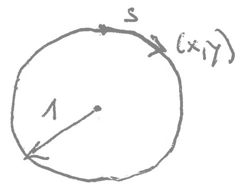
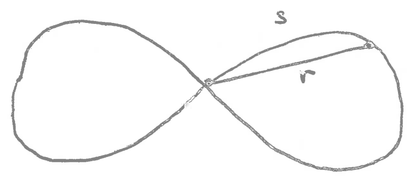
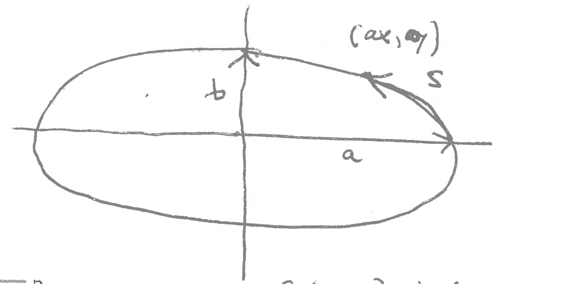
Integrals of algebraic functions of one variable

Ex: computing the length of a $\left\{\begin{array}{l} \text{circle} \\ \text{lemniscate} \\ \text{ellipse} \end{array}\right\}$ $\qquad a > b\ ,\ k^2 = 1 - \frac{b^2}{a^2}$







$s = \int \frac{dx}{\sqrt{1-x^2}}$ $\qquad s = \int \frac{dr}{\sqrt{1-r^4}}$ $\qquad ds = 2a \int \frac{(1-k^2x^2)\,dx}{\sqrt{(1-x^2)(1-k^2x^2)}}$

<u>General problem</u>: study $\boxed{\int R(x,y)\,dx = \int \omega,}$ $\qquad$ $\boxed{\text{Above}: \ y^2 - f(x) = 0}$

$\boxed{F(x,y) = 0}$ $\quad$ ($F \in \mathbb{C}[x,y]$ non-constant irreducible)

$R \in \mathbb{C}(x,y)$ rational function

<u>Geometric formulation</u>: $C: F(x,y) = 0$ irreducible plane curve (affine)

$\omega = R(x,y)\,dx$ rational differential on C

study $P \longmapsto \int_O^P \omega = I(P)$, $\quad O \in C(\mathbb{C})$ fixed base point
$\qquad P \in C(\mathbb{C})$ variable point

integrate along a chosen path in $C(\mathbb{C}) \setminus \{\text{singularities of } \omega\}$

<u>Choice of another path</u>: $\int_{\gamma_1} \omega = \int_{\gamma_2} \omega + \int_{\gamma} \omega$, $\quad \gamma = \gamma_1 - \gamma_2$, $\partial\gamma = 0$ <u>closed path</u>

$I(P)$ is defined modulo <u>periods</u> of ω:

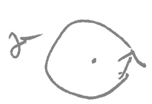
$\left\{ \int_\gamma \omega \ \middle|\ \gamma \text{ closed path in } C(\mathbb{C}) \setminus \mathrm{Sing}(\omega) \right\}$

$= \left\{ \int_\gamma \omega \ \middle|\ [\gamma] \in H_1(C(\mathbb{C}) \setminus \mathrm{Sing}(\omega)) \right\}$

<u>Ex</u>: $C: y = 0$

, $\omega = \frac{dx}{x}$, $\mathrm{Sing}(\omega) = \{0\}$

$\int_1^P \frac{dx}{x} = \log(P) \in \mathbb{C}/(2\pi i\mathbb{Z})$



$\int_\gamma \frac{dx}{x} = 2\pi i$

<u>Variant</u>: embed $C \subset \tilde{C}$ suitable <u>projective curve</u>

consider also $P \in \tilde{C}(\mathbb{C}) \setminus \mathrm{Sing}(\omega)$

Elementary case: integrals of rational functions

$$R(t)\in\mathbb{C}(t) \Rightarrow R(t) = \underset{\mathbb{C}[t]}{\underset{\uparrow}{A(t)}} + \sum_{j=1}^{m}\left(c_j\cdot\left(\frac{1}{t-a_j}\right) + \sum_{k=2}^{n} b_{jk}\frac{1}{(t-a_j)^k}\right)$$

$$\Rightarrow \underset{\text{(multivalued fn)}}{\int R(t)\,dt} = F(t) + \sum_{j=1}^{m} c_j \log(F_j(t)), \qquad F, F_j \in \mathbb{C}(t) \text{ rational fns}, \ c_j\in\mathbb{C}$$

Exercise: if $F(t_1,\dots,t_n)$ is a multivalued fn such that

$$\forall j \ \frac{\partial F}{\partial t_j}\in\mathbb{C}(t_1,\dots,t_n) \text{ (rational fn)} \Rightarrow F = G + \sum_{k=1}^{m} c_k \log(G_k), \quad c_k\in\mathbb{C}$$

("F = elementary function") $\qquad G, G_k \in \mathbb{C}(t_1,\dots,t_n)$ rational fns

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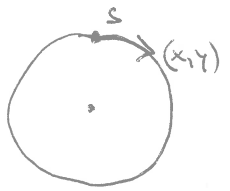
Addition formulas



$$s = \int_0^x \omega, \qquad \omega = \frac{dx}{\sqrt{1-x^2}}$$

Inverse fn: $x = \sin(s)$, $\quad \sin' = \sqrt{1-\sin^2}$

$$\sin(s_1+s_2) = \sin(s_1)\sqrt{1-\sin^2(s_2)} + \sin(s_2)\sqrt{1-\sin^2(s_1)}$$

$$\Leftrightarrow \int_0^u \frac{dx}{\sqrt{1-x^2}} + \int_0^v \frac{dx}{\sqrt{1-x^2}} = \int_0^w \frac{dx}{\sqrt{1-x^2}}, \quad w = u\sqrt{1-v^2} + v\sqrt{1-u^2}$$

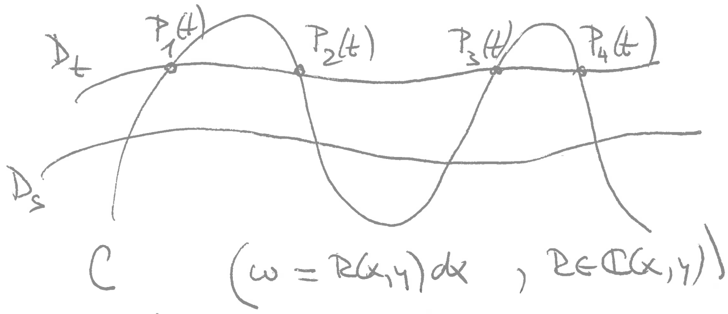
Euler's generalisation: $\quad f(x) = 1 + mx^2 + nx^4$

$$\int_0^u \frac{dx}{\sqrt{f(x)}} + \int_0^v \frac{dx}{\sqrt{f(x)}} = \int_0^w \frac{dx}{\sqrt{f(x)}}, \qquad w = \frac{u\sqrt{f(v)} + v\sqrt{f(u)}}{1-nu^2v^2}$$

Abel's approach to addition formulas

Consider Abel's sums: intersect $C: F(x,y)=0$ $\quad (F\in\mathbb{C}[x,y])$

with a family of curves $D_t: G(x,y,t)=0$ $\quad \left(G\in\mathbb{C}[x,y,t],\ t=(t_1,\dots,t_n)\right)$



$(C\cap D_t)(\mathbb{C}) = \{P_k(t)\}$ (with multiplicities)

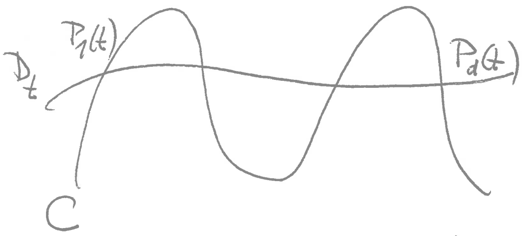
and let $I(t) = \sum_k \int_0^{P_k(t)} \omega$ (modulo periods)

$(\omega = R(x,y)\,dx, \quad R\in\mathbb{C}(x,y))$

Abel's Thm on Elementary Functions: (modulo periods)

$$\boxed{\begin{array}{c} I(t) = F(t) + \sum_{\ell} c_\ell \log(F_\ell(t)) \\ F, F_\ell \in \mathbb{C}(t_1,\dots,t_n),\ c_\ell\in\mathbb{C}\end{array}}$$

Pf : $P_k(t) = (x_k(t), y_k(t))$



$C : F(x,y) = 0$, $D_t : G(x,y,t) = 0$

$\omega = R(x,y)\,dx$, $I(t) = \sum_k \int_O^{P_k(t)} \omega$

$(C \cap D_t)(\mathbb{C}) = \{P_k(t)\}$

$\underline{F_x' dx + F_y' dy = 0}$, $G_x' dx + G_y' dy + \underbrace{G_t' dt}_{\sum G_{t_j}' \cdot dt_j} = 0$

eliminate dy : $\begin{vmatrix} F_x' & F_y' \\ G_x' & G_y' \end{vmatrix} dx = F_y' G_t' dt$ $\qquad \left(F_x' = \frac{\partial F}{\partial x}\right)$

$$\frac{\partial I}{\partial t_j} = \sum_k R(x_k(t), y_k(t)) \frac{\partial x_k(t)}{\partial t_j} = \underbrace{\sum_k \left(\frac{R F_y'}{\begin{vmatrix} F_x' & F_y' \\ G_x' & G_y' \end{vmatrix}} G_{t_j}' \right)(x_k(t), y_k(t), t)}_{\text{rational fn of } \{t_j\} \text{ and } \{(x_k(t), y_k(t))\},}$$

rational fn of $\{t_j\}$ $\overset{\text{Thm on}}{\underset{\text{symmetric fns}}{\Longleftarrow}}$ invariant under permutations of $\{k\}$

So $\forall j \; \frac{\partial I}{\partial t_j} \in \mathbb{C}(t_1, \dots, t_n) \underset{\text{Exercise above}}{\Longrightarrow} I = F + \sum c_\ell \log(F_\ell)$

$F, F_\ell \in \mathbb{C}(t_1, \dots, t_n)$, $c_\ell \in \mathbb{C}$

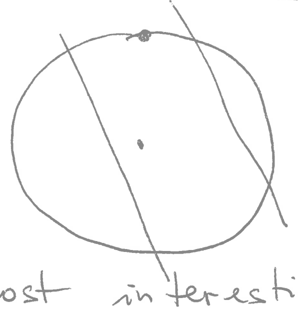
Rmk : need some "genericity assumptions" to ensure $\begin{vmatrix} F_x' & F_y' \\ G_x' & G_y' \end{vmatrix} \neq 0$ at all $P_k(t)$

Interesting case : when $\forall j \; \frac{\partial I}{\partial t_j} = 0$ for the given family $\{D_t\}$

$\Longrightarrow I(t)$ (modulo periods) is constant —— " ——

(terminology : $\underline{\int \omega \text{ satisfies Abel's principle w.r.t. } \{D_t\}}$)

Ex :



$C : x^2 + y^2 = 1$, $\omega = \frac{dx}{y} \left(= \frac{dx}{\sqrt{1-x^2}} \right)$

D_t : system of parallel lines

$y - cx - t = 0$, $\underline{c \in \mathbb{C} \text{ constant}}$

The most interesting case : $C \subset \tilde{C}$ non-singular projective curve

ω is non-singular ($\Leftrightarrow$ holomorphic on $\tilde{C}(\mathbb{C})$) on $\tilde{C}$

$\Longrightarrow_{P \mapsto} \int_O^P \omega$ (mod periods) is defined (and finite) everywhere on $\tilde{C}$

$\Longrightarrow I(t)$ —— " ——

$\underset{\text{El. Fn Thm}}{\overset{\text{Abel's}}{\Longrightarrow}}$ $\underline{I(t) = \text{constant (mod periods)} \quad \text{for every family } \{D_t\}}$

Reference: S. L. Kleiman, *What is Abel's Theorem Anyway?* in The Legacy of Niels Henrik Abel (Laudal, Piene, eds.), Springer, 2004.

Ex: (hyperelliptic curves $C: y^2 = f(x)$, $f \in \mathbb{C}[x]$ with distinct roots, $n = \deg(f) > 2$

($n = 1, 2$: elementary)

$\boxed{n = 2k \text{ or } 2k-1}$, $k \geq 2$ $\qquad \boxed{f(x) = a_0 \prod_{j=1}^{n} (x - \alpha_j)}$,

change of variables : $\boxed{x = \frac{A\tilde{x}+B}{C\tilde{x}+D} = h(\tilde{x})}$, $h = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in GL_2(\mathbb{C})$, $\boxed{y = \frac{\tilde{y}}{(C\tilde{x}+D)^k}}$

$\Rightarrow \boxed{\tilde{y}^2 = \tilde{f}(\tilde{x})}$, $\tilde{n} = \deg(\tilde{f}) \in \{2k-1, 2k\}$, $\tilde{f}(\tilde{x}) = \tilde{a}_0 \prod_{j=1}^{\tilde{n}} (\tilde{x} - \tilde{\alpha}_j)$

$\{\text{roots of } f\}$ ($\cup \{\infty\}$ if $n = 2k-1$) $= \{h(\text{roots of } \tilde{f})\}$ ($\cup \{\infty\}$ if $\tilde{n} = 2k-1$)

$dx = \frac{(AD - BC)}{(C\tilde{x}+D)^2} d\tilde{x}$; if $R(x,y)$ is a rational function of x, y, then

$$R(x,y)\,dx = \underbrace{\tilde{R}(\tilde{x}, \tilde{y})}_{\text{rational fn of } \tilde{x}, \tilde{y}}\,d\tilde{x}$$

Elliptic case: $k = 2$, $\deg(f) \in \{3, 4\}$

Legendre's normal form (original form): $y^2 = (1-x^2)(1-\lambda x^2)$ $\qquad (\lambda = k^2 \neq 0, 1)$

change of variables $x^2 = z$:

$$\omega = \frac{dx}{y} = \frac{dx}{\sqrt{(1-x^2)(1-\lambda x^2)}} = \frac{1}{2} \frac{dz}{\sqrt{z(1-z)(1-\lambda z)}}$$

Riemann's ~~form~~ normal form : $y^2 = x(1-x)(1-\lambda x)$ $\qquad (*)$

(called Legendre's normal form these days)

Weierstrass normal form : $y^2 = 4x^3 - g_2 x - g_3$

General case : $y^2 = a_0(x-\alpha_1)(x-\alpha_2)(x-\alpha_3)(x-\alpha_4)$ $\qquad \alpha_j$ distinct

$\exists!\, h \in GL_2(\mathbb{C})$, $h = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ $\qquad \alpha_2, \alpha_3, \alpha_4 \xmapsto{h^{-1}} 1, 0, \infty$

(λ = cross-ratio $r(\alpha_1, \ldots, \alpha_4)$) $\qquad \alpha_1 \longmapsto \lambda \in \mathbb{C} \setminus \{0, 1\}$

$x = h(\tilde{x}) = \frac{A\tilde{x}+B}{C\tilde{x}+D}$, $y = \frac{\tilde{y}}{C\tilde{x}+D}$ $\qquad \tilde{y}^2 = \tilde{a}_0 \tilde{x}(1-\tilde{x})(\lambda - \tilde{x})$

(replacing $\tilde{x}$ by $1/\tilde{x}$ and $\tilde{y}$ by $c\tilde{y}/\tilde{x}^2$ leads to $(*)$)

λ depends on the ordering of the roots of $f(x)$.

Fix α_4; then $\{$permutations of $\alpha_1, \alpha_2, \alpha_3\} \longleftrightarrow \left\{\lambda, \frac{1}{\lambda}, 1-\lambda, \frac{1}{1-\lambda}, \frac{\lambda}{\lambda-1}, \frac{\lambda-1}{\lambda}\right\}$

Interlude: algebraic curves and Riemann surfaces

affine plane curve: $C: g(x,y)=0$ $\qquad g\in\mathbb{C}[x,y]$ non-constant

Def: $P=(x_0,y_0)\in C(\mathbb{C})$ is a singular point of C if $g'_x(P)=g'_y(P)=0$

C is non-singular if the system $g=g'_x=g'_y=0$ has no solution in $\mathbb{C}^2$
($\Rightarrow$ g is irreducible (exercise!))

projective plane curve: $\widetilde{C}: G(X,Y,Z)=0$, $G\in\mathbb{C}[X,Y,Z]$ homogeneous of $\deg = d>0$

Def: $P=(X_0:Y_0:Z_0)\in\widetilde{C}(\mathbb{C})$ is a singular pt of $\widetilde{C}$
if $G'_X(P)=G'_Y(P)=G'_Z(P)=0$. $\qquad \widetilde{C}\subset\mathbb{P}^2_{\mathbb{C}}$

Exercise: these two definitions are equivalent for

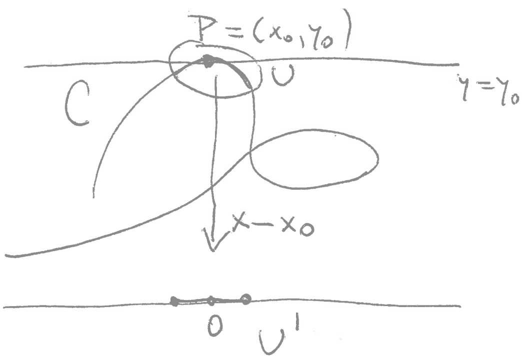
$P=(x,y)=(x:y:1)\in\widetilde{C}(\mathbb{C})\cap\mathbb{C}^2$, $\quad \mathbb{C}^2=\mathbb{P}^2(\mathbb{C})\setminus\{Z=0\}$
$\left(\frac{X}{Z},\frac{Y}{Z}\right)\longleftrightarrow(X:Y:Z)$

Irreducible plane curves: $C: g(x,y)=0$, $g\in\mathbb{C}[x,y]$ non-constant irreducible

$C\subset\mathbb{A}^2_{\mathbb{C}}$ irreducible curve

singular locus of C: $C_{sing}=\{g=g'_x=g'_y=0\}$ (finitely many points)

$C(\mathbb{C})\setminus C_{sing}(\mathbb{C})$ is a Riemann surface (connected! – exercise):



(a) if $g'_y(P)\neq 0 \Rightarrow \exists\, U\ni P$ open, $x-x_0: C(\mathbb{C})\cap U\longrightarrow U'$ bijection

$\mathbb{C}\supset U'\ni 0$ open

and $C(\mathbb{C})\cap U=\{(x_0+t, A(t))\mid t\in U'\}$

where $A: U'\longrightarrow\mathbb{C}$ is holomorphic

(inverse fn thm – holomorphic version)

(b) if $g'_x(P)\neq 0$: idem for $y-y_0$

Conclusion: $x-x_0$ (resp. $y-y_0$) can be taken as a local coordinate on $C(\mathbb{C})\cap U$ if $g'_y(P)\neq 0$ (resp. $g'_x(P)\neq 0$).

These local coordinates (for all $P\in(C\setminus C_{sing})(\mathbb{C})$) will be compatible $\Longrightarrow$ make $C(\mathbb{C})\setminus C_{sing}(\mathbb{C})$ into a Riemann surface.

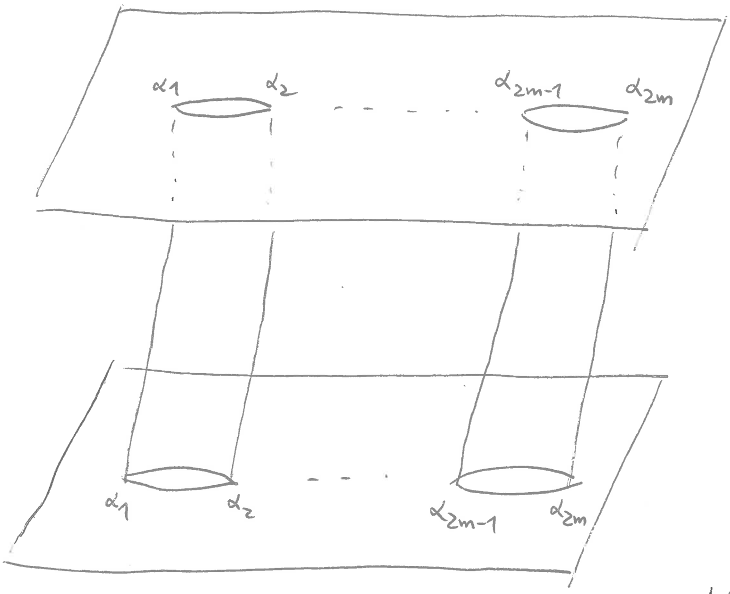
The Riemann surface of $\sqrt{f(x)}$

$y^2 = f(x)$, $f(x) = a_0x^n + \dots + a_n = a_0\prod_{k=1}^{n}(x-\alpha_k)$ $\qquad$ $\alpha_k \in \mathbb{C}$ distinct

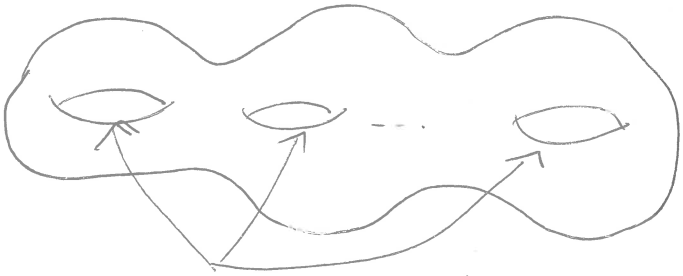
$n \in \{2m-1, 2m\}$, $m \geq 2$ $\qquad$ $(n \geq 3)$

Classical construction (with $\alpha_{2m} = \infty \in \mathbb{P}^1(\mathbb{C})$ if $n = 2m-1$)

Glue together appropriately two copies of $\underbrace{\mathbb{P}^1(\mathbb{C})}_{\simeq S^2}$ with cuts between α_1 — α_2 - - - - α_{2m-1} — α_{2m}



two copies of S^2 joined by m handles



$m-1$ holes $\qquad$ (genus $= m-1$)

Naive algebraic construction:

complete the affine curve $C: y^2 - f(x) = 0$ to a projective curve $\tilde{C}: \underbrace{Z^n\left(\left(\frac{Y}{Z}\right)^2 - f\left(\frac{X}{Z}\right)\right)}_{G(X,Y,Z) = Y^2Z^{n-2} - (a_0X^n + a_1X^{n-1}Z + \dots + a_nZ^n)} = 0$

$\underbrace{\tilde{C}(\mathbb{C}) \setminus \{Z=0\}}_{C(\mathbb{C})} = \{O = (0:1:0)\}$ unique pt at ∞

In affine coordinates $u = \frac{X}{Y} = \frac{x}{y}$, $v = \frac{Z}{Y} = \frac{1}{y}$

$\tilde{C} \setminus \{Y=0\}: \underbrace{v^{n-2} - (a_0u^n + a_1u^{n-1}v + \dots + a_nv^n)}_{h(u,v)} = 0$

$O \Leftrightarrow u = v = 0$

$\frac{\partial h}{\partial u}(0,0) = 0$, $\frac{\partial h}{\partial v}(0,0) = \begin{cases} 1 & n=3 \\ 0 & n \geq 4 \end{cases}$

$O \in \tilde{C}_{sing} \iff n \geq 4$

Conclusion: $\tilde{C}$ non-singular $\iff n = 3$

the affine curve C is non-singular: $C: g(x,y) = 0$

$g = y^2 - f(x)$, $g'_x = -f'_x$, $g'_y = 2y$

If $g = g'_x = g'_y\,(x_0,y_0) = 0$

$\Rightarrow f(x_0) = f'(x_0) = 0$ — impossible

(f has distinct roots)

Riemann surface of $\sqrt{f(x)}$ – intelligent construction

$$f(x) = a_0 \prod_{k=1}^{2m} (x-\alpha_k) \in \mathbb{C}[x], \quad \alpha_k \in \mathbb{C} \text{ distinct}, \quad m \geq 1.$$
$$= a_0 x^{2m} + \dots + a_{2m} \quad) \qquad \underline{a_0 \neq 0}$$

Recall: $\mathbb{P}^1(\mathbb{C})$ is glued together from two copies of $\mathbb{C}$ with respective coordinates z, z' along

$$\boxed{\begin{matrix} \mathbb{C} & \supset & \mathbb{C}\setminus\{0\} & \overset{\sim}{\longleftrightarrow} & \mathbb{C}\setminus\{0\} \subset \mathbb{C} \\ & & z & \longleftrightarrow & \boxed{z' = \frac{1}{z}} \end{matrix}}$$

(z = local coordinate at 0, $z' = \frac{1}{z}$ = local coordinate at ∞)

affine plane curves:

$$C: y^2 - (a_0 x^{2m} + \dots + a_{2m}) = 0$$
$$C': y'^2 - (a_0 + \dots + a_{2m} x'^{2m}) = 0$$

$$\begin{pmatrix} x = \frac{1}{x'}, & y = \frac{y'}{x'^m} \\ x' = \frac{1}{x}, & y' = \frac{y}{x^m} \end{pmatrix}$$

Glue them together:

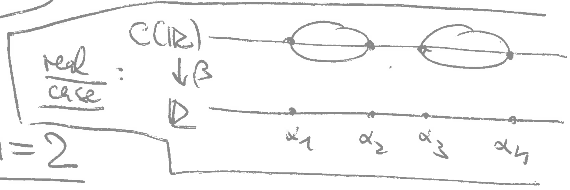
\begin{tikzcd}
 & & S & & \\
C(\mathbb{C}) \arrow[dd, "\beta"'] & C(\mathbb{C})\cap\{x\neq0\} \arrow[l, hook] \arrow[rr, leftrightarrow, "\sim", "{\boxed{x'=\frac{1}{x},\ y'=\frac{y}{x^m}}}"'] & & C'(\mathbb{C})\cap\{x'\neq0\} \arrow[r, hook] & C'(\mathbb{C}) \arrow[dd, "\beta"] \\
 & \mathbb{C}\setminus\{0\} \arrow[ld, hook] \arrow[rr, leftrightarrow, "\sim", "\boxed{x'=\frac{1}{x}}"'] \arrow[rd, "\beta"'] & & \mathbb{C}\setminus\{0\} \arrow[rd, hook] & \\
\mathbb{C} & & \mathbb{P}^1(\mathbb{C}) & & \mathbb{C}
\end{tikzcd}

$(x,y) \in C(\mathbb{C})$, $(x,y) \mapsto x \in \mathbb{C}$

$C'(\mathbb{C}) \ni (x',y')$, $(x',y') \mapsto x'$, $\mathbb{C} \ni x'$

$(0, \pm\sqrt{a_0}) \mapsto x' = 0$; $(0,\pm\sqrt{a_0}) = O_+, O_-$ = 2 pts at ∞ of C

$$\begin{matrix} S & \ni (x,y) \\ \beta \downarrow & \downarrow \\ \mathbb{P}^1(\mathbb{C}) \supset \mathbb{C} \ni & x \end{matrix} \qquad y^2 = f(x), \qquad \underline{\deg(\beta) = 2}$$



$$|\beta^{-1}(P)| = \begin{cases} 1 & P = (\alpha_k, 0) \quad k = 1, \dots, 2m \\ 0 & \text{otherwise} \end{cases}$$

Riemann – Hurwitz formula:

$$2g_S - 2 = 2(2\cdot 0 - 2) + 2m(2-1) = 2m - 4 \Longrightarrow \boxed{g_S = m - 1}$$

Riemann surfaces (terminology)

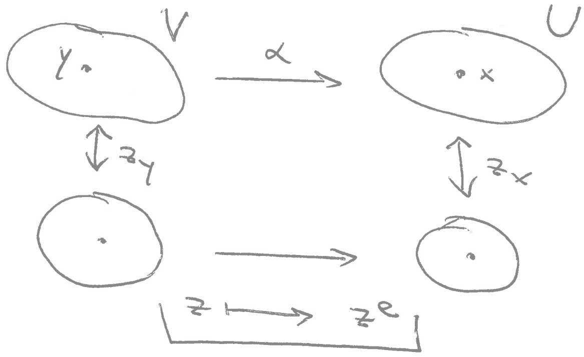
X, Y Riemann surfaces (connected)

- $x \in X$

local coordinate

- $0 \neq f \in \mathcal{M}(U)$ (meromorphic fn) $\quad f = \sum_{n \geq n_0} a_n z_x^n$, $a_{n_0} \neq 0$; $\boxed{\mathrm{ord}_x(f) := n_0}$
- $0 \neq \omega \in \Omega^1_{mer}(U)$ (meromorphic differential) $\boxed{\omega = f\, dz_x}$, $\boxed{\mathrm{ord}_x(\omega) := \mathrm{ord}_x(f)}$
- $Y \xrightarrow{\alpha} X$, $y \in Y \implies \exists$ local coordinates

holomorphic non-const.



$x = \alpha(y)$

$$\boxed{z_x \circ \alpha = z_y^e}$$

$\underline{e = e_y \geq 1}$ ramification index of α at y

$0 \neq f \in \mathcal{M}(U)$
$0 \neq \omega \in \Omega^1_{mer}(U)$ $\quad \underline{\mathrm{ord}_y(f \circ \alpha) = e_y\, \mathrm{ord}_x(f)}$

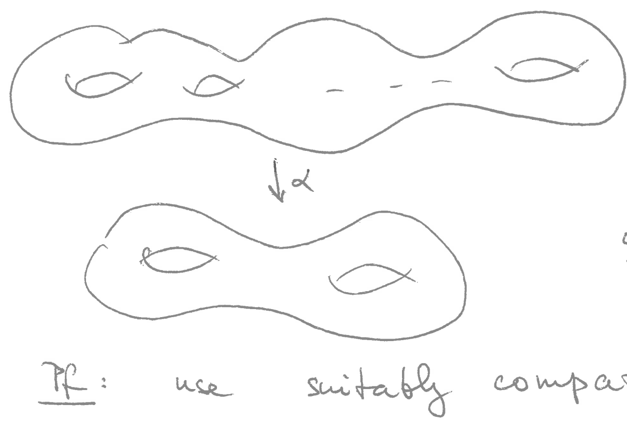
$$\underline{\alpha^*(dz_x) = d(z_y^e) = e\, z_y^{e-1} dz_y \implies \mathrm{ord}_y(\alpha^*\omega) = e_y - 1 + e_y\, \mathrm{ord}_x(\omega)}$$

- if X, Y cpt: $\quad \forall x \in X \quad \sum_{\alpha(y) = x} e_y = \deg(\alpha)$ $\quad$ (degree of α)

- Riemann – Hurwitz formula: if X, Y cpt

$$\boxed{2g_Y - 2 = (2g_X - 2)\deg(\alpha) + \sum_{y \in Y} (e_y - 1)}$$

Y



g_Y holes

X

g_X holes

Pf: use suitably compatible triangulations and the Euler – Poincaré formula $2 - 2g = s_0 - s_1 + s_2$

s_k = number of simplices of dim $= k$ in the given triangulation

Differentials

X – Riemann surface

$\mathrm{Div}(X) = \{ D = \sum_{x\in X} n_x (x) \mid n_x \in \mathbb{Z}$, $\forall K \subset X$ compact $|\{x\in K \mid n_x \neq 0\}| < \infty \}$ ("locally finite sums")

$0 \neq f \in \mathcal{M}(X)$ $\quad \mathrm{div}(f) = \sum_{x\in X} \mathrm{ord}_x(f)(x) \in \mathrm{Div}(X)$

$0 \neq \omega \in \Omega^1_{mer}(X)$ $\quad \mathrm{div}(\omega) = \sum_{x\in X} \mathrm{ord}_x(\omega)(x) \in \mathrm{Div}(X)$

$f \in \mathcal{O}(X)$ (holomorphic) $\iff \mathrm{div}(f) \geq 0$ $\quad (\forall x\in X \;\; n_x \geq 0)$

$\omega \in \Omega^1(X)$ (—"—) $\iff \mathrm{div}(\omega) \geq 0$

- if X is compact: (a) $\mathrm{Div}(X) = \{ D = \sum_{x\in X} n_x(x) \mid n_x \in \mathbb{Z}$, the sum is finite $\}$

  (b) $0 \neq f \in \mathcal{M}(X)$ $\quad \deg(\mathrm{div}(f)) = 0$ $\quad (\deg(D) = \sum n_x)$

Ex: $X = \mathbb{C}/L$ (c) $\mathcal{O}(X) = \mathbb{C}$, (d) $\mathrm{div}(f) \geq 0 \iff f \in \mathbb{C}^\times \iff \mathrm{div}(f) = 0$.

Ex: $X = \mathbb{P}^1(\mathbb{C})$, $\omega = dz$; $\forall a \in \mathbb{C}$ $\quad \omega = 1\cdot d\underbrace{(z-a)} \implies \mathrm{ord}_a(dz) = 0$

(local coordinate at a)

at $\infty \in \mathbb{P}^1(\mathbb{C})$ local coordinate $z_\infty = \frac{1}{z}$

$$dz = -\frac{dz_\infty}{z_\infty^2} \implies \mathrm{ord}_\infty(dz) = -2, \quad \mathrm{div}(dz) = -2(\infty),\ \deg(\mathrm{div}(dz)) = -2$$

Ex: $C: g(x,y) = 0$, $g \in \mathbb{C}[x,y]$ non-constant, irreducible

$X = (C \setminus C_{sing})(\mathbb{C})$ is a Riemann surface

$$\boxed{\omega = \frac{dx}{g'_y} = -\frac{dy}{g'_x} \in \Omega^1_{mer}(X)} \qquad (0 = g'_x\, dx + g'_y\, dy \text{ on } X)$$

$x, y \in \mathcal{M}(X)$

Fact: $\mathrm{div}(\omega) = 0$ on X $\quad (\implies \omega \in \Omega^1(X)$ holomorphic on $X)$

Pf: for $P = (x_P, y_P) \in (C \setminus C_{sing})(\mathbb{C})$,

if $g'_x(P) \neq 0 \implies y - y_P$ loc. coord. at P, $\quad \omega = \dfrac{-d(y-y_P)}{g'_x}$

$\implies \mathrm{ord}_P(\omega) = 0$.

if $g'_y(P) \neq 0 \implies x - x_P$ loc. coord. at P, $\quad \omega = \dfrac{d(x-x_P)}{g'_y} \implies \mathrm{ord}_P(\omega) = 0$.

Hyperelliptic affine curves: $C: y^2 = f(x)$, $f \in \mathbb{C}[x]$, $\deg(f) = 2m$ with distinct roots $(m \geq 1)$

$$\omega = \frac{dx}{y} = \frac{2\,dy}{f'(x)}$$

$g(x,y) = y^2 - f(x) \implies 2y\,dy = f'(x)\,dx$ on $C(\mathbb{C})$

$$\boxed{\mathrm{div}\left(\frac{dx}{y}\right) = 0 \text{ on } C(\mathbb{C})}$$

C non-singular

Hyperelliptic Riemann surfaces again

$S \supset C(\mathbb{C})$, $C: y^2 = f(x)$, $\deg(f) = 2m$, $m \geq 1$
distinct roots

$\omega = \frac{dx}{y} = \frac{2dy}{f'(x)} \in \Omega^1_{mer}(S) \cap \mathcal{O}(C(\mathbb{C}))$

$f = a_0 x^{2m} + \cdots + a_{2m}$, $a_0 \neq 0$

$S \setminus C(\mathbb{C})$: in coordinates $x' = 1/x$, $y' = y/x^m$,

$S \setminus \{\underbrace{(0, \pm\sqrt{a_{2m}})}_{P_+, P_-}\} = C'(\mathbb{C})$, $C': y'^2 = a_0 + a_1 x' + \cdots + a_{2m} x'^{2m}$

(distinct if $f(0) \neq 0$)

$S \setminus C(\mathbb{C}) = \{O_+, O_-\}$

$O_+, O_- \longleftrightarrow (x', y') = (0, \pm\sqrt{a_0})$
(distinct) $a_0 \neq 0$

local coordinate at $O_\pm$: $x' = \frac{1}{x}$, $y'(O_\pm) \neq 0$

$\mathrm{div}(x) = (P_+) + (P_-) - (O_+) - (O_-)$

$\boxed{\omega = \frac{dx}{y}} = \frac{d(x'^{-1})}{y' x'^{-m}} = -\frac{x'^{m-2}}{y'} dx' \Longrightarrow \mathrm{ord}_{O_\pm}(\omega) = m-2$

Conclusion : $\mathrm{div}\left(\frac{dx}{y}\right) = (m-2)(O_+) + (m-2)(O_-)$

$\forall k \in \mathbb{Z}$ $\mathrm{div}\left(\frac{x^k dx}{y}\right) = k(P_+) + k(P_-) + (m-2-k)(O_+) + (m-2-k)(O_-)$

Cor : $\frac{x^k dx}{y} \in \Omega^1(S)$ (holomorphic differential) $\Longleftrightarrow k = 0, 1, \dots, m-2$

Exercise : $\boxed{\Omega^1(S) = \bigoplus_{k=0}^{m-2} \mathbb{C} \cdot \frac{x^k dx}{y}}$ $(\Longrightarrow \dim \Omega^1(S) = m-1 = g(S))$

Rmk : if $m = 2$ ($\Longleftrightarrow \deg(f) = 4$), then $\mathrm{div}(\omega) = 0$.

If $0 \neq \eta \in \Omega^1_{mer}(S)$, then $\eta = f\omega$, $0 \neq f \in \mathcal{M}(X)$

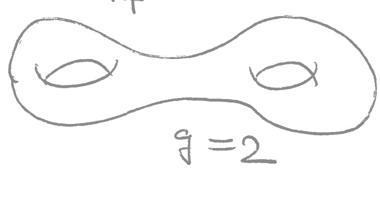
$\eta \in \Omega^1(S) \Longleftrightarrow 0 \leq \mathrm{div}(\eta) = \mathrm{div}(f) + \underbrace{\mathrm{div}(\omega)}_{0} \Longleftrightarrow f \in \mathbb{C}^\times$

So $\Omega^1(S) = \mathbb{C} \cdot \frac{dx}{y}$ if $m = 2$ $(\Longleftrightarrow g(S) = 1)$

General fact : X compact Riemann surface

$$\underbrace{\dim \Omega^1(X)}_{\text{"analytic genus" } g_{an}(X)} = \underbrace{\text{genus of } X}_{\text{topological genus } g_{top}(X) = \text{the number of holes}}$$

($g_{an} \leq g_{top}$ is easy)



$g = 2$

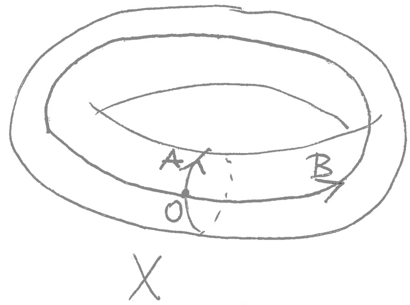
The Abel–Jacobi map

Toy model: $\omega = \frac{dz}{z} \in \mathcal{O}(\mathbb{C}\setminus\{0\})$, $\quad \mathbb{C}\setminus\{0\} \xrightarrow{\log} \mathbb{C}/2\pi i\mathbb{Z}$

$$\int_\gamma \frac{dz}{z} = 2\pi i \qquad\qquad \begin{array}{ccc} \psi & & \psi \\ a & \longmapsto & \displaystyle\int_1^a \frac{dz}{z} \end{array} \ \Big(\text{mod } \underbrace{\text{periods of } \tfrac{dz}{z}}_{2\pi i\mathbb{Z}}\Big)$$

The case $g=1$

Data: $X=$ compact Riemann surface of (topological) genus $g=1$
$0 \neq \omega \in \Omega^1(X)$ holomorphic differential $\quad (\Rightarrow \Omega^1(X) = \mathbb{C}\omega$; see below$)$



Fix closed cycles A, B on X as in the picture:

$$H_1(X,\mathbb{Z}) = \mathbb{Z}\cdot\underbrace{[A]}_{\text{homology class of } A} \oplus \mathbb{Z}[B]$$

orientation: intersection pairing

$$I : H_1(X,\mathbb{Z}) \times H_1(X,\mathbb{Z}) \longrightarrow \mathbb{Z}$$

$$I(A,B) = -1 \quad (\Longleftrightarrow I(B,A) = 1)$$

Periods of ω: $L = \left\{ \int_\gamma \omega \;\middle|\; \gamma \text{ closed path} \right\} = \left\{ m\int_A \omega + n\int_B \omega \;\middle|\; m,n\in\mathbb{Z} \right\}$

$[\gamma] = m[A] + n[B] \quad (m,n\in\mathbb{Z})$

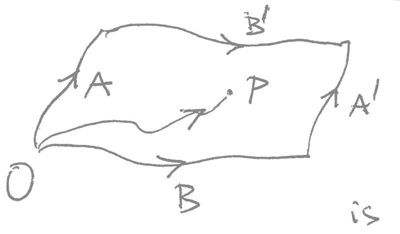
$$= \mathbb{Z}\Big(\int_A \omega\Big) + \mathbb{Z}\Big(\int_B \omega\Big)$$

Note: $\int_\gamma \omega$ depends only on the homology class $[\gamma]$ of γ:

locally $\omega = f(z)\,dz$, f holomorphic $\Rightarrow d\omega = \frac{\partial f}{\partial z}\underbrace{dz\wedge dz}_{=0} + \underbrace{\frac{\partial f}{\partial \bar z}}_{=0 \text{ (Cauchy–Riemann)}} d\bar z\wedge dz = 0$
$\left(\frac{\partial f}{\partial \bar z} = 0\right)$

so if $\gamma = \partial\Delta$, then $\displaystyle\int_\gamma \omega = \int_{\partial\Delta} \omega \overset{\text{Stokes}}{=} \int_\Delta d\omega = 0.$

Cut X along A and B: $\quad U = X\setminus(A\cup B)$ is simply connected



On X, A' is identified with A
 B' — " — B

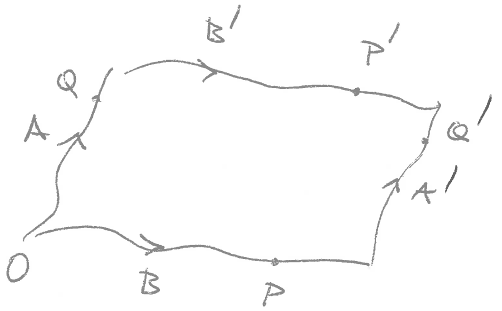
For $P\in U$, $\displaystyle f(P) := \int_O^P \omega = \int_\gamma \omega \quad$ γ any path from O to P

is well-defined, $\quad f\in\mathcal{O}(U)$, $\quad df = \omega|_U$

$\Rightarrow \bar\partial f = 0$, $\quad \omega = \partial f = \frac{\partial f}{\partial z}dz$, $\quad \bar\omega = \overline{\left(\frac{\partial f}{\partial z}\right)} d\bar z$, $\quad i\,\omega\wedge\bar\omega = \left|\frac{\partial f}{\partial z}\right|^2 \underbrace{i\,dz\wedge d\bar z}_{2dx\wedge dy}$, $z = x+iy$

$$\Rightarrow i\int_X \omega\wedge\bar\omega = i\int_U \omega\wedge\bar\omega > 0$$

Note: $d(f\bar{\omega}) = df \wedge \bar{\omega} + f \underbrace{d\omega}_{=0} = \omega \wedge \bar{\omega} \overset{\text{Stokes}}{\Longrightarrow} i \int_{\partial U} f\bar{\omega} = i \int_U \omega \wedge \bar{\omega} > 0$



$$f(P) = \int_O^P \omega$$

$$f(P') - f(P) = \int_A \omega =: \omega_A$$

$$f(Q') - f(Q) = \int_B \omega =: \omega_B$$

$\Rightarrow \left(\int_{A'} - \int_A\right)(f\bar{\omega}) = \left(\int_B \omega\right)\left(\int_A \bar{\omega}\right) = \omega_B \overline{\omega_A}$, $\left(\int_B - \int_{B'}\right)(f\bar{\omega}) = -\left(\int_A \omega\right)\left(\int_B \bar{\omega}\right) = -\omega_A \overline{\omega_B}$

$$\Rightarrow 0 < \frac{i}{2}\left(\int_B - \int_{B'} + \int_{A'} - \int_A\right) f\bar{\omega} = \frac{1}{2i} \begin{vmatrix} \int_A \omega & \int_A \bar{\omega} \\ \int_B \omega & \int_B \bar{\omega} \end{vmatrix} = \mathrm{Im}(\omega_A \overline{\omega_B})$$

Cor: $L = \mathbb{Z}\omega_A + \mathbb{Z}\omega_B \subset \mathbb{C}$ is a lattice and $\mathrm{Im}\left(\frac{\omega_A}{\omega_B}\right) > 0$.

Cor: For any fixed $O \in X$, the Abel-Jacobi map

$$\begin{aligned} X &\xrightarrow{\alpha} \mathbb{C}/L \\ P &\longmapsto \int_O^P \omega \pmod{L} \end{aligned}$$

is well-defined (independent of a chosen path of integration $O \longrightarrow P$) and holomorphic.

By definition, $\alpha^*(dz) = \omega$.

Thm. α is an isomorphism of compact Riemann surfaces.

Pf: X compact $\Rightarrow$ α proper.

Riemann-Hurwitz formula: $\underbrace{2g(X)-2}_{0} = \deg(\alpha)\big(\underbrace{2g(\mathbb{C}/L)-2}_{0}\big) + \sum_{x \in X}(e_x - 1)$

$\Rightarrow \forall x \in X \quad e_x = 1 \Rightarrow \alpha$ is unramified everywhere.

α proper $\Rightarrow$ α is an unramified covering. It corresponds to the map

$$\begin{array}{ccc} \pi_1(X, O) & \xrightarrow{\alpha_*} & \pi_1(\mathbb{C}/L, 0) \\ \| & & \| \\ H_1(X, \mathbb{Z}) & \xrightarrow{\alpha_*} & H_1(\mathbb{C}/L, \mathbb{Z}) \end{array}$$

given by $\int_{\alpha_*[\gamma]} dz = \int_{[\gamma]} \alpha^*(dz) = \int_{[\gamma]} \omega$.

As $\left\{ \int_{[\gamma]} \omega \mid [\gamma] \in H_1(X, \mathbb{Z}) \right\} = L = \left\{ \int_{[\gamma']} dz \mid [\gamma'] \in H_1(\mathbb{C}/L, \mathbb{Z}) \right\}$ $\Rightarrow$ α_* surjective $\Downarrow$ α bijective.

Invariants of lattices $L \subset \mathbb{C}$

$L \subset \mathbb{C}$ lattice, $g_2(L) = 60\, G_4(L)$, $g_3(L) = 140\, G_6(L)$

$G_k(L) = \sum_{0 \neq u \in L} u^{-k}$ $(k > 2,\ 2 | k)$, $\Delta(L) = g_2^3(L) - 27 g_3^2(L) \neq 0$

$$J(L) = \frac{g_2^3(L)}{\Delta(L)}$$

__Prop.__ $\{L \subset \mathbb{C} \text{ lattice}\} \xrightarrow{(g_2, g_3)} \{(g_2, g_3) \in \mathbb{C}^2 \mid g_2^3 - 27 g_3^2 \neq 0\}$

est une __bijection.__

__Proof.__ Given $g_2, g_3 \in \mathbb{C}$ with $g_2^3 - 27 g_3^2 \neq 0$, the Abel-Jacobi map for $E : Y^2 Z = 4X^3 - g_2 X Z^2 - g_3 Z^3$, $O = (0:1:0)$ and $\omega = \frac{dx}{y}$ $(x = X/Z,\ y = Y/Z)$ is a holomorphic isomorphism

$$\alpha : E(\mathbb{C}) \xrightarrow{\sim} \mathbb{C}/L(E), \qquad L(E) = \left\{ \int_\gamma \omega \;\middle|\; \gamma \in H_1(E(\mathbb{C}), \mathbb{Z}) \right\}$$

$$P \longmapsto \int_O^P \omega \pmod{L(E)}$$

Weierstrass isomorphism: function $\wp(z) = \wp(z, L(E))$ defines a holomorphic

$$\beta : \mathbb{C}/L(E) \xrightarrow{\sim} \tilde{E}(\mathbb{C}) \quad : \tilde{Y}^2 \tilde{Z} = 4\tilde{X}^3 - \tilde{g}_2 \tilde{X} \tilde{Z}^2 - \tilde{g}_3 \tilde{Z}^3,$$

$$z \pmod{L(E)} \longmapsto (\wp(z) : \wp'(z) : 1) \qquad \tilde{x} = \frac{\tilde{X}}{\tilde{Z}},\ \tilde{y} = \frac{\tilde{Y}}{\tilde{Z}}$$

$$0 \longmapsto \tilde{O}$$

$\tilde{g}_2 = g_2(L(E))$, $\tilde{g}_3 = g_3(L(E))$. Under these isomorphisms,

$$\mathbb{C}(x, y) \xrightarrow{\sim} \mathcal{M}(\mathbb{C}/L(E)) = \mathbb{C}(\wp(z), \wp'(z)) \xrightarrow{\sim} \mathbb{C}(\tilde{x}, \tilde{y})$$

$y^2 = 4x^3 - g_2 x - g_3$ $\qquad\qquad$ $\tilde{y}^2 = 4\tilde{x}^3 - \tilde{g}_2 \tilde{x} - \tilde{g}_3$

$$x \longleftrightarrow \wp(z) \longleftrightarrow \tilde{x}$$

$$y \longleftrightarrow \wp'(z) \longleftrightarrow \tilde{y}$$

these are isomorphisms of $\mathbb{C}$-algebras, and so $\tilde{y}^2 = 4\tilde{x}^3 - \tilde{g}_2 \tilde{x} - \tilde{g}_3 \Rightarrow y^2 = 4x^3 - \tilde{g}_2 x - \tilde{g}_3$. But $y^2 = 4x^3 - g_2 x - g_3$ for all $(x, y) \in E(\mathbb{C}) \Rightarrow \tilde{g}_2 = g_2,\ \tilde{g}_3 = g_3 \Rightarrow \tilde{E} = E$ and $\beta = \alpha^{-1}$.

Therefore (g_2, g_3) is __surjective.__

Moreover, given $L \subset \mathbb{C}$ lattice, then we have

$E_L : Y^2 Z = 4X^3 - g_2(L) X Z^2 - g_3(L) Z^3$ and $\mathbb{C}/L \underset{\text{Abel-Jacobi}}{\overset{(\wp, \wp')}{\xrightarrow{\sim}}} E_L(\mathbb{C})$,

then $L = \left\{ \int_\gamma dz \mid \gamma \in H_1(\mathbb{C}/L, \mathbb{Z}) \right\}$ $\qquad$ $dz \longleftrightarrow dx/y$

$$= \left\{ \int_{\gamma_L} \frac{dx}{y} \;\middle|\; \gamma_L \in H_1(E_L(\mathbb{C}), \mathbb{Z}) \right\} = L(E_L) \Rightarrow (g_2, g_3) \text{ is } \underline{\text{injective}}.$$

Application to the modular invariant $j = 12^3 J$, $J = \dfrac{g_2^3}{g_2^3 - 27 g_3^2}$

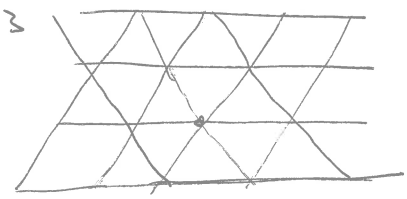
Recall: $t \in \mathbb{C}^\times \Rightarrow g_2(tL) = t^{-4} g_2(L)$, $g_3(L) = t^{-6} g_3(L)$

$\Rightarrow$ $J(tL) = J(L)$, so $J : \{\text{lattices } L \subset \mathbb{C}\}/\mathbb{C}^\times \longrightarrow \mathbb{C}$

Square lattice: $L = t\cdot(\mathbb{Z}i + \mathbb{Z}) = iL \Rightarrow g_3(L) = 0 \Rightarrow J(L) = 1$.

Honeycomb lattice: $L = t\cdot(\mathbb{Z}\rho + \mathbb{Z}) = \rho L \Rightarrow g_2(L) = 0 \Rightarrow J(L) = 0$.

$\rho = e^{2\pi i/3}$



Thm: $J : \{\text{lattices } L \subset \mathbb{C}\}/\mathbb{C}^\times \longrightarrow \mathbb{C}$ is bijective.

Pf: Surjectivity: if $J \in \mathbb{C} \setminus \{0,1\}$, let $g_2 = g_3 := g$, where $g = \dfrac{27J}{J-1}$

$\Rightarrow J = g_2^3/(g_2^3 - 27 g_3^2)$ and $g_2^3 - 27 g_3^2 \neq 0$. We know that (by Prop. above)

$\exists L \subset \mathbb{C}$ lattice s.t. $g_k = g_k(L)$ $(k = 2,3) \Rightarrow J = J(L)$.

Injectivity: if $J(L) = J(L') \Rightarrow \exists t \in \mathbb{C}^\times$ $g_2(L') = t^{-4} g_2(L)$, $g_3(L') = t^{-6} g_3(L)$

replace L by tL $\Rightarrow$ can assume $g_k(L') = g_k(L) =: g_k$ $(k = 2,3)$.

We know that $L = \{ \int_\gamma \omega \mid \gamma \in H_1(X, \mathbb{Z}) \}$ depends only on g_2, g_3)

where $X = \overset{\text{cpt}}{\text{Riemann}}$ surface attached to $y^2 = 4x^3 - g_2 x - g_3$, $\omega = \dfrac{dx}{y}$

$\Rightarrow L' = L$.

Rmk: the set $\{\text{lattices } L \subset \mathbb{C}\}/\mathbb{C}^\times$ has the following equivalent descriptions:

(1) $SL_2(\mathbb{Z}) \backslash \mathcal{H}$: $\tau \longleftrightarrow \mathbb{Z}\tau + \mathbb{Z} = L_\tau$

(2) $\{\mathbb{C}/L\}$ / (holomorphic isomorphisms preserving 0)

(3) $\{(X, O) \mid X$ cpt Riemann surface, $g(X) = 1$, $O \in X\}$ / $\left(\begin{array}{l}\text{holomorphic isom.}\\ \text{preserving } O\end{array}\right)$

(via $X \xrightarrow[\sim]{\alpha} \mathbb{C}/L$, $P \longmapsto \int_0^P \omega$ for some $0 \neq \omega \in \underbrace{\Omega^1(X)}_{\dim = 1 \text{ over } \mathbb{C}}$)

(4) $\{(E, O) \mid E$ smooth projective curve over $\mathbb{C}$, $g(E) = 1$, $O \in E(\mathbb{C})\}$ / $\left(\begin{array}{l}\text{algebraic isom.}\\ \text{preserving } O\end{array}\right)$

(via $X = E(\mathbb{C})$)

Exercise. The map

$(g_2, g_3) : \{\text{lattices } L \subset \mathbb{C}\} \longrightarrow \{(g_2, g_3) \in \mathbb{C}^2 \mid g_2^3 - 27 g_3^2 \neq 0\}$ is bijective.

$g_2(L) = 60\, G_4(L)$, $g_3(L) = 140\, G_6(L)$.

Applications to the function λ

Let $L \subset \mathbb{C}$ be a lattice.

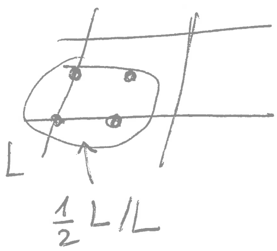
Def. For an integer $N \geq 1$, the full level N structure on $\mathbb{C}/L$ (or on L) is an isomorphism of abelian groups

$$(\mathbb{Z}/N\mathbb{Z})^2 \xrightarrow{\sim} \underbrace{(\mathbb{C}/L)[N]}_{\{x \in \mathbb{C}/L \mid Nx = 0\}} = \tfrac{1}{N}L/L.$$

Weierstrass map: $(\wp, \wp') : \mathbb{C}/L \xrightarrow{\sim} \tilde{E}(\mathbb{C}) \xrightarrow{p} \mathbb{P}^1(\mathbb{C})$ $\quad p(x,y) = x$

$\tilde{E} \setminus \{0\}$ = affine curve $y^2 = f(x) = 4x^3 - g_2 x - g_3 = 4\prod_{j=1}^{3}(x - e_j)$.

We know: $\{e_1, e_2, e_3\} = \{\wp(u) \mid u \in \tfrac{1}{2}L/L \setminus \{0\}\}$



So $\tfrac{1}{2}L/L \longleftrightarrow \{0\} \cup \{(e_j, 0) \mid j = 1, 2, 3\}$

Cor: full level 2 structure on $\mathbb{C}/L$

$\Updownarrow$

choice of an ordering of the roots of $f(x)$

Def: $\lambda(L, \text{full level 2 structure}) :=$ cross-ratio $r(e_1, e_2, e_3, \infty)$

$$= \frac{e_1 - e_3}{e_2 - e_3} \in \mathbb{C} \setminus \{0, 1\}$$

= cross-ratio (ordered ramification points of $p : \tilde{E}(\mathbb{C}) \longrightarrow \mathbb{P}^1(\mathbb{C})$)

Note: $e_j = \wp\left(\frac{\omega_j}{2}\right)$ for some basis $\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 = L$, $\operatorname{Im}\left(\frac{\omega_1}{\omega_2}\right) > 0$

$\omega_3 = \omega_1 + \omega_2$

Classical normalisation is different ($\omega_1 \longleftrightarrow \omega_2$, I think)

Change of 2-level structure: action of $SL_2(\mathbb{Z})/\Gamma(2) \xrightarrow{\sim} SL_2(\mathbb{Z}/2\mathbb{Z})$

$\Gamma(2) = \{g \in SL_2(\mathbb{Z}) \mid g \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod 2\}$

$\Updownarrow$

action of S_3 on $\{e_1, e_2, e_3\}$

$SL_2(\mathbb{Z}/2\mathbb{Z}) \xrightarrow{\sim} S_3$ (action on $(\mathbb{Z}/2\mathbb{Z})^2 \setminus \{(0,0)\}$)

$$\begin{tikzcd}
\Gamma(2)\backslash\mathcal{H} \arrow[r, leftrightarrow, "\text{bijection}"] \arrow[d] & \{(L \subset \mathbb{C} \text{ lattice, full 2-level structure on } \mathbb{C}/L)\}/\mathbb{C}^\times \arrow[d] \arrow[rd, "\lambda"] & \\
SL_2(\mathbb{Z})\backslash\mathcal{H} \arrow[r, leftrightarrow, "\text{bijection}"] & \{L \subset \mathbb{C} \text{ lattice}\}/\mathbb{C}^\times \arrow[rd, "j"] & \mathbb{C} \setminus \{0,1\} \\
& & \mathbb{C}
\end{tikzcd}$$

Top arrow: canonical 2-level structure on $L_\tau = \mathbb{Z}\tau + \mathbb{Z}$ given by $\omega_1 = \tau$, $\omega_2 = 1$.

$\left(e_1 = \wp\left(\frac{\tau}{2}\right), e_2 = \wp\left(\frac{1}{2}\right), e_3 = \wp\left(\frac{\tau+1}{2}\right)\right)$

Exercise: Show that λ is a bijection. Express j in terms of λ.

[Hint: Legendre family $y^2 = x(x-1)(x-\lambda)$ gives a section of λ]

Differentials of the first (resp. second) kind

$X =$ compact Riemann surface

Def: $\{$differentials of the 1^{st} kind on $X\} := \Omega^1(X)$ (holomorphic diff.)

$\{$ ——— " ——— 2^{nd} — " — $\} := \Omega^1_{mer}(X)^{res=0}$ (contains $\Omega^1(X)$)

$= \{\omega \in \Omega^1_{mer}(X) \mid \forall x \in X \ \ res_x(\omega) = 0\}$

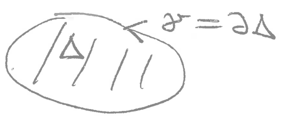
Recall: if  is a local coordinate at $x \in X$

$\omega = \left(\sum_{n \geq n_0} a_n z_x^n\right) dz_x$

$res_x(\omega) := a_{-1}$ (independent of z_x)

Note: $f \in \mathcal{M}(X) \implies res_x(df) = 0$. **Cor:** $d\mathcal{M}(X) \subset \Omega^1_{mer}(X)^{res=0}$

Periods:



if $\omega \in \Omega^1_{mer}(X)$, $\gamma = \partial\Delta$ path avoiding the singularities of ω

$$\implies \int_{\partial\Delta} \omega = 2\pi i \sum_{x \in \Delta} res_x(\omega)$$

Cor: if $\omega \in \Omega^1_{mer}(X)^{res=0}$ is of 2^{nd} kind, if  γ $(\partial\gamma = 0)$ is a closed path avoiding the singularities of ω, then the period $\int_\gamma \omega$ depends only on the homology class $[\gamma] \in H_1(X, \mathbb{Z})$.

Note: if $\omega = df$ $(f \in \mathcal{M}(X))$ γ closed path $(\partial\gamma = 0)$ $\implies \int_\gamma \omega = \int_{\partial\gamma} f = 0$.

Cor: the periods define a map

$$(*) \qquad H_1(X, \mathbb{Z}) \times \left(\Omega^1_{mer}(X)^{res=0} / d\mathcal{M}(X)\right) \longrightarrow \mathbb{C}$$

$$[\gamma] \ , \ \omega \longmapsto \int_\gamma \omega$$

Exercise:

$$\Omega^1_{mer}(X)^{res=0} / d\mathcal{M}(X) = \begin{cases} 0, & \text{if } X = \mathbb{P}^1(\mathbb{C}) \\ \mathbb{C} \cdot dz + \mathbb{C} \cdot \wp(z) dz, & \text{if } X = \mathbb{C}/L. \end{cases}$$

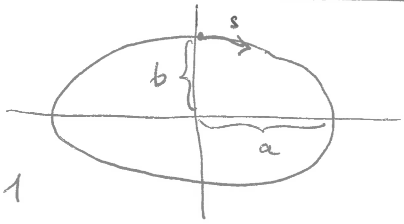
Cor: the matrix of $(*)$ for $X = \mathbb{C}/L$ in the bases ω_1, ω_2 (of L) and $dz, -\wp(z)dz$ (of $\Omega^1_{mer}(\mathbb{C}/L)^{res=0}/d\mathcal{M}(\mathbb{C}/L)$) is $\begin{pmatrix} \omega_1 & \eta_1 \\ \omega_2 & \eta_2 \end{pmatrix}$.

Periods in families and differential equations

Ex: ellipse

$a > b$

$0 < 1 - \frac{b^2}{a^2} = k^2 = \lambda < 1$



$x = a\sin(t)$, $y = b\cos(t)$

$ds^2 = dx^2 + dy^2 = (a^2\cos^2(t) + b^2\sin^2(t))dt^2$

$= a^2(1 - \lambda\sin^2(t))\, dt^2$

$s = a\int_0^t \sqrt{1-\lambda\sin^2(t)}\, dt$

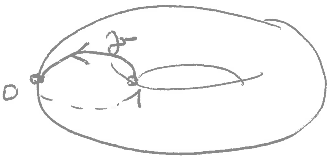
Length of the ellipse: $4a \underbrace{\int_0^{\pi/2} \sqrt{1-\lambda\sin^2(t)}\, dt}_{E(k)}$

$(\lambda = k^2)$

Change of variables: $x = \sin^2(t)$, $1-x = \cos^2(t)$, $dx = 2\sqrt{x(1-x)}\, dt$

$f(x) = f_\lambda(x) := x(1-x)(1-\lambda x)$, C_λ: $y^2 = x(1-x)(1-\lambda x)$ ($\cup$ pt ∞)

$$E(k) = \frac{1}{2}\underbrace{\int_0^1 \frac{(1-\lambda x)\,dx}{\sqrt{f_\lambda(x)}}}_{} = \frac{1}{4}\int_\gamma \frac{(1-\lambda x)dx}{y}$$



$C_\lambda(\mathbb{C})$

differential of 2^{nd} kind on $C_\lambda(\mathbb{C})$

differential of 1^{st} kind on $C_\lambda(\mathbb{C})$: $\frac{dx}{y}$

$$K(k) := \frac{1}{2}\int_0^1 \frac{dx}{\sqrt{f_\lambda(x)}} = \frac{1}{4}\int_\gamma \frac{dx}{y}$$

Power series expansions: $(\lambda = k^2)$

$$4K(k) = \int_\gamma \frac{dx}{y} = 2\int_0^1 \frac{dx}{\sqrt{x(1-x)(1-\lambda x)}} = 4\int_0^{\pi/2} \frac{dt}{\sqrt{1-\lambda\sin^2(t)}} =$$

$$= 4\sum_{n=0}^\infty \binom{-1/2}{n}(-\lambda)^n \underbrace{\int_0^{\pi/2} \sin^{2n}(t)\,dt}_{\frac{\pi}{2}(-1)^n\binom{-1/2}{n}} = 2\pi\sum_{n=0}^\infty \binom{-1/2}{n}^2 \lambda^n$$

$$4E(k) = \int_\gamma \frac{(1-\lambda x)\,dx}{y} = 2\int_0^1 \frac{(1-\lambda x)\,dx}{\sqrt{x(1-x)(1-\lambda x)}} = 4\int_0^{\pi/2} \sqrt{1-\lambda\sin^2(t)}\,dt =$$

$$= 4\sum_{n=0}^\infty \binom{1/2}{n}(-\lambda)^n \cdot \frac{\pi}{2}(-1)^n\binom{-1/2}{n} = 2\pi\sum_{n=0}^\infty \binom{-1/2}{n}\binom{1/2}{n}\lambda^n$$

Notation: $(a)_n := a(a+1)\cdots(a+n-1)$; $(-1)^n\binom{-1/2}{n} = \frac{(1/2)_n}{n!} = \frac{(1/2)_n}{(1)_n}$

$(-1)^n\binom{1/2}{n} = \frac{(-1/2)_n}{n!}$

$$\boxed{4K(k) = 2\pi\sum_{n\geq 0} \frac{(1/2)_n(1/2)_n}{(1)_n}\frac{\lambda^n}{n!}}$$

$$\boxed{4E(k) = 2\pi\sum_{n\geq 0} \frac{(-1/2)_n(1/2)_n}{(1)_n}\frac{\lambda^n}{n!}}$$

Hypergeometric series

General principle: if $f, g \in \mathbb{C}[z]$, then the formal power series

$$\boxed{F(z) := 1 + \frac{f(0)}{g(1)} z + \frac{f(0)f(1)}{g(1)g(2)} z^2 + \cdots}$$ (if defined) satisfies

$$z f\left(z\frac{d}{dz}\right) : F(z) \longmapsto f(0) z + \frac{f(0)f(1)}{g(1)} z^2 + \frac{f(0)f(1)f(2)}{g(1)g(2)} z^3 + \cdots$$

(since $f\left(z\frac{d}{dz}\right) : z^n \longmapsto f(n) z^n$) and

$$g\left(z\frac{d}{dz}\right) : F(z) \longmapsto g(0) + f(0) z + \frac{f(0)f(1)}{g(1)} z^2 + \cdots,$$ and so:

Prop. If $f, g \in \mathbb{C}[z]$ and $g(0)=0$, then $(z f(zD) - g(zD))F(z) = 0$

$\boxed{D = \frac{d}{dz}}$ $\begin{cases} \text{if } \deg(f) < \deg(g) \implies F \text{ converges in } \mathbb{C} \\ \text{if } \deg(f) = \deg(g) \implies \quad \text{in some } \{|z| < R\} \end{cases}$

Factorise f and g: $f(z) = \prod_{j=1}^{p} (z + a_j)$, $g(z) = z \prod_{k=1}^{q} (z + b_k - 1)$, $p \le q+1$

$$\boxed{F(z) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{z^n}{n!} =: {}_pF_q(\underline{a}, \underline{b}; z)}$$

$(a)_n = a(a+1)\cdots(a+n-1)$

$$\boxed{L_{\underline{a},\underline{b}} F = 0, \quad L_{\underline{a},\underline{b}} = \prod_{j=1}^{p} (zD + a_j) - D \prod_{k=1}^{q} (zD + b_k - 1), \quad \boxed{D = \frac{d}{dz}}}$$

Ex: (0) ${}_0F_0(-,-;z) = \sum_{n\ge0} \frac{z^n}{n!} = e^z$, (1) ${}_1F_0(a,-;z) = \sum_{n\ge0} \binom{-a}{n}(-z)^n = (1-z)^{-a}$

(2) ${}_1F_1(a,c;z) = \sum_{n\ge0} \frac{(a)_n}{(c)_n} \frac{z^n}{n!}$ satisfies $(zD^2 + (c-z)D - a)F = 0$

(3) ${}_2F_1(a,b;c;z) = \sum_{n\ge0} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}$ satisfies $(z(z-1)D^2 + ((a+b+1)z - c)D + ab)F = 0$

Cor: $\int_\gamma \frac{dx}{y} = 4K(k) = (2\pi)\, {}_2F_1(\tfrac12, \tfrac12; 1; \lambda)$ satisfies $(\lambda(\lambda-1)D^2 + (2\lambda-1)D + \tfrac14)F = 0$

$D = d/d\lambda$ (Gauss)

$\int_\gamma \frac{(1-\lambda x)dx}{y} = 4E(k) = (2\pi)\, {}_2F_1(-\tfrac12, \tfrac12; 1; \lambda)$ satisfies $(\lambda(\lambda-1)D^2 + (\lambda-1)D - \tfrac14)F = 0$

Contiguity relation: $(a+1)_n - (a)_n = n\,(a+1)_{n-1} \implies$

$$\frac{{}_2F_1(a+1,b;c;z) - {}_2F_1(a,b;c;z)}{z} = (b/c)\, {}_2F_1(a+1, b+1; c+1; z)$$

Cor: $\int_\gamma \frac{x\,dx}{y} = \frac{4(K(k) - E(k))}{\lambda} = \pi\, {}_2F_1(\tfrac12, \tfrac32; 2; \lambda)$

satisfies $(\lambda(\lambda-1)D^2 + (3\lambda-2)D + \tfrac34)F = 0$

Differential equation for periods — general case

Data: • family of non-singular projective curves C_λ depending analytically on a parameter $\lambda \in U \overset{\text{open}}{\subseteq} \mathbb{C}$:

$C_\lambda \setminus \{O\} : y^2 = f(x)$, $\quad f(x) \in \mathbb{C}[x]$, $\deg(f) = 3$, f has distinct roots

$f = f(x,\lambda)$ depends analytically on λ (its coefficients lie in $\mathcal{O}(U)$)

• closed cycles γ_λ on C_λ, varying naturally in λ

$$\Longrightarrow \quad \underbrace{\omega = \frac{dx}{y} \in \Omega^1(X_\lambda)}_{\text{of } 1^{st} \text{ kind}}, \quad \underbrace{\eta = \frac{-x\,dx}{y} = -x\omega \in \Omega^1_{mer}(X_\lambda)^{res=0}}_{\text{of } 2^{nd} \text{ kind}} \qquad (X_\lambda = C_\lambda(\mathbb{C}))$$

Periods: $\int_{\gamma_\lambda} \omega$, $\int_{\gamma_\lambda} \eta$ depend only on $[\gamma_\lambda] \in H_1(X_\lambda, \mathbb{Z})$ and the classes $[\omega], [\eta] \in \Omega^1_{mer}(X_\lambda)^{res=0} / d\mathcal{M}(X_\lambda) =: H(X_\lambda)$

In fact: $H(X_\lambda) = \mathbb{C}\cdot[\omega] \oplus \mathbb{C}\cdot[\eta]$, but we are not going to use it.

Apply $D_\lambda = \frac{d}{d\lambda}$ (considering x, λ as independent variables):

$$D_\lambda \omega = -\frac{(D_\lambda y)\,dx}{y^2} = -\frac{(D_\lambda f)\,dx}{2y^3}, \quad D_\lambda \eta = -x D_\lambda \omega \quad (y^2 = f(x) \Rightarrow 2y D_\lambda y = D_\lambda f)$$

Goal: find linear relations between $[\omega], [\eta], [D_\lambda\omega], [D_\lambda\eta]$.

Such relations exist: $\omega, \eta, D_\lambda\omega, D_\lambda\eta \in \underbrace{\left\{ \frac{P(x)\,dx}{y^3} \;\middle|\; P \in \mathbb{C}[x],\ \deg(P) \le 4 \right\}}_{\dim = 5}$

If $Q(x) \in \mathbb{C}[x]$, $\deg(Q) \le 2$

$$\Longrightarrow \quad d\left(\frac{Q(x)\cancel{dx}}{y}\right) = \frac{(2fQ' - f'Q)\,dx}{2y^3}$$ lies in ↗, and $d\left(\frac{1}{y}\right), d\left(\frac{x}{y}\right), d\left(\frac{x^2}{y}\right)$ are linearly independent

Cor: $[\omega], [\eta], [D_\lambda\omega], [D_\lambda\eta]$ lie in a vector space of $\dim \le 5 - 3 = 2$.

Ex: Legendre family (slightly reparameterised):

$C_\lambda \setminus \{O\} : y^2 = x(1-x)(1-\lambda x)$, $\qquad \lambda \in \mathbb{C} \setminus \{0,1\}$

ramification points of $p : C_\lambda \longrightarrow \mathbb{P}^1(\mathbb{C})$, $(x,y) \longmapsto x$: $\quad 0, 1, \lambda^{-1}, \infty$

If $0 < |\lambda| < 1$, we can take, e.g., γ_λ

$$\Longrightarrow \quad \int_{\gamma_\lambda} \omega = 2 \int_0^1 \frac{dx}{\sqrt{x(1-x)(1-\lambda x)}}, \quad \int_{\gamma_\lambda} \eta = 2 \int_0^1 \frac{-x\,dx}{\sqrt{x(1-x)(1-\lambda x)}} \quad \text{(as above)}$$

linear algebra calculations inside $\left\{ \frac{P(x)dx}{y^3} \mid \deg(P) \le 4 \right\}$

$\Longrightarrow$ (1) $\underline{\eta + 2D_\lambda \omega + 2\lambda D_\lambda \eta = 0}$ ($\Longleftrightarrow 2(1-\lambda x) D_\lambda \omega = x\omega$, which follows directly from $(D_\lambda y)\omega + y(D_\lambda \omega) = D_\lambda(dx) = 0 \Longrightarrow -\frac{D_\lambda \omega}{\omega} = \frac{D_\lambda y}{y} = \frac{D_\lambda f}{2y^2} = \frac{D_\lambda f}{2f}$)

and (2) $\underline{\omega + \eta + 2(\lambda-1) D_\lambda \omega = d\left(\frac{2(x^2-x)}{y}\right)}$. The classes in $H(X_\lambda)$ satisfy

$$2\begin{pmatrix} 1 & \lambda \\ \lambda-1 & 0 \end{pmatrix}\begin{pmatrix} [D_\lambda \omega] \\ [D_\lambda \eta] \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} [\omega] \\ [\eta] \end{pmatrix} = 0 \Longrightarrow \underline{\begin{pmatrix} \lambda-2 & -\lambda \\ 1 & \lambda \end{pmatrix}\begin{pmatrix} [D_\lambda \omega] \\ [D_\lambda \eta] \end{pmatrix} + \frac{1}{2}\begin{pmatrix} [\omega] \\ [\eta] \end{pmatrix} = 0}$$

Applying D_λ and combining it with $\nearrow$ yields the final result:

$$\left(\lambda(\lambda-1) D_\lambda^2 + (2\lambda-1) D_\lambda + \frac{1}{4}\right)[\omega] = 0, \quad \left(\lambda(\lambda-1) D_\lambda^2 + (3\lambda-2) D_\lambda + \frac{3}{4}\right)[\eta] = 0$$

$\Longrightarrow$ the same equations for $\int_0^1 \omega$ resp. $\int_0^1 \eta$ (proved directly above).

However, the same equations hold for $\int_{\gamma_\lambda'} \omega$ resp. $\int_{\gamma_\lambda'} \eta$, for any family of closed cycles γ_λ' in C_λ varying holomorphically in $\lambda \in U$.

<u>Exercise</u>: Consider the family $y^2 = 4x^3 - t(x+1)$, where

$t = \frac{27J}{J-1}$ $\quad \left(J = \frac{t^3}{t^3 - 27t^2} \in \mathbb{C} \setminus \{0, 1\} \right)$,

$\omega = \frac{dx}{y}$, $\eta = \frac{-x\,dx}{y} = -x\omega$. Show that

$$36J(J-1)\frac{d}{dJ}\begin{pmatrix} [\omega] \\ [\eta] \end{pmatrix} = \begin{pmatrix} 3(J+2) & -2(J-1) \\ 9J/2 & -3(J+2) \end{pmatrix}\begin{pmatrix} [\omega] \\ [\eta] \end{pmatrix}$$

$$\Longrightarrow \quad \underline{\left(\frac{d}{dJ}\right)^2 [\omega] + \frac{1}{J}\frac{d}{dJ}[\omega] + \frac{31J-4}{144J^2(J-1)^2}[\omega] = 0}$$

Find A, B such that $\frac{[\omega]}{J^A(J-1)^B}$ satisfies a suitable hypergeometric diff. equation $\left(J(J-1)\left(\frac{d}{dJ}\right)^2 + ((a+b+1)J - c)\frac{d}{dJ} + ab\right)(\cdot) = 0$.

<u>Manin</u>: Periods of differentials of 2^{nd} kind on a family $\{X_\lambda\}$ of $\underline{\text{genus } g}$:
$H(X_\lambda)$ has $\dim = 2g \Longrightarrow \exists$ linear relation between $\left(\frac{d}{d\lambda}\right)^k [\omega]$ $(k = 0, \dots, 2g)$
($\Longrightarrow$ linear diff. equation for periods of order $2g$)

General theory in arbitrary dimension: "$\underline{\text{Gauss–Manin connection}}$".

Period relations for compact Riemann surfaces

X = compact Riemann surface of genus $g > 0$

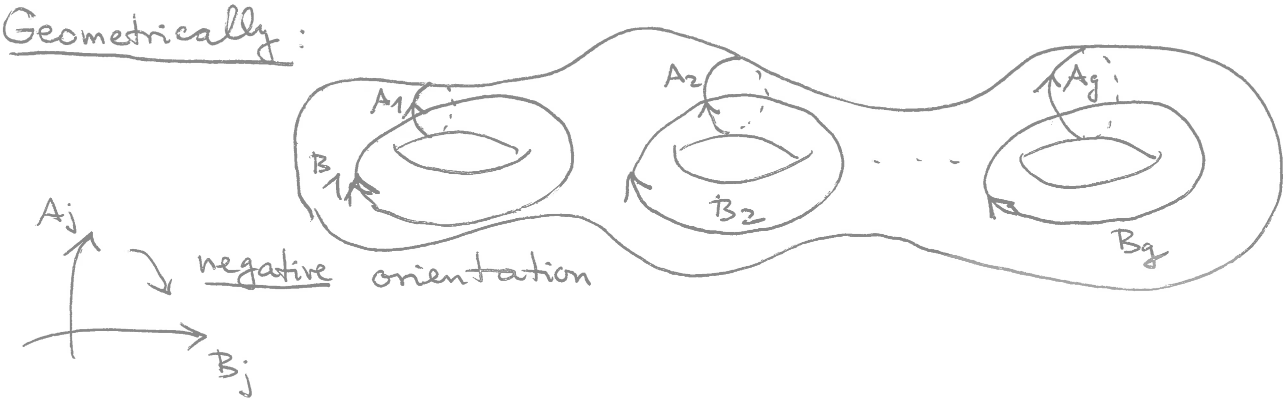
$H_1(X,\mathbb{Z}) \simeq \mathbb{Z}^{2g}$

Intersection pairing $\quad H_1(X,\mathbb{Z}) \times H_1(X,\mathbb{Z}) \xrightarrow{I} \mathbb{Z}$ (skew-symmetric)

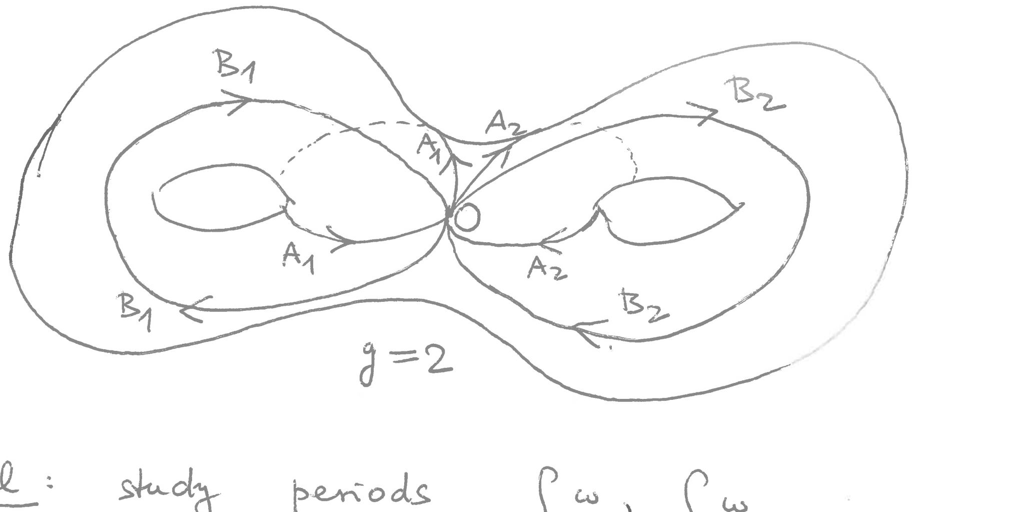
choose a symplectic basis: $\quad H_1(X,\mathbb{Z}) = \bigoplus_{j=1}^{g} \left(\mathbb{Z}[A_j] \oplus \mathbb{Z}[B_j]\right)$

such that $\quad I(A_j, B_k) = -\delta_{jk} = -I(B_k, A_j)$

Geometrically:



The classes $[A_j]$, $[B_k]$ can be represented by cycles passing through a chosen point $O \in X$:



$g=2$

Goal: study periods $\displaystyle\int_{A_j} \omega, \int_{B_k} \omega$

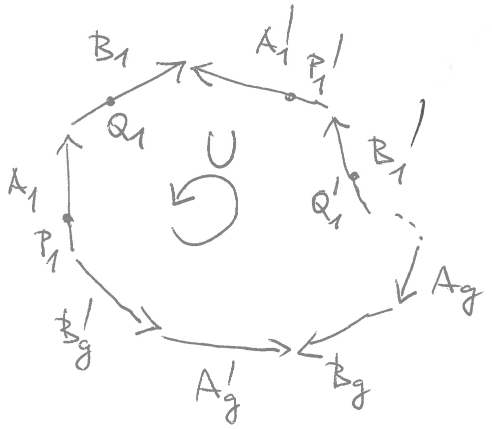
of holomorphic differentials $\quad \omega \in \Omega^1(X)$

(more generally, of differentials of 2^{nd} kind
$\eta \in \Omega^1_{mer}(X)^{res=0} / dM(X)$)

Thm. If α, β are C^∞ 1-forms on X such that $d\alpha = d\beta = 0$, then

$$\sum_{j=1}^{g} \begin{vmatrix} \int_{B_j}\alpha & \int_{B_j}\beta \\ \int_{A_j}\alpha & \int_{A_j}\beta \end{vmatrix} = \int_X \alpha\wedge\beta$$

Pf. The open subset $U = X \setminus \bigcup_{j=1}^{g}(A_j \cup B_j)$ is simply connected
$\overset{d\alpha=0}{\Longrightarrow} \exists\, f_\alpha \in C^\infty(U)$ $df_\alpha = \alpha|_U$. The boundary of U is as follows:



A_j is glued to A_j' so that
$$f_\alpha(P_j') - f_\alpha(P_j) = \int_{B_j}\alpha$$

B_k is glued to B_k' so that
$$f_\alpha(Q_k') - f_\alpha(Q_k) = -\int_{A_k}\alpha$$

On U, $d(f_\alpha\beta) = (\alpha\wedge\beta)|_U$

$$\Longrightarrow \int_X \alpha\wedge\beta = \int_U \alpha\wedge\beta = \int_{\partial U} f_\alpha\beta = \sum_{j=1}^{g} \underbrace{\int_{A_j}(f_\alpha(P') - f_\alpha(P))\beta}_{\left(\int_{B_j}\alpha\right)\left(\int_{A_j}\beta\right)} + \underbrace{\int_{B_j}(f_\alpha(Q') - f_\alpha(Q))\beta}_{\left(-\int_{A_j}\alpha\right)\left(\int_{B_j}\beta\right)}$$

Abstract formulation: define $PD(\alpha) := \sum_{j=1}^{g}\left(\left(\int_{B_j}\alpha\right)A_j - \left(\int_{A_j}\alpha\right)B_j\right)$

Then

$$\int_{PD(\alpha)}\beta = \underbrace{\sum_{j=1}^{g} \begin{vmatrix} \int_{B_j}\alpha & \int_{B_j}\beta \\ \int_{A_j}\alpha & \int_{A_j}\beta \end{vmatrix}}_{= I(PD(\alpha),\, PD(\beta))} \overset{\text{Thm}}{=} \int_X \alpha\wedge\beta$$

PD (= Poincaré duality): $(R = \mathbb{R}, \mathbb{C})$

$$H^1_{dR}(X, R) \xrightarrow{PD} H_1(X, R)$$

Period matrices: for α, β 1-forms on X s.t. $d\alpha = 0 = d\beta$, write

$$P_A(\alpha) := \begin{pmatrix} \int_{A_1}\alpha \\ \vdots \\ \int_{A_g}\alpha \end{pmatrix}, \quad P_B(\alpha) := \begin{pmatrix} \int_{B_1}\alpha \\ \vdots \\ \int_{B_g}\alpha \end{pmatrix} \in \mathbb{C}^g \qquad J := \begin{pmatrix} 0 & -I_g \\ I_g & 0 \end{pmatrix}$$

standard symplectic form

$$\boxed{P(\alpha) := \begin{pmatrix} P_A(\alpha) \\ P_B(\alpha) \end{pmatrix} \in \mathbb{C}^{2g}}$$

Thm above $\iff$ $\boxed{{}^tP(\alpha)\, J\, P(\beta) = \int_X \alpha\wedge\beta}$

Application to holomorphic 1-forms

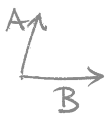
Thm (Riemann bilinear relations) If $\alpha, \beta \in \Omega^1(X)$, then

(1) ${}^tP(\alpha)\, J\, P(\beta) = 0$. (2) If $\alpha \neq 0$, $i\, {}^tP(\alpha)\, J\, \overline{P(\alpha)} > 0$.

Pf: (1) locally $\alpha = f(z)dz$, $\beta = g(z)dz \implies \alpha\wedge\beta = fg\, dz\wedge dz = 0$.

(2) $i\, \alpha\wedge\bar\alpha = |f(z)|^2 \underbrace{i\, dz\wedge d\bar z}_{2\, dx\wedge dy} > 0$

Ex: $g=1$: (2) $\iff i \begin{vmatrix} \omega_B & \overline{\omega_B} \\ \omega_A & \overline{\omega_A} \end{vmatrix} > 0 \iff \operatorname{Im}(\omega_A \overline{\omega_B}) > 0$



$\omega_A = \int_A \alpha$, $\omega_B = \int_B \alpha$

Period matrices

Fact: $\boxed{\dim_{\mathbb{C}} \Omega^1(X) = g}$ Fix a basis $\alpha_1, \dots, \alpha_g$ of $\Omega^1(X)$

Period matrix of $H_1(X, \mathbb{C}) \times \Omega^1(X) \xrightarrow{\int} \mathbb{C}$:

$$M := \left(P(\alpha_1) \middle| \cdots \middle| P(\alpha_g) \right) \in M_{2g,g}(\mathbb{C})$$

change of basis of $\Omega^1(X)$: replace M by Mh, $h \in GL_g(\mathbb{C})$

(case $g=1$: replace $\begin{pmatrix}\omega_A \\ \omega_B\end{pmatrix}$ by $\begin{pmatrix} t\omega_A \\ t\omega_B \end{pmatrix}$, $t \in \mathbb{C}^\times$)

notation: for $x, y \in \mathbb{C}^g$, let $\alpha(x) := \sum_{j=1}^{g} x_j \alpha_j \in \Omega^1(X)$

Riemann bilinear relations for $\alpha(x)$ and $\alpha(y)$:

(1) ${}^tx\, ({}^tMJM)\, y = 0 \quad \forall x, y \in \mathbb{C}^g \iff {}^tMJM = 0$

(2) $i\, {}^tx\, ({}^tMJ\overline{M})\, \bar x > 0 \quad \forall\, 0 \neq x \in \mathbb{C}^g \iff$ the hermitian matrix $i\, {}^tMJ\overline{M}$ is positive definite

The period domain

Def: $\mathrm{Per}_g := \{ M \in M_{2g,g}(\mathbb{C}) \mid M$ satisfies Riemann relations $\ {}^tMJM = 0,\ \underbrace{i\,{}^tMJ\overline{M} > 0}_{\text{positive definite hermitian}} \}$

What is this space?

$$\mathrm{Per}_1 = \left\{ \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} \in \mathbb{C}^2 \;\middle|\; \mathrm{Im}(\omega_1 \overline{\omega_2}) > 0 \right\} = \left\{ t \begin{pmatrix} \tau \\ 1 \end{pmatrix} \;\middle|\; t \in \mathbb{C}^\times,\ \tau \in \mathcal{H} \right\}$$

Special case: $M = \begin{pmatrix} T \\ I_g \end{pmatrix}$, $T \in M_g(\mathbb{C})$

(1) $\begin{pmatrix} {}^tT & I \end{pmatrix} \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \begin{pmatrix} T \\ I \end{pmatrix} = 0 \iff$ $T = {}^tT$ is symmetric

(2) $i\begin{pmatrix} \underset{T}{\underbrace{{}^tT}} & I \end{pmatrix} \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \begin{pmatrix} \overline{T} \\ I \end{pmatrix} > 0 \iff \underset{M_g(\mathbb{R})^{sym}}{\underset{\cap}{\mathrm{Im}(T)}} > 0$ positive definite real quadratic form

Def: the Siegel upper half space

$$\mathcal{H}_g := \{ T = {}^tT \in M_g(\mathbb{C}) \mid \mathrm{Im}(T) \text{ positive definite} \}$$

Thm: the map

$$\mathcal{H}_g \times GL_g(\mathbb{C}) \longrightarrow \mathrm{Per}_g \qquad (T, h) \longmapsto \begin{pmatrix} T \\ I_g \end{pmatrix} h$$

is bijective.

It commutes with the right action of $GL_g(\mathbb{C})$ given by $(T,h)h' = (T, hh')$ resp. right multiplication by h' on Per_g.

Pf: If $M = \begin{pmatrix} M_A \\ M_B \end{pmatrix} \in \mathrm{Per}_g \Rightarrow {}^tM_B M_A$ is symmetric and $\mathrm{Im}({}^tM_A \overline{M_B}) > 0$ is positive definite

$\Rightarrow M_B \in GL_g(\mathbb{C})$. Write $M = \begin{pmatrix} T \\ I \end{pmatrix} h$, $h = M_B$, $T = M_A M_B^{-1}$.

Then $T \in \mathcal{H}_g$. Conversely, if $T \in \mathcal{H}_g$, then $\begin{pmatrix} T \\ I_g \end{pmatrix} \in \mathrm{Per}_g \Rightarrow \begin{pmatrix} T \\ I_g \end{pmatrix} h \in \mathrm{Per}_g$ $\forall h \in GL_g(\mathbb{C})$

Rmk: $Sp_{2g}(\mathbb{R}) = \{ U \in GL_{2g}(\mathbb{R}) \mid {}^tUJU = J \}$ acts on Per_g by $M \longmapsto UM$ (this commutes with the right action of $GL_g(\mathbb{C})$, and changes $\{(A_j), (B_j)\}$ to another symplectic basis of $H_1(X,\mathbb{R})$). Formula: if $\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) \in Sp_{2g}(\mathbb{R})$ $(A,B,C,D \in M_g(\mathbb{R}))$,

then $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} T \\ I \end{pmatrix} = \begin{pmatrix} T' \\ I \end{pmatrix} (CT+D)$, $T' = (AT+B)(CT+D)^{-1} \in \mathcal{H}_g$

Coordinate-free definition of $\mathcal{P}er_g$

Def. A symplectic vector space over a field K is a pair (V, B), where V = finite-dimensional vector space over K

$B : V \times V \longrightarrow K$ non-degenerate bilinear $\underbrace{\text{alternating}}_{\forall x \in V\ B(x,x)=0}$ pairing $(\Rightarrow \dim V = 2n)$

$(\Rightarrow \forall x,y \in V\ B(y,x) = -B(x,y))$

A subspace $W \subset V$ is isotropic if $\forall x, y \in W\ B(x,y) = 0$.

$(\Leftrightarrow W \subset W^{\perp} := \{x \in V \mid B(x, W) = 0\}) \Rightarrow \dim(W) \leq n = \frac{1}{2}\dim(V)$

A lagrangian subspace of V is an isotropic subspace $W \subset V$ of maximal dimension $\dim(W) = \frac{1}{2}\dim(V)$ $(\Leftrightarrow W = W^{\perp})$.

Ex: If $\{P_j, Q_k\}$ is a symplectic basis of V: $B(P_j, P_k) = 0 = B(Q_j, Q_k)$, $B(P_j, Q_k) = \delta_{jk}$

$\Rightarrow \bigoplus_1^n K P_j$ and $\bigoplus_1^n K Q_j$ are langrangian subspaces.

Abstract version of $\mathcal{P}er_g$: (V, B) symplectic space over $\mathbb{R}$, $\dim V = 2g$

B extends $\mathbb{C}$-linearly to a symplectic space $(\underbrace{V_{\mathbb{C}}}_{V \otimes_{\mathbb{R}} \mathbb{C}}, B_{\mathbb{C}})$ over $\mathbb{C}$

$$\mathcal{P}er(V, B) = \{W \subset V_{\mathbb{C}} \text{ lagrangian subspace} \mid \forall 0 \neq x \in W\ \ i B_{\mathbb{C}}(x, \bar{x}) > 0\}$$

Ex: X compact Riemann surface of genus $g > 0$

$V = H^1_{dR}(X, \mathbb{R})$, $B(\alpha, \beta) = \int_X \alpha \wedge \beta$

$W = \Omega^1(X) \subset V_{\mathbb{C}}$ lies in $\mathcal{P}er(V, B)$ (by Riemann's relations)

The Cayley transform for the Siegel space $\mathcal{H}_g$:

Let $c = \begin{pmatrix} I_g & -iI_g \\ I_g & iI_g \end{pmatrix}$. For $T \in \mathcal{H}_g$, $c(T) = (T - iI_g)(T + iI_g)^{-1}$ belongs to

$\{W \in M_g(\mathbb{C}) \mid W = {}^tW,\ \underbrace{I_g - W\overline{W}}_{\text{positive definite hermitian}} > 0\}$

Facts: (1) $c : \mathcal{H}_g \xrightarrow{\sim} \{W \in M_g(\mathbb{C}) \mid W = {}^tW,\ I_g - W\overline{W} > 0\}$ is a bijection.

(2) $c\, Sp_{2g}(\mathbb{R})\, c^{-1} = \left\{\begin{pmatrix} U & \overline{V} \\ V & \overline{U} \end{pmatrix} \mid U, V \in M_g(\mathbb{C})\right\} \cap Sp_{2g}(\mathbb{C}) = U(g,g) \cap Sp_{2g}(\mathbb{C})$

For $g = 1$ we recover $\{w \in \mathbb{C} \mid 1 - w\overline{w} > 0\} = D$ and $c = \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$, with

$$c\, SL_2(\mathbb{R})\, c^{-1} = \left\{\begin{pmatrix} u & \overline{v} \\ v & \overline{u} \end{pmatrix} \mid |u|^2 - |v|^2 = 1\right\} = SU(1,1).$$

The Abel-Jacobi map ($g \geq 1$ arbitrary)

Rmk. If $M = \begin{pmatrix} M_A \\ M_B \end{pmatrix} \in \mathrm{Per}_g$, then the rows of M are linearly independent over $\mathbb{R}$ $\Rightarrow$ they generate a lattice $L_M \subset (\mathbb{C}^g)^* = \{(w_1, \dots, w_g) \mid w_j \in \mathbb{C}\}$ (the dual space of $\mathbb{C}^g$)

Pf: $\forall T \in \mathcal{H}_g$ $\mathrm{Im}(T) \in GL_g(\mathbb{R}) \Rightarrow \begin{pmatrix} T & \bar{T} \\ I_g & I_g \end{pmatrix} \in GL_{2g}(\mathbb{C})$.

Data: • $X =$ compact Riemann surface of genus $g \geq 1$

• $\alpha_1, \dots, \alpha_g$ – a basis of $\Omega^1(X)$ (global holomorphic differentials)

• $O \in X$ (base point)

Abel-Jacobi map:
$$\alpha_O : X \longrightarrow (\mathbb{C}^g)^*/L$$
$$P \longmapsto \left(\int_O^P \alpha_1, \dots, \int_O^P \alpha_g\right) (\mathrm{mod}\, L)$$

$L =$ group of periods $= \left\{\left(\int_\gamma \alpha_1, \dots, \int_\gamma \alpha_g\right) \mid [\gamma] \in H_1(X, \mathbb{Z})\right\}$

Rmk above $\Rightarrow$ L is a lattice in $(\mathbb{C}^g)^*$.

Abstract formulation: the pairing
$$H_1(X, \mathbb{Z}) \times \Omega^1(X) \xrightarrow{\int} \mathbb{C}$$
$$[\gamma], \omega \longmapsto \int_\gamma \omega$$
defines a map $H_1(X, \mathbb{Z}) \longrightarrow \Omega^1(X)^*$ (dual space) which is injective and whose image is a lattice.

The Abel-Jacobi map with base point O is then
$$\alpha_O : X \longrightarrow \boxed{\Omega^1(X)^* / H_1(X, \mathbb{Z}) =: J(X)} \quad \text{Jacobian variety of } X$$
$$P \longmapsto \left(\left(\omega \mapsto \int_O^P \omega\right) \bmod H_1(X, \mathbb{Z})\right)$$

$\mathbb{Z}$-linear extensions of α_O:
$$\alpha_O : \mathrm{Div}(X) \longrightarrow J(X)$$
$$\sum n_P(P) \longmapsto \sum n_P \alpha_O(P)$$

restriction $\alpha : \mathrm{Div}^0(X) \longrightarrow J(X)$ does not depend on O.

Fundamental results (Abel, Jacobi, Riemann, ...):

(1) $\forall\ 0 \neq f \in \mathcal{M}(X)$ $\quad \alpha(\mathrm{div}(f)) = 0$ $\quad$ ($\Leftarrow$ Abel's Thm)

(2) α induces an isomorphism of abelian groups

$$Cl^0(X) = \mathrm{Div}^0(X)/P(X) \xrightarrow{\sim} J(X)$$

(3) the restriction of α_0 to

$$\mathrm{Div}^g_+(X) := \{(P_1) + \cdots (P_g) \mid P_j \in X \text{ (not necessarily distinct)}\} \subset \mathrm{Div}(X)$$

is a generically bijective surjection

$$\alpha_{0,g,+} : \mathrm{Div}^g_+(X) \longrightarrow J(X)$$

(4) the inverse map to the restriction of $\alpha_{0,g,+}$ to the subset of $\mathrm{Div}^g_+(X)$ on which it is injective can be described explicitly in terms of the Riemann theta function attached to the period matrix of X.

Functoriality of Jacobians

$X \xrightarrow{\alpha} Y$ non-constant holomorphic map

X, Y = compact Riemann surfaces

covariant (= Albanese) functoriality:

$\mathrm{Div}(X) \xrightarrow{\alpha_*} \mathrm{Div}(Y)$, $\sum n_x (x) \longmapsto \sum n_x (\alpha(x))$

$H_1(X,\mathbb{Z}) \xrightarrow{\alpha_*} H_1(Y,\mathbb{Z})$

$\Omega^1(X) \xleftarrow{\alpha^*} \Omega^1(Y)$ $\overset{\text{dual}}{\Longrightarrow}$ $\Omega^1(X)^* \xrightarrow{\alpha_*} \Omega^1(Y)^*$

$$\Rightarrow \quad \begin{array}{ccc} \mathrm{Div}^0(X) & \xrightarrow{\alpha_*} & \mathrm{Div}^0(Y) \\ AJ_X \downarrow & & \downarrow AJ_Y \\ J(X) & \longrightarrow & J(Y) \end{array} \qquad \underbrace{\frac{\Omega^1(X)^*}{H_1(X,\mathbb{Z})}}_{J(X)} \longrightarrow \underbrace{\frac{\Omega^1(Y)^*}{H_1(Y,\mathbb{Z})}}_{J(Y)}$$

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contravariant (= Picard) functoriality:

$\mathrm{Div}(X) \xleftarrow{\alpha^*} \mathrm{Div}(Y)$ $\quad \sum_x e_x n_{\alpha(x)} (x) \longleftarrow\!\shortmid \sum n_y (y)$, $\alpha_*\alpha^* = \deg(\alpha)$

$H_1(X,\mathbb{Z}) \xleftarrow{\alpha^*} H_1(Y,\mathbb{Z})$ (dual to α_* via the intersection pairings)

$\Omega^1(X) \xrightarrow{\alpha_*} \Omega^1(Y)$ trace map

$$\begin{array}{ccc} \mathrm{Div}^0(X) & \xleftarrow{\alpha^*} & \mathrm{Div}^0(Y) \\ AJ_X \downarrow & & \downarrow AJ_Y \\ J(X) & \xleftarrow{\alpha^*} & J(Y) \end{array}$$

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correspondences:

$$\begin{tikzcd} & X \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ Y \arrow[rr, dashed, "T"] & & Z \end{tikzcd}$$

α, β as above (X, Y, Z cpt Riemann surfaces)

$T_* = \beta_* \alpha^* : J(Y) \longrightarrow J(Z)$

$T^* = \alpha_* \beta^* : J(Z) \longrightarrow J(Y)$

Algebraic version

X smooth projective curve over a field K

$\Longrightarrow$ abelian variety $J(X)$ over K parameterising $\mathrm{Cl}^0(X)$ (over all fields $L \supset K$)