \gamma \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \pmod 2 \right\}.$$

(4) Determine the subgroup $\Gamma = \langle a, b\rangle \subset SL_2(\mathbf{Z})$ generated by $b = T^2$ and $a = ST^2S^{-1} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$.

By considering the action of $a^{\pm 1}$ and $b^{\pm 1}$ on the discs $\{|z \pm 1/2| < 1/2\}$ and the half-planes $\{\mathrm{Re}(z) > 1\}$, $\{\mathrm{Re}(z) < -1\}$ (and their complements), show that the group Γ is freely generated by a and b.

Why does the existence of such a free subgroup of finite index $\Gamma \subset SL_2(\mathbf{Z})$ imply that there exist subgroups of finite index in Γ that are not congruence subgroups of $SL_2(\mathbf{Z})$ (i.e., they do not contain any principal congruence subgroup $\Gamma(N) = \{\alpha \in SL_2(\mathbf{Z}) \mid \alpha \equiv I_2 \pmod N\}$)?

(5) Why is the group $SO(2)$ isomorphic to $U(1)$, via the map $\begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix} \mapsto e^{i\alpha}$? Does this statement generalise to other situations?

(6) Show that $\{g \in SL_2(\mathbf{C}) \mid g(\mathcal{H}) = \mathcal{H}\} = SL_2(\mathbf{R})$. Show that the Cayley map $\tau \mapsto (\tau - i)/(\tau + i) = w$ is a holomorphic isomorphism between $\mathcal{H}$ and the open unit disc $D = \{|w| < 1\}$. Determine the groups $\widetilde{G} = \{g \in SL_2(\mathbf{C}) \mid g(D) = D\}$ and $\widetilde{K} = \{g \in \widetilde{G} \mid g(0) = 0\}$. For $a, b \in D$ give an explicit element $g \in \tilde{G}$ such that $g(a) = b$. Determine all holomorphic isomorphisms $f : D \xrightarrow{\sim} D$ (hint: use the Schwarz Lemma from complex analysis to treat the case $f(0) = 0$). Idem for holomorphic isomorphisms $\mathcal{H} \xrightarrow{\sim} \mathcal{H}$.

(7) What are natural higher-dimensional analogues of $\mathcal{H}$, D, the Cayley map, the groups $SL_2(\mathbf{R}) \supset SO(2)$ and $\widetilde{G} \supset \widetilde{K}$?

(8) Let $a, b, N \geq 1$ be integers satisfying $(a, b, N) = 1$. Show that there exists an integer $b' \equiv b \pmod N$ such that $(a, b') = 1$. Deduce that the natural morphism

$SL_2(\mathbf{Z}) \longrightarrow SL_2(\mathbf{Z}/N\mathbf{Z})$ is surjective. What happens if $\mathbf{Z}$ (resp. N) is replaced by a Dedekind ring and N by a non-zero ideal of A?

(9) **(Lattices with full level structures)** Let $N \geq 1$ be an integer. A full level N structure on a lattice $L \subset \mathbf{C}$ is an isomorphism of abelian groups $\lambda : L/NL \xrightarrow{\sim} (\mathbf{Z}/N\mathbf{Z})^2$ (where we consider elements of $(\mathbf{Z}/N\mathbf{Z})^2$ as row vectors). Show that, for any integer $k \in \mathbf{Z}$, there is a natural bijection between the space of functions $F : \{(\text{lattices } L \subset \mathbf{C}, \lambda : L/NL \xrightarrow{\sim} (\mathbf{Z}/N\mathbf{Z})^2\} \longrightarrow \mathbf{C}$ satisfying $\forall t \in \mathbf{C}^\times \quad F(tL, \lambda \circ t^{-1}) = t^{-k} F(L, \lambda)$, the space of functions $\tilde{f} : (\mathbf{C} \setminus \mathbf{R}) \times GL_2(\mathbf{Z}/N\mathbf{Z}) \longrightarrow \mathbf{C}$ satisfying

$$\forall \alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbf{Z}) \quad \forall \beta \in GL_2(\mathbf{Z}/N/\mathbf{Z}) \quad \tilde{f}\left(\frac{a\tau + b}{c\tau + d}, \alpha\beta\right) = (c\tau + d)^k \, \tilde{f}(\tau, \beta)$$

and an analogous space of functions $f : \mathcal{H} \times GL_2(\mathbf{Z}/N\mathbf{Z}) \longrightarrow \mathbf{C}$, this time with $\alpha \in SL_2(\mathbf{Z})$. Describe f in terms of a collection of independent functions $f_u : \mathcal{H} \longrightarrow \mathbf{C}$ ($u \in (\mathbf{Z}/N\mathbf{Z})^\times$) satisfying appropriate functional equations.

(10) **(Basic theta function)** Deduce from Poisson's formula that

$$\sum_{n \in \mathbf{Z}} e^{-\pi t(n+x)^2} = t^{-1/2} \sum_{m \in \mathbf{Z}} e^{-\pi m^2/t + 2\pi i m x} \qquad (x, t \in \mathbf{R},\ t > 0).$$

Let $\phi : \mathbf{Z}/N\mathbf{Z} \longrightarrow \mathbf{C}$ be a function. Denote by $\widehat{\phi}$ its Fourier transform: $\widehat{\phi}(b) = \sum_{a \in \mathbf{Z}/N\mathbf{Z}} \phi(a) e^{2\pi i ab/N}$. Relate the theta function $\theta_\phi(\tau) = \sum_{n \in \mathbf{Z}} \phi(n) e^{\pi i n^2 \tau}$ to $\theta_{\widehat{\phi}}(-1/N^2\tau)$.

For any rational number a/c ($a, c \in \mathbf{Z}$), find an expression for the limit $S(a/c) = \lim_{t \to 0+} t^{1/2} \theta(a/c + it)$. Deduce a formula for the eighth root of unity in the transformation formula for θ under an arbitrary element of Γ_θ. What happens if one compares $S(a/c)$ to $S(-c/a)$?

Show that Euler's pentagonal number formula implies that

$$\eta(24\tau) = q \prod_{n=1}^{\infty} (1 - q^{24n}) = \theta_\phi(\tau),$$

for a suitable function $\phi : \mathbf{Z}/12\mathbf{Z} \longrightarrow \mathbf{C}$. Deduce from the previous discussion that $\eta(-1/\tau) = \sqrt{\tau/i}\, \eta(\tau)$.

(11) **(Hyperbolic geometry in $\mathcal{H}$ and D)** Show that the hyperbolic (Poincaré) metric $ds^2 = (dx^2 + dy^2)/y^2$ on $\mathcal{H}$ ($\tau = x + iy$) is invariant under the action of $SL_2(\mathbf{R})$.

Show that the vertical half-lines $x = c$ are geodesics and compute the corresponding distance $d(ia, ib)$ in this metric ($0 < a < b$). Interpret this distance in terms of a suitable cross ratio.

Determine geometrically all geodesics and isometries with respect to the metric ds^2.

Compute the area of a geodesic triangle.

What metric on D corresponds to ds^2 under the Cayley map? Show that hyperbolic circles $d(\cdot, P) = r$ in $\mathcal{H}$ and in D are Euclidean circles, and vice versa. Determine the centre and the hyperbolic radius of the hyperbolic circle in $\mathcal{H}$ containing ia and ib ($0 < a < b$). Determine its hyperbolic circumference and area.

Determine the set of points at a fixed distance from a given geodesic, and the set of points equidistant from two points.

(12) **(The modular tessellation of $\mathcal{H}$)** Apply to the geodesic triangle $\Delta \subset \mathcal{H}$ with vertices $\infty, 0, 1$ all elements of $SL_2(\mathbf{Z})$. Which triangles are obtained in this way, and what kind of a pattern do they form?

(13) **(Free subgroups of $PGL_2(\mathbf{C})$ from geometry)** Is there a general construction of free subgroups of $PGL_2(\mathbf{C})$ encoded in geometry of several pairs of discs in $\mathbf{C} \cup \{\infty\}$ (as we saw in a special case in (4))?

(14) **(The upper half-plane $\mathcal{H}$ and quadratic forms)** The action of $G = SL_2(\mathbf{R})$ on $\mathcal{H}$ is transitive and the stabiliser G_i of $i \in \mathcal{H}$ is the special orthogonal group $K = SO(2)$. On the other hand, the standard linear action of $SL_n(\mathbf{R})$ on $\mathbf{R}^n$ gives rise to a transitive action on the set of all positive definite quadratic forms on $\mathbf{R}^n$ of fixed determinant, and the stabiliser of the standard form $x_1^2 + \cdots + x_n^2$ is the subgroup $SO(n)$. Describe explicitly a natural correspondence between points of $\mathcal{H}$ and positive definite quadratic forms on $\mathbf{R}^2$ of determinant equal to 1, which is compatible with the action of $SL_2(\mathbf{R})$. What sets of quadratic forms do geodesics in $\mathcal{H}$ correspond to?

(15) **(Möbius geometry)** The group of transformations of $\mathbf{C} \cup \{\infty\}$ given by Möbius maps $z \mapsto w = (az + b)/(cz + d)$ ($a, b, c, d \in \mathbf{C}$, $ad - bc \neq 0$) and their complex conjugates $z \mapsto \overline{w}$ can be described geometrically as the group generated by symmetries (inverses) with respect to Euclidean circles and affine lines.

What can one say about the group generated by analogous symmetries in a higher-dimensional Euclidean space (completed by a one point $\{\infty\}$ at infinity)?

One can identify $\mathbf{C} \cup \{\infty\}$ with a sphere $S^2 \subset \mathbf{R}^3$ via stereographic projection of S^2 from the north pole to the equatorial plane. Can the Möbius transformations (and their complex conjugates) acting on $\mathbf{C} \cup \{\infty\}$ be described in terms of geometry of $\mathbf{R}^3 \supset S^2$? If yes, what is the relation to the hyperbolic geometry of $\mathcal{H}$ equipped with the action of $SL_2(\mathbf{R})$?