Introduction to Modular Forms — Questions and exercises 2
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(1) **(Modular forms on $\Gamma(N)$ and $\Gamma_0(N)$)** Let $N \geq 1$, $\alpha = \begin{pmatrix} N & 0 \\ 0 & 1 \end{pmatrix}$. Show that $\alpha^{-1}\Gamma(N)\alpha \supset \Gamma_1(N)$. Deduce that any $f(\tau) \in M_k(\Gamma(N))$ can be written as $f(\tau) = \sum_{j=1}^{M} c_j f_j(\tau/N)$, where $c_j \in \mathbf{C}$ and $f_j(\tau) \in M_k(\Gamma_0(N^2), \chi_j)$ for a suitable Dirichlet character $\chi_j : (\mathbf{Z}/N^2\mathbf{Z})^\times \longrightarrow \mathbf{C}^\times$.

(2) **(Hauptmodules for $\Gamma_0(p)$)** Let $N \geq 1$ be an integer. Show that $f_N(\tau) = \Delta(N\tau)/\Delta(\tau)$ is a meromorphic function on $X_0(N)$; compute its divisor. If $N \in \{2, 3, 5, 7, 13\}$, show that $g_N = f_N^{1/(N-1)}$ is also a meromorphic function on $X_0(N)$ and that g_N defines an isomorphism $X_0(N) \xrightarrow{\sim} \mathbf{P}^1(\mathbf{C})$.

(3) **(Ford circles)** All examples of fundamental domains that we have seen (for $\Gamma = SL_2(\mathbf{Z}), \Gamma(2), \Gamma_\theta, \Gamma^0(p)$, $p > 2$ prime) were of the form $\{u < \mathrm{Re}(\tau) < u + h\} \cap \bigcap D_j$, where D_j is the exterior of a suitable circle $|c_j\tau + d_j| = 1$. Why was that the case? Is there a general statement along these lines?

In the examples above, $h = 1, 2, 2, p$ and the circles are given by $|\tau + n| = 1$ ($n \in \mathbf{Z}$), $|2\tau + (2n+1)| = 1$ ($n \in \mathbf{Z}$), $|\tau + 2n| = 1$ ($n \in \mathbf{Z}$) and $|\tau + n| = 1$ ($n \in \mathbf{Z}$, $p \nmid n$), respectively.

(4) **(Rankin–Selberg L-function for Hecke eigenforms)** If

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s} = \prod_p \frac{1}{(1 - \alpha_p p^{-s})(1 - \alpha'_p p^{-s})} \tag{1}$$

$$\sum_{n=1}^{\infty} \frac{b_n}{n^s} = \prod_p \frac{1}{(1 - \beta_p p^{-s})(1 - \beta'_p p^{-s})}, \tag{2}$$

$$\tag{3}$$

what can one say about $\sum_{n=1}^{\infty} a_n b_n / n^s$?

Let $k, l > 2$ be even integers and $f \in S_{k+l}(SL_2(\mathbf{Z}))$ a normalised Hecke eigenform. Compute the Petersson scalar product

$$(f, E_k E_l)_{SL_2(\mathbf{Z})} = \iint_{SL_2(\mathbf{Z})\backslash\mathcal{H}} f(\tau)\, \overline{E_k(\tau) E_l(\tau)}\, y^{k+l} \, \frac{dx\,dy}{y^2}.$$

(5) A compact Riemann surface of genus $g \geq 2$ is called a **Hurwitz Riemann surface** (or a **Hurwitz curve** when considered as a non-singular projective complex curve) if its group of holomorphic automorphisms (which are all algebraic) has maximal possible order $84(g-1)$ (see Exercise 6 below).

Show that $X(7)$ is a Hurwitz curve of genus 3. Make explicit the ramification points of $X(7) \longrightarrow X(1) \xrightarrow{\sim} \mathbf{P}^1(\mathbf{C})$ and the corresponding ramification indices.

(6) **(Theorem of Hurwitz)** Let X be a compact Riemann surface of genus $g \geq 2$, let G be the group of holomorphic automorphisms of X.
(a) By considering the actions of G on $H_1(X, \mathbf{Z})$, $(H^1_{dR}(X, \mathbf{R}), \cup)$ and $\Omega^1(X)$ respectively, show that the image of G in $\mathrm{Aut}_{\mathbf{Z}}(H_1(X, \mathbf{Z})) \xrightarrow{\sim} GL_{2g}(\mathbf{Z})$ is finite and the action of G on $\Omega^1(X)$ is faithful; deduce that G is finite.
(b) Show that, for each $x \in X$, the stabiliser $G_x = \{g \in G \mid g(x) = x\}$ is a finite cyclic group and there exists a local coordinate z_x at x such that $z_x \circ g = \chi(g) z_x$ for all $g \in G_x$, where $\chi : G_x \longrightarrow \mathbf{C}^\times$ an injective character. Deduce that $Y = G\backslash X$ has a natural structure of a compact Riemann surface.
(c) Show that, for each $x \in X$, the ramification index of the projection $\pi : X \longrightarrow Y$ at x is equal to $e_x = |G_x|$ and depends only on $y = \pi(x)$; denote it by e_y and let $\mathrm{Ram} = \{y \in Y \mid e_y \neq 1\}$ be the ramification locus of π.
(d) Deduce from the Riemann–Hurwitz formula that

$$\frac{2g-2}{|G|} = 2g(Y) - 2 + \sum_{y \in \mathrm{Ram}} \left(1 - \frac{1}{e_y}\right).$$

(e) Show that $|G| \leq 84(g-1)$, with equality if and only if $g(Y) = 0$, $|\mathrm{Ram}| = 3$ and $\{e_y\} = \{2, 3, 7\}$.

(7) **(Theorem of Siegel)** If $\Gamma \subset SL_2(\mathbf{R})$ is a Fuchsian subgroup of the first kind, then $\mathrm{vol}(\Gamma\backslash\mathcal{H}) \geq \pi/21$. Is there any relation to Exercise 6?

(8) **(Eisenstein series and modular units)** Modular forms without zeroes in $\mathcal{H}$ (and therefore with divisor supported on the cusps) are very interesting objects (example: $\Delta(N\tau)$). Quotients of such modular forms are **modular units**: meromorphic functions on X_Γ with divisor supported on the cusps of X_Γ (example: $\Delta(N\tau)/\Delta(\tau)$).

Recall that, for any integer $k \geq 3$,

$$\wp^{(k-2)}(z; \mathbf{Z}\tau + \mathbf{Z}) = (-1)^k (k-1)!\, G_k(z, \tau), \quad G_k(z, \tau) = \sum_{m,n \in \mathbf{Z}} \frac{1}{(z + m\tau + n)^k}.$$

We know that, for each $(u, v) = (\frac{a}{N}, \frac{b}{N}) \in (\frac{1}{N}\mathbf{Z}/\mathbf{Z})^2 \setminus \{(0,0)\}$,

$$G_{k;(u,v)}(\tau) = G_k\left(\frac{a\tau + b}{N}, \tau\right) = N^k \sum_{\substack{m \equiv a \pmod N \\ n \equiv b \pmod N}} \frac{1}{(m\tau + n)^k} \in M_k(\Gamma(N)).$$

Show that:

(a) If $(u,v) \in (\frac{1}{N}\mathbf{Z}/\mathbf{Z})^2 \setminus (\frac{1}{2}\mathbf{Z}/\mathbf{Z})^2$, then the function $G_{3;(u,v)}(\tau)$ has no zeroes in $\mathcal{H}$.
(b) If $v \in (\frac{1}{N}\mathbf{Z}/\mathbf{Z}) \setminus \{0\}$, then $G_{k;(0,v)}(\tau) \in M_k(\Gamma_1(N))$ and

$$\forall \alpha = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_0(N) \qquad G_{k;(0,v)}\big|_k \alpha = G_{k;(0,Dv)}.$$

(c) If $v = \frac{b}{N} \in (\frac{1}{N}\mathbf{Z}/\mathbf{Z}) \setminus (\frac{1}{2}\mathbf{Z}/\mathbf{Z})$, then

$$\lim_{\mathrm{Im}(\tau)\to+\infty} G_{k;(0,v)}(\tau) = N^k \sum_{\substack{n\in\mathbf{Z} \\ n\equiv b[N]}} \frac{1}{n^k} \neq 0.$$

(d) If $u = \frac{a}{N}$, where $0 < a < N/2$, then $G_{k;(u,0)} = cq_N^a + \cdots$ as $\mathrm{Im}(\tau) \to +\infty$, where $q_N = e^{2\pi i\tau/N}$ and $c \neq 0$.
(e) If $p > 2$ is a prime, then

$$f(\tau) = \prod_{b=1}^{\frac{p-1}{2}} G_{3;(0,\frac{b}{p})}(\tau) \in M_{\frac{3(p-1)}{2}}\left(\Gamma_0(p), \left(\frac{\cdot}{p}\right)\right).$$

This function has the same divisor as $(\eta^p(\tau)/\eta(p\tau))^3$, hence

$$(\eta^p(\tau)/\eta(p\tau))^3 = cf(\tau) \in M_{\frac{3(p-1)}{2}}\left(\Gamma_0(p), \left(\frac{\cdot}{p}\right)\right) \qquad (c \neq 0).$$

(f) Replace $\wp'$ in the previous construction by $\wp$ to show that, for a prime p,

$$(\eta^p(\tau)/\eta(p\tau))^2 \in M_{p-1}(\Gamma_0(p)), \qquad \eta^p(\tau)/\eta(p\tau) \in M_{\frac{p-1}{2}}\left(\Gamma_0(p), \left(\frac{\cdot}{p}\right)\right).$$

(9) **(Examples of cusp forms)** Let $k, N \geq 1$ be integers such that $k(N+1) = 24$. Let $f_k(\tau) = (\eta(\tau)\eta(N\tau))^k$. Show that:
(a) If $S_k(\Gamma_1(N)) \neq 0$, then $S_k(\Gamma_1(N)) = \mathbf{C}f_k$.
(b) If $S_k(\Gamma_0(N)) \neq 0$, then $S_k(\Gamma_0(N)) = S_k(\Gamma_1(N)) = \mathbf{C}f_k$.
(c) Deduce from Exercise 8 that

$$(\eta(\tau)\eta(11\tau))^2 \in S_2(\Gamma_0(11)), \qquad (\eta(\tau)\eta(5\tau))^4 \in S_4(\Gamma_0(5)),$$
$$\eta(\tau)\eta(23\tau) \in M_1(\Gamma_0(23), \left(\frac{\cdot}{23}\right)), \qquad \eta(\tau)\eta(7\tau) \in S_3(\Gamma_0(7), \left(\frac{\cdot}{7}\right)).$$

(10) Let $p > 2$ be a prime, let $d = (p-1, 12)$. Show that:
(a) The function $(\eta(p\tau)/\eta(\tau))^{24/d}$ is a meromorphic function on $X_0(p)$. Its divisor is equal to $\frac{p-1}{d}((\infty) - (0))$.
(b) The result in (a) is optimal: the class of the divisor $(\infty) - (0)$ in $Cl^0(X_0(p))$ is torsion, of order $\frac{p-1}{d}$ (which is equal to the numerator of $\frac{p-1}{12}$).

It turns out that, according to a theorem of Manin and Drinfeld, all divisors of the form $(x) - (y)$ (where x, y are arbitrary cusps of $X(N)$) have torsion class in $Cl^0(X(N))$.

(11) Is it just a coincidence that the coefficients of

$$D\Delta/\Delta = E_2(\tau) = 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n) q^n$$

are given by $\sigma_1(n) = \deg T(n)$?