Eisenstein series and differential operators

Examples: $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$

$k \geq 3$
$$\sum_{u\in L} \frac{1}{(z+u)^k} = \sum_{m,n\in\mathbb{Z}} \frac{1}{(z+m\omega_1+n\omega_2)^k} = \frac{(-1)^k}{(k-1)!}\,\wp^{(k)}(z;L)$$

$$\sum_{m,n\in\mathbb{Z}}{}' \frac{\phi(m,n)}{(m\omega_1+n\omega_2)^k}, \quad \phi : (\mathbb{Z}/N\mathbb{Z})^2 \longrightarrow \mathbb{C}$$

$$\zeta_{\mathbb{Q}(i)}(s) = \sum_{\substack{(0)\neq I \subset \mathbb{Z}[i] \\ \text{ideal}}} N(I)^{-s} = \frac{1}{4}\sum_{m,n\in\mathbb{Z}}{}' \frac{1}{|mi+n|^{2s}}, \quad I = (\underbrace{mi+n}_{})$$
unique up to $\mathbb{Z}[i]^\times = \{\pm 1, \pm i\}$

$$f(\tau) = \sum_{m,n\in\mathbb{Z}}{}' \frac{\phi(m,n)}{(m\tau+n)^k\,|m\tau+n|^{2s}}, \quad \phi : (\mathbb{Z}/N\mathbb{Z})^2 \longrightarrow \mathbb{C}$$

satisfies $\forall\, \alpha = \begin{pmatrix} a & b \\ c & d\end{pmatrix} \in \Gamma(N)$ $\quad f\Big(\underbrace{\frac{a\tau+b}{c\tau+d}}_{\alpha(\tau)}\Big) = (c\tau+d)^k\,|c\tau+d|^{2s} f(\tau)$

But $\mathrm{Im}(\alpha(\tau))/\mathrm{Im}(\tau) = |c\tau+d|^{-2}$, hence

$$g(\tau) = \mathrm{Im}(\tau)^s f(\tau) = \sum_{m,n\in\mathbb{Z}}{}' \frac{\phi(m,n)}{(m\tau+n)^k}\left(\frac{\mathrm{Im}(\tau)}{|m\tau+n|^2}\right)^s \quad \text{satisfies}$$

$$g\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k g(\tau) \quad \Big(\Longleftrightarrow \forall \alpha\in\Gamma(N) \quad \underbrace{g|_k\alpha}_{\tau \longmapsto (c\tau+d)^{-k} g\left(\frac{a\tau+b}{c\tau+d}\right)} = g\Big)$$

Goal: investigate the relationship between these functions for varying $k\in\mathbb{Z}$ and $s\in\mathbb{C}$ $\quad (\underbrace{k+2\,\mathrm{Re}(s) > 2}_{\text{ensures absolute convergence}})$

Differential operators: $\partial/\partial\tau$, $\partial/\partial\overline{\tau}$ $\qquad \tau - \overline{\tau} = 2i\,\mathrm{Im}(\tau)$

basic relations: $-\frac{1}{c}\left(\frac{1}{c\tau+d} - \frac{1}{c\overline{\tau}+d}\right) = \frac{\tau-\overline{\tau}}{|c\tau+d|^2}$ $\qquad (c,d\in\mathbb{R},\ c\neq 0)$

fix c, d; let
$$\boxed{\phi_{k,s} = \frac{1}{(c\tau+d)^k}\cdot\left(\frac{\mathrm{Im}(\tau)}{|c\tau+d|^2}\right)^s} \qquad (k\in\mathbb{Z},\ s\in\mathbb{C})$$

$$\frac{\partial}{\partial\tau} : \frac{1}{(c\tau+d)^k} \longmapsto \frac{1}{(c\tau+d)^k}\cdot\frac{-kc}{c\tau+d}, \qquad \frac{k}{\tau-\overline{\tau}} : \frac{1}{(c\tau+d)^k} \longmapsto \frac{1}{(c\tau+d)^k}\cdot\frac{k}{\tau-\overline{\tau}}$$

$$\left(\frac{\partial}{\partial\tau} + \frac{k}{\tau-\overline{\tau}}\right) : \underbrace{\frac{1}{(c\tau+d)^k}}_{\phi_{k,0}} \longmapsto \frac{k}{(c\tau+d)^k}\underbrace{\left(\frac{1}{\tau-\overline{\tau}} - \frac{c}{c\tau+d}\right)}_{\frac{c\overline{\tau}+d}{(\tau-\overline{\tau})(c\tau+d)}} = \underbrace{\frac{k(c\overline{\tau}+d)}{(\tau-\overline{\tau})(c\tau+d)^{k+1}}}_{\frac{k}{2i}\phi_{k+2,-1}}$$

$$\frac{\mathrm{Im}(\tau)}{|c\tau+d|^2} = \frac{i}{2}\cdot\frac{1}{c}\left(\frac{1}{c\tau+d} - \frac{1}{c\bar\tau+d}\right) \quad \begin{cases} \xrightarrow{\partial/\partial\tau} -\frac{i}{2}\,\frac{1}{(c\tau+d)^2} \\ \xrightarrow{\partial/\partial\bar\tau} \frac{i}{2}\,\frac{1}{(c\bar\tau+d)^2} \end{cases}$$

$$\phi_{k,s} = \phi_{k,0}\left(\frac{\mathrm{Im}(\tau)}{|c\tau+d|^2}\right)^s$$

$$\underbrace{\frac{\partial}{\partial\tau} + \frac{k}{\tau-\bar\tau}}_{\frac{1}{2i}R_k} : \phi_{k,s} \longmapsto \underbrace{(R_k\phi_{k,0})\left(\frac{\mathrm{Im}(\tau)}{|c\tau+d|^2}\right)^s + \phi_{k,0}\, s\left(\frac{\mathrm{Im}(\tau)}{|c\tau+d|^2}\right)^{s-1}\frac{\partial}{\partial\tau}\left(\frac{\mathrm{Im}(\tau)}{|c\tau+d|^2}\right)}_{-\frac{i}{2}(k+s)\phi_{k+2,s-1}}$$

$$\underbrace{-(\tau-\bar\tau)^2\frac{\partial}{\partial\bar\tau}}_{2i\,L_k} : \phi_{k,s} \longmapsto \phi_{k,0}\, s\left(\frac{\mathrm{Im}(\tau)}{|c\tau+d|^2}\right)^{s-1} \underbrace{L_k\left(\frac{\mathrm{Im}(\tau)}{|c\tau+d|^2}\right)}_{2i\left(\frac{\mathrm{Im}(\tau)}{c\bar\tau+d}\right)^2} = 2is\,\phi_{k-2,s+1}$$

Relations : $\quad H_k \text{``}=\text{''} [R_k, L_k] \text{''}= R_{k-2}\circ L_k - L_{k+2}\circ R_k$

explanation :

$$\begin{tikzcd}[column sep=large]
 & \phi_{k-2,s+1} & \phi_{k,s} & \phi_{k+2,s-1} \\
\text{weight} & k-2 \arrow[r, bend left, "R_{k-2}"] & k \arrow[l, bend left, "L_k"] \arrow[r, bend left, "R_k"] & k+2 \arrow[l, bend left, "L_{k+2}"]
\end{tikzcd}$$

$\begin{pmatrix} L = \text{left} \\ R = \text{right}\end{pmatrix}$

$$H_k(\phi_{k,s}) = \Big(\underbrace{-2i(s-1)\left(-\tfrac{i}{2}(k+s)\right) + \left(-\tfrac{i}{2}(k+s-1)\right)2is}_{(k+s-1)s-(k+s)(s-1)\ =\ k}\Big)\phi_{k,s}$$

$$H_k\circ R_{k-2} - R_{k-2}\circ H_{k-2} = 2R_{k-2}, \quad H_k\circ L_{k+2} - L_{k+2}\circ H_{k+2} = -2L_{k+2}$$

Cor : action of the Lie algebra $sl(2) = \mathrm{Lie}(SL_2(\mathbb{R}))$

$X = \begin{pmatrix}0&1\\0&0\end{pmatrix}$ acts by R_k

$Y = \begin{pmatrix}0&0\\1&0\end{pmatrix}$ — " — L_k on $\phi_{k,s}$

$H = \begin{pmatrix}1&0\\0&-1\end{pmatrix}$ — " — H_k

Casimir operator

$$\Omega = XY + YX + \frac{H^2}{2} = 2YX + \left(\frac{H^2}{2}+H\right) = 2XY + \left(\frac{H^2}{2}-H\right)$$

(commutes with X, Y, H)

$$\Omega(\phi_{k,s}) = \left(2L_{k+2}R_k + \frac{k^2}{2} + k\right)\phi_{k,s}$$

$$= \left(2(s-1)(k+s) + \frac{k^2}{2} + k\right)\phi_{k,s} = \left(\frac{(k+2s-1)^2-1}{2}\right)\phi_{k,s}$$

General formulas for $\boxed{\frac{R_k}{2i} = \frac{\partial}{\partial \tau} + \frac{k}{\tau - \bar\tau} \ , \ 2iL_k = -(\tau-\bar\tau)^2 \frac{\partial}{\partial \bar\tau}}$

$$H_k := R_{k-2}\circ L_k - L_{k+2}\circ R_k = -\frac{\partial}{\partial\tau}(\tau-\bar\tau)^2\frac{\partial}{\partial\bar\tau} + (\tau-\bar\tau)^2\frac{\partial}{\partial\bar\tau}\frac{\partial}{\partial\tau} +$$

$$+ (k-2)\left(-(\tau-\bar\tau)\frac{\partial}{\partial\bar\tau}\right) + k(\tau-\bar\tau)^2\frac{\partial}{\partial\bar\tau}\frac{1}{\tau-\bar\tau} = \underbrace{\left[(\tau-\bar\tau)^2, \frac{\partial}{\partial\tau}\right]}_{-2(\tau-\bar\tau)}\frac{\partial}{\partial\bar\tau} +$$

$$+ 2(\tau-\bar\tau)\frac{\partial}{\partial\bar\tau} + k(\tau-\bar\tau)^2\underbrace{\left[\frac{\partial}{\partial\bar\tau}, \frac{1}{\tau-\bar\tau}\right]}_{1/(\tau-\bar\tau)^2} = k$$

$$\Omega_k = R_{k-2}\circ L_k + L_{k+2}\circ R_k + \frac{H_k^2}{2} = 2L_{k+2}\circ R_k + \frac{H_k^2}{2} + H_k =$$

$$= -2(\tau-\bar\tau)^2\left(\frac{\partial}{\partial\bar\tau}\frac{\partial}{\partial\tau} + \frac{k}{\tau-\bar\tau}\frac{\partial}{\partial\bar\tau} + \frac{k}{(\tau-\bar\tau)^2}\right) + \frac{k^2}{2} + k$$

$$= -2(\tau-\bar\tau)^2\left(\frac{\partial}{\partial\bar\tau}\frac{\partial}{\partial\tau}\right) - 2k(\tau-\bar\tau)\frac{\partial}{\partial\bar\tau} + \left(\frac{k^2}{2} - k\right)$$

Back to Eisenstein series: fix $N \geq 1$, $\phi : (\mathbb{Z}/N\mathbb{Z})^2 \longrightarrow \mathbb{C}$

Let $$G_{k,s}(\tau, \phi) := \sum_{m,n\in\mathbb{Z}}{}' \frac{\phi(m,n)}{(m\tau+n)^k}\left(\frac{\mathrm{Im}(\tau)}{|m\tau+n|^2}\right)^s \qquad \left(\begin{matrix} k\in\mathbb{Z},\ s\in\mathbb{C} \\ k+2\,\mathrm{Re}(s) > 2\end{matrix}\right)$$

Properties: (1) $\forall\ \alpha = \begin{pmatrix} a & b \\ c & d\end{pmatrix} \in SL_2(\mathbb{Z})$

$$G_{k,s}(\cdot, \phi)\big|_k \alpha = G_{k,s}(\cdot, \alpha^{-1} * \phi) \qquad (\alpha * \phi)(m,n) = \phi((m,n)\alpha)$$

$$(\Longrightarrow \forall\ \alpha \in \Gamma(N) \qquad G_{k,s}(\cdot,\phi)\big|_k\alpha = G_{k,s}(\cdot,\phi))$$

(2)
$$L_k : G_{k,s}(\cdot,\phi) \longmapsto s\ G_{k-2,s+1}(\cdot,\phi)$$
$$R_k : G_{k,s}(\cdot,\phi) \longmapsto (k+s)\ G_{k+2,s-1}(\cdot,\phi)$$
$$H_k : G_{k,s}(\cdot,\phi) \longmapsto k\ G_{k,s}(\cdot,\phi)$$
$$\Omega_k : G_{k,s}(\cdot,\phi) \longmapsto \frac{(k+2s-1)^2-1}{2}\ G_{k,s}(\cdot,\phi)$$

The Space $V := \bigoplus_{n\in\mathbb{Z}} \mathbb{C}\cdot G_{k+2n,s-n}(\cdot,\phi)$ (depends also on ϕ)

is stable under the action of $\mathfrak{sl}(2)$ defined above.
It depends only on $k+2s$; denote it by $V_{k+2s,\phi}$.
This is an example of a Harish-Chandra module.
Ω acts on $V_{t,\phi}$ by the scalar $\frac{(t-1)^2-1}{2}$.

Special case $s\in\mathbb{Z}$: write $\alpha=k+s,\ \beta=s\in\mathbb{Z}$ $(\alpha+\beta>2)$
$(k=\alpha-\beta,\ s=\beta)$

$$G_{k,s}(\cdot,\phi)=G_{\alpha-\beta,\beta}(\cdot,\phi)=\mathrm{Im}(\tau)^{\beta}\underbrace{\sum_{m,n}{}'\frac{\phi(m,n)}{(m\tau+n)^{\alpha}(m\bar\tau+n)^{\beta}}}_{F_{\alpha,\beta,\phi}}\qquad \text{weight } \alpha-\beta$$

$$\beta F_{\alpha-1,\beta+1}\xleftarrow{\overbrace{L_{\alpha-\beta}}^{Y}} F_{\alpha,\beta}\xrightarrow{\overbrace{R_{\alpha-\beta}}^{X}}\alpha F_{\alpha+1,\beta-1},\qquad \underline{H(F_{\alpha,\beta})=(\alpha-\beta)F_{\alpha,\beta}}$$

$$\Omega(F_{\alpha,\beta})=\frac{(\alpha+\beta-1)^2-1}{2}$$

Structure of the Harish-Chandra module $V_{\alpha+\beta,\phi}=\bigoplus_{n\in\mathbb{Z}}\mathbb{C}F_{\alpha+n,\beta-n,\phi}$ (under the action of $\mathbb{C}X\oplus\mathbb{C}Y\oplus\mathbb{C}H$)

(1) $Y\underbrace{F_{\alpha+n,\beta-n}}_{F}=0\iff\beta-n=0\iff F=c\cdot F_{\alpha+\beta,0}=c\cdot G_{\alpha+\beta,0,\phi}$ holomorphic

(2) $X\underbrace{F_{\alpha+n,\beta-n}}_{F}=0\iff\alpha=0\iff F=c\cdot F_{0,\alpha+\beta}=c\,\mathrm{Im}(\tau)^{\alpha+\beta}\underbrace{\overline{G_{\alpha+\beta,0,\bar\phi}}}_{\text{antiholomorphic}}$

$$F_{\alpha+\beta-1,1}\xleftarrow{Y=0}F_{\alpha+\beta,0}\xrightarrow{X}F_{\alpha+\beta+1,-1}\xrightarrow{X}\cdots$$

$$F_{-1,\alpha+\beta+1}\qquad F_{0,\alpha+\beta}\xrightarrow{X=0}F_{1,\alpha+\beta-1}\longrightarrow\cdots\qquad F_{\alpha+\beta-1,1}\xleftarrow{Y=0}F_{\alpha+\beta,0}\qquad F_{\alpha+\beta+1,-1}$$

weights: ~~$\ell=\alpha+\beta$~~ (= eigenvalues of H)

$$-\ell\qquad 2-\ell\qquad\cdots\qquad \ell-2\qquad \ell\qquad \ell+2$$

$$\boxed{\overline{F_{\alpha,\beta,\phi}}=\mathrm{Im}(\tau)^{\beta-\alpha}F_{\beta,\alpha,\bar\phi}}\qquad \underline{\ell=\alpha+\beta>2}$$

Fourier expansions: we know the answer for $F_{\ell,0,\bar\phi}=\sum_{m,n}{}'\frac{\phi(m,n)}{(m\tau+n)^{\ell}}$ (resp. for $F_{0,\ell,\phi}=\mathrm{Im}(\tau)^{\ell}\overline{F_{\ell,0,\bar\phi}}$); applying

$\underline{R_\ell^r:=R_{\ell+2r-2}\circ\cdots\circ R_{\ell+2}\circ R_\ell}$ (resp. $\underline{L_{-\ell}^r=L_{-\ell-2r+2}\circ\cdots\circ L_{-\ell-2}\circ L_{-\ell}}$)

to $F_{\ell,0,\phi}$ (resp. to $F_{0,\ell,\phi}$) we obtain the expansions of $F_{\alpha,\beta,\phi}$ for $\alpha,\beta\in\mathbb{Z}$ satisfying $\underline{\alpha\le 0 \text{ or } \beta\le 0.}$ What about the case $\alpha,\beta>0$??

Prop. For each $m \geq 0$, the iterated differential operator

$$R_k^m = R_{k+2m-2} \circ R_{k+2m-4} \circ \cdots \circ R_{k+2} \circ R_k \qquad (\text{weight } k \longmapsto k+2m)$$

is given by

$$\boxed{\frac{R_k^m}{(2i)^m} = \sum_{j=0}^{m} \binom{m}{j} (k+j)_{m-j} \frac{1}{(\tau-\bar\tau)^{m-j}} \left(\frac{\partial}{\partial\tau}\right)^j}$$

$(a)_n = a(a+1)\cdots(a+n-1)$

$(n = 0, 1, 2, \dots)$

Pf: Induction on m. $\left(\frac{1}{2i} R_k = \frac{\partial}{\partial\tau} + \frac{k}{\tau-\bar\tau}\right)$

Cor: For $q = e^{2\pi i\tau}$ and $n \in \mathbb{Z} \setminus \{0\}$, $(y = \mathrm{Im}(\tau))$

$$\frac{1}{(2\pi i)^m} \frac{R_k^m(q^n)}{(2i)^m} = \left(\sum_{j=0}^{m} \binom{m}{j} (k+j)_{m-j} \left(-\frac{1}{4\pi n y}\right)^{m-j} \right) n^m q^n$$

Thm (Bol's formula) For any integer $m \geq 1$,

$$\boxed{\frac{1}{(2i)^{m-1}} R_{2-m}^{m-1} = \left(\frac{\partial}{\partial\tau}\right)^{m-1}} \qquad (\text{weight } 2-m \longmapsto m)$$

Back to $(\ell \geq 3)$

$$\begin{tikzcd} F_{0,\ell} & F_{1,\ell-1} \arrow[l, "Y"'] \arrow[r, bend left, "X"] \arrow[rrrr, bend right, "(\partial/\partial\tau)^{\ell-1} \quad (\text{Bol})"'] & F_{2,\ell-2} \arrow[r, bend left, "X"] & \cdots & F_{\ell-1,1} \arrow[r, bend left, "X"] & F_{\ell,0} \end{tikzcd}$$

What can we say about $f(\tau) := F_{1,\ell-1,\phi} = \sum_{m,n}{}' \frac{\phi(m,n) \cdot \mathrm{Im}(\tau)^{\ell-1}}{(m\tau+n)(m\bar\tau+n)^{\ell-1}}$?

(0) $\forall \alpha \in \Gamma(N)$ $\quad f|_{2-\ell}\,\alpha = f$ $\qquad$ (weight $2-\ell$)

(1) $Yf = L_{2-\ell} f = (\ell-1) F_{0,\ell,\phi}$, $\qquad F_{0,\ell,\phi} = \mathrm{Im}(\tau)^{\ell} \underbrace{\overline{F_{\ell,0,\bar\phi}}}_{\overline{G_{\ell,0,\bar\phi}}}$

$$\boxed{\text{Operator} \quad \xi_{2-\ell} f := \overline{\mathrm{Im}(\tau)^{-\ell} L_{2-\ell} f}}$$

$\frac{\xi_{2-\ell} f}{(\ell-1)} = G_{\ell,0,\bar\phi}$ usual holomorphic Eisenstein series $\sum_{m,n}{}' \frac{\overline{\phi(m,n)}}{(m\tau+n)^{\ell}}$

(2) $(\partial/\partial\tau)^{\ell-1} f = \frac{1}{(2i)^{\ell-1}} R_{2-\ell}^{\ell-1} f = c\, F_{\ell,0,\phi} = c\, G_{\ell,0,\phi}$, $\quad c \neq 0$ explicit constant

__Question__ : what can one say about a $\mathcal{C}^\infty$ function $f : \mathcal{H} \to \mathbb{C}$

$(\ell \in \mathbb{Z}, \ell \geq 2)$ if one knows $\overline{\left((\tau - \bar\tau)^{2-\ell} \dfrac{\partial f}{\partial \bar\tau} \right)} = g(\tau)$

and $(\partial/\partial\tau)^{\ell-1} f = h(\tau)$?

__Answer__ : fix $\tau_0 \in \mathcal{H}$. the function

$$G(\tau) := \int_{\tau_0}^{\tau} (\bar\tau - z)^{\ell-2} g(z)\, dz \quad \text{satisfies} \quad \frac{\partial G}{\partial \tau} = (\bar\tau - \tau)^{\ell-2} g(\tau),$$

hence $\partial \bar G / \partial \bar\tau = (\tau - \bar\tau)^{\ell-2}\, \overline{g(\tau)} \implies (\partial/\partial\bar\tau)(f - \bar G) = 0$

$\implies f - \bar G$ is holomorphic.

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the function $F(\tau) = \int_{\tau_0}^{\tau} (\tau - z)^{\ell-2} h(z)\, dz$ satisfies

$\frac{1}{(\ell-2)!} (\partial/\partial\tau)^{\ell-1} F = h(\tau)$ $\implies P := f - \bar G - F$ is a __polynomial__ of degree $\leq \ell - 2$ $\left((\partial/\partial\tau)^{\ell-1} P = 0 \right)$.

__Reformulation__ : for any polynomial $Q(\tau)$ with $\deg(Q) \leq \ell - 2$, the function

$$f(\tau) = Q(\tau) + \underbrace{\frac{1}{(\ell-2)!} \int_{\tau_0}^{\tau} (\tau - z)^{\ell-2} h(z)\, dz}_{\text{Eichler integral}} + \underbrace{\overline{\int_{\tau_0}^{\tau} (\bar\tau - z)^{\ell-2} g(z)\, dz}}_{\text{Niebur integral}}$$

satisfies the two conditions above.

__Special cases__ : (1) __Eichler integral__, $\tau_0 \to i\infty$:

If $h(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i n z}$, $a_n = O(n^c)$, then we can let $\tau_0 \to i\infty$ and integrate for $z = \tau + it$, $0 \leq t < +\infty$:

$$\frac{1}{(\ell-2)!} \int_{i\infty}^{\tau} (\tau - z)^{\ell-2} h(z)\, dz = \frac{1}{(\ell-2)!} \int_{+\infty}^{0} (-it)^{\ell-2} \left(\sum_{n=1}^{\infty} a_n e^{2\pi i n \tau} e^{-2\pi n t} \right) i\, dt$$

$$= \frac{(-i)^{\ell-1}}{\Gamma(\ell-1)} \sum_{n=1}^{\infty} a_n e^{2\pi i n \tau} \underbrace{\int_0^{+\infty} t^{\ell-1} e^{-2\pi n t} \frac{dt}{t}}_{\Gamma(\ell-1)/(2\pi n)^{\ell-1}} = \frac{1}{(2\pi i)^{\ell-1}} \sum_{n=1}^{\infty} \frac{a_n}{n^{\ell-1}} e^{2\pi i n \tau}$$

(= naive $(\ell-1)$-fold integral of h)

(2) <u>Niebur integral</u>, $\tau_0 \to i\infty$:

If $g(z) = \sum_{n=1}^{\infty} b_n e^{2\pi i n z}$, $b_n = O(n^c)$,

we let again $\tau_0 \to +i\infty$, $z = \tau + it$, $0 \le t < +\infty$:

$$\int_{i\infty}^{\tau} (\overline{\tau} - z)^{\ell-2} g(z)\, dz = \int_{+\infty}^{0} (-2iy - it)^{\ell-2} g(x + i(y+t))\, i\,dt = \qquad \begin{matrix}(\tau = x+iy)\\ \underline{y>0}\\ (t+y=u)\end{matrix}$$

$$= (-i)^{\ell-1} \int_{y}^{+\infty} (u+y)^{\ell-2} \sum_{n\ge 1} b_n e^{2\pi i n x} e^{-2\pi n u}\, du$$

$$= (-i)^{\ell-1} \sum_{n=1}^{\infty} b_n e^{2\pi i n x} \underbrace{\int_{y}^{+\infty} (u+y)^{\ell-2} e^{-2\pi n u}\, du}_{e^{2\pi n y} \frac{1}{(2\pi n)^{\ell-1}} \underbrace{\int_{4\pi n y}^{+\infty} v^{\ell-1} e^{-v} \frac{dv}{v}}_{\Gamma(\ell-1,\, 4\pi n y)}} \qquad u+y = \frac{v}{2\pi n}$$

<u>incomplete</u> <u>Γ-function</u>

$$= \frac{1}{(2\pi i)^{\ell-1}} \sum_{n=1}^{\infty} b_n e^{2\pi i n x} e^{2\pi n y}\, \Gamma(\ell-1, 4\pi n y)$$

<u>More on incomplete Γ-function</u>: $m \in \mathbb{Z}$, $m \ge 0$

$$\underbrace{\int_{t}^{+\infty} x^m e^{-x}\, dx}_{\Gamma(m+1,\, t)} = \left(\sum_{j=0}^{m} j! \binom{m}{j} t^{m-j} \right) e^{-t} \qquad (t>0)$$

$\Longrightarrow$ above, the Niebur integral is equal to

$$\frac{1}{(2\pi i)^{\ell-1}} \sum_{n=1}^{\infty} b_n e^{2\pi i n \tau} \left(\sum_{j=0}^{\ell-2} j! \binom{\ell-2}{j} (4\pi n y)^{\ell-2-j} \right) \qquad (\underline{y = \mathrm{Im}\,\tau})$$

Basic non-holomorphic Eisenstein series

$\Gamma = SL_2(\mathbb{Z})$, $N = \begin{pmatrix} 1 & \mathbb{R} \\ 0 & 1 \end{pmatrix} \supset \Gamma_\infty = \Gamma \cap N = \begin{pmatrix} 1 & \mathbb{Z} \\ 0 & 1 \end{pmatrix}$

The function $f_s : \tau = x+iy \longmapsto y^s = \operatorname{Im}(\tau)^s$ is invariant under N. Put

$$E^*(\tau, s) = \frac{1}{2} \sum_{\pm\Gamma_\infty \backslash \Gamma} f_s(\gamma\tau) = \frac{1}{2} \sum_{\substack{m,n \in \mathbb{Z} \\ (m,n)=1}} \frac{y^s}{|m\tau+n|^{2s}} \qquad (\operatorname{Re}(s) > 1)$$

$$E(\tau, s) = \zeta(2s) E^*(\tau, s) = \frac{1}{2} \sum_{m,n\in\mathbb{Z}}{}' \frac{y^s}{|m\tau+n|^{2s}} \qquad \Bigg|\ \forall \gamma \in \Gamma \quad E(\gamma(\tau), s) = E(\tau, s)$$

Fourier expansion: $\underline{m=0}$: $y^s \zeta(2s)$

$\underline{m \neq 0}$:
$$\frac{\Gamma(s)}{\pi^s |m\tau+n|^{2s}} = \int_0^\infty e^{-\pi[(mx+n)^2 + m^2y^2]t}\, t^s \frac{dt}{t}$$

$$y^{-s}\Gamma(s)E(\tau,s) = \Gamma(s)\zeta(2s) + \frac{\pi^s}{2} \sum_{m\neq 0} \int_0^\infty e^{-\pi m^2y^2 t} \Big(\underbrace{\sum_{n\in\mathbb{Z}} e^{-\pi(mx+n)^2 t}}_{t^{-1/2} \sum_{n\in\mathbb{Z}} e^{-\pi n^2/t - 2\pi i m n x}} \Big) t^s \frac{dt}{t}$$

$$= \Gamma(s)\zeta(2s) + \pi^s \sum_{m=1}^\infty \sum_{n\neq 0} e^{-2\pi i m n x} \underbrace{\int_0^\infty e^{-\pi(m^2y^2t + n^2/t)}\, t^{s-\frac{1}{2}} \frac{dt}{t}}_{2\left(\frac{|n|}{|m|y}\right)^{s-\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi|mn|y)} + \pi^s \sum_{m=1}^\infty \frac{\Gamma(s-\frac{1}{2})}{(\pi m^2 y^2)^{s-\frac{1}{2}}}$$

Summary: $\pi^{-s}\Gamma(s)E(\tau,s) = y^s \pi^{-s}\Gamma(s)\zeta(2s) + y^{1-s}\pi^{\frac{1}{2}-s}\Gamma(s-\frac{1}{2})\zeta(2s-1) +$

(symmetric under $s \longleftrightarrow 1-s$)

$$+ 2\pi^s \sum_{k=1}^\infty \frac{\sigma_{2s-1}(k)}{k^{s-1/2}} \left(e^{2\pi i k x} + e^{-2\pi i k x}\right) y^{1/2} K_{s-\frac{1}{2}}(2\pi k y)$$

Above: $\boxed{K_s(x) = \frac{1}{2} \int_0^\infty e^{-(t+t^{-1})x/2}\, t^s \frac{dt}{t}}$ $\ \xrightarrow{x\to+\infty} 0$

$t := u^{-1} \Rightarrow K_{-s} = K_s$

$\underline{K_s' = -\frac{1}{2}(K_{s+1} + K_{s-1})}$

$$d\left(e^{-(t+t^{-1})x/2} t^s\right) = s e^{-(t+t^{-1})x/2} t^s \frac{dt}{t} - \frac{x}{2}(1-t^{-2}) e^{-(t+t^{-1})x/2} t^s dt$$

$\Rightarrow \underline{0 = sK_s - \frac{x}{2}(K_{s+1} - K_{s-1})} \Rightarrow \underline{K_{s+1} = -K_s' + \frac{s}{x}K_s}$, $\underline{K_s' = -K_{s-1} - \frac{s}{x}K_s}$

$\Rightarrow K_{s+1}' = -K_s'' + \frac{s}{x}K_s' - \frac{sK_s}{x^2} = -K_s - \frac{s+1}{x}K_{s+1} = -K_s - \frac{s+1}{x}\left(-K_s' + \frac{s}{x}K_s\right)$

$\Rightarrow \underline{K_s'' + \frac{1}{x}K_s' - \left(1 + \frac{s^2}{x^2}\right)K_s = 0}\ \Big|\ (\sqrt{x}K_s(x))'' + \left(-1 + \frac{(\frac{1}{4}-s^2)}{x^2}\right)(\sqrt{x}K_s(x)) = 0$

$\Rightarrow \underline{\sqrt{x}K_{1/2}(x) = Ce^{-x}}$

$t = \frac{2u}{x} \Rightarrow t = \sqrt{\frac{2u}{x}}$, $\sqrt{\frac{x}{2}} K_{1/2}(x) = \sqrt{\frac{x}{2}} K_{1/2}(x) = \frac{1}{2}\int_0^\infty e^{-(u + x^2/4u)} \sqrt{u}\frac{du}{u}$

$x \to 0$, $\frac{1}{2}\Gamma(\frac{1}{2}) = \frac{\sqrt{\pi}}{2}$

$$\Rightarrow \boxed{K_{1/2}(x) = \sqrt{\frac{\pi}{2x}}\, e^{-x}}$$

Modular forms and group theory

Iwasawa decomposition of $G = SL_2(\mathbb{R})$:

$$G = NAK = \underbrace{\left\{\begin{pmatrix}1&x\\0&1\end{pmatrix}\right\}}_{N} \cdot \underbrace{\left\{\begin{pmatrix}a&0\\0&a^{-1}\end{pmatrix} \,\middle|\, a>0\right\}}_{A} \cdot \underbrace{\left\{\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix} = h_\theta\right\}}_{K=SO(2)}$$

$$K = SO(2) \xrightarrow[\sim]{r} U(1) \subset \mathbb{C}, \qquad h_\theta \longmapsto e^{i\theta}$$

Action on $G/K \xrightarrow{\sim} \mathcal{H}$, $gK \mapsto g(i)$

$$g = \begin{pmatrix}a&b\\c&d\end{pmatrix}\in G : \tau \longmapsto \frac{a\tau+b}{c\tau+d}; \qquad NA \xrightarrow{\sim} G/K \xrightarrow{\sim} \mathcal{H}$$

$$g(i) = \underbrace{\frac{ac+bd}{c^2+d^2}}_{x} + i\,\underbrace{\frac{1}{c^2+d^2}}_{y}$$

$$na = \begin{pmatrix}1&x\\0&1\end{pmatrix}\begin{pmatrix}y^{1/2}&0\\0&y^{-1/2}\end{pmatrix} \longmapsto (na)(i) = x+iy = \tau$$

$$\begin{pmatrix}1&x\\0&1\end{pmatrix}\begin{pmatrix}y^{1/2}&0\\0&y^{-1/2}\end{pmatrix} h_\theta = \begin{pmatrix} * & * \\ \underbrace{y^{-1/2}\sin\theta}_{c} & \underbrace{y^{-1/2}\cos\theta}_{d}\end{pmatrix}$$

$$e^{i\theta} = y^{1/2}(ci+d) = \frac{ci+d}{\sqrt{c^2+d^2}} = y^{1/2} J(g,i)$$

Bijection: $J\left(\begin{pmatrix}a&b\\c&d\end{pmatrix},\tau\right) = c\tau+d$

$$\left\{f:\mathcal{H}\to\mathbb{C} \,\middle|\, \begin{array}{l} f(\gamma\tau) = J(\gamma,\tau)^k f(\tau) \\ \forall \gamma\in\Gamma \end{array}\right\} \xleftrightarrow{1-1} \left\{\begin{array}{ll} F: G\to\mathbb{C} & \\ \forall h\in K & F(gh) = r(h)^{-k}F(g) \\ \forall\gamma\in\Gamma & F(\gamma g) = F(g)\end{array}\right\}$$

$$f(\overbrace{x+iy}^{\tau}) = y^{-k/2}\, F\Big(\overbrace{\begin{pmatrix}1&x\\0&1\end{pmatrix}\begin{pmatrix}y^{1/2}&0\\0&y^{-1/2}\end{pmatrix}}^{g_\tau}\Big)$$

$$f(\tau) = \mathrm{Im}(\tau)^{-k/2}\, F(g_\tau)$$

$$F\Big(\underbrace{\begin{pmatrix}1&x\\0&1\end{pmatrix}\begin{pmatrix}y^{1/2}&0\\0&y^{-1/2}\end{pmatrix} h_\theta}_{nak = h\in H_\varepsilon}\Big) = \underbrace{\left(y^{1/2}e^{-i\theta}\right)^k f(x+iy)}_{\mathrm{Im}(g(i))^{k/2}\, r(h_\theta)^{-k} f(g(i))}$$

$$F(g) = J(g,i)^{-k} f(g(i))$$

Differential operators: the right regular action $(g * F)(g') = F(g'g)$ induces an action of the Lie algebra $\mathrm{Lie}(G) = \mathfrak{sl}(2)$:

$$X = \begin{pmatrix}0&1\\0&0\end{pmatrix},\quad Y = \begin{pmatrix}0&0\\1&0\end{pmatrix},\quad H = \begin{pmatrix}1&0\\0&-1\end{pmatrix}, \qquad [X,Y] = H, \quad [H,X] = 2X, \quad [H,Y] = -2Y$$

$$(XF)(g) = \frac{d}{dt} F(g e^{tX})\Big|_{t=0} = \frac{d}{dt} F\left(g\begin{pmatrix}1&t\\0&1\end{pmatrix}\right)\Big|_{t=0} \quad \text{etc.}$$

$$g = \begin{pmatrix}a&b\\c&d\end{pmatrix}\in G, \qquad g e^{tX} = \begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}1&t\\0&1\end{pmatrix} = \begin{pmatrix}a & at+b\\ c & ct+d\end{pmatrix}$$

$$g e^{tY} = \begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}1&0\\t&1\end{pmatrix} = \begin{pmatrix}a+tb & b\\ c+td & d\end{pmatrix}, \qquad g e^{tH} = \begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}e^t&0\\0&e^{-t}\end{pmatrix} = \begin{pmatrix}ae^t & be^{-t}\\ ce^t & de^{-t}\end{pmatrix}$$

$$X = a\frac{\partial}{\partial b} + c\frac{\partial}{\partial d}, \quad Y = b\frac{\partial}{\partial a} + d\frac{\partial}{\partial c}, \quad H = a\frac{\partial}{\partial a} - b\frac{\partial}{\partial b} + c\frac{\partial}{\partial c} - d\frac{\partial}{\partial d}$$

Invariant differential operators

G = connected Lie group, V complex vector space ($\dim V < \infty$)

G acts on $C^\infty(G,V)$ by left and right regular actions:

$$(L(g)F)(g') = F(g^{-1}g'), \quad (R(g)F)(g') = F(g'g) \qquad L(g_1)\circ R(g_2) = R(g_2)\circ L(g_1)$$

$\Rightarrow$ actions of the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$ (and of $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}\otimes_{\mathbb{R}}\mathbb{C}$)

$$(L(X)F) = \frac{d}{dt} L(e^{tX})F\Big|_{t=0}, \quad (R(X)F = \frac{d}{dt} R(e^{tX})F\Big|_{t=0}$$

(1st order differential operators)

$\Rightarrow$ actions of the universal enveloping algebra $U(\mathfrak{g})$

(contains $X_1 \cdots X_k$ ($X_i \in \mathfrak{g}$), with the rule $XY - YX = [X,Y]$)

($X_1 \cdots X_k$ gives a differential operator of order k)

Facts: (1) A differential operator $D : C^\infty(G,V) \circlearrowleft$ is invariant under $L(g)$ $\forall g \in G$ $\iff$ $D = R(\text{element of } U(\mathfrak{g})) \in R(U(\mathfrak{g}))$

(2) D is invariant under $L(g)$ and $R(g)$ for all $g \in G$

$\iff$ $D = R(u)$ for $u \in \underbrace{U(\mathfrak{g})^{Ad(G)-\text{invariants}}}_{Z(U(\mathfrak{g}))}$ (centre of $U(\mathfrak{g})$) (since G connected)

Ex: $G = SL_2(\mathbb{R})$, $\mathfrak{g} = \mathfrak{sl}(2) = \mathbb{R}H \oplus \mathbb{R}X \oplus \mathbb{R}Y$

$[X,Y] = H$, $[H,X] = 2X$, $[H,Y] = -2Y$

$$Z(U(\mathfrak{g})) = \mathbb{R}[\Omega], \quad \underbrace{\Omega = XY + YX + \frac{H^2}{2}}_{\text{the Casimir element}} = 2YX + \left(\frac{H^2}{2} + H\right) = 2XY + \left(\frac{H^2}{2} - H\right)$$

Vector space identification: $S(\mathfrak{g}) \xrightarrow{\sim} U(\mathfrak{g})$

$\boxed{S(\mathfrak{g}) = \text{symmetric algebra of } \mathfrak{g}}$

$$X_1 \cdots X_k \longmapsto \left(F \mapsto \frac{\partial}{\partial t_1}\cdots\frac{\partial}{\partial t_k} R(e^{t_1X_1 + \cdots + t_kX_k})F\Big|_{t_j=0}\right)$$

(ex: $X_1X_2 \longmapsto \left(F \longmapsto R\left(\frac{X_1X_2 + X_2X_1}{2}\right)F\right)$

Commutes with the adjoint action $Ad_G(g) : X \longmapsto gXg^{-1}$

Back to $G = SL_2(\mathbb{R})$ and $\mathfrak{g} = \mathrm{Lie}(G) = \mathfrak{sl}(2) = \{A \in M_2(\mathbb{R}) \mid \forall t \in \mathbb{R}\ e^{tA} \in G\}$

$\mathfrak{g} = \mathbb{R}X \oplus \mathbb{R}Y \oplus \mathbb{R}H$, $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ $= \{ \text{—"—} \mid \mathrm{Tr}(A) = 0 \}$

$[H, X] = 2X$, $[H, Y] = -2Y$, $[X, Y] = H$

Right regular action of $\mathfrak{g}$ on $\mathcal{C}^\infty(G, V)$ in coordinates coming from the Iwasawa decomposition $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = nak = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix} h_\theta = g_\tau h_\theta$

$h_\theta = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} = e^{\theta \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}} = e^{\theta(-X+Y)}$, $n = e^{xX}$, $a = e^{(\log(y)/2)H}$

$g(i) = x + iy = \tau$, $J(h_\theta, i) = e^{i\theta}$, $ci + d = J(g, i) = J(g_\tau, \underbrace{h_\theta(i)}_{i}) J(h_\theta, i)$ with $\underbrace{J(g_\tau, i)}_{y^{-1/2}}$, $\underbrace{J(h_\theta, i)}_{e^{i\theta}}$

$\cos\theta = \dfrac{d}{\sqrt{c^2+d^2}}$, $\sin\theta = \dfrac{c}{\sqrt{c^2+d^2}}$, $e^{i\theta} = y^{1/2}(ci+d)$

$x = \dfrac{ac+bd}{c^2+d^2}$, $y = \dfrac{1}{c^2+d^2}$

We know: $X = a\dfrac{\partial}{\partial b} + c\dfrac{\partial}{\partial d}$, $Y = b\dfrac{\partial}{\partial a} + d\dfrac{\partial}{\partial c}$, $H = a\dfrac{\partial}{\partial a} - b\dfrac{\partial}{\partial b} + c\dfrac{\partial}{\partial c} - d\dfrac{\partial}{\partial d}$

(under $R(g)$). So:

$$\boxed{\begin{aligned} Xx &= Yx = \frac{d^2-c^2}{(c^2+d^2)^2} = y\cos(2\theta), & Hx &= \frac{4cd}{(c^2+d^2)^2} = 2y\sin(2\theta) \\ Xy &= Yy = \frac{-2cd}{(c^2+d^2)^2} = -y\sin(2\theta), & Hy &= \frac{2(d^2-c^2)}{(c^2+d^2)^2} = 2y\cos(2\theta) \end{aligned}}$$

$X(ci+d) = ic \implies e^{i\theta} X(i\theta) = Xe^{i\theta} = X(y^{1/2}(ci+d)) = y^{1/2}\left(ic + \dfrac{ci+d}{2y} Xy\right)$

idem for Y $\implies$ $X\theta = -\sin^2\theta$, $Y\theta = \cos^2\theta$, $H\theta = \sin(2\theta)$

Summary:

$$\boxed{\begin{aligned} X &= y\left(\cos(2\theta)\frac{\partial}{\partial x} - \sin(2\theta)\frac{\partial}{\partial y}\right) - (\sin^2\theta)\frac{\partial}{\partial \theta} \\ Y &= y\left(\cos(2\theta)\frac{\partial}{\partial x} - \sin(2\theta)\frac{\partial}{\partial y}\right) + (\cos^2\theta)\frac{\partial}{\partial \theta} \\ H &= 2y\left(\sin(2\theta)\frac{\partial}{\partial x} + \cos(2\theta)\frac{\partial}{\partial y}\right) + \sin(2\theta)\frac{\partial}{\partial \theta} \end{aligned}}$$

$$\boxed{\begin{aligned} &-X+Y = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \frac{\partial}{\partial\theta} \quad (= \text{generator of } \mathrm{Lie}(SO(2)),\ e^{\theta(-X+Y)} = h_\theta) \\ &X + Y \pm iH = e^{\pm 2i\theta}\left(2y\left(\frac{\partial}{\partial x} \pm i\frac{\partial}{\partial y}\right) + \frac{\partial}{\partial\theta}\right) \end{aligned}}$$

Casimir operator: $\Omega = XY + YX + \dfrac{H^2}{2} = 2\left(y^2\left(\left(\dfrac{\partial}{\partial x}\right)^2 + \left(\dfrac{\partial}{\partial y}\right)^2\right) + y\dfrac{\partial}{\partial x}\dfrac{\partial}{\partial\theta}\right)$

Laplacian: $\Delta := -\frac{1}{2}\Omega = -y^2\left(\left(\dfrac{\partial}{\partial x}\right)^2 + \left(\dfrac{\partial}{\partial y}\right)^2\right) - y\dfrac{\partial}{\partial x}\dfrac{\partial}{\partial\theta}$

$\Delta(y^s) = s(1-s)y^s$

Correspondence $f \longleftrightarrow F$ (twisted action of weight $k \in \mathbb{Z}$):

$f \in C^\infty(\mathcal{H})$ $\qquad$ $F \in C^\infty(G)$, $\quad F(g h_\theta) = e^{-ik\theta} F(g)$

$$\boxed{f(\tau) = f(g(i)) = J(g,i)^k F(g) = \mathrm{Im}(\tau)^{-k/2} F(g_\tau)} \qquad F(g) = J(g,i)^{-k} f(g(i))$$

$$\boxed{J\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tau\right) = c\tau + d} \qquad g_\tau = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix} \qquad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix},\ J(g,i) = ci + d$$

$\tau = x + iy = g(i)$ $\qquad$ $g = g_\tau h_\theta$

$$F(\gamma g) = J(g,i)^{-k} \underbrace{J(\gamma, g(i))^{-k} f(g(i))}_{(f|_k \gamma)(g(i))}$$

$$\boxed{F(g) = f(x+iy)\, y^{k/2} e^{-ik\theta}, \quad g(i) = x + iy}$$

$$\begin{cases} (-X+Y) F = -ik F \\ (X + Y \pm iH) F = 2 e^{\mp 2i\theta} y\left(\left(\frac{\partial}{\partial x} \pm i \frac{\partial}{\partial y}\right) f\right) y^{k/2} e^{-ik\theta} \end{cases}$$

If $F \longleftrightarrow f$, then $(g \longmapsto F(\gamma g))$ corresponds to $f|_k \gamma$ $(\gamma \in G)$

Cor: f is holomorphic $\Longleftrightarrow (X + Y + iH) F = 0$

$$(X + Y - iH)(X + Y + iH) = (Y - X)^2 + 2\Omega + 2i(Y - X)$$

Cor: if f is holomorphic $\Longrightarrow 2\Omega F + ((-ik)^2 + 2i(-ik)) F = 0$

$$\Longrightarrow \Delta F = -\frac{\Omega}{2} F = \frac{k}{2}\left(1 - \frac{k}{2}\right) F$$

Passage from H to $-X+Y$

Standard Cartan subalgebra of $\mathfrak{sl}(2)$: $\mathbb{R}H = \mathrm{Lie}(A)$, $A = \{e^{tH} \mid t \in \mathbb{R}\}$
$= \left\{\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix},\ a > 0\right\}$

Compact Cartan subalgebra: $\mathrm{Lie}\, SO(2) = \mathbb{R}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \mathbb{R}(-X+Y)$

$$SO(2) = \{ e^{\theta(-X+Y)} = h_\theta \mid \theta \in \mathbb{R} \}$$

Over $\mathbb{C}$: two Cartan subalgebras are conjugate

change of basis between standard coordinates x, y in $\mathbb{R}^2$ and the coordinates $z = x + iy$, $\bar{z} = x - iy$: $\begin{pmatrix} z \\ \bar{z} \end{pmatrix} = \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

standard basis (X, Y, H) of $\mathfrak{sl}(2)$ is conjugated by $g = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$ to $g(X, Y, H) g^{-1} = (X_+, X_-, H_0)$, where

$$\boxed{X_\pm = \frac{H \pm i(X+Y)}{2}, \quad H_0 = i(Y - X)}$$

Again, $[X_+, X_-] = H_0$, $\quad [H_0, X_\pm] = \pm 2 X_\pm$

Summary of the formulas: $f : \mathcal{H} \longrightarrow \mathbb{C}$

lift of weight k: $F = \tilde{f} : G \longrightarrow \mathbb{C}$, $\tilde{f}(g) = f(x+iy)\, y^{k/2} e^{-ik\theta}$

$\tau = x+iy = g(i)$

$$g = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix} h_\theta, \quad h_\theta = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \qquad \tilde{f}(g h_\theta) = e^{-ik\theta} \tilde{f}(g)$$

Action of $\mathfrak{sl}(2)_{\mathbb{C}}$ on $\tilde{f}$:

$$\boxed{X_\pm = e^{\mp 2i\theta}\left(y(\partial_y \pm i\partial_x) \pm \frac{i}{2}\partial_\theta\right) \\ H_0 = i(-X+Y) = i\,\partial_\theta}$$

$$\boxed{X_+ = i e^{-2i\theta}\left(2y\,\partial_\tau + \frac{1}{2}\partial_\theta\right) \\ X_- = -i e^{2i\theta}\left(2y\,\partial_{\bar\tau} + \frac{1}{2}\partial_\theta\right)}$$

$$\boxed{H_0 \tilde{f} = k\tilde{f}} \Longrightarrow \boxed{H_0(X_\pm \tilde{f}) = (k\pm 2)(X_\pm \tilde{f})}$$

Action of $X_\pm$ changes weight: it replaces k by $k\pm 2$

(*)
$$X_+\tilde{f} = \widetilde{\left(2i\,\partial_\tau f + \frac{k}{y} f\right)} = \widetilde{R_k f}, \qquad R_k = 2i\,\partial_\tau + \frac{k}{y}$$

lift of weight $k+2$

$$X_-\tilde{f} = \widetilde{\left(-2iy^2\,\partial_{\bar\tau} f\right)} = \widetilde{L_k f}, \qquad L_k = -2iy^2\,\partial_{\bar\tau}$$

lift from $\mathcal{H}$ to G of weight $k-2$

$$\Delta = \Delta_k = -y^2\left(\left(\frac{\partial}{\partial x}\right)^2 + \left(\frac{\partial}{\partial y}\right)^2\right) + 2iky\frac{\partial}{\partial \bar\tau} + \frac{k}{2}\left(1-\frac{k}{2}\right)$$

$$L_{k-2}\circ L_k f \curvearrowleft L_k f \curvearrowleft f \curvearrowright R_k f \curvearrowright R_{k+2}\circ R_k f \curvearrowright \cdots$$

weights: $k-4$ $\quad k-2 \quad k \quad k+2 \quad k+4$

Note: $R(g)$ $(g \in G)$ commutes with left action $L(g')$, so if $\tilde{f} : G \longrightarrow \mathbb{C}$ satisfies $\forall \gamma \in \Gamma$ $\tilde{f}(\gamma g) = \tilde{f}(g)$ $\Longrightarrow$ so do $X_\pm \tilde{f}$

$\Longrightarrow$ linear differential operators of 1st order $(K = SO(2))$

$$\begin{array}{ccccc}
\tilde{M}_{k-2}(\Gamma) & \xleftarrow{X_-} & \tilde{M}_k(\Gamma) := \{\tilde{f} : G \longrightarrow V \mid \tilde{f}(\gamma g h_\theta) = e^{-ik\theta}\tilde{f}(g)\ \forall \gamma\in\Gamma, \forall h_\theta \in K\} & \xrightarrow{X_+} & \tilde{M}_{k+2}(\Gamma) \\
\updownarrow \text{ lift of wt} = k-2 & & \updownarrow \text{ lift of weight } k & & \updownarrow \\
M_{k-2}(\Gamma) & \xleftarrow{L_k} & M_k(\Gamma) := \{f : \mathcal{H} \longrightarrow V \mid \forall \gamma \in \Gamma \ f(\gamma(\tau)) = \underbrace{J(\gamma,\tau)^k}_{(c\tau+d)^k} f(\tau)\} & \xrightarrow{R_k} & M_{k+2}(\Gamma)
\end{array}$$

Note: $\boxed{f \text{ is holomorphic} \Longleftrightarrow X_-\tilde{f} = 0}$

Maass–Shimura differential operators and Taylor expansions

Recall: $\frac{1}{2i} R_k = \frac{\partial}{\partial \tau} + \frac{k}{\tau - \bar{\tau}}$, $\quad R_k^r = R_{k+2(r-1)} \circ \cdots \circ R_{k+2} \circ R_k$

($R_k = \delta_k$, $\quad R_k^r = \delta_k^r$ — older notation)

Fix $k \in \mathbb{Z}$, $\tau \in \mathcal{H}$. Given: $f : \mathcal{H} \to \mathbb{C}$ holomorphic

(in practice, $f \in M_k(\Gamma)$, $\Gamma \subset SL_2(\mathbb{R})$)

Goal: write the Taylor expansion of f around τ in an invariant way, compatible with $\mathcal{H} \longrightarrow \Gamma \backslash \mathcal{H}$.

Cayley map: $\mathcal{H} \xrightarrow{\sim} D = \{ |w| < 1 \}$

$$z \longmapsto \alpha_\tau(z) = \frac{z - \tau}{z - \bar{\tau}} = w \qquad \alpha_\tau = \begin{pmatrix} 1 & -\tau \\ 1 & -\bar{\tau} \end{pmatrix}$$

$$\beta_\tau(w) = \frac{-\bar{\tau} w + \tau}{-w + 1} \longleftarrow w \qquad \beta_\tau = \begin{pmatrix} -\bar{\tau} & \tau \\ -1 & 1 \end{pmatrix} = \alpha_\tau^2$$

$$\begin{tikzcd}
\mathbb{C} & \mathcal{H} \ni \tau \arrow[l, "f"'] \arrow[r, bend left, "\alpha_\tau"] & D \ni 0 \arrow[l, bend left, "\beta_\tau"] \arrow[ll, bend left=40, "f \circ \beta_\tau"]
\end{tikzcd}$$

$\det(\beta_\tau) = \tau - \bar{\tau}$

$J(\beta, w) = 1 - w$

Let $g = f|_k \beta_\tau$: $\quad g(w) = (f \circ \beta_\tau)(w) \underbrace{J(\beta, w)^{-k}}_{(1-w)^{-k}} \underbrace{\det(\beta)^{k/2}}_{(\tau - \bar{\tau})^{k/2}}$

Formulas:

$$\frac{dg}{dw} = \underbrace{\left(\left(\frac{df}{dz} \right) \circ \beta_\tau \right) \underbrace{\frac{d\beta_\tau}{dw}}_{(\tau - \bar{\tau})(1-w)^{-2}} (1-w)^{-k} (\tau - \bar{\tau})^{k/2}}_{\left(\frac{df}{dz} \big|_{k+2} \beta_\tau \right)(w)} + \underbrace{(f \circ \beta_\tau)\, k\, (1-w)^{-k-1} (\tau - \bar{\tau})^{k/2}}_{\underbrace{\frac{k(1-w)}{\tau - \bar{\tau}}}_{\frac{k}{z - \bar{\tau}}} (f|_{k+2} \beta_\tau)(w)}$$

So:

$$\frac{d}{dw} (f|_k \beta_\tau) = \Bigg(\Big(\underbrace{\frac{d}{dz} + \frac{k}{z - \bar{\tau}}}_{\text{“} \frac{1}{2i} R_k \text{”}} \Big) f \Bigg) \Bigg|_{k+2} \beta_\tau$$

in the variable z, but keeping $\bar{\tau}$

$$\Longrightarrow \quad \left(\left(\frac{d}{dw} \right)^r (f|_k \beta_\tau) \right)(w = 0) = \frac{1}{(2i)^r} \left((R_k^r f) \big|_{k+2r} \beta_\tau \right)(w = 0)$$

$$\boxed{\left(\frac{d}{dw} \right)^r \left(\frac{(f \circ \beta_\tau)(w)}{(1-w)^k} \right) \Bigg|_{w=0} = \operatorname{Im}(\tau)^r \, (R_k^r f)(\tau)}$$

The case of half-integral weight

Work on $\widetilde{G} = Mp_2(\mathbb{R}) \xrightarrow{p} G$ two-fold covering $(u+vi=\tau\in\mathcal{H})$

p has natural splitting over $NA = \underbrace{\left\{ \begin{pmatrix}1&u\\0&1\end{pmatrix}\begin{pmatrix}v^{1/2}&0\\0&v^{-1/2}\end{pmatrix} \right\}}_{g_\tau} \subset G$:

$\widetilde{G}$ acts on $\mathcal{S}(\mathbb{R}) \subset L^2(\mathbb{R})$, and we take

$$\left(\widetilde{\begin{pmatrix}1&u\\0&1\end{pmatrix}}\phi\right)(x) := e^{\pi i u x^2}\phi(x), \quad \left(\widetilde{\begin{pmatrix}v^{1/2}&0\\0&v^{-1/2}\end{pmatrix}}\phi\right)(x) = v^{1/4}\phi(v^{1/2}x)$$

Under the ~~...~~ the action of $\mathrm{Lie}(\widetilde{G}) \xrightarrow[\sim]{p} \mathrm{Lie}(G)$, this gives

$$X\phi = \frac{d}{dt}\widetilde{\begin{pmatrix}1&t\\0&1\end{pmatrix}}\phi\Big|_{t=0} = \cancel{\pi i}x^2\phi = \frac{2\pi i x^2}{2}\phi, \quad H\phi = \frac{d}{dt}\widetilde{\begin{pmatrix}e^t&0\\0&e^{-t}\end{pmatrix}}\phi\Big|_{t=0} = \left(\frac{1}{2}+x\frac{d}{dx}\right)\phi = \frac{1}{2}\left(x\frac{d}{dx}+\frac{d}{dx}x\right)\phi$$

As $p(\pm e^{-2\pi i/8}\mathcal{F}) = J = \begin{pmatrix}0&-1\\1&0\end{pmatrix}$, ($\mathcal{F}$ = Fourier transform), conjugation by $\mathcal{F}$ defines a natural splitting of p over

$$JNAJ^{-1} = \left\{ \begin{pmatrix}a&0\\ *&a^{-1}\end{pmatrix} \,\Big|\, a>0 \right\}, \text{ and } JXJ^{-1} = -Y$$

$$\Longrightarrow \quad Y\phi = \frac{-2\pi i(\mathcal{F}x\mathcal{F}^{-1})^2}{2}\phi = \frac{1}{2}\left(-\frac{1}{2\pi i}\left(\frac{d}{dx}\right)^2\right)\phi$$

$$\boxed{H_0 = i(-X+Y) = \pi x^2 - \frac{1}{4\pi}\left(\frac{d}{dx}\right)^2} \qquad \boxed{\widetilde{g}_\tau\phi_i = \mathrm{Im}(\tau)^{1/4}\phi_\tau}$$

Action on the Gaussian: $\phi_\tau(x) = e^{\pi i\tau x^2}$, $\phi_i(x) = e^{-\pi x^2}$

$$\frac{d}{dx}\phi_i = (-2\pi x)\phi_i \Rightarrow \left(\frac{d}{dx}\right)^2\phi_i = -2\pi\phi_i + (2\pi x)^2\phi_i \Longrightarrow \boxed{H_0\phi_i = \frac{1}{2}\phi_i}$$

Restriction of $p: \widetilde{G} \longrightarrow G$ to $SO(2)=K$: (weight $\frac{1}{2}$)

$$\boxed{(-X+Y)\phi_i = -\frac{i}{2}\phi_i}$$

$$\begin{array}{ccc}\widetilde{K} & \longrightarrow & K\\ \| & & \| \\ \{\widetilde{h}_t = e^{t(H_0/i)} \mid t\in\mathbb{R}\} & \longrightarrow & \left\{ h_t = \begin{pmatrix}\cos t & -\sin t\\ \sin t & \cos t\end{pmatrix}\right\}\end{array}$$

$$\boxed{e^{t(H_0/i)}\phi_i = e^{-it/2}\phi_i}$$ corresponds to $k=\frac{1}{2}$ (half-integral weight)

Producing the theta function $\theta(\tau)$:

$$\widetilde{\theta}(g) = \langle \delta_{\mathbb{Z}}, g\phi_i\rangle \quad \left(\langle\delta_{\mathbb{Z}},\phi\rangle = \sum_{n\in\mathbb{Z}}\phi(n)\right) \quad (g\in\widetilde{G})$$

$$\boxed{\widetilde{\theta} = \text{lift of wt} = \tfrac{1}{2} \text{ of } \theta}$$

$$\theta(\tau) = \mathrm{Im}(\tau)^{-1/4}\langle\delta_{\mathbb{Z}}, \underbrace{\widetilde{g}_\tau\phi_i}_{\mathrm{Im}(\tau)^{1/4}\phi_\tau}\rangle = \sum_{n\in\mathbb{Z}}\phi_\tau(n) = \sum_{n\in\mathbb{Z}} e^{\pi i\tau n^2}$$

Applying the weight raising operators X_+ :

$\forall m \geq 0$ $\quad X_+^m \phi_i$ satisfies $(-X+Y)\, X_+^m \phi_i = \frac{-i\left(\frac{1}{2}+2m\right)}{2} X_+^m \phi_i$

$\Longrightarrow$ get $\tilde{\theta}_m(g) \doteq \langle \delta_{\mathbb{Z}}, g X_+^m \phi_i \rangle$, which is a lift (of weight $2m+\frac{1}{2}$)

of $\theta_m(\tau) = \operatorname{Im}(\tau)^{-1/4} \langle \delta_{\mathbb{Z}}, \tilde{g}_\tau X_+^m \phi_i \rangle$

As $\delta_{\mathbb{Z}}$ is "invariant" under any $\tilde{\gamma} \in p^{-1}(\Gamma_\theta)$, $\quad \left(\delta_{\mathbb{Z}} \tilde{\gamma} = \eta(\tilde{\gamma}) \delta_{\mathbb{Z}},\ \eta(\tilde{\gamma})^8 = 1 \right)$

$\forall\ \alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_\theta \qquad \theta_m\left(\frac{a\tau+b}{c\tau+d}\right) = \varepsilon_\alpha\, (c\tau+d)^{2m+1/2}\, \theta_m(\tau) \qquad (*)$

$\varepsilon_\alpha^8 = 1$

Exercise: compute $X_+^m \phi_i$ in terms of Hermite polynomials.

Rmks: (1) Clearly, $(X_+^m \phi_i)(x) = P_m(x)\, \phi_i(x)$ for some (even) polynomial $P_m(x)$ of degree $2m \Longrightarrow (\tilde{g}_\tau X_+^m \phi_i)(x) = \operatorname{Im}(\tau)^{1/4} P_m(\operatorname{Im}(\tau)^{1/2} x)\, \phi_\tau(x)$

$$\Longrightarrow \boxed{\theta_m(\tau) = \sum_{n \in \mathbb{Z}} \operatorname{Im}(\tau)^{-m} P_m(\operatorname{Im}(\tau)^{1/2} n)\, e^{\pi i n^2 \tau}}$$

(2) Given $\phi_{\mathbb{Z}} : \mathbb{Z}/N\mathbb{Z} \longrightarrow \mathbb{C}$, the distribution $\underset{\mathcal{S}(\mathbb{R})}{\phi} \longmapsto \sum_{n \in \mathbb{Z}} \phi_{\mathbb{Z}}(n) \phi(n)$ is "invariant" under $p^{-1}(\Gamma_\theta \cap \Gamma(N))$

$$\Longrightarrow \quad \sum_{n \in \mathbb{Z}} \phi_{\mathbb{Z}}(n)\, \operatorname{Im}(\tau)^{-m} P_m(\operatorname{Im}(\tau)^{1/2} n)\, e^{\pi i n^2 \tau}$$

satisfies $(*)$ for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_\theta \cap \underbrace{\Gamma(N)}_{\left\{ \alpha \in SL_2(\mathbb{Z}) \,\middle|\, \alpha \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} (\operatorname{mod} N) \right\}}$

---

(3) Above: for any tempered distribution $T : \mathcal{S}(\mathbb{R}) \longrightarrow \mathbb{C}$ and any vector $\phi \in \mathcal{S}(\mathbb{R})$ such that $H_0 \phi = k\phi$ (where $k \in \frac{1}{2}\mathbb{Z}$), $\forall t \in \mathbb{R}$ $\tilde{h}_t \phi = e^{-ikt} \phi \Longrightarrow$ the function $F(g) := \langle T, g\phi \rangle$ $(g \in \tilde{G})$ satisfies $\forall t \in \mathbb{R}$ $F(g \tilde{h}_t) = e^{-ikt} F(g) \Longrightarrow$ it gives rise to $f : \mathcal{H} \longrightarrow \mathbb{C}$ $f(\tau) := \operatorname{Im}(\tau)^{-k/2} F(\tilde{g}_\tau)$. If T is "invariant" under $p^{-1}(\Gamma)$, $\Gamma \subset G$, so is F (under $L(p^{-1}(\Gamma))$) $\Longrightarrow$ f satisfies $(*)$ with $(c\tau+d)^k$.

General notion of a modular form

$G = SL_2(\mathbb{R})$, $K = SO(2) = \{ h_\theta \mid \theta \in \mathbb{R}/2\pi\mathbb{Z} \}$, $G/K \xrightarrow{\sim} \mathcal{H}$

$k \in \mathbb{Z}$

$g = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix} h_\theta$, $\quad F(g) = y^{k/2} e^{-ik\theta} f(x+iy)$

$\Gamma \subset G$ discrete subgroup (such that $\mathrm{vol}(\Gamma \backslash \mathcal{H}) < \infty$)

(e.g., $\Gamma \subset SL_2(\mathbb{Z})$ of finite index).

Interesting functions: satisfy $F(\gamma g h_\theta) = e^{-ik\theta} F(g)$

$F : G \longrightarrow \mathbb{C} = V$ $\qquad \gamma \in \Gamma$, $h_\theta \in K$, $\quad h_\theta \mapsto e^{ik\theta}$ representation

$K \longrightarrow GL_1(\mathbb{C}) = GL(V)$

AND $\quad P(\Omega) F = 0$

for some non-zero polynomial P (Ω = Casimir operator)

Rmk: (1) One often imposes additional growth conditions on F.

(2) the simplest case: $\deg(P) = 1$: $\Delta F = \lambda F$ $\quad (\Delta = -\frac{\Omega}{2})$

Assume: $\boxed{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \Gamma}$ $\quad (\Longleftrightarrow \ f(\tau+1) = f(\tau))$

$$\Longrightarrow \quad F\Big(\underbrace{\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix} h_\theta}_{g}\Big) = \sum_{n \in \mathbb{Z}} a_n(g) e^{2\pi i n x}$$

$$a_n(g) = \int_{\mathbb{R}/\mathbb{Z}} F\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g \right) e^{-2\pi i n x} \, dx$$

Properties: $\Delta a_n = \lambda a_n$, $\quad a_n\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix} h_\theta \right) = e^{2\pi i n x} e^{-ik\theta} A_n(y)$

$$\Delta a_n = 4\pi^2 n^2 y^2 a_n - 2\pi k n y \, a_n - e^{2\pi i n x} e^{-ik\theta} y^2 A_n''(y)$$

So: $\Delta a_n = \lambda a_n \Longleftrightarrow \boxed{A_n''(y) + \left(-4\pi^2 n^2 + \frac{2\pi k n}{y} + \frac{\lambda}{y^2} \right) A_n(y) = 0}$

Whittaker differential equation (related to the hypergeometric equation for ${}_1F_1$)

In fact: $A_n(y) = A(2\pi n y)$ ($\forall$ fixed n), $\quad A''(y) + \left(-1 + \frac{k}{y} + \frac{\lambda}{y^2} \right) A(y) = 0$

Case $\lambda = \frac{k}{2}\left(1 - \frac{k}{2}\right)$: one solution $A(y) = c \cdot y^{k/2} e^{-y}$ (decreases fast as $y \to +\infty$)

this contributes holomorphic terms to $f(\tau)$:

$$\sum_{n \in \mathbb{Z}} b_n q^n, \quad q = e^{2\pi i \tau}.$$

Non-holomorphic case with $k=0$

Ex: $f(\tau) = \sum_{m,n}{}' \frac{y^s}{|m\tau+n|^{2s}} = \zeta(2s) \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \operatorname{Im}(\gamma\tau)^s$, $\quad \Gamma = SL_2(\mathbb{Z}) \cup \Gamma_\infty = \left\{ \pm\begin{pmatrix}1&*\\0&1\end{pmatrix}\right\}$

$\tau \mapsto \operatorname{Im}(\tau)^s = y^s$ satisfies $\Delta y^s = s(1-s)y^s$

In general: if $F(g) = f(x+iy)$, $\Delta F = s(1-s)F$, $x+iy = g(i)$
$f(\tau+1) = f(\tau)$

$\Longrightarrow$ Fourier coefficients of F $\quad a_n(g) = \int_{\mathbb{R}/\mathbb{Z}} F\left(\begin{pmatrix}1&x\\0&1\end{pmatrix}g\right) e^{-2\pi i n x}\,dx \quad (n\in\mathbb{Z})$

$a_n\left(\begin{pmatrix}1&x\\0&1\end{pmatrix}\begin{pmatrix}y^{1/2}&0\\0&y^{-1/2}\end{pmatrix} h_\theta\right) = e^{2\pi i n x} A_n(y)$

As above (for $k=0$, $\lambda = s(1-s)$): $\quad A_n(y) = A(2\pi n) \quad$ (fix $n\in\mathbb{Z}$)

$$A''(y) + \left(-1 + \frac{s(1-s)}{y^2}\right) A(y) = 0$$

Put $A(y) = y^{1/2} B(y)$:

$$\boxed{B''(y) + \frac{1}{y} B(y) - \left(1 + \frac{(s-\frac12)^2}{y^2}\right) B(y) = 0}$$

variant of the Bessel differential equation (with index $i(s-\frac12)$)

"Good solution" with exponential decay at $y \to +\infty$:

$$B(y) = (\text{const.})\, K_{s-\frac12}(y)$$

So if f (and F) satisfy an appropriate growth condition,

$$\boxed{f(\tau) = b_0 + \sum_{0\neq n\in\mathbb{Z}} b_n(f)\, y^{1/2} K_{s-1/2}(2\pi n y)\, e^{2\pi i n x} \qquad (\tau = x+iy)}$$

Confluent hypergeometric functions ${}_1F_1(a,c;z)$

The function $\displaystyle {}_1F_1(a,c;z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n}\frac{z^n}{n!}$ $\qquad (a)_n = a(a+1)\cdots(a+n-1)$

$(a, c \in \mathbb{C}, \quad c \notin \mathbb{Z}_{\leq 0})$

is holomorphic in $\mathbb{C}$ and satisfies the confluent hypergeometric differential equation

$$(*) \qquad (xD^2 + (c-x)D - a)u = 0 \qquad D = \frac{d}{dx}$$

This equation has 2 singularities: regular at 0, irregular at ∞. It is obtained from the hypergeometric differential equation for ${}_2F_1$ (which has regular singularities at $0, 1$ and ∞) by letting two singular points coalesce.

Transformations between differential equations $(pD^2 + qD + r)u = 0$: $(p = p(x), \ldots)$

(1) Gauge transformation $u = \varphi v$:

If $L = pD^2 + qD + r$, then $\dfrac{L(\varphi v)}{\varphi} = L_\varphi v$, where

$$L_\varphi = pD^2 + \left(2p\frac{D\varphi}{\varphi} + q\right)D + \left(p\frac{D^2\varphi}{\varphi} + q\frac{D\varphi}{\varphi} + r\right)$$

Special form: when $L_\varphi = pD^2 + r_\varphi \iff 2p\varphi' + q = 0 \iff \varphi(x) = \exp\left(\int_{x_0}^{x} -\frac{q}{2p}\right)$

Rmk: In general, L is self-adjoint with respect to the scalar product $(f,g)_w := \int_a^b f(x)g(x)w(x)\,dx$ given by a weight function w

(i.e., $(Lf,g)_w = (f,Lg)_w$ if $f(a)=f(b)=0$ and $g(a)=g(b)=0$)

$\iff (pw)' = qw \iff w(x) = \exp\left(-p + \int_{x_0}^{x}\frac{q}{p}\right)$

Pf: $(Lf,g)_w - (f,Lg)_w = \int_a^b (f'g - fg')(qw - (pw)')\,dx$ for such f,g

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(2) Change of variables: $u = v\circ y$, $\quad u(x) = v(y(x))$ for some $y(x)$

$$Lu = py'^2(v''\circ y) + (qy' + py'')(v'\circ y) + r(v\circ y)$$

One can keep r fixed but replace p by 1 if one takes y such that $y' = p^{-1/2}$.

Back to (*) $(xD^2+(c-x)D-a)u=0$.

Bessel's equation reduces to a special case of (*):

$f=J_{\pm\nu}(x)$ are solutions of $(x^2D^2+xD+(x^2-\nu^2))f=0$

$\Rightarrow$ $g(x)=e^{\pm ix}x^{\nu}f(x)$ is a solution of $(xD^2+((2\nu+1)\pm 2ix)D\pm i(2\nu+1))g=0$

$\Rightarrow$ $u(t)=g\left(\pm\frac{it}{2}\right)$ is a solution of $\left(t\left(\frac{d}{dt}\right)^2+((2\nu+1)-t)\frac{d}{dt}-(\nu+\frac{1}{2})\right)u=0$,

which is (*) for $c=2\nu+1$, $a=-(\nu+\frac{1}{2})=-\frac{c}{2}$.

Applying gauge tranformation $u=\varphi v$ to (*) to get rid of D:

need $2x\frac{\varphi'}{\varphi}+(c-x)=0 \iff \varphi=(\text{const})\,e^{x/2}x^{-c/2}$

then $L=xD^2+(c-x)D-a$ satisfies

$L_\varphi v=\frac{L(\varphi v)}{\varphi}=(D^2+r_\varphi)v$, $r_\varphi=-\frac{1}{4}+\frac{\kappa}{x}+\frac{1-4\mu^2}{4x^2}$, $a=\mu-\kappa+\frac{1}{2}$, $c=1+2\mu$

So: u is a solution of (*) $\iff$ $v=e^{-x/2}x^{c/2}u$ is a solution of Whittaker's equation $v''+\left(-\frac{1}{4}+\frac{\kappa}{x}+\frac{1-4\mu^2}{4x^2}\right)v=0$.

Symmetry of the differential operator $L_{a,c}=\underbrace{(xD)(xD+c-1)-x(xD+a)}_{x\,(xD^2+(c-x)D-a)}$:

$x^{-b}\circ(xD)\circ x^{b}=xD+b$

$\Rightarrow$ $x^{c-1}\circ L_{a,c}\circ x^{1-c}=L_{a+1-c,\,2-c}$

Standard solution of (*): $M(a,c;x)={}_1F_1(a,c;x)=\sum_{n=0}^{\infty}\frac{(a)_n}{(c)_n}\frac{x^n}{n!}$ $(c\notin\mathbb{Z}_{\le 0})$

Cor: if $c\notin\mathbb{Z}_{\ge 2}$, then $x^{1-c}M(a+1-c,2-c;x)$ is also a solution of (*).

Rmk: the exponents of (*) at the regular singular point 0 are, indeed, 0 and $1-c$.

Integral representation of $M(a,c;x)$: if $\mathrm{Re}(c)>\mathrm{Re}(a)>0$

$$\frac{(a)_n}{(c)_n}=\frac{\Gamma(a+n)/\Gamma(a)}{\Gamma(c+n)/\Gamma(c)}=\underbrace{\frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}}_{1/B(a,c-a)}\underbrace{\frac{\Gamma(a+n)\Gamma(c-a)}{\Gamma(c+n)}}_{B(a+n,c-a)}=\frac{1}{B(a,c-a)}\int_0^1 t^{n+a-1}(1-t)^{c-a-1}\,dt$$

$$\Longrightarrow\quad M(a,c;x)=\frac{1}{B(a,c-a)}\int_0^1 t^{a-1}(1-t)^{c-a-1}e^{tx}\,dt\,.$$

Asymptotics for $\mathrm{Re}(x)\to+\infty$: $\quad(u''-u')+\left(\frac{c}{x}u'-\frac{a}{x}u\right)=0 \qquad v''-v'=0 \;\Downarrow\; v=\lambda_1+\lambda_2e^x$

$(1-t)x=s$

$$B(a,c-a)M(a,c;x)=\int_0^x\left(1-\frac{s}{x}\right)^{a-1}\left(\frac{s}{x}\right)^{c-a-1}e^{x-s}\frac{ds}{x}=x^{a-c}e^x\int_0^x\left(1-\frac{s}{x}\right)^{a-1}s^{c-a-1}e^{-s}\,ds$$

$$\downarrow\; x\to+\infty \qquad \int_0^\infty s^{c-a-1}e^{-s}\,ds=\Gamma(c-a)$$

Conclusion: $\displaystyle\lim_{\mathrm{Re}(x)\to+\infty}\frac{M(a,c;x)}{e^x x^{a-c}}=\frac{\Gamma(c)}{\Gamma(a)}$ (if $\mathrm{Re}(c)>\mathrm{Re}(a)>0$)

Solution of (*) not growing exponentially as $\mathrm{Re}(x)\to+\infty$:

Write $u(x)=\int_0^\infty e^{-xt}\varphi(t)\,dt$. Then $0=xu''+(c-x)u'-au=$

$$=\int_0^\infty (xe^{-tx})(t^2+t)\varphi\,dt-\int_0^\infty e^{-xt}(a+ct)\varphi\,dt=\int_0^\infty e^{-xt}\{((t^2+t)\varphi)'-(a+ct)\varphi\}\,dt$$

(if $\varphi'\in L^1$ and $\lim_{t\to0+}t\varphi(t)=0$) $\Longrightarrow\quad ((t^2+t)\varphi)'-(a+ct)\varphi=0$

$$\varphi'/\varphi=\frac{(c-2)t+(a-1)}{t^2+t}=\frac{c-a-1}{t+1}+\frac{a-1}{t}\Longrightarrow\varphi=(\mathrm{const})\,t^{a-1}(1+t)^{c-a-1}$$ (need $\mathrm{Re}(a)>0$)

Def: if $\mathrm{Re}(a)>0$, let $\displaystyle U(a,c;x):=\frac{1}{\Gamma(a)}\int_0^\infty e^{-xt}t^{a-1}(1+t)^{c-a-1}\,dt$

$$=\frac{x^{-a}}{\Gamma(a)}\int_0^\infty e^{-s}s^{a-1}\left(1+\frac{s}{x}\right)^{c-a-1}ds$$

($\Longrightarrow \lim_{\mathrm{Re}(x)\to+\infty}x^aU(a,c;x)=1$)

Prop. If $\mathrm{Re}(a)>0$ and $c\notin\mathbb{Z}$, then

$$U(a,c;x)=A(a,c)M(a,c;x)+B(a,c)x^{1-c}M(a+1-c,2-c;x)$$ (we already know this)

with $$A(a,c)\underset{x\to0}{=}\frac{1}{\Gamma(a)}\int_0^\infty t^{a-1}(1+t)^{c-a-1}=\frac{B(a,1-c)}{\Gamma(a)}=\frac{\Gamma(1-c)}{\Gamma(a+1-c)}$$

$$B(a,c)=\lim_{x\to0+}x^{c-1}U(a,c;x)=\lim_{x\to0+}\frac{1}{\Gamma(a)}\int_0^\infty e^{-s}s^{a-1}(x+s)^{c-a-1}\,ds=\frac{\Gamma(c-1)}{\Gamma(a)}$$

(if $\mathrm{Re}(c)>1$; general case – analytic cont.)

Kummer: (1) $v(x)$ solution of (*) for $(c-a, c) \Longrightarrow u(x) = e^x v(-x)$ solution of (*) for (a, c)

Letting $x \to 0$, we obtain $\underline{M(a,c;x) = e^x M(c-a, c; -x)}$

(2) $\underline{M(a, 2a; 4x) = e^{2x} {}_0F_1(a+\tfrac{1}{2}; x^2)}$, ${}_0F_1(b; x) = \sum_{n=0}^{\infty} \frac{1}{(b)_n} \frac{x^n}{n!}$.

Contiguous functions : $M := M(a, c; x)$, $M(a\pm) := M(a\pm 1, c; x)$
$M(c\pm) := M(a, c\pm 1; x)$

$$\left(x\frac{d}{dx} - x\right) M = (a-c)(M - M(a-))$$
$$(a-c+1) \quad M = a M(a+) - (c-1) M(c-)$$
$$c M = c M(a-) + x M(c+)$$

Similarly for U:

$$\frac{d}{dx} U(a,c) = -a\, U(a+1, c+1) =$$
$$= U(a,c) - U(a, c+1)$$
$$U = a U(a+) + U(c-)$$
$$(c-a) U = x U(c+) - U(a-)$$

Special cases: (1) $M(c, c; x) = e^x$

(2) $\underline{\operatorname{erf}(x)} := \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt = \underline{\frac{2x}{\sqrt{\pi}} M(\tfrac{1}{2}, \tfrac{3}{2}; -x^2)}$

(3) Incomplete Γ-function : (for $x > 0$)

$$\underline{\gamma(a,x)} := \int_0^x e^{-t} t^a \frac{dt}{t} = \underline{\frac{x^a}{a} M(a, a+1; -x)} = \frac{x^a e^{-x}}{a} M(1, a+1; x)$$

$$\underline{\Gamma(a,x)} := \int_x^\infty e^{-t} t^a \frac{dt}{t} = \underline{x^a e^{-x} U(1, a+1; x)}$$

(check: $U(1, a+1; x) = \underbrace{\frac{\Gamma(-a)}{\underbrace{\Gamma(1-a)}_{1/(-a)}}}\, M(1, a+1; x) + \Gamma(a) x^{-a} \underbrace{M(1-a, 1-a; x)}_{e^x}$,

but $M(1, a+1; x) = e^x M(a, a+1; -x)$ (Kummer), so, indeed,
$\gamma(a,x) + \Gamma(a,x) = \Gamma(a)$)

(4) Special case $a = \frac{1}{2}$ of (3) :

$$\gamma(\tfrac{1}{2}, y^2) = \int_0^{y^2} e^{-t} t^{-1/2} dt = 2\int_0^y e^{-s^2} ds = \sqrt{\pi}\, \operatorname{erf}(y) = \underline{2y M(\tfrac{1}{2}, \tfrac{3}{2}; -y^2)}$$

$$\Gamma(\tfrac{1}{2}, y^2) = \Gamma(\tfrac{1}{2}) - \gamma(\tfrac{1}{2}, y^2) = \sqrt{\pi}\, \operatorname{erfc}(y) = y e^{-y^2} U(1, \tfrac{3}{2}; y^2) \qquad (y > 0)$$

(5) $\operatorname{Ei}(z) = \int_{-\infty}^{z} \frac{e^t dt}{t}$ $\quad (z \in [0, +\infty))$, $\operatorname{Ei}(z) = -e^z U(1, 1; -z)$

Back to the Whittaker equation.

(**) $\quad v'' + \left(\frac{1}{4} + \frac{\kappa}{x} + \frac{1-4\mu^2}{4x^2}\right)v = 0$: solutions are $v = e^{-x/2}x^{c/2}u$, where u is a solution of (*) for

$(xu'' + (c-x)u' - au = 0)$ $\quad (a,c) = (\mu - \kappa + \frac{1}{2}, 1+2\mu)$

Symmetry: replacing (a,c) by $(a+1-c, 2-c) = (-\mu-\kappa+\frac{1}{2}, 1-\mu)$ amounts to replacing μ by $-\mu$.

If $2\mu \notin \mathbb{Z}$: (**) has two linearly independent solutions

$$M_{\kappa,\mu}(x) = e^{-x/2}x^{\mu+1/2}M(\mu-\kappa+\tfrac{1}{2}, 1+2\mu; x)$$

$$M_{\kappa,-\mu}(x) = e^{-x/2}x^{-\mu+1/2}M(-\mu-\kappa+\tfrac{1}{2}, 1-2\mu; x)$$

Asymptotics: as $\mathrm{Re}(x) \to +\infty$, $\quad M_{\kappa,\mu}(x) \sim \frac{\Gamma(1+2\mu)}{\Gamma(\mu-\kappa+\frac{1}{2})} x^{-\kappa}e^{x/2}$

Solution with exponential decay as $\mathrm{Re}(x) \to +\infty$,

$$W_{\kappa,\mu}(x) := e^{-x/2}x^{\mu+1/2}U(\mu-\kappa+\tfrac{1}{2}, 1+2\mu; x)$$

$$= \frac{\Gamma(-2\mu)}{\Gamma(-\mu-\kappa+\frac{1}{2})} M_{\kappa,\mu}(x) + \frac{\Gamma(2\mu)}{\Gamma(\mu-\kappa+\frac{1}{2})} M_{\kappa,-\mu}(x) = W_{\kappa,-\mu}(x)$$

$W_{\kappa,\mu}(x) \sim x^{\kappa}e^{-x/2}$ as $\mathrm{Re}(x) \to +\infty$.

Relation to Bessel functions: solutions of $x^2f'' + xf' + (x^2-\nu^2)f = 0$

Gauge transformation to get rid of ∂: $f = x^{-1/2}g$, $\quad g'' + \left(1 + \frac{1-4\nu^2}{4x^2}\right)g = 0$

Change of variables $t = \pm 2ix$:

$g(x) = u(\pm 2ix)$

$u(t) = g(\mp \frac{i}{2}t)$

$$\boxed{\left(\frac{d}{dt}\right)^2 u + \left(-\frac{1}{4} + \frac{1-4\nu^2}{4t^2}\right)u = 0}$$

Whittaker equation for $\kappa = 0$, $\mu = \nu$ ($\Leftrightarrow a = c/2$)

$\Rightarrow$ $x^{-1/2}M_{0,\pm\nu}(2ix)$ are solutions of Bessel's equation

$$\underbrace{\text{(const.)}\, e^{-ix}x^{\pm\nu}\underbrace{M(\pm\nu+\tfrac{1}{2}, \pm 2\nu+1; 2ix)}_{e^{ix}\,{}_0F_1(\pm\nu+1; -(x/2)^2)}}_{x^{\pm\nu}\sum_{m=0}^{\infty}\frac{(-1)^m}{(\pm\nu+1)_m\, m!}\left(\frac{x}{2}\right)^{2m}} = \Gamma(\pm\nu+1)\,2^{\pm\nu}J_{\pm\nu}(x)$$

?$(2i)^{\pm\nu}$?

Modified Bessel functions :

$$I_\nu(x) := e^{-\pi i\nu/2} J_\nu(ix) = \sum_{m\geq 0} \frac{1}{\Gamma(\nu+m+1)\, m!} \left(\frac{x}{2}\right)^{\nu+2m}$$

is a solution of $x^2u'' + xu' - (x^2+\nu^2)u = 0$.

Another solution : $K_\nu(x) := \frac{\pi}{2} \frac{I_{-\nu}(x) - I_\nu(x)}{\sin(\pi\nu)}$

decreases exponentially as $x \to +\infty$, $K_\nu(x) \sim \sqrt{\frac{\pi}{2}}\, e^{-x} x^{-1/2}$

$$\sqrt{\pi}\,(2x)^{-1/2} W_{0,\nu}(2x) = K_\nu(x)$$

Non-holomorphic regularisation of $\zeta(z,L)$ and $G_2(L)$

Weierstrass ζ-function of a lattice $L\subset\mathbb{C}$:

$$\zeta(z,L)=\frac{1}{z}+\sum_{0\neq u\in L}\left(\frac{1}{z+u}-\frac{1}{u}+\frac{z}{u^2}\right)$$

$\exists$ $\mathbb{R}$-linear $\eta:\mathbb{C}\to\mathbb{C}$ such that $\forall u\in L\ \forall z\in\mathbb{C}$ $\zeta(z+u)=\zeta(z)+\eta(u)$

s-regularisation: ($z\notin L$ fixed)

$$(z+u)^{-\alpha}=z^{-\alpha}\left(1+\frac{u}{z}\right)^{-\alpha}=u^{-\alpha}\left(1+\frac{z}{u}\right)^{-\alpha}=u^{-\alpha}-\alpha z u^{-\alpha-1}+O(|u|^{-\alpha-2})$$

$$\overline{(z+u)}^{-\beta}=\bar u^{-\beta}-\beta\bar z\,\bar u^{-\beta-1}+O(|u|^{-\beta-2})$$

$$\frac{1}{(z+u)^\alpha\,\overline{(z+u)}^\beta}=\frac{1}{u^\alpha\bar u^\beta}-\frac{\alpha z}{u^{\alpha+1}\bar u^\beta}-\frac{\beta\bar z}{u^\alpha\bar u^{\beta+1}}+O(|u|^{-\alpha-\beta-2})$$

$\alpha=s,\ \beta=s-1$:

$$\frac{\overline{z+u}}{|z+u|^{2s}}=\frac{\bar u}{|u|^{2s}}-\frac{sz}{|u|^{2s-2}u^2}-\frac{(s-1)\bar z}{|u|^{2s}}+O(|u|^{-2s-1})$$

Def: $$\zeta_s(z,L):=\frac{\bar z}{|z|^{2s}}+\sum_{0\neq u\in L}\left(\frac{\overline{z+u}}{|z+u|^{2s}}-\frac{\bar u}{|u|^{2s}}+\frac{sz}{|u|^{2s-2}u^2}+\frac{(s-1)\bar z}{|u|^{2s}}\right)$$
($z\notin L$

(absolutely convergent if $\mathrm{Re}(s)>\frac{1}{2}$, by the above)

Other interesting functions: $$E(\tau,s)=\sum_{m,n\in\mathbb{Z}}{}^{'}\frac{y^s}{|m\tau+n|^{2s}}\quad(\mathrm{Re}(s)>1)\ (\tau=x+iy)$$

Write $L=\mathbb{Z}\omega_1+\mathbb{Z}\omega_2$, $\tau=\frac{\omega_1}{\omega_2}\in\mathcal{H}$

$$E(L,s):=\sum_{0\neq u\in L}\frac{1}{|u|^{2s}}\quad(\mathrm{Re}(s)>1)$$

$$G(z,L,s):=\sum_{0\neq u\in L}\frac{\overline{z+u}}{|z+u|^{2s}}\quad\left(\mathrm{Re}(s)>\frac{3}{2}\right)$$

Facts: (1) $$E(\tau,s)=\frac{\pi}{s-1}+O(1)\quad\text{as } s\to1$$

$$\Rightarrow\quad E(L,s)=\frac{\pi}{\mathrm{vol}(\mathbb{C}/L)}\cdot\frac{1}{s-1}+O(1)\quad(\text{— " —})$$

(2) The functions $\left\{\begin{matrix}\sum_{0\neq u\in L}\frac{1}{u^2|u|^{2s}}\\ G(z,L,s)\end{matrix}\right\}$ have holomorphic cont. to $\left\{\begin{matrix}\mathbb{C}\\ \mathrm{Re}(s)>\frac{1}{2}\end{matrix}\right.$

Let $$G_2(L):=\lim_{s\to0}\sum_{0\neq u\in L}\frac{1}{u^2|u|^{2s}}\neq G_2^*(L).\quad(\text{Pf. Poisson summation})$$

Prop. $\zeta(z,L) = \frac{1}{z} + G(z,L,1) + G_2^*(L) z + \frac{\pi}{\mathrm{vol}(\mathbb{C}/L)} \bar{z}$.

Pf. If $\mathrm{Re}(s) > \frac{3}{2}$, one can rearrange terms in the series defining $\zeta_s(z,L) \Longrightarrow \sum_{u\neq 0} \frac{\bar{u}}{|u|^{2s}} = 0$. We know that

$$\lim_{s\to 1}\left(\sum_{u\neq 0} \frac{\bar{z}}{|u|^{2s}}(s-1)\right) = \frac{\bar{z}\pi}{\mathrm{vol}(\mathbb{C}/L)} \Longrightarrow \sum_{0\neq u\in L}\left(\frac{\overline{z+u}}{|z+u|^{2s}} + \frac{sz}{|u|^{2s-2}u^2}\right)$$

is holomorphic at s.

Letting $s\to 1$, we obtain Prop..

Prop. $\boxed{\eta(z,L) = G_2^*(L) z + \frac{\pi}{\mathrm{vol}(\mathbb{C}/L)} \bar{z}}$

Pf. The function $z \longmapsto \underbrace{\left(\frac{1}{z} + G(z,L,1)\right)}_{\left(\frac{\bar{z}}{|z|^{2s}} + G(z,L,s)\right)_{s=1}}$ is L-periodic

$$\Longrightarrow \forall u\in L \quad \underbrace{\zeta(z+u, L) - \zeta(z,L)}_{\eta(u)} = G_2^*(L) u + \frac{\pi}{\mathrm{vol}(\mathbb{C}/L)} \bar{u}$$

$\left(\Longrightarrow \text{Legendre's relation } \begin{vmatrix} \omega_1 & \eta(\omega_1) \\ \omega_2 & \eta(\omega_2)\end{vmatrix} = \frac{\pi \begin{vmatrix} \omega_1 & \omega_2 \\ \bar{\omega}_1 & \bar{\omega}_2 \end{vmatrix}}{\mathrm{vol}(\mathbb{C}/L)} = 2\pi i\right)$

For $L = \mathbb{Z}\tau + \mathbb{Z}$:

$$\eta(1) = \frac{2\zeta(2)}{\cancel{\pi^2 / 3}} + \sum_{m=1}^{\infty} \sum_{n\in\mathbb{Z}} \frac{1}{(m\tau+n)^2} =$$

$$= \frac{\pi^2}{3}\underbrace{\left(1 - 24\sum_{n=1}^{\infty} \sigma_1(n) q^n\right)}_{E_2(\tau)} \qquad (q = e^{2\pi i\tau})$$

Write $G_2^*(L) = 2\zeta(2) E_2^*(L)$ and take $z = 1$ in Prop.:

$$\frac{\pi^2}{3} E_2(\tau) = \frac{\pi^2}{3} E_2^*(\tau) + \frac{\pi}{\mathrm{Im}(\tau)}$$

$$\Downarrow$$

$$\boxed{E_2^*(\tau) = E_2(\tau) - \frac{3}{\pi\,\mathrm{Im}(\tau)}}$$

Non-holomorphic

$\forall \begin{pmatrix} a & b \\ c & d\end{pmatrix} \in SL_2(\mathbb{Z})$

$$E_2^*\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^2 E_2^*(\tau)$$

Fourier expansion of $G_{\alpha,\beta}(\tau) = \sum'_{m,n\in\mathbb{Z}} \frac{1}{(m\tau+n)^{\alpha}\,\overline{(m\tau+n)}^{\beta}}$

If $\alpha+\beta=s$, $\mathrm{Re}(s)>2$, then $G_{\alpha,\beta}(\tau) = \mathrm{Im}(\tau)^{-\beta} \sum'_{m,n\in\mathbb{Z}} \frac{\cancel{\mathrm{Im}(\tau)^{\beta}}\ \mathrm{Im}(\tau)^{\beta}}{(m\tau+n)^{\alpha-\beta}\,|m\tau+n|^{2\beta}}$

If $k=\alpha-\beta\in\mathbb{Z}$, then $\frac{1}{2} G_{\alpha,\beta}(\tau) = \mathrm{Im}(\tau)^{-\beta}\zeta(\alpha+\beta) \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \underbrace{\mathrm{Im}(\gamma(\tau))^{\beta} J(\gamma,\tau)^{-k}}_{(y^{\beta}|_k\gamma)(\tau)}$

$$= \mathrm{Im}(\tau)^{-\beta}\zeta(\alpha+\beta)\, G^{\mathrm{prim}}_{\alpha,\beta}(\tau)$$

$y = \mathrm{Im}(\tau)$

Differential operators: $L_k = -2iy^2\partial_{\bar\tau}$, $R_k = 2i\partial_\tau + \frac{k}{2}$

$R_k(y^{\beta}) = \alpha y^{\beta-1}$, $L_k(y^{\beta}) = \cancel{(\alpha+\beta)}\ \beta y^{\beta+1}$

$\Rightarrow$ $R_k : G^{\mathrm{prim}}_{\alpha,\beta} \longmapsto \alpha\, G^{\mathrm{prim}}_{\alpha+1,\beta-1}$, $L_k : G^{\mathrm{prim}}_{\alpha,\beta} \longmapsto \beta\, G^{\mathrm{prim}}_{\alpha-1,\beta+1}$

$$\Delta_k = -y^2\left(\left(\frac{\partial}{\partial x}\right)^2 + \left(\frac{\partial}{\partial y}\right)^2\right) + 2iy\,\partial_{\bar\tau} + \frac{k}{2}\left(1-\frac{k}{2}\right) \Rightarrow \Delta_k(y^{\beta}) = \frac{s}{2}\left(1-\frac{s}{2}\right)y^{\beta}$$

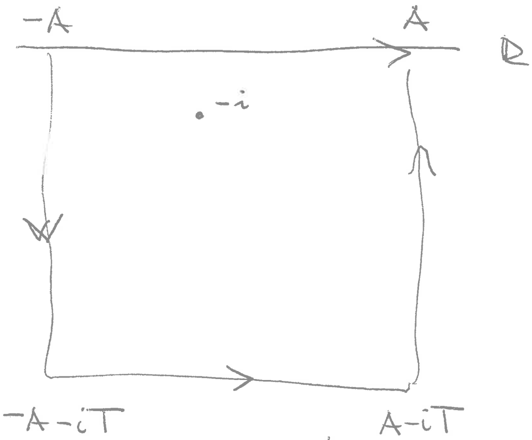
Fourier expansion: if $f_{\alpha,\beta}(x) = \frac{1}{(x+i)^{\alpha}}\frac{1}{(x-i)^{\beta}}$ $(x\in\mathbb{R})$,

then $G_{\alpha,\beta}(\tau) = \underbrace{2\zeta(\alpha+\beta)}_{\text{corr. to } m=0} + \sum_{m=1}^{\infty} \frac{(1-(-1)^{\alpha+\beta})}{(mv)^{\alpha+\beta}} \underbrace{\sum_{n\in\mathbb{Z}} f_{\alpha,\beta}\left(\frac{n+mu}{mv}\right)}_{\text{Poisson: } (mv)\sum_{n\in\mathbb{Z}} e^{2\pi i mvn}(\mathcal{F}f_{\alpha,\beta})(mvn)}$

$\tau = u+iv$

Computing $\mathcal{F}(f_{\alpha,\beta})$: (1) Toy model: $\beta=0$, $\alpha=k\in\mathbb{Z}_{\geq 2}$

$$(\mathcal{F}f_{k,0})(y) = \int_{\mathbb{R}} \frac{e^{-2\pi i x y}}{(x+i)^k}\,dx$$



If $y\geq 0$ is fixed: replace $x\in\mathbb{R}$ by a complex variable z and compare

$$I_1 = \int_{-A}^{A} \frac{e^{-2\pi i y z}}{(z+i)^k}\,dz \quad \text{to} \quad \underbrace{\int_{-A}^{-A-iT} + \int_{-A-iT}^{A-iT} + \int_{A-iT}^{A}}_{I_2}$$

Residue theorem: for $A,T \geq 1$,

$$I_1 - I_2 = -2\pi i \sum_{\mathrm{Im}(z)<0} \mathrm{Res}_z \frac{e^{-2\pi i y z}\,dz}{(z+i)^k} = -2\pi i\, \mathrm{Res}_{z'=0} \frac{e^{-2\pi i y(z'-i)}}{z'^k} = -(2\pi i)\frac{e^{-2\pi y}(-2\pi i y)^{k-1}}{(k-1)!}$$

$I_2 \to 0$ as $A,T\to+\infty$ $\Rightarrow$ $(\mathcal{F}_{k,0})(y) = \frac{(-2\pi i)^k e^{-2\pi y} y^{k-1}}{(k-1)!}$

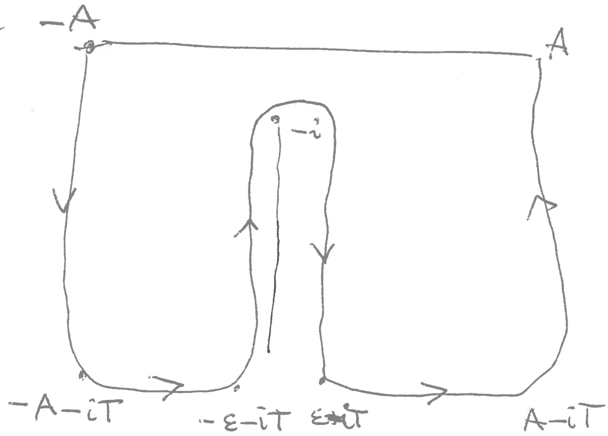
If $y<0$: again, $(\mathcal{F}f_{k,0})(y) = 2\pi i \sum_{\mathrm{Im}(z)>0} \mathrm{Res}_z \frac{e^{-2\pi i y z}}{(z+i)^k}\,dz = 0$

(2) $\underline{\mathcal{F}(f_{\alpha,0})}$: $\mathrm{Re}(\alpha)>1$ (~~not~~ assuming $\alpha\in\mathbb{Z}$)

fix a branch of $(z+i)^{\alpha}$ on the complement of $i(-\infty,-1]$ (e.g., the one equal to $e^{\pi i\alpha/2}$ at $x=0$).

If $y<0$, again $(\mathcal{F} f_{\alpha,0})(y)=2\pi i \sum\limits_{\mathrm{Im}(z)>0} \mathrm{Res}_z \dfrac{e^{-2\pi i y z}}{(z+i)^{\alpha}}\,dz=0$

$\underline{\text{If } y\geq 0}$: integrate along



and let $A,T\to+\infty$.

this reduces the integral to the following one:

Writing $1-iz=w/2\pi y$ gives

$$(\mathcal{F}f_{\alpha,0})(y)=\int \frac{e^{-2\pi i y z}}{(z+i)^{\alpha}}\,dz = e^{-2\pi y}(-2\pi i y)^{\alpha-1}\int e^{w}w^{-\alpha}\,dw = \frac{e^{-2\pi y}(-2\pi i y)^{\alpha-1}(-2\pi i)}{\Gamma(\alpha)}$$

$\dfrac{-2\pi i}{\Gamma(\alpha)}$ $\underline{\text{Hankel's integral}}$

(3) $\underline{\mathcal{F}(f_{\alpha,\beta})(y),\ y>0}$: $\underline{f_{\alpha,\beta}(x)=(x+i)^{-\alpha}(x-i)^{-\beta}}$

$$Df_{\alpha\beta}=\left(-\frac{\alpha}{x+i}-\frac{\beta}{x-i}\right)f_{\alpha\beta} \Longrightarrow \left((x^2+1)D+(\alpha+\beta)x+i(\beta-\alpha)\right)f_{\alpha\beta}=0$$

$g=g_{\alpha\beta}=\mathcal{F}(f_{\alpha\beta})$ satisfies, therefore,

$$\left(\left(\left(\frac{-1}{2\pi i}D\right)^2+1\right)(2\pi i x)+(\alpha+\beta)\left(-\frac{D}{2\pi i}\right)+i(\beta-\alpha)\right)g=0$$

$$\underline{\left(xD^2+(2-\alpha-\beta)D+(-4\pi^2x+2\pi(\alpha-\beta))\right)g=0}$$

$\underline{\text{Gauge transformation}}$ $\quad g=x^{\frac{\alpha+\beta}{2}-1}h$

$$\underline{h''+\left(-4\pi^2+\frac{2\pi(\alpha-\beta)}{x}-\frac{(\alpha+\beta)(\alpha+\beta-2)}{4x^2}\right)h=0}$$

$\underline{4\pi x=t:}$
$\underline{h(x)=v(t)}$ $\quad \underline{\left(\left(\frac{d}{dt}\right)^2+\left(-\frac{1}{4}+\frac{(\alpha-\beta)/2}{t}-\frac{1-4\left(\frac{\alpha+\beta-1}{2}\right)^2}{4t^2}\right)\right)v=0}$

Whittaker equation with $\kappa=\dfrac{\alpha-\beta}{2}$, $\mu=\dfrac{\alpha+\beta-1}{2}$ $\longleftrightarrow$ related to $M(a,c;t)$ with $\boxed{\begin{array}{l} a=\mu-\kappa+\frac{1}{2}=\beta \\ c=2\mu+1=\alpha+\beta \end{array}}$

$v(t) = e^{-t/2}\, t^{(\alpha+\beta)/2}\, F(t)$, F solution of $(*)_{a,c}$ $\qquad t = 4\pi y$, $a=\beta,\ c=\alpha+\beta$

$$\Longrightarrow (\mathcal{F}f_{\alpha,\beta})(y) = y^{\frac{\alpha+\beta}{2}-1}\, v(4\pi y) = (4\pi)^{\frac{\alpha+\beta}{2}}\, e^{-2\pi y}\, y^{\alpha+\beta-1}\, F(4\pi y)$$

Riemann – Lebesgue Lemma : $(\mathcal{F}f_{\alpha,\beta})(y) \longrightarrow 0$ as $y \longrightarrow +\infty$

$$M(\beta, \alpha+\beta; 4\pi y) \sim \frac{\Gamma(\alpha+\beta)}{\Gamma(\beta)}\, e^{4\pi y}\, (4\pi y)^{-\alpha}$$

$$\Longrightarrow \quad F(y) = (\text{const.})\ U(\beta, \alpha+\beta, 4\pi y)$$

Value of the constant: $(\mathcal{F}f_{\alpha,\beta})(y) = \dfrac{i^{\beta-\alpha}(2\pi)}{\Gamma(\alpha)\Gamma(\beta)}\, e^{-2\pi y}\, (2\pi y)^{\alpha+\beta-1}\, U(\beta, \alpha+\beta, 4\pi y)$ if $y>0$

(for $y<0$: replace y by $|y|$ and interchange $\alpha \longleftrightarrow \beta$ in $U(\cdot, \cdot, 4\pi y)$)

$$(\mathcal{F}f_{\alpha,\beta})(0) = \frac{i^{\beta-\alpha}(2\pi)}{\Gamma(\alpha)\Gamma(\beta)}\, \frac{\Gamma(\alpha+\beta-1)}{2^{\alpha+\beta-1}}.$$

Conclusion :

$(q = e^{2\pi i\tau})$

$$\frac{1}{2}\frac{G_{\alpha,\beta}(\tau)}{\mathrm{Im}(\tau)^{\beta}} = \zeta(\alpha+\beta) + \frac{(2\pi)\, i^{\beta-\alpha}}{\Gamma(\alpha)\Gamma(\beta)}\, \frac{\Gamma(\alpha+\beta-1)\,\zeta(\alpha+\beta-1)}{2^{\alpha+\beta-1}\ \mathrm{Im}(\tau)^{\alpha+\beta-1}} +$$

$$+\, i^{\beta-\alpha}\frac{(2\pi)^{\alpha+\beta}}{\Gamma(\alpha)} \sum_{m=1}^{\infty} \sigma_{\alpha+\beta-1}(m)\, U(\beta, \alpha+\beta, 4\pi m\, \mathrm{Im}(\tau))\, q^m +$$

$$+\, i^{\beta-\alpha}\frac{(2\pi)^{\alpha+\beta}}{\Gamma(\beta)} \sum_{m=1}^{\infty} \sigma_{\alpha+\beta-1}(m)\, U(\alpha, \alpha+\beta, 4\pi m\, \mathrm{Im}(\tau))\, \overline{q^m}$$

Very interesting special case : $\alpha = 1,\ \beta = m+1$ $\quad, m \geq 1$

up to constants :

$$\begin{tikzcd}
 & 0 & & & & 0 & \\
G_{0,m+2} \arrow[ur, bend left, "R_{-m}=0"] & G_{1,m+1} \arrow[l, bend left, "L_{-m}"] \arrow[r, bend left, "R_{-m}"] & G_{2,m} \arrow[r, bend left] & \cdots \arrow[r, bend left] & G_{m+1,1} \arrow[r, bend left] & \underbrace{G_{m+2,0}}_{\text{holomorphic}} \arrow[u, bend right, "L_{m+2}=0"']
\end{tikzcd}$$

$\mathrm{wt} = -m \qquad \mathrm{wt} = 2-m \qquad\qquad \mathrm{wt} = m+2$

$$U(m+1, m+2; x) = \frac{1}{\Gamma(m+1)}\int_0^\infty e^{-xt}\, t^m\, dt = x^{-m-1}$$

$$U(1, m+2; x) = e^{x}\int_1^\infty e^{-xt}\, t^m\, dt = x^{-m-1} e^{x}\int_0^\infty e^{-t}\, t^m\, dt$$

$$= x^{-m-1} e^{x}\, \Gamma(m+1, x)$$

$$\frac{G_{1,m+1}(\tau)}{2} = \mathrm{Im}(\tau)^{m+1}\,\zeta(m+2) + \frac{(2\pi)\,i^m}{2^{m+1}}\,\zeta(m+1) +$$

$$+ \frac{(2\pi)\,i^m}{2^{m+1}} \sum_{n=1}^{\infty} \sigma_{-m-1}(n)\, q^n + \frac{(2\pi)\,i^m}{2^{m+1}\, m!} \sum_{n=1}^{\infty} \sigma_{-m-1}(n)\, \underbrace{\beta_{-m}(4\pi n\,\mathrm{Im}(\tau))}_{\Gamma(m+1,\,4\pi n\,\mathrm{Im}(\tau))}\, \bar{q}^{-n}$$

Bol's identity: $(m \geq 0)$

$$\text{wt}\quad \underbrace{-m \xrightarrow{R_{-m}} -m+2 \xrightarrow{R_{2-m}} \cdots \quad m \xrightarrow{R_m}}_{m+1 \text{ operators}} m+2$$

$R_k = 2i\,\partial_\tau + \frac{k}{y}$

$$\boxed{R_m \circ R_{m-2} \circ \cdots \circ R_{-m} = (2i\,\partial_\tau)^{m+1}}$$

(non-holomorphic terms disappear !!)

Pf: Induction on m (exercise).

Example above: $\frac{1}{2\pi i}\partial_\tau = q\frac{d}{dq}$

$$G_{1,m+1} \curvearrowright \cdots \curvearrowright G_{m+2,0}$$

$0 \leftarrow G_{m+2,0}$ holomorphic

$$\left(q\frac{d}{dq}\right)^{m+1} \sum_{n=1}^{\infty} \sigma_{-m-1}(n)\, q^n = \sum_{n=1}^{\infty} \underbrace{n^{m+1}\sigma_{-m-1}(n)}_{\sigma_{m+1}(n)}\, q^n$$

$$\left(q\frac{d}{dq}\right)^{m+1} (\text{the non-holomorphic term in } G_{1,m+1}) = 0$$
