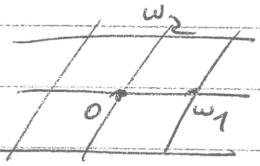
Elliptic functions

Given : a lattice $L \subset \mathbb{C}$

i.e. $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$

s.t. ω_1, ω_2 are L/ over $\mathbb{R}$

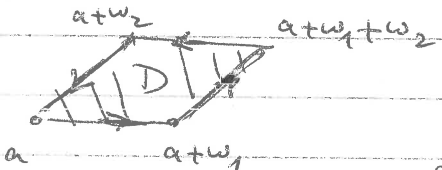


$$\left\{ \begin{matrix} \text{Elliptic functions} \\ \text{with } \cancel{\text{w.r.t.}} \text{ period lattice } L \end{matrix} \right\} = \left\{ \begin{matrix} \text{meromorphic } f : \mathbb{C} \to \mathbb{C} \cup \{\infty\} \\ \text{s.t. } f(z+u) = f(z) \quad \forall u \in L \end{matrix} \right\}$$

$$\| $$

$$\left\{ \begin{matrix} \text{meromorphic functions} \\ \mathbb{C}/L \longrightarrow \mathbb{C} \cup \{\infty\} \end{matrix} \right\} = \mathcal{M}(\mathbb{C}/L)$$

Complex torus $\mathbb{C}/L$: fix $a \in \mathbb{C}$, ω_1, ω_2 s.t. $\mathrm{Im}\left(\frac{\omega_2}{\omega_1}\right) > 0$



fundamental parallelogram of L

glue ∂D

Prop. If $f \in \mathcal{M}(\mathbb{C}/L)$ ~~then~~ has no zeros or poles on ∂D, then

(1) $\displaystyle\sum_{x \in D} \mathrm{ord}_x(f) = 0$

(2) $\displaystyle\sum_{x \in D} x \, \mathrm{ord}_x(f) \in L$

Recall : if z is a local coordinate at x, then we can write f in a neighb. of x as $z^n(a_0 + a_1 z + \dots)$, $a_0 \neq 0$, then $\mathrm{ord}_x(f) = n \in \mathbb{Z}$

~~Pf. (1)~~

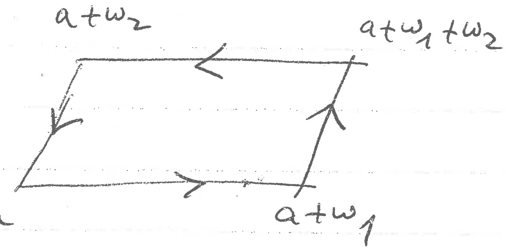
Reformulation : If $0 \neq f \in \mathcal{M}(\mathbb{C}/L)$, then

(1) $\displaystyle\sum_{x \in \mathbb{C}/L} \mathrm{ord}_x(f) = 0$

(2) $\displaystyle\sum_{x \in \mathbb{C}/L} \mathrm{ord}_x(f) x = 0$ in $\underline{\mathbb{C}/L}$.

Pf: (1) True on any cpt Riemann surface X: put $\omega = \frac{df}{f}$ ($f \in \mathcal{M}(X)$, $f \neq 0$), then $0 = \displaystyle\sum_{x \in X} \underbrace{\mathrm{res}_x(\omega)}_{\mathrm{ord}_x(f)}$.

(2)



$$\text{LHS} = \frac{1}{2\pi i}\int_{\partial D} z\,\frac{f'(z)}{f(z)}\,dz = \frac{1}{2\pi i}\int_a^{a+\omega_1}\big(z-(z+\omega_2)\big)\frac{f'}{f}(z)\,dz \; +$$

$$+\;\frac{1}{2\pi i}\int_a^{a+\omega_2}\big(-z+(z+\omega_1)\big)\frac{f'}{f}(z)\,dz = -\frac{\omega_2}{2\pi i}\underbrace{\int_a^{a+\omega_1} d\log f}_{\in 2\pi i\mathbb{Z}} + \frac{\omega_1}{2\pi i}\underbrace{\int_a^{a+\omega_2} d\log f}_{\in 2\pi i\mathbb{Z}} \in L$$

Reformulation : X cpt RS (connected)

Def: Divisor group : $\mathrm{Div}(X) = \left\{ \sum_{i=1}^{k} n_i (P_i) \;\middle|\; k\geq 0,\ n_i\in\mathbb{Z},\ P_i\in X \right\}$

$\deg : \mathrm{Div}(X) \longrightarrow \mathbb{Z}$, $\deg\left(\sum n_i (P_i)\right) = \sum n_i$

$\mathrm{Div}^0(X) = \{ D \in \mathrm{Div}(X) \mid \deg(D) = 0 \}$

Def: $f\in \mathcal{M}(X)$, $f\not\equiv 0$ $\rightsquigarrow$ principal divisor

$$\mathrm{div}(f) = \sum_{P\in X} \mathrm{ord}_P(f)\,(P)$$

Principal divisors $P(X) = \{\mathrm{div}(f)\} \subset \mathrm{Div}(X)$

Cor: (1) ($\forall$ cpt RS X) $P(X) \subseteq \mathrm{Div}^0(X)$

(2) If $X = \mathbb{C}/L$, define $\boxplus : \mathrm{Div}(X) \longrightarrow X$

$$\sum n_i (P_i) \longmapsto \underbrace{\sum n_i P_i}_{\text{using group structure on } X}$$

Then $\boxplus(\mathrm{div}(f)) = 0$ $\forall f$.

Fact: X cpt RS (connected) ~~+ the div(f) = 0~~

$\Downarrow$

$\mathcal{O}(X) = \mathbb{C}$ $\Rightarrow$ [$\mathrm{div}(f) = \mathrm{div}(g) \Leftrightarrow f = cg,\ c\in\mathbb{C}^*$]

$\boxed{\text{FORMULAS}}$

$$\sigma(z,L) = \sigma(z) := z \prod_{u\in L}{}' \left(1-\frac{z}{u}\right) \exp\left(\frac{z}{u} + \frac{1}{2}\left(\frac{z}{u}\right)^2\right) \qquad \Big\| \qquad \sigma(\lambda z, \lambda L) = \lambda \sigma(z,L)$$

$$\zeta(z,L) := \frac{\sigma'(z)}{\sigma(z)} = \frac{1}{z} + \sum_{u\in L}{}' \underbrace{\left(\frac{1}{z-u} + \frac{1}{u} + \frac{z}{u^2}\right)} = \frac{1}{z} - G_4(L)z^3 - G_6(L)z^5 - \dots$$

$$-\frac{1}{u}\cdot\frac{1}{1-\frac{z}{u}} + \frac{1}{u} + \frac{z}{u^2} = -\frac{z^2}{u^3} - \frac{z^3}{u^4} - \frac{z^4}{u^5} - \dots$$

$$G_{2k}(L) = \sum_{u\in L}{}' \frac{1}{u^{2k}} \qquad (k \geq 2)$$

$$\wp(z,L) := -\zeta'(z,L) = \frac{1}{z^2} + \sum_{u\in L}{}' \left(\frac{1}{(z-u)^2} - \frac{1}{u^2}\right) = \frac{1}{z^2} + 3G_4(L)z^2 + 5G_6(L)z^4 + \dots$$

even

$$\wp'(z,L) := -2\sum_{u\in L} \frac{1}{(z-u)^3} = -\frac{2}{z^3} + 6G_4(L)z + 20\,G_6(L)z^3 + \dots$$

odd

$$G_{2k}(\lambda L) = \lambda^{-2k} G_{2k}(L), \qquad \zeta(\lambda z, \lambda L) = \lambda^{-1}\zeta(z,L), \qquad \wp^{(n)}(\lambda z, \lambda L) = \lambda^{-2-n}\wp(z,L)$$

$$\wp'(z)^2 = \frac{4}{z^6} - \frac{24G_4}{z^2} - 80G_6 + \dots$$

$$\wp(z)^3 = \frac{1}{z^6} + \frac{9G_4}{z^2} + 15G_6 + \dots$$

$$\wp(z)^2 = \frac{1}{z^4} + 6G_4 + 10G_6 z^2 + \dots$$

$$\wp'(z)^2 - 4\wp(z)^3 + 60G_4\wp(z) + 140G_6 = 0 + a_2 z^2 + a_4 z^4 + \dots$$

$$\wp'^2 = 4\wp^3 - g_2\wp - g_3, \qquad g_2(L) = 60G_4(L), \quad g_3(L) = 140G_6(L)$$

$$4X^3 - g_2X - g_3 = 4(X^3 + aX + b) = 4(X-e_1)(X-e_2)(X-e_3)$$

$$\operatorname{disc}(X^3+aX+b) = -4a^3 - 27b^2 = \prod_{i<j}(e_i-e_j)^2, \qquad a = -\frac{g_2}{4},\ b = -\frac{g_3}{4}$$

$$\Delta(L) = 16\prod_{i<j}(e_i-e_j)^2 = 16\left(-4\left(\frac{-g_2}{4}\right)^3 - 27\left(\frac{-g_3}{4}\right)^2\right) = g_2^3 - 27g_3^2$$

$$j(L) = \frac{(12g_2)^3}{\Delta} = \frac{1728\,g_2^3(L)}{\Delta(L)}$$

$$\boxed{\wp'' = 6\wp^2 - \frac{g_2}{2} = 6\wp^2 - 30G_4}$$

$$\wp(z) = \frac{1}{z^2} + \sum_{n=1}^{\infty} (2n+1)\, G_{2n+2}(L)\, z^{2n}$$

$$\wp'(z) = -\frac{2}{z^3} + \sum_{n=1}^{\infty} 2n(2n+1)\, G_{2n+2}\, z^{2n-1} \qquad = -\frac{2}{z^3} + 6G_4 z + 20G_6 z^3 + \dots$$

$$\wp''(z) = \frac{6}{z^4} + \sum_{n=1}^{\infty} 2n(4n^2-1)\, G_{2n+2}\, z^{2n-2} \qquad = \frac{6}{z^4} + 6G_4 + 60G_6 z^2 + \dots$$

$$\wp(z)^2 = \frac{1}{z^4} + 6G_4 + 10G_6 z^2 + \dots \quad +$$

$$+ \left[2(2n+1)\, G_{2n+2} + 3\cdot(2n-3)\, G_4 \cdot G_{2n-2} + \dots + (2n-3)\cdot 3\, G_{2n-2}\cdot G_4 \right] z^{2n} + \dots$$

$$\big(\underbrace{2n(4n^2-1) - 12(2n+1)}_{2(n-2)(2n+1)(2n+3)}\big)\, G_{2n+2} = 6 \sum_{i=2}^{n-1} (2i-1)(2n+1-2i)\, G_{2i}\, G_{2n+2-2i}$$

$(n \geq 2)$

$\underline{n=2}$:
$$0 \cdot G_6 = 0$$

$\underline{n=3}$:
$$2\cdot 7\cdot 9\, G_8 = 6\cdot 3^2\, G_4^2 \implies 7G_8 = 3G_4^2$$
$$7\cdot(2s_8) = 3(2s_4)^2 \quad \Big\} \implies \underline{\underline{E_8 = E_4^2}}$$

$\underline{n=4}$:
$$2^2\cdot 9\cdot 11\, G_{10} = 6\cdot 2\cdot 3\cdot 5\, G_4 G_6 \implies \underline{\underline{E_{10} = E_4 E_6}}$$

$\underline{n=5}$:
$$2\cdot 3\cdot 11\cdot 13\, G_{12} = 6\cdot(2\cdot 3\cdot 7\, G_4 G_8 + 5^2\, G_6^2)$$

$$2\cdot 3\cdot 7\, (2s_4)(2s_8) = \frac{3^2\cdot 7^2}{2^2\cdot 5}(2s_6)^2$$

$$\implies E_{12} = (\text{const})\left(\frac{3^2\cdot 7^2}{2\cdot 5} E_4 E_8 + 5^2 E_6^2\right) \qquad \text{etc.}$$

$\underline{n=6}$:
$$2\cdot 4\cdot 13\cdot 15\, G_{14} = 6\cdot 2\,(3\cdot 9\cdot G_4\cdot G_{10} + 5\cdot 7\, G_6\cdot G_8) =$$
$$= (\text{const})\, G_4^2 G_6$$

$$\implies \underline{\underline{E_{14} = E_4^2 E_6 = E_8 E_6 = E_4 E_{10}}}$$

$$\sin(\pi z) = \pi z \prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right)$$

$$\underbrace{\frac{d}{dz}\log\sin(\pi z)}_{\pi \operatorname{cotg} \pi z} = \frac{1}{z} + \sum_{n=1}^{\infty}\underbrace{\left(\frac{1}{z-n}+\frac{1}{z+n}\right)}_{-2\left(\frac{z}{n^2}+\frac{z^3}{n^4}+\frac{z^5}{n^6}+\cdots\right)} =$$

$$= \frac{1}{z} - 2s_2 z - 2s_4 z^3 - 2s_6 z^5 - \cdots$$

$$s_{2k} = \sum_{n=1}^{\infty}\frac{1}{n^{2k}} \quad (=\zeta(2k)) \qquad ; \quad s_2 = \frac{\pi^2}{6}, \quad s_4 = \frac{\pi^4}{90}, \quad s_6 = \frac{\pi^6}{3^3\cdot 5\cdot 7} = \frac{\pi^6}{945}$$

$$q = e^{2\pi i\tau}$$

$$\underbrace{\pi\operatorname{cotg}(\pi\tau)}_{\text{“}\sum_{n\in\mathbb{Z}}\frac{1}{\tau+n}\text{”}} = \pi i\,\frac{e^{\pi i\tau}+e^{-\pi i\tau}}{e^{\pi i\tau}-e^{-\pi i\tau}} = \pi i\,\frac{q+1}{q-1} = \pi i\left(-1-2\sum_{n=1} q^n\right)$$

$$\text{Eisenstein sum} = \lim_{N\to\infty}\sum_{n=-N}^{N}\left(\quad\right)$$

$$\left(\frac{d}{d\tau}\right)^{k-1}(\;-\;): \qquad (k\geq 2)$$

$$(-1)^k (k-1)! \sum_{n\in\mathbb{Z}}\frac{1}{(\tau+n)^k} = (2\pi i)^k\sum_{n=1}^{\infty} n^{k-1}q^n \qquad (k\geq 2)$$

$$(k\geq 2) \qquad L = \mathbb{Z}\tau+\mathbb{Z}, \quad \operatorname{Im}(\tau) > 0$$

$$G_{2k}(\tau) = G_{2k}(\mathbb{Z}\tau+\mathbb{Z}) = \sum_{m,n}{}' \frac{1}{(m\tau+n)^{2k}} = \sum_{n\neq 0}\frac{1}{n^{2k}} + 2\sum_{m=1}^{\infty}\sum_{n\in\mathbb{Z}}\frac{1}{(m\tau+n)^{2k}} =$$

$$= 2\zeta(2k) + 2\frac{(2\pi i)^{2k}}{(2k-1)!}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} n^{2k-1}q^{mn} =$$

$$= 2\zeta(2k) + 2\frac{(2\pi i)^{2k}}{(2k-1)!}\sum_{n=1}^{\infty}\frac{n^{2k-1}q^n}{1-q^n} =$$

$$= 2\zeta(2k) + \frac{2(2\pi i)^{2k}}{(2k-1)!}\sum_{n=1}^{\infty}\underbrace{\left(\sum_{d|n} d^{2k-1}\right)}_{\sigma_{2k-1}(n)} q^n = 2\zeta(2k)\,E_{2k}(\tau)$$

$$\boxed{E_{2k}(\tau) = 1 + \frac{(2\pi i)^{2k}}{(2k-1)!\,\zeta(2k)}\sum_{n=1}^{\infty}\sigma_{2k-1}(n)\,q^n}$$

Bernoulli numbers:

$$\frac{t}{e^t-1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}$$

$$\Rightarrow \qquad t = \left(\sum_{i=0}^{\infty} B_i \frac{t^i}{i!}\right)\left(\sum_{j=1}^{\infty} \frac{t^j}{j!}\right)$$

$$\Rightarrow \qquad B_0 = 1 \,, \quad \binom{n}{0} B_0 + \binom{n}{1} B_1 + \dots + \binom{n}{n-1} B_{n-1} = 0 \qquad (\forall n \geq 2)$$

$$\underline{B_{2k-1} = 0 \quad \forall k \geq 2} \,, \quad \underline{B_1 = -\frac{1}{2}}$$

$2n$	2	4	6	8	10	12	14	16
B_{2n}	$\frac{1}{6}$	$-\frac{1}{30}$	$\frac{1}{42}$	$-\frac{1}{30}$	$\frac{5}{66}$	$-\frac{691}{2730}$	$\frac{7}{6}$	$-\frac{3617}{510}$

Relation to ζ-function:

$$\underbrace{\pi \cot \pi z} = \frac{1}{z} - 2\sum_{n=1}^{\infty} \underbrace{\zeta(2n)}_{s_{2n}} z^{2n-1}$$

$$\pi i \frac{e^{\pi i z}+e^{-\pi i z}}{e^{\pi i z}-e^{-\pi i z}} = \pi i \frac{e^{2\pi i z}+1}{e^{2\pi i z}-1} = \pi i \left(1 + \frac{2}{e^{2\pi i z}-1}\right)$$

$$\Rightarrow \qquad \pi z \cot \pi z = 1 - 2\sum_{n=1}^{\infty} \zeta(2n) z^{2n}$$

$$= \pi i \left(\frac{2z}{e^{2\pi i z}-1} + z\right) = 1 + \sum_{n=2}^{\infty} B_n \frac{(2\pi i)^n}{n!} z^n$$

$$\Rightarrow \qquad \boxed{2\zeta(2n) = -\frac{(2\pi i)^{2n}}{(2n)!} B_{2n}} \qquad (n \geq 1)$$

$$E_4(\tau) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n$$

$$E_6(\tau) = 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n$$

$$E_8(\tau) = 1 + 480 \sum_{n=1}^{\infty} \sigma_7(n) q^n$$

$$E_{10}(\tau) = 1 - 264 \sum_{n=1}^{\infty} \sigma_9(n) q^n$$

$$E_{12}(\tau) = 1 + \frac{65520}{691} \sum_{n=1}^{\infty} \sigma_{11}(n) q^n$$

$$E_{14}(\tau) = 1 - 24 \sum_{n=1}^{\infty} \sigma_{13}(n) q^n$$

$$E_8 = E_4^2$$

$$E_{10} = E_4 E_6$$

$$E_{14} = E_4^2 E_6 \quad (= E_8 E_6 = E_4 E_{10}$$

$$\zeta(2k) = -\frac{(2\pi i)^{2k}}{2(2k)!} B_{2k} \quad , \quad \zeta(1-2k) = -\frac{B_{2k}}{2k}$$

$$\frac{(2\pi i)^{2k}}{(2k-1)!\,\zeta(2k)} = -\frac{4k}{B_{2k}}$$

$$E_{2k}(\tau) = 1 - \frac{4k}{B_{2k}} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n$$

$2k$	2	4	6	8	10	12	14	16
B_{2k}	$\frac{1}{6}$	$-\frac{1}{30}$	$\frac{1}{42}$	$-\frac{1}{30}$	$\frac{5}{66}$	$-\frac{691}{2730}$	$\frac{7}{6}$	$-\frac{3617}{510}$

Another normalization:

$$\frac{1}{2}\zeta(1-2k)\, E_{2k}(\tau) = \frac{1}{2}\zeta(1-2k) + \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n$$

$$G_4 = (2\zeta_4) E_4 = \frac{\pi^4}{45} E_4 \quad , \quad G_6 = (2\zeta_6) E_6 = \frac{2\pi^6}{945} E_6$$

$$g_2 = 60 G_4 = \frac{4}{3}\pi^4 E_4 \quad , \quad g_3 = 140 G_6 = \frac{8}{27}\pi^6 E_6$$

$$\Delta = g_2^3 - 27 g_3^2 = \frac{2^6 \pi^{12}}{3^3}(E_4^3 - E_6^2) \quad , \quad j = \frac{12^3 E_4^3}{E_4^3 - E_6^2} = \frac{1}{q} + \cdots$$

$$3 \cdot 240 - 2(-504) = 1728 = 12^3$$

$-\zeta'(z,L) = \wp(z,L)$ is periodic $\Longrightarrow$

$$\zeta(z+u) = \zeta(z) + \underbrace{\eta(u)}_{\in\mathbb{C}} \qquad \eta(u;L) \qquad u\in L$$

$$\eta(u+v) = \eta(u) + \eta(v) \qquad u,v\in L$$

Fixing ω_1, ω_2 s.t. $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$, $\mathrm{Im}\left(\frac{\omega_1}{\omega_2}\right) > 0$

put $\eta_i = \eta(\omega_i)$.

Legendre's relation: $\begin{vmatrix} \omega_1 & \eta_1 \\ \omega_2 & \eta_2 \end{vmatrix} = 2\pi i$

Pf. we first show

Lemma. $\sigma(z+u) = \sigma(z)\,\psi(u)\,e^{\eta(u)(z+\frac{u}{2})}$ $(u\in L)$ $(*)$

with $\psi(u) = \begin{cases} 1 & \frac{u}{2}\in L \\ -1 & \frac{u}{2}\notin L \end{cases}$

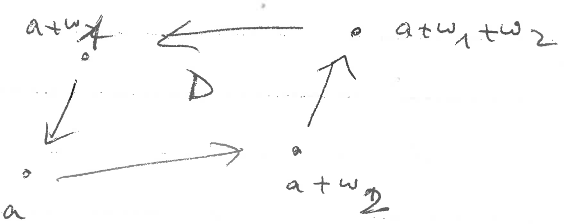
Pf. $\frac{\sigma'(z)}{\sigma(z)} = \zeta(z,L) \Longrightarrow \exists$ relation $(*)$ with some $\psi(u)$.

If $\frac{u}{2}\notin L$, put $z = -\frac{u}{2} \Longrightarrow -1 = \frac{\sigma(\frac{u}{2})}{\sigma(-\frac{u}{2})} = \psi(u)$.

As $\psi(2u) = \psi(\frac{u}{2})^2$, ~~we get~~ writing $u = 2^n v$, $v\in L$, $\frac{v}{2}\notin L$

we get the rest. Lemma is proved.

Pf of Legendre's relation:



$$\underbrace{\int_{\partial D} \zeta(z,L)\,dz}\ = 2\pi i \sum_{x\in D} \mathrm{res}_x\, \zeta(z,L) = 2\pi i$$

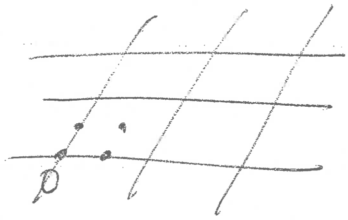
$$\int_a^{a+\omega_2} \underbrace{d\log\sigma(z) - d\log\sigma(z+\omega_1)}_{-\eta_1\, d(z+\frac{\omega_1}{2})} + \int_a^{a+\omega_1} \underbrace{d\log\sigma(z+\omega_2) - d\log\sigma(z)}_{\eta_2\, d(z+\frac{\omega_2}{2})} = -\omega_2\eta_1 + \omega_1\eta_2.$$

Properties of $\wp(z)$

Points of order 2 on $\mathbb{C}/L$:

Fix ω_1, ω_2 s.t. $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$

put $\omega_3 = \omega_1 + \omega_2$



$$(\mathbb{C}/L)_2 = \{0\} \cup \Big\{ \frac{\omega_1}{2}, \frac{\omega_2}{2}, \underbrace{\frac{\omega_1+\omega_2}{2}}_{\frac{\omega_3}{2}} \pmod L \Big\}$$

Put $e_i = \wp\left(\frac{\omega_i}{2}\right)$ $\quad (i = 1, 2, 3)$.

Proposition: (1) $e_1, e_2, e_3 \in \mathbb{C}$ are distinct.

(2) $\wp'(z)^2 = 4\,(\wp(z)-e_1)(\wp(z)-e_2)(\wp(z)-e_3)$

Pf. (1) $\wp'\left(\frac{\omega_i}{2}\right) = -\wp'\left(-\frac{\omega_i}{2}\right) = -\wp'\left(-\frac{\omega_i}{2} + \omega_i\right) \implies \wp'\left(\frac{\omega_i}{2}\right) = 0$

$\implies \operatorname{div}(\wp(z) - e_i) = 2\left(\frac{\omega_i}{2}\right) - 2(0) \implies \{e_i\}$ are distinct.

(2)
$$\left.\begin{aligned} \operatorname{div}(\mathrm{RHS}) &= \sum_{i=1}^{3} 2\left(\frac{\omega_i}{2}\right) - 6(0) \\ \operatorname{div}(\wp'(z)) &= \sum_{i=1}^{3} \left(\frac{\omega_i}{2}\right) - 3(0) \end{aligned}\right\} \implies \mathrm{LHS} = (\text{const})\, \mathrm{RHS}$$

For $z \to 0$, $\wp(z) \sim \frac{1}{z^2}$, $\wp'(z) \sim -\frac{2}{z^3} \implies$ const. $= 1$. ■

Proposition: $\wp'(z)^2 = 4\wp^3(z) - g_2\wp(z) - g_3$

$g_2 = g_2(L) = 60\,G_4(L)$, $\quad g_3 = g_3(L) = 140\,G_6(L)$

Pf:
$$\wp(z) = \frac{1}{z^2} + 3G_4 z^2 + 5G_6 z^4 + \cdots$$
$$\wp'(z) = -\frac{2}{z^3} + 6G_4 z + 20G_6 z^3 + \cdots$$
$$\wp'(z)^2 = \frac{4}{z^6} - \frac{24G_4}{z^2} - 80G_6 + \cdots$$
$$\wp(z)^2 = \frac{1}{z^4} + 6G_4 + 10G_6 z^2 + \cdots$$
$$\wp(z)^3 = \frac{1}{z^6} + \frac{9G_4}{z^2} + 15G_6 + \cdots$$

$\implies \wp'(z)^2 - (4\wp(z)^3 - 60G_4\wp(z) - 140G_6) = 0 + a_2 z^2 + \cdots \in \mathcal{M}(\mathbb{C}/L)$

has no poles $\implies$ it is constant $\implies$ equal to 0.

Discriminant: $4x^3 - g_2x - g_3 = 4(x-e_1)(x-e_2)(x-e_3) = 4(x^3+ax+b)$

$$\operatorname{disc}(x^3+ax+b) = \prod_{i<j}(e_i-e_j)^2 = -4a^3 - 27b^2$$

Def: $\Delta(L) = 16\prod_{i<j}(e_i-e_j)^2 = 16\left(-4\left(\frac{-g_2}{4}\right)^3 - 27\left(\frac{-g_3}{4}\right)^2\right) = g_2^3 - 27g_3^2 \neq 0$

$$j(L) = \frac{(12g_2(L))^3}{\Delta(L)} = \frac{1728\,g_2^3}{g_2^3 - 27g_3^2} \qquad (j(\lambda L) = j(L))$$

Fourier expansion of Eisenstein series

<u>Fact</u>: $\sin(\pi z) = \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)$ (Euler)

<u>$\frac{d}{dz} \circ \log$</u>: $\pi \operatorname{cotg} \pi z = \frac{1}{z} + \sum_{n=1}^{\infty} \left(\frac{1}{z+n} + \frac{1}{z-n} \right) = \lim_{N \to +\infty} \sum_{n=-N}^{N} \frac{1}{z+n}$

$(z \in \mathbb{C} \setminus \mathbb{Z})$

$$= \frac{1}{z} + \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{1}{1+\frac{z}{n}} - \frac{1}{1-\frac{z}{n}} \right)$$

$$= \frac{1}{z} + \sum_{n=1}^{\infty} \frac{1}{n} \left(\sum_{k=0}^{\infty} (-1)^k \left(\frac{z}{n}\right)^k - \left(\frac{z}{n}\right)^k \right) \qquad k = 2m-1$$

$$= \frac{1}{z} - 2 \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{z}{n}\right)^{2m-1} = \frac{1}{z} - 2 \sum_{m=1}^{\infty} \zeta(2m) z^{2m-1}$$

<u>Bernoulli numbers</u>: $\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}$ $\qquad (B_0 = 1)$

$$\frac{t}{e^t-1} + \frac{t}{2} = \frac{t}{2} \cdot \frac{e^t+1}{e^t-1} = \frac{t}{2} \coth\left(\frac{t}{2}\right) \quad (\underline{\text{even}} \text{ function}) = \sum_{m=0}^{\infty} B_{2m} \frac{t^{2m}}{(2m)!}$$

$$\Longrightarrow \quad B_1 = -\frac{1}{2}, \qquad B_{2m+1} = 0 \qquad \forall m \geq 1.$$

<u>$t = 2\pi i z$</u>: $\underbrace{\frac{2\pi i z}{e^{2\pi i z} - 1}}_{\sum_{m=0}^{\infty} B_{2m} \frac{(2\pi i z)^{2m}}{(2m)!}} + \pi i z = \pi i z \frac{e^{\pi i z} + e^{-\pi i z}}{e^{\pi i z} - e^{-\pi i z}} = \underbrace{\pi z \operatorname{cotg}(\pi z)}_{1 - 2\sum_{m=1}^{\infty} \zeta(2m) z^{2m}}$

<u>Cor</u>: $-2\zeta(2m) = \frac{(2\pi i)^{2m}}{(2m)!} B_{2m} \qquad \forall m \geq 1$

<u>$q = e^{2\pi i \tau}$</u>: $\underbrace{\pi \operatorname{cotg}(\pi \tau)}_{} = \frac{1}{\tau} + \sum_{n=1}^{\infty} \left(\frac{1}{\tau+n} + \frac{1}{\tau-n} \right)$

$\tau \in \mathcal{H}$

$|q| < 1$ $\qquad \frac{2\pi i}{q-1} + \pi i = \pi i \left(1 - \frac{2}{1-q}\right) = \pi i \left(1 - 2\sum_{k=0}^{\infty} q^k\right)$

$q \frac{d}{dq} = \frac{1}{2\pi i} \frac{d}{d\tau}$. <u>Apply $\left(\frac{d}{d\tau}\right)^{\ell-1}$</u>: $(\ell \geq 2)$

$$(-1)^{\ell-1} (\ell-1)! \sum_{n \in \mathbb{Z}} \frac{1}{(\tau+n)^\ell} = -(2\pi i)^\ell \sum_{k=1}^{\infty} k^{\ell-1} q^k$$

<u>$\ell = 2$</u>: $\sum_{n \in \mathbb{Z}} \frac{1}{(\tau+n)^2} = (2\pi i)^2 \sum_{k=1}^{\infty} k q^k$

$$\sum_{n\in\mathbb{Z}} \frac{1}{(\tau+n)^\ell} = \frac{(-1)^\ell}{(\ell-1)!}(2\pi i)^\ell \sum_{k=1}^\infty k^{\ell-1} q^k \qquad (\ell\geq 2)$$

$$\sum_{m,n\in\mathbb{Z}}{}' \frac{1}{(m\tau+n)^\ell} = \underbrace{\sum_{m=0}} + \underbrace{\sum_{m\neq 0}}$$

$$(1+(-1)^\ell)\left(\zeta(\ell) + \frac{(-1)^\ell}{(\ell-1)!}(2\pi i)^\ell \overbrace{\underbrace{\sum_{m=1}^\infty \sum_{k=1}^\infty k^{\ell-1} q^{mk}}_{\sum_{n=1}^\infty \sigma_{\ell-1}(n) q^n}}^{\sum_{n=1}^\infty \frac{n^{\ell-1}q^n}{1-q^n}}\right)$$

$\underline{\ell\geq 2 \text{ even}}$: $\qquad \zeta(\ell) = -\frac{1}{2}\cdot\frac{(2\pi i)^\ell}{\ell!} B_\ell$

$$\sum_{m,n\in\mathbb{Z}}{}' \frac{1}{(m\tau+n)^\ell} = 2\zeta(\ell)\underbrace{\left(1-\frac{2\ell}{B_\ell}\sum_{n=1}^\infty \sigma_{\ell-1}(n) q^n\right)}_{E_\ell(\tau)}$$

$$E_4(\tau) = 1+240\sum_{n=1}^\infty \sigma_3(n) q^n \qquad \sigma_r(n) = \sum_{d\mid n} d^r$$

$$E_6(\tau) = 1-504\sum_{n=1}^\infty \sigma_5(n) q^n$$

$$E_8(\tau) = 1+480\sum_{n=1}^\infty \sigma_7(n) q^n$$

$$E_{10}(\tau) = 1-264\sum_{n=1}^\infty \sigma_9(n) q^n$$

$$E_{12}(\tau) = 1+\frac{65520}{691}\sum_{n=1}^\infty \sigma_{11}(n) q^n$$

$$E_{14}(\tau) = 1-24\sum_{n=1}^\infty \sigma_{13}(n) q^n$$

$$E_{16}(\tau) = 1+\frac{16320}{3617}\sum_{n=1}^\infty \sigma_{15}(n) q^n$$

<u>Define</u>:

$$G_2(\tau) := \sum_{\substack{m\neq 0\\ m\in\mathbb{Z}}}\left(\sum_{n\in\mathbb{Z}}\frac{1}{(m\tau+n)^2}\right) + \sum_{n\neq 0}\frac{1}{n^2}. \quad \text{Then}$$

$$G_2(\tau) = \underbrace{2\zeta(2)}_{\pi^2/3} E_2(\tau),$$

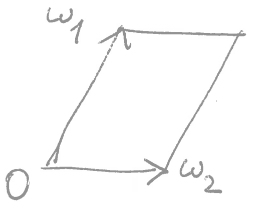
$$E_2(\tau) = 1-24\sum_{n=1}^\infty \sigma_1(n) q^n.$$

<u>Note</u>: $240+240=480$, $240-504=-264$, $240-264=-24$

? Is it true that $E_4^2 = E_8$, $E_4E_6 = E_{10}$, $E_4E_{10} = E_{14}$?

Weierstrass functions and Eisenstein series

Lattice $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 \subset \mathbb{C}$, $\quad \mathrm{Im}(\omega_1/\omega_2) > 0$



covolume $\quad \mathrm{vol}(\mathbb{C}/L) = |\omega_2|^2 \underbrace{\mathrm{vol}\left(\mathbb{C}/(\mathbb{Z}\tfrac{\omega_1}{\omega_2} + \mathbb{Z})\right)}_{\mathrm{Im}\left(\frac{\omega_1}{\omega_2}\right)}$

$$= \frac{1}{2i}\begin{vmatrix} \omega_1 & \overline{\omega_1} \\ \omega_2 & \overline{\omega_2} \end{vmatrix}.$$

Weierstrass functions:

$$\sigma(z,L) = z\prod_{u\in L}{}' \left(1 - \frac{z}{u}\right) e^{z/u + z^2/2u^2}, \quad \zeta(z,L) = \frac{\sigma'}{\sigma}(z,L) = \frac{1}{z} + \sum_{u\in L}{}' \left(\frac{1}{z-u} + \frac{1}{u} + \frac{z}{u^2}\right)$$

$$\wp(z,L) = -\zeta'(z,L) = \frac{1}{z^2} + \sum_{u\in L}{}' \left(\frac{1}{(z-u)^2} - \frac{1}{u^2}\right),$$

$$\wp^{(r)}(z,L) = (-1)^r r! \sum_{u\in L} \frac{1}{(z-u)^{r+2}} \qquad (r \geq 1)$$

Quasi-periods: $\quad \zeta(z+\omega_j, L) - \zeta(z,L) = \eta_j \qquad (j=1,2)$

Legendre's relation: $\quad \begin{vmatrix} \omega_1 & \eta_1 \\ \omega_2 & \eta_2 \end{vmatrix} = 2\pi i$

Invariant formulation: $\forall u \in L \ \exists!\ \eta(L,u) \in \mathbb{C} \ \forall z \in \mathbb{C}$ $\quad \zeta(z+u,L) = \zeta(z) + \eta(L,u)$

$$\begin{array}{ccc} u & \longmapsto & \eta(L,u) \\ \cap & & \cap \\ L & \longrightarrow & \mathbb{C} \end{array}$$

is $\mathbb{Z}$-linear $\Longrightarrow$ it extends (uniquely) to an $\mathbb{R}$-linear function $L_{\mathbb{R}} = \mathbb{C} \longrightarrow \mathbb{C}$

$\Longrightarrow \exists!\ \alpha(L), \beta(L) \in \mathbb{C} \quad \forall u \in L \quad \eta(u,L) = \alpha(L)u + \beta(L)\overline{u}$

Properties of $\alpha(L), \beta(L)$: $\forall t \in \mathbb{C}^\times \quad \zeta(tL, tz) = t^{-1}\zeta(L,z)$

$$\Longrightarrow \alpha(tL) = t^{-2}\alpha(L), \quad \beta(tL) = \beta(L)/t\overline{t}$$

Legendre's relation: $\quad (\alpha = \alpha(L), \ \beta = \beta(L))$

$$2\pi i = \begin{vmatrix} \omega_1 & \alpha\omega_1 + \beta\overline{\omega_1} \\ \omega_2 & \alpha\omega_2 + \beta\overline{\omega_2} \end{vmatrix} = \beta \begin{vmatrix} \omega_1 & \overline{\omega_1} \\ \omega_2 & \overline{\omega_2} \end{vmatrix} \Longrightarrow \boxed{\beta(L) = \frac{2\pi i}{2i\,\mathrm{vol}(\mathbb{C}/L)} = \frac{\pi}{\mathrm{vol}(\mathbb{C}/L)}}$$

Another notation: $\quad s_2^*(L) = \alpha(L)$

Goal: compute $s_2^*(L)$ for $L = L_\tau = \mathbb{Z}\tau + \mathbb{Z}$, $\quad \tau \in \mathcal{H}$

$(\omega_1 = \tau, \ \omega_2 = 1)$

$$\zeta(z) = \zeta(z, L_\tau) = \frac{1}{z} + \sum_{m,n}{}' \left(\frac{1}{z+m\tau+n} - \frac{1}{m\tau+n} + \frac{z}{(m\tau+n)^2} \right)$$

$$\eta_1 = \zeta(z+\tau) - \zeta(z) = \frac{1}{z+\tau} - \frac{1}{z} + \sum_{m,n}{}' \left(\frac{1}{z+(m+1)\tau+n} - \frac{1}{z+m\tau+n} + \frac{\tau}{(m\tau+n)^2} \right)$$

Write this as $\left(\sum_{n=0} + \sum_{n\neq 0} \right) \sum_m$:

$$\eta_1 = \underbrace{\sum_{m\in\mathbb{Z}} \left(\frac{1}{z+(m+1)\tau} - \frac{1}{z+m\tau} \right)}_{0} + \frac{1}{\tau} \sum_{m\neq 0} \frac{1}{m^2} + \sum_{n\neq 0} \sum_{m\in\mathbb{Z}} \frac{\tau}{(m\tau+n)^2}$$

$$\eta_2 = \zeta(z+1) - \zeta(z) = \frac{1}{z+1} - \frac{1}{z} + \sum_{m,n}{}' \left(\frac{1}{z+m\tau+(n+1)} - \frac{1}{z+m\tau+n} + \frac{1}{(m\tau+n)^2} \right)$$

$$= \underbrace{\sum_{n\in\mathbb{Z}} \left(\frac{1}{z+(n+1)} - \frac{1}{z+n} \right)}_{0} + \sum_{n\neq 0} \frac{1}{n^2} + \sum_{m\neq 0} \sum_{n\in\mathbb{Z}} \frac{1}{(m\tau+n)^2} =: G_2(\tau)$$

Fourier expansion : $\frac{1}{\tau} + \sum_{n=1}^{\infty} \left(\frac{1}{\tau+n} + \frac{1}{\tau-n} \right) = \pi \cot(\pi\tau) = \pi i \left(1 - 2\sum_{n=1}^{\infty} q^n\right)$

$$\sum_{n\in\mathbb{Z}} \frac{1}{(\tau+n)^2} = (2\pi i) \sum_{n=1}^{\infty} (2\pi i n) q^n \qquad (q = e^{2\pi i\tau})$$

$$\Longrightarrow G_2(\tau) = 2\sum_{n=1}^{\infty} \frac{1}{n^2} + 2(2\pi i)^n \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} n q^{mn} = \frac{\pi^2}{3} \underbrace{\left(1 - 24 \sum_{n=1}^{\infty} \sigma_1(n) q^n \right)}_{E_2(\tau)}$$

$$\left(\Longrightarrow \eta_1 = \tau^{-1} G_2\left(-\frac{1}{\tau}\right) \right)$$

Summary : $\eta(\mathbb{Z}\tau + \mathbb{Z}, 1) = G_2(\tau) = \frac{\pi^2}{3} E_2(\tau)$

Recall : $\eta(\mathbb{Z}\tau + \mathbb{Z}, 1) = \alpha(\mathbb{Z}\tau + \mathbb{Z}) + \underbrace{\beta(\mathbb{Z}\tau + \mathbb{Z})}_{\pi/\mathrm{Im}(\tau) = \frac{2\pi i}{\tau - \bar\tau}}$

Prop. $\forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ $\quad \underbrace{\left(G_2(\tau) - \frac{2\pi i}{\tau - \bar\tau} \right)}_{G_2^*(\tau) = \alpha(\mathbb{Z}\tau+\mathbb{Z})} \Big|_2 \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \underbrace{G_2(\tau) - \frac{2\pi i}{\tau - \bar\tau}}_{G_2^*(\tau)}$

Pf. $G_2^*(\tau)$ depends only on $\mathbb{Z}\tau + \mathbb{Z}$ and $\alpha(tL) = t^{-2}\alpha(L)$.

Notation : $s_2^*(\tau) = s_2^*(\mathbb{Z}\tau + \mathbb{Z}) = \alpha(\mathbb{Z}\tau + \mathbb{Z}) \quad (= G_2^*(\tau))$

Rmk. $G_2^*(\tau) = \lim_{s\to 0+} \sum_{m,n}{}' \frac{1}{(m\tau+n)^2 |m\tau+n|^s}$ (Hecke's regularisation)

Cor. $E_2^*(\tau) := E_2(\tau) - \frac{3}{\pi \mathrm{Im}(\tau)}$ satisfies $E_2^* \big|_2 \begin{pmatrix} a & b \\ c & d \end{pmatrix} = E_2^*$

$\forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$

Back to Weierstrass functions: $\eta(L,u) = \alpha u + \beta \bar{u}$,

$$\alpha = \alpha(L) = G_2^*(L) := \omega_2^{-2} G_2^*\left(\frac{\omega_1}{\omega_2}\right) = s_2^*(L)$$

$$\beta = \beta(L) = \frac{\pi}{\mathrm{vol}(\mathbb{C}/L)} = 2\pi i \Big/ \begin{vmatrix} \omega_1 & \overline{\omega_1} \\ \omega_2 & \overline{\omega_2} \end{vmatrix}$$

Define: $G_1^*(z,L) := \zeta(z,L) - \eta(L,z) = \zeta(z,L) - \alpha z - \beta \bar{z}$

$$G_2^*(z,L) := -\frac{\partial}{\partial z} G_1^*(z,L) = \wp(z,L) + \alpha$$

$\forall k \geq 3$ $\quad G_k(z,L) = G_k^*(z,L) := \dfrac{(-1)^{k-2}}{(k-1)!}(\partial/\partial z)^{k-2}\wp(z,L) = \displaystyle\sum_{u\in L} \frac{1}{(z+u)^k}$

$(z \notin L)$

Consider $z, \bar{z}, \omega_1, \overline{\omega_1}, \omega_2, \overline{\omega_2}$ as independent variables

Differential operators: $\partial_L = \overline{\omega_1}\dfrac{\partial}{\partial \omega_1} + \overline{\omega_2}\dfrac{\partial}{\partial \omega_2}$, $\quad \mathcal{D} = \partial_L + \bar{z}\dfrac{\partial}{\partial z}$

Note: $\forall k \geq 3, \forall j \geq 0$ $\quad \mathcal{D}^j G_k(z,L) = (-1)^j \underbrace{k(k+1)\cdots(k+j-1)}_{(k)_j} \displaystyle\sum_{u\in L} \frac{(\overline{z+u})^j}{(z+u)^{k+j}}$

Goal: find relations between various $\mathcal{D}^j G_k^*(z,L)$

and $G_k(L) = \displaystyle\sum_{u\in L}{}' u^{-k}$ (here $k>2$, $2|k$)

Laurent expansions: $\zeta(z,L) = \dfrac{1}{z} - \displaystyle\sum_{k=2}^{\infty} G_{2k}(L) z^{2k-1}$

$\wp(z,L) = \dfrac{1}{z^2} + \displaystyle\sum_{k=2}^{\infty}(2k-1)G_{2k}(L) z^{2k-2}$, $\quad G_m(z,L) = \dfrac{1}{z^m} + (-1)^m \displaystyle\sum_{k=\left[\frac{m-1}{2}\right]}^{\infty} G_{2k}(L)\binom{2k-1}{m-1} z^{2k-m}$

$(m \geq 3)$

Differential equations: $\dfrac{\partial}{\partial z}\wp = \wp'(z,L) = -2G_3(z,L)$

$\wp'^2 = 4\wp^3 - 60G_4(L)\wp - 140G_6(L) \implies G_3(z,L)^2 = \wp(z,L)^3 - 15G_4(L)\wp(z,L) - 35G_6(L)$

$\wp'' = 6\wp^2 - 30G_4(L) \implies G_4(z,L) = \wp(z,L)^2 - 5G_4(L)$

Comparing coefficients at z^{2k-4} $(k \geq 4)$:

$$\binom{2k-1}{3} G_{2k}(L) = 2(2k-1)G_{2k}(L) + \sum_{\substack{i+j=k \\ i,j\geq 2}} (2i-1)G_{2i}(L)(2j-1)G_{2j}(L)$$

$\forall k \geq 4$

$$\implies \frac{(k-3)(2k-1)(2k+1)}{3} G_{2k}(L) = \sum_{\substack{i+j=k \\ i,j\geq 2}} (2i-1)(2j-1)G_{2i}(L)G_{2j}(L)$$

$\boxed{\text{Ex: } G_8 = \frac{3}{7}G_4^2}$

Cor: $\forall k \geq 4$ $\quad G_{2k}(L) \in \mathbb{Q}[\underbrace{G_4(L)}_{\deg=4}, \underbrace{G_6(L)}_{\deg=6}]_{\deg=2k}$

Key differential equation:

$$(\ast) \qquad \beta\, \mathcal{D} G_1^{\ast}(z,L) = G_1^{\ast}(z,L)\, G_2^{\ast}(z,L) - G_3(z,L) \qquad \left(\beta = \frac{\pi}{\mathrm{vol}(\mathbb{C}/L)}\right)$$

Note: $\mathcal{D} \begin{vmatrix} \omega_1 & \overline{\omega_1} \\ \omega_2 & \overline{\omega_2} \end{vmatrix} = 0 \implies \partial_L \beta = 0 = \mathcal{D}\beta$.

Thm 1. $\forall k \geq 1 \ \forall \ell \geq 0$

$$\underbrace{\beta^{\ell}(k-1)!\, \mathcal{D}^{\ell} G_k^{\ast}(z,L)}_{(-\beta)^{\ell}(k+\ell-1)! \sum_{u \in L} \frac{\overline{(z+u)}^{\ell}}{(z+u)^{k+\ell}} \text{ if } k \geq 3} \in \mathbb{Z}\left[\{(j-1)!\, G_j^{\ast}(z,L)\}_{j \leq k+2\ell}\right] \qquad (z \notin L)$$

Pf. $\ell = 1$: apply $(\partial/\partial z)^{k-1}$ to $(\ast)$

general $\ell > 1$: induction on ℓ.

Thm 2. $\beta(\partial_L(G_2^{\ast}(L))) = G_2^{\ast}(L)^2 - 5G_4(L)$

$\forall k \geq 2 \qquad \beta(\partial_L(G_{2k}(L))) \in \mathbb{Z}[\underbrace{G_2^{\ast}(L)}_{\deg = 2}, \underbrace{G_4(L)}_{\deg = 4}, \dots, G_{2k+2}(L)]_{\deg = 2k+2}$

Cor. $\forall \ell \geq 0 \ \forall k \geq 2 \qquad \beta^{\ell}(\partial_L^{\ell}(G_2^{\ast}(L))) \in \mathbb{Z}[G_2^{\ast}(L), G_4(L), \dots, G_{2+2\ell}(L)]_{\deg = 2+2\ell}$

$$\beta^{\ell}(\partial_L^{\ell}(G_{2k}(L))) \in \mathbb{Z}[G_2^{\ast}(L), G_4(L), \dots, G_{2k+2\ell}]_{\deg = 2k+2\ell}$$

Pf of thm 2. Compare the coefficients in the Laurent expansions of both sides of $(\ast)$.

Rmk. $\forall k \geq 2 \ \forall \ell \geq 0 \qquad \partial_L^{\ell}(G_{2k}(L)) = (-1)^{\ell} (2k)_{\ell} \sum_{u \in L}' \frac{\overline{u}^{\ell}}{u^{2k+\ell}}$.

Rmk. $G_1^{\ast}(z,L) = \lim_{s \to 1} \left(\sum_{u \in L} \frac{\overline{z+u}}{|z+u|^{2s}} \right)$

Proof of (∗): Step 1. Compute $F(z) = -\partial_L \wp'(z,L) = 6 \sum_{u\in L} \frac{\bar{u}}{(z-u)^4}$

$\forall u \in L \quad F(z+u) - F(z) = \bar{u}\, \wp''(z)$

$$\left.\begin{array}{l} \forall u\in L \quad F(z+u)-F(z)=\bar u\,\wp''(z) \\ \frac{(\zeta(z,L)-\alpha z)}{\beta}\, \wp''(z) \text{ has the same property} \end{array}\right\} \Rightarrow F(z) - \frac{\zeta(z)-\alpha z}{\beta}\wp''(z) = G(z) \in \mathcal{M}(\mathbb{C}/L)$$

$G(z)$ is odd, the only pole is at $z=0 \pmod L$

$\Rightarrow \quad G(z) = \wp'(z)(c_0 + c_1 \wp(z))$. Comparison of Laurent expansions: $\Rightarrow$ values of c_0, c_1

$$F = -\partial_L \wp' = \frac{1}{\beta}(\zeta(z) - \alpha z)\wp''(z) + \frac{3}{\beta}(\wp(z) - \alpha)\wp'(z)$$

<u>Note</u>: comparison of coefficients $\Rightarrow \forall k \geq 2 \quad \beta(\partial_L(G_{2k}(L))) \in \mathbb{Q}[\alpha, G_4(L), G_6(L), \dots, G_{2k+2}(L)]$

<u>Step 2</u>. Compute $F_1(z) = -\partial_L \wp(z,L)$: $\frac{\partial}{\partial z} F_1 = F$, $\quad F_1(z) = c_2 z^2 + \dots$

$$\Rightarrow F_0 = \int F(z)\,dz = (\text{const.}) + \frac{1}{\beta}(\zeta - \alpha z)\wp' + \frac{2}{\beta}\left(\frac{\wp''}{6} - \alpha\wp\right)$$

the Laurent expansion determines the constant $\Rightarrow$

$$\beta F_1 = \beta(-\partial_L \wp) = -10\,G_4(L) + (\zeta(z) - \alpha z)\wp'(z) + 2\left(\frac{\wp''(z)}{6} - \alpha\wp(z)\right)$$

<u>Step 3</u>. Compute $F_2(z) = \partial_L(\zeta(z,L))$: $\frac{\partial}{\partial z} F_2 = F_1 = c_2 z^2 + \dots$
(even function)

$$\Rightarrow \beta(\partial_L \zeta(z,L)) = (\zeta(z) - \alpha z)\wp(z) + \frac{\wp'(z)}{2} - 5G_4(L)\,z + \alpha\zeta(z)$$

(since $\wp'' = 6\wp^2 - 30\,G_4(L)$)

<u>Step 4</u>. $\partial_L(\beta) = 0$, $\quad \zeta(z+u) - \zeta(z) = \alpha u + \beta\bar{u} \quad \forall u \in L$

Apply ∂_L: $\underbrace{(\partial_L\zeta)(z+u) - (\partial_L\zeta)(z)}_{\frac{1}{\beta}(\underbrace{\eta(u)-\alpha u}_{\beta\bar u})\wp(z) - \frac{5G_4(L)}{\beta}u + \frac{\alpha}{\beta}(\alpha u+\beta\bar u)} + \bar{u}\,\underbrace{\zeta'(z)}_{-\wp(z)} = (\partial_L\alpha)u + \alpha\bar{u}$

$$\Rightarrow \beta(\partial_L \alpha) = \alpha^2 - 5G_4(L)$$

<u>Step 5</u>. $\mathcal{D} = \partial_L + \bar{z}\frac{\partial}{\partial z}$ and $\beta(\partial_L\zeta) = (\zeta - \alpha z)(\wp + \alpha) + \frac{\wp'}{2} + \beta(\partial_L\alpha)z$

$\mathcal{D}(\beta\bar{z}) = 0 \Rightarrow \beta(\mathcal{D}G_1^*(z,L)) = \beta((\partial_L(\zeta - \alpha z)) - \bar{z}(\wp + \alpha)) =$

$$= (\wp + \alpha)(\zeta - \alpha z - \beta\bar{z}) + \frac{\wp'}{2} = G_2^*(z,L)\,G_1^*(z,L) - G_3(z,L).$$

Non-holomorphic regularisation of $\zeta(z,L)$ and $G_2(L)$

Weierstrass ζ-function of a lattice $L\subset\mathbb{C}$:

$$\zeta(z,L)=\frac{1}{z}+\sum_{0\neq u\in L}\left(\frac{1}{z+u}-\frac{1}{u}+\frac{z}{u^2}\right)$$

$\exists$ $\mathbb{R}$-linear $\eta:\mathbb{C}\to\mathbb{C}$ such that $\forall u\in L\ \forall z\in\mathbb{C}\ \ \zeta(z+u)=\zeta(z)+\eta(u)$

s-regularisation: ($z\notin L$ fixed)

$$(z+u)^{-\alpha}=z^{-\alpha}\left(1+\frac{u}{z}\right)^{-\alpha}=u^{-\alpha}\left(1+\frac{z}{u}\right)^{-\alpha}=u^{-\alpha}-\alpha z u^{-\alpha-1}+O(|u|^{-\alpha-2})$$

$$\overline{(z+u)}^{-\beta}=\bar u^{-\beta}-\beta\bar z\,\bar u^{-\beta-1}+O(|u|^{-\beta-2})$$

$$\frac{1}{(z+u)^\alpha\,\overline{(z+u)}^\beta}=\frac{1}{u^\alpha\bar u^\beta}-\frac{\alpha z}{u^{\alpha+1}\bar u^\beta}-\frac{\beta\bar z}{u^\alpha\bar u^{\beta+1}}+O(|u|^{-\alpha-\beta-2})$$

$\alpha=s,\ \beta=s-1$:

$$\frac{\overline{z+u}}{|z+u|^{2s}}=\frac{\bar u}{|u|^{2s}}-\frac{sz}{|u|^{2s-2}u^2}-\frac{(s-1)\bar z}{|u|^{2s}}+O(|u|^{-2s-1})$$

Def: ($z\notin L$)

$$\zeta_s(z,L):=\frac{\bar z}{|z|^{2s}}+\sum_{0\neq u\in L}\left(\frac{\overline{z+u}}{|z+u|^{2s}}-\frac{\bar u}{|u|^{2s}}+\frac{sz}{|u|^{2s-2}u^2}+\frac{(s-1)\bar z}{|u|^{2s}}\right)$$

(absolutely convergent if $\mathrm{Re}(s)>\frac{1}{2}$, by the above)

Other interesting functions: $E(\tau,s)=\sum_{m,n\in\mathbb{Z}}{}'\frac{y^s}{|m\tau+n|^{2s}}$ $(\mathrm{Re}(s)>1)$ $(\tau=x+iy)$

Write $L=\mathbb{Z}\omega_1+\mathbb{Z}\omega_2$, $\tau=\frac{\omega_1}{\omega_2}\in\mathcal{H}$

$$E(L,s):=\sum_{0\neq u\in L}\frac{1}{|u|^{2s}}\qquad(\mathrm{Re}(s)>1)$$

$$G(z,L,s):=\sum_{0\neq u\in L}\frac{\overline{z+u}}{|z+u|^{2s}}\qquad\left(\mathrm{Re}(s)>\frac{3}{2}\right)$$

Facts: (1)

$$E(\tau,s)=\frac{\pi}{s-1}+O(1)\quad\text{as } s\to 1$$

$$\Rightarrow\quad E(L,s)=\frac{\pi}{\mathrm{vol}(\mathbb{C}/L)}\cdot\frac{1}{s-1}+O(1)\qquad(\text{— " —})$$

(2) The functions $\left\{\begin{array}{l}\sum_{0\neq u\in L}\frac{1}{u^2|u|^{2s}}\\ G(z,L,s)\end{array}\right\}$ have holomorphic cont. to $\left\{\begin{array}{l}\mathbb{C}\\ \mathrm{Re}(s)>\frac{1}{2}\end{array}\right.$

Let

$$\varepsilon_2^*(L):=\lim_{s\to 0}\sum_{0\neq u\in L}\frac{1}{u^2|u|^{2s}}\doteq G_2^*(L).\qquad\text{(Pf. Poisson summation)}$$

Prop. $\zeta(z,L) = \frac{1}{z} + G(z,L,1) + G_2^*(L)z + \frac{\pi}{\mathrm{vol}(\mathbb{C}/L)}\bar{z}$.

Pf. If $\mathrm{Re}(s) > \frac{3}{2}$, one can rearrange terms in the series defining $\zeta_s(z,L) \Longrightarrow \sum_{u\neq 0} \frac{\bar{u}}{|u|^{2s}} = 0$. We know that

$$\lim_{s\to 1}\left(\sum_{u\neq 0}\frac{\bar{z}}{|u|^{2s}}(s-1)\right) = \frac{\bar{z}\pi}{\mathrm{vol}(\mathbb{C}/L)} \Longrightarrow \sum_{0\neq u\in L}\left(\frac{\overline{z+u}}{|z+u|^{2s}} + \frac{sz}{|u|^{2s-2}u^2}\right)$$

is holomorphic at s.

Letting $s\to 1$, we obtain Prop.

Prop. $$\boxed{\eta(z,L) = G_2^*(L)z + \frac{\pi}{\mathrm{vol}(\mathbb{C}/L)}\bar{z}}$$

Pf. The function $z \longmapsto \underbrace{\left(\frac{1}{z} + G(z,L,1)\right)}_{\left(\frac{\bar{z}}{|z|^{2s}} + G(z,L,s)\right)_{s=1}}$ is L-periodic

$$\Longrightarrow \forall u\in L \quad \underbrace{\zeta(z+u,L) - \zeta(z,L)}_{\eta(u)} = G_2^*(L)u + \frac{\pi}{\mathrm{vol}(\mathbb{C}/L)}\bar{u}$$

$\left(\Longrightarrow \text{Legendre's relation } \begin{vmatrix}\omega_1 & \eta(\omega_1)\\ \omega_2 & \eta(\omega_2)\end{vmatrix} = \frac{\pi\begin{vmatrix}\omega_1 & \omega_2\\ \bar{\omega}_1 & \bar{\omega}_2\end{vmatrix}}{\mathrm{vol}(\mathbb{C}/L)} = 2\pi i\right)$

For $L = \mathbb{Z}\tau + \mathbb{Z}$:

$$\eta(1) = \frac{2\zeta(2)}{\cancel{\pi^2/3}} + \sum_{m=1}^{\infty}\sum_{n\in\mathbb{Z}}\frac{1}{(m\tau+n)^2} = \frac{\pi^2}{3}\underbrace{\left(1 - 24\sum_{n=1}^{\infty}\sigma_1(n)q^n\right)}_{E_2(\tau)} \quad (q = e^{2\pi i\tau})$$

Write $G_2^*(L) = 2\zeta(2)E_2^*(L)$ and take $z = 1$ in Prop.:

$$\frac{\pi^2}{3}E_2(\tau) = \frac{\pi^2}{3}E_2^*(\tau) + \frac{\pi}{\mathrm{Im}(\tau)}$$

$$\Downarrow$$

$$\boxed{E_2^*(\tau) = E_2(\tau) - \frac{3}{\pi\,\mathrm{Im}(\tau)}}$$

Non-holomorphic

$\forall \begin{pmatrix}a & b\\ c & d\end{pmatrix}\in SL_2(\mathbb{Z})$

$$E_2^*\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^2 E_2^*(\tau)$$