Hecke operators on $SL_2(\mathbb{Z})$

Recall: Ramanujan's observation (proved by Mordell):
the coefficients of $\Delta(\tau) = q\prod_{n=1}^{\infty}(1-q^n)^{24} = \sum_{n=1}^{\infty}\tau(n)q^n$
satisfy

(1) $(m,n)=1 \Longrightarrow \tau(mn) = \tau(m)\tau(n)$

(2) p prime, $n \geq 1 \Longrightarrow \tau(p)\tau(p^n) = \tau(p^{n+1}) + p^{11}\tau(p^{n-1})$

$$\Longleftrightarrow \quad L(\Delta,s) = \sum_{n=1}^{\infty}\frac{\tau(n)}{n^s} = \prod_{p \text{ prime}}\left(1-\frac{\tau(p)}{p^s}+\frac{p^{11}}{p^{2s}}\right)^{-1}$$

n	1	2	3	4	5	6	7
$\tau(n)$	1	-24	252	-1472	4830	-6048	-16744

How can one prove this?

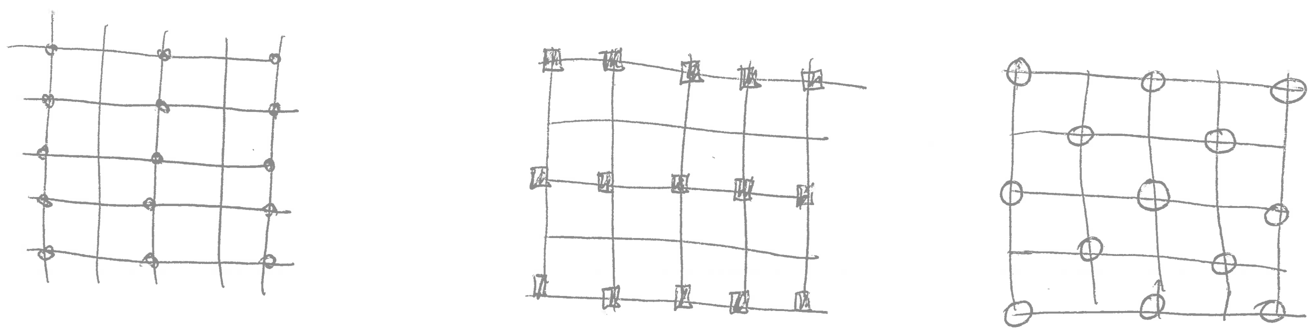
Goal: relate $\tau(p)\underbrace{\sum_n \tau(n)q^{pn}}_{\Delta(p\tau)}$ to $\underbrace{\sum \tau(n)q^n}_{\Delta(\tau)}$

$\Delta \longleftrightarrow F:\{\text{lattices } L\subset\mathbb{C}\}\longrightarrow\mathbb{C}$, $\quad F(tL) = t^{-12}F(L)$

$\Delta(\tau) = F(\underbrace{\mathbb{Z}\tau+\mathbb{Z}}_{L})$, $\quad \Delta(p\tau) = F(\underbrace{\mathbb{Z}p\tau+\mathbb{Z}}_{L'})$ sublattice of L of index p

Idea: consider averages over all sublattices of a given index.

Ex: $\mathbb{Z}^2$ has 3 sublattices of index 2



Hecke operators on lattices (of rank 2)

Fix an abelian group L isomorphic to $\mathbb{Z}^2$.

Def: $\mathcal{L} = \left\{ \sum_{k=1}^{n} c_k [L_k] \;\middle|\; \text{finite formal sum, } c_k \in \mathbb{Z}, \ L_k \subset L \text{ subgroup of finite index} \right\}$

"sublattice of L"

Def: For $n \geq 1$, define operators $T(n)$, $T(n,n) : \mathcal{L} \to \mathcal{L}$ by

$$T(n)[L_0] = \sum_{(L_0 : L_1) = n} [L_1], \qquad T(n,n)[L_0] = [nL_0]$$ (and by $\mathbb{Z}$-linearity)

Observe: (1) $T(m,m)\,T(n,n) = T(mn, mn)$

(2) $T(m)\,T(n,n) = T(n,n)\,T(m)$

Proposition: $(m,n) = 1 \Longrightarrow T(m)T(n) = T(mn) = T(n)T(m)$.

Pf. Fix $L_0 \subset L$ (sublattice)

If $L_0 \supset L_1 \supset L_2$ with $(L_0 : L_1) = m$, $(L_1 : L_2) = n \Longrightarrow (L_0 : L_2) = mn$.

Conversely, if $L_0 \supset L_2$ with $(L_0 : L_2) = mn$, fix a Bezout relation $mu + nv = 1$ ($u, v \in \mathbb{Z}$). Then L_0/L_2 is a direct sum of $\underbrace{M = (nL_0 + L_2)/L_2}_{\text{the unique subgroup of } L_0/L_2 \text{ of order } m}$ and $\underbrace{N = (mL_0 + L_2)/L_2}_{(\text{—''—}) \text{ of order } n}$

therefore $\left\{ \begin{matrix} nL_0 + L_2 \\ mL_0 + L_2 \end{matrix} \right\}$ is the unique subgroup of L_0 of index $\left\{ \begin{matrix} n \\ m \end{matrix} \right\}$.

hence $$T(mn)[L_0] = \sum_{(L_0 : L_2) = mn} [L_2] = \sum_{\substack{L_1 \\ (L_0 : L_1) = n}} \Bigg(\sum_{\substack{L_2 \\ (L_1 : L_2) = m}} [L_2] \Bigg) = T(m)T(n)[L_0]$$

Def. $L' \subset L$ is a primitive sublattice of index n if $L/L' \simeq \mathbb{Z}/n\mathbb{Z}$.

Theorem on elementary divisors

$\forall$ sublattice $L' \subset L$ (i.e., a subgroup of finite index)
$\exists$ $\mathbb{Z}$-bases $L = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$ for a unique pair
$L' = \mathbb{Z}d_1e_1 \oplus \mathbb{Z}d_2e_2$ $\quad 0 < d_1, d_2 \in \mathbb{N}$ such that $d_1 | d_2$

Cor: $L \supset d_1L = \mathbb{Z}d_1e_1 \oplus \mathbb{Z}d_1e_2 \supset L' = \mathbb{Z}d_1e_1 \oplus \mathbb{Z}d_2e_2$

$L/L' \simeq \mathbb{Z}/d_1\mathbb{Z} \oplus \mathbb{Z}/d_2\mathbb{Z}$

primitive sublattice:
$d_1L/L' \simeq d_1\mathbb{Z}/d_2\mathbb{Z} \simeq \mathbb{Z}/\frac{d_2}{d_1}\mathbb{Z}$.

Def. $T(n)_{prim} : \mathcal{L} \longrightarrow \mathcal{L}$ is defined by

$$T(n)_{prim}[L_0] = \sum_{L_0/L_1 \simeq \mathbb{Z}/n\mathbb{Z}} [L_1]$$

(and extended by $\mathbb{Z}$-linearity)

Cor. $T(n) = \sum_{d^2|n} T(\frac{n}{d^2})_{prim}$ $\quad T(d,d) = \sum_{d^2|n} T(d,d)\, T(\frac{n}{d^2})_{prim}$

Special case: p prime, $n \geq 0$: $T(p^n) = \sum_{i=0}^{[n/2]} T(p,p)^i \, T(p^{n-2i})_{prim}$

Sublattices of index p^k (p prime)

Fix $L_0 \subset L$, $(L : L_0) < \infty$.

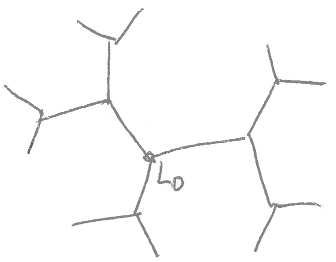
Sublattices of index p: bijections

$$\{L_1 \subset L_0 \mid (L_0 : L_1) = p\} \longleftrightarrow \{\underbrace{L_1/pL_0}_{\text{1-dim. subspace}} \subset L_0/pL_0 \simeq \mathbb{F}_p^2\} = \mathbb{P}^1(\mathbb{F}_p) = \\ = \mathbb{F}_p \cup \{\infty\}$$

Goal: geometric visualisation of sublattices of L of index p^k ($k \geq 0$).

Observation: if $L/L_0' \simeq \mathbb{Z}/p^n\mathbb{Z}$, then there is a unique chain of intermediate lattices $L = L_0 \supset L_1 \supset L_2 \supset \ldots \supset L_n = L'$ such that $(L_i : L_{i+1}) = p$. Moreover, $L_i/L_{i+k} \simeq \mathbb{Z}/p^k\mathbb{Z}$

the Bruhat – Tits tree : infinite tree of degree $p+1$



$\{\text{vertices}\} = \{\text{primitive sublattices } L' \subset L_0 \text{ of index } p^n \ (n \geq 0)\}$

$= \{\text{all sublattices } L' \subset L_0 \text{ of index } p^n \ (n \geq 0)\} / (\text{scaler multiple})$

$\{\text{edges}\}$ $\quad L' \circ\!\!-\!\!\circ L'' \iff L' \subset L''$ of index p

or $\quad L'' \subset L'$ — " — .

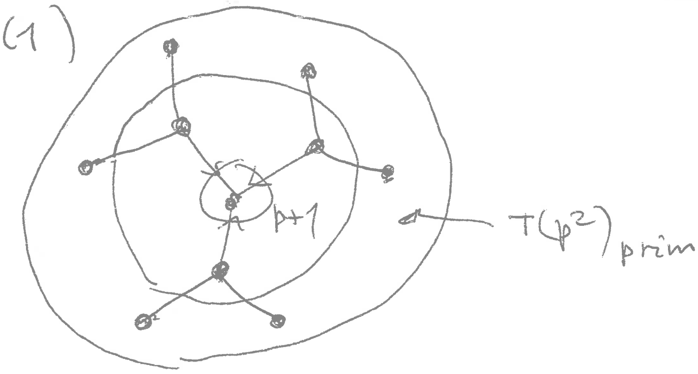
Observe : $\quad T(p^n)_{prim}[L_0] = \sum\limits_{\underbrace{dist(L_0, L')}_{\text{distance on the tree}} = n} [L']$

Prop. (1) $\quad T(p)\,T(p) = T(p^2)_{prim} + (p+1)\,T(p,p)$.

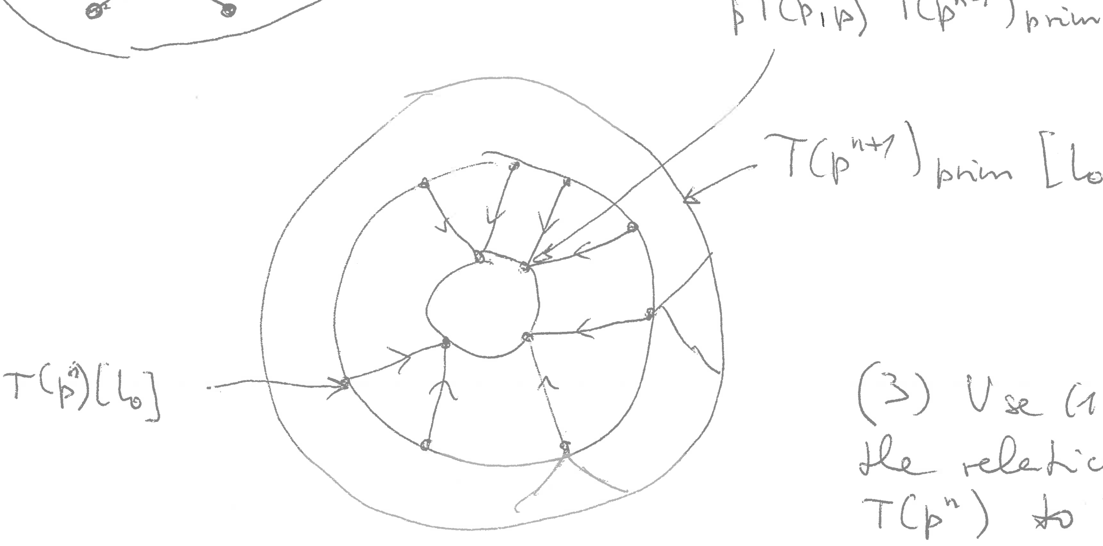
(2) $\forall n \geq 2 \quad T(p)\,T(p^n)_{prim} = T(p^{n+1})_{prim} + p\,T(p,p)\,T(p^{n-1})_{prim}$

(3) $\forall n \geq 1 \quad T(p)\,T(p^n) = T(p^{n+1}) + p\,T(p,p)\,T(p^{n-1})$

Pf. (1)



(2)



(3) Use (1), (2) and the relation ~~of~~ $T(p^n)$ ~~to~~ $T(p^{n-2i})_{prim}$

Cor. $T(m)\,T(n) = \sum_{d \mid (m,n)} d\, T(d,d)\, T\!\left(\frac{mn}{d^2}\right) = T(n)\,T(m)$

Pf: Write m, n as products of prime powers. Multiplicativity (when $(m,n)=1$) reduces to the case $m = p^a$, $n = p^b$. Then use induction on a, the case $a=1$ being proved above.

Generating functions (identities between $T(n) \in \mathrm{End}(\mathcal{L})$)

$$\left(\sum_{n=0}^{\infty} T(p^n) X^n\right)\left(1 - T(p)X + p\,T(p,p)X^2\right) = 1$$

$$\sum_{n=1}^{\infty} \frac{T(n)}{n^s} = \prod_{p \text{ prime}} \left(1 + \frac{T(p)}{p^s} + \frac{T(p^2)}{p^{2s}} + \cdots\right) = \prod_p \left(1 - T(p)p^{-s} + T(p,p)\,p^{1-2s}\right.$$

Degrees: put $\deg T(n) := \sum_{\substack{L_1 \subset L_0 \\ (L_0 : L_1) = n}} 1$ (similarly for $T(n)_{\text{prim}}$)

Cor. (1) $\sum_{n=0}^{\infty} \deg T(p^n) X^n = \frac{1}{1-(p+1)X + pX^2} = \frac{1}{(1-X)(1-pX)} \Rightarrow \deg T(p^n) = 1 + p + \cdots + p^n$

(2) $(m, n = 1) \Longrightarrow \deg(mn)$

Hecke operators on $M_k(SL_2(\mathbb{Z}))$

Def. For a function $F : \{\text{lattices } L \subset \mathbb{C}\} \longrightarrow \mathbb{C}$ satisfying $F(tL) = t^{-k} F(L)$ $(\Rightarrow F \circ T(n;n) = n^{-k} F)$

Put $\underbrace{(F \mid T(n))}(L) := n^{k-1} F(T(n)[L]) = n^{k-1} \sum_{\substack{L' \subset L \\ (L : L') = n}} F(L')$

(also has degree $-k$ under $L \mapsto tL$)

Prop. The operators $T(n)$ acting on functions F as above satisfy

$$\sum_{n=1}^{\infty} \frac{T(n)}{n^s} = \prod_p \frac{1}{1-T(p)p^{-s}+p^{k-1-2s}}$$

(i.e.) $(m,n)=1 \Longrightarrow T(mn)=T(m)T(n)$

p prime, $n\geq 1 \Longrightarrow T(p)T(p^n)=T(p^{n+1})+p^{k-1}T(p^{n-1})$

$$T(m)T(n) = \sum_{d\mid(m,n)} d^{k-1}\,T\!\left(\frac{mn}{d^2}\right)$$

Pf. In the identities for $T(n)\in\mathrm{End}(\mathcal{L})$, replace $T(n)$ by $n^{1-k}T(n)$ and $T(n,n)$ by n^{-k}.

Explicit description of sublattices of a given index

Def. For $n\geq 1$, let $M(n)=\left\{\begin{pmatrix} A & B\\ C & D\end{pmatrix} \;\middle|\; \begin{array}{l} A,B,C,D\in\mathbb{Z}\\ AD-BC=n\end{array}\right\}$

Let $L=\mathbb{Z}\omega_1+\mathbb{Z}\omega_2$

Bijection $\quad SL_2(\mathbb{Z})\backslash M(n) \longleftrightarrow \{L'\subset L \mid (L:L')=n\}$

$$SL_2(\mathbb{Z})\begin{pmatrix} A & B\\ C & D\end{pmatrix} \longmapsto L'=\mathbb{Z}(A\omega_1+B\omega_2)+\mathbb{Z}(C\omega_1+D\omega_2)$$

Lemma. $\left\{\begin{pmatrix} a & b\\ 0 & d\end{pmatrix} \;\middle|\; a,d\in\mathbb{N},\ ad=n,\ b\in\mathbb{Z}/d\mathbb{Z}\right\}$ are representatives of $SL_2(\mathbb{Z})\backslash M(n)$.

Pf. $\forall \begin{pmatrix} A & B\\ C & D\end{pmatrix}\in M(n)$, let $e:=\gcd(A,C)$.

Then $\exists\ \gamma=\begin{pmatrix} * & *\\ -C/e & A/e\end{pmatrix}\in SL_2(\mathbb{Z}) \Longrightarrow \pm\gamma g=\begin{pmatrix} a & b\\ 0 & d\end{pmatrix}$, $a,d\in\mathbb{N}$, $ad=n$.

$\begin{pmatrix} a & b\\ 0 & d\end{pmatrix}=\gamma\begin{pmatrix} a' & b'\\ 0 & d'\end{pmatrix}$ for some $\gamma\in SL_2(\mathbb{Z}) \Longleftrightarrow \gamma=\begin{pmatrix} 1 & t\\ 0 & 1\end{pmatrix}$

$\Longleftrightarrow a=a',\ d=d',\ b'\equiv b \pmod d$

Cor. $(F|T(n))(\mathbb{Z}\omega_1+\mathbb{Z}\omega_2) = n^{k-1}\sum_{\substack{ad=n\\ b\in\mathbb{Z}/d\mathbb{Z}}} F(\mathbb{Z}(a\omega_1+b\omega_2)+\mathbb{Z}d\omega_2)$

Action on modular forms

Def. For $f\in M_k = M_k(SL_2(\mathbb{Z}))$, define $F(\mathbb{Z}\omega_1+\mathbb{Z}\omega_2) = \omega_2^{-k} f\left(\frac{\omega_1}{\omega_2}\right)$

$(f|T(n))(\tau) := (F|T(n))(\mathbb{Z}\tau+\mathbb{Z})$

Cor. $(f|T(n))(\tau) = n^{k-1}\sum_{\substack{ad=n\\ b\in\mathbb{Z}/d\mathbb{Z}}} d^{-k} f\left(\frac{a\tau+b}{d}\right)$

Prop. (1) If $f(\tau) = \sum_{m=0}^{\infty} a(m)q^m \in M_k$, then

$(f|T(n))(\tau) = \sum_{m=0}^{\infty} b(m)q^m$, where $b(m) = \sum_{d|(m,n)} d^{k-1} a\left(\frac{mn}{d^2}\right)$

(2) $\forall n\geq 1$ $T(n)$ maps $M_k\longrightarrow M_k$, $S_k\longrightarrow S_k$.

Pf (1) $f|T(n) = n^{k-1}\sum_{ad=n} d^{-k}\sum_{b(\mathrm{mod}\ d)}\sum_{m=0}^{\infty} a(m) e^{2\pi i m\left(\frac{a\tau}{d}+\frac{b}{d}\right)}$

$$= n^{k-1}\sum_{ad=n} d^{-k}\sum_{m=0}^{\infty} a(m) e^{2\pi i m\frac{a\tau}{d}} \underbrace{\sum_{b(\mathrm{mod}\ d)} \left(e^{2\pi i m/d}\right)^b}_{=\begin{cases} d & d|m\\ 0 & d\nmid m\end{cases}}$$

$$\overset{m:=md}{=} \sum_{ad=n}\left(\frac{n}{d}\right)^{k-1}\sum_{m=0}^{\infty} a(md)q^{ma} \overset{d:=\frac{n}{d}}{=} \sum_{m=0}^{\infty}\sum_{d|n} d^{k-1} a\left(\frac{mn}{d}\right) q^{md}$$

$$\overset{m:=m/d}{=} \sum_{m=0}^{\infty}\left(\sum_{d|(m,n)} d^{k-1} a\left(\frac{mn}{d^2}\right)\right) q^m$$

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(2) $f|T(n)$ is holomorphic in $\mathcal{H}$ and transforms appropriately under $SL_2(\mathbb{Z})$. The Fourier expansion is given by (1).

Special cases: (i) $a_0(f|T(n)) = \sigma_{k-1}(n)\, a_0(f)$
(ii) $a_1(f|T(n)) = a_n(f)$ $\quad (*)$

Lemma. $k \geq 2$ even $\Longrightarrow G_k | T(n) = \sigma_{k-1}(n)\, G_k \quad \forall n \geq 1$

Pf. $n=p$ prime: $$(G_k|T(p))(L) = p^{k-1} \sum_{\substack{L' \subset L \\ (L:L')=p}} \sum_{u \in L' \setminus \{0\}} u^{-k}$$

For $u \in L \setminus \{0\}$: either $u \in$ exactly one L'
or $u \in$ at least two $L' \Longleftrightarrow u \in pL \Longleftrightarrow u \in$ all L'

So: $(G_k|T(p))(L) = p^{k-1}(G_k(L) + p\, G_k(pL)) = (p^{k-1}+1)\, G_k(L)$

$$\frac{1}{1-(p^{k-1}+1)X + p^{k-1}X^2} = \frac{1}{(1-X)(1-p^k X)} \Longrightarrow G_k | T(p^n) = \underbrace{(1+p^{k-1}+\dots+p^{(k-1)n})}_{\sigma_{k-1}(p^n)}\, G_k$$

General n – use $T(mn) = T(m)T(n)$ for $(m,n)=1$.

Thm Let $f = \sum_{n=0}^{\infty} a(n) q^n \in M_k$ be an eigenform for all $T(n)$:

$\forall n \geq 1 \qquad f|T(n) = \lambda(n) f$.

then: (i) $\forall n \geq 1 \quad a(n) = \lambda(n)\, a(1)$

(ii) If $a(0) \neq 0$, ~~tak se si zoho dat~~ then $\lambda(n) = \sigma_{k-1}(n)$ and $f = c \cdot G_k$.

(iii) $$\boxed{\lambda(m)\lambda(n) = \sum_{d|(m,n)} d^{k-1} \lambda\left(\frac{mn}{d^2}\right)}$$

(in particular, $(m,n)=1 \Longrightarrow \lambda(mn) = \lambda(m)\lambda(n)$)

(iv) $$\sum_{n=1}^{\infty} \frac{a(n)}{n^s} = a(1) \sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s} = a(1) \prod_{p \text{ prime}} \left(1 - \frac{\lambda(p)}{p^s} + p^{k-1-2s}\right)^{-1}$$

Pf. (i) Follows from $(*)$. (ii) $\lambda(n) = \sigma_{k-1}(n)$ follows from $(*)$,
$f - a(1) G_k^* = \text{const.} \in M_k \quad \left(G_k^* = -\frac{B_k}{2k} + \sum_{n \geq 1} \sigma_{k-1}(n) q^n\right)$
$\Longrightarrow \text{const.} = 0$.

(iii) (iv) Follows from the corresponding results for $T(n)$.

Def. $f = \sum_{n=0}^{\infty} a_n q^n \in M_k$ is a normalised (Hecke-) eigenform if $a_1 = 1$ and $\forall n \geq 1$ $f | T(n) = \lambda(n) f$.

then $\lambda(n) = a_n$ and $L(f,s) := \sum_{1}^{\infty} \frac{a_n}{n^s} = \prod_{p \text{ prime}} (1 - a_p p^{-s} + p^{k-1-2s})^{-1}$

Ex: (1) $f = G_k$, $\lambda(n) = \sigma_{k-1}(n)$

$$\sum_{n=1}^{\infty} \frac{\sigma_{k-1}(n)}{n^s} = \prod_p \frac{1}{(1-p^{-s})(1-p^{k-1-s})} = \zeta(s)\zeta(s-k+1)$$

$(k > 2 \text{ even})$

(2) $\Delta_k = \sum_{n=1}^{\infty} \tau_k(n) q^n \in \underbrace{S_k}_{\dim = 1}$ $\qquad k \in \{12, 16, 18, 20, 22, 26\}$

$\tau_k(1) = 1$

$\Rightarrow \Delta_k | T(n) = \tau_k(n) \Delta_k$

$$\sum_{n=1}^{\infty} \frac{\tau_k(n)}{n^s} = \prod_p (1 - \tau_k(p) p^{-s} + p^{k-1-2s})^{-1}$$

Exercise let $M_k(\mathbb{Z}) = \{ f = \sum_{n=0}^{\infty} a_n q^n \in M_k \mid \forall n \geq 0 \; a_n \in \mathbb{Z} \}$.

$M_k(\mathbb{Z})$ has a $\mathbb{Z}$-basis of the form

$$\left\{ \begin{array}{lll} E_4^a \Delta^b & 4a + 12b = k, \; a, b \geq 0 & \text{if } k \equiv 0 \pmod 4 \\ E_6 E_4^a \Delta^b & 4a + 12b = k - 6, \; a, b \geq 0 & \text{if } k \equiv 2 \pmod 4 \end{array} \right\},$$

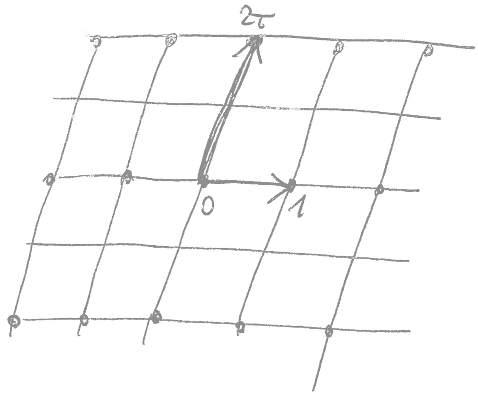
which is also a $\mathbb{C}$-basis of M_k.

Cor. In the above $\mathbb{Z}$-basis, each $T(n)$ acts by a matrix with coefficient in $\mathbb{Z}$. Thus all eigenvalues of $T(n)$ are algebraic integers.

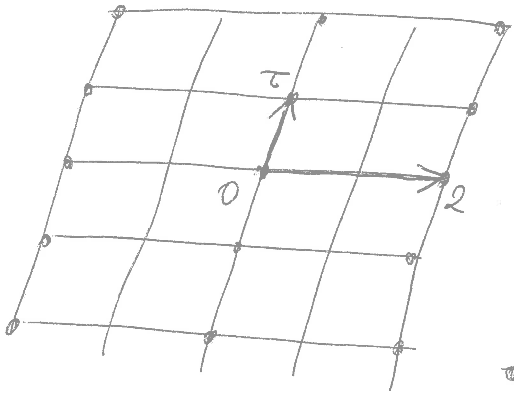
Goal. M_k has a $\mathbb{C}$-basis consisting of Hecke eigenforms

Hecke operators and group theory

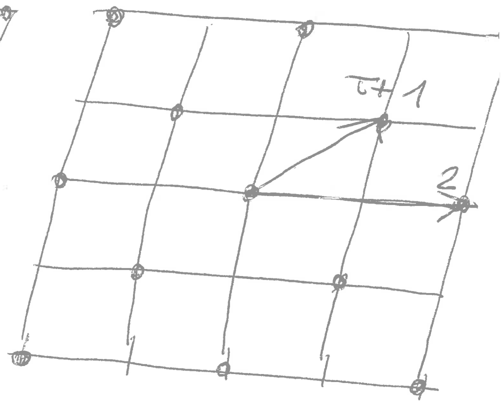
Ex: $L = \mathbb{Z}\tau + \mathbb{Z}\cdot 1$ has 3 sublattices of index 2



$L_0 = \mathbb{Z}(2\tau) + \mathbb{Z}\cdot 1$



$L_1 = \mathbb{Z}\tau + \mathbb{Z}\cdot 2$



$L_2 = \mathbb{Z}(\tau+1) + \mathbb{Z}\cdot 2$

$j(\tau) = j(L)$ does not determine $j(2\tau) = j(L_0)$, but it does determine the set $\{ j(2\tau), j(\frac{\tau}{2}), j(\frac{\tau+1}{2}) \}$

$$\{ j(L') \mid (L : L') = 2 \} = \{ j(L_0), j(L_1), j(L_2) \}$$

Above: $j(2\tau) = (j|_0 \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix})(\tau)$

Action of the matrix $\alpha = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \in GL_2(\mathbb{Q})^+$ on $\mathcal{H}$

$\alpha(\tau) = 2\tau$ does not define a map

$$SL_2(\mathbb{Z})\backslash\mathcal{H} \longrightarrow SL_2(\mathbb{Z})\backslash\mathcal{H},$$

only a correspondence (a "multivalued map"):

$$\begin{array}{ccc}
\mathcal{H} \ni \tau & \longmapsto & 2\tau \in \mathcal{H} \\
\mathcal{H} & \xrightarrow{\alpha} & \mathcal{H} \\
\downarrow p & & \downarrow p \\
SL_2(\mathbb{Z})\backslash\mathcal{H} & \dashrightarrow & SL_2(\mathbb{Z})\backslash\mathcal{H} \\
\Psi & & \\
x = SL_2(\mathbb{Z})\cdot\tau & \dashrightarrow & \{ p(\alpha(p^{-1}(x))) \}
\end{array}$$

$$\{ p(\alpha(p^{-1}(x))) \} = \{ SL_2(\mathbb{Z}).2\tau,\ SL_2(\mathbb{Z})\cdot\frac{\tau}{2},\ SL_2(\mathbb{Z})\cdot\frac{1+\tau}{2} \}$$

Abstract situation: Y space

G group acting on Y (on the left)

$\Gamma, \Gamma', \ldots \subset G$ class of subgroups

Notation: $Y_\Gamma = \Gamma\backslash Y$, $\quad Y \xrightarrow{p_\Gamma} \Gamma\backslash Y = Y_\Gamma$ projection

$y \longmapsto p_\Gamma(y) = [y]_\Gamma$

Formal properties: (1) $\Gamma' < \Gamma$ subgroup: $\Gamma = \coprod \Gamma' \alpha_i$

$$\begin{tikzcd} Y \arrow[d, "p_{\Gamma'}"] \arrow[dd, bend right=60, "p_\Gamma"'] \\ Y_{\Gamma'} \arrow[d, "p"] \\ Y_\Gamma \end{tikzcd} \qquad \begin{tikzcd} y \arrow[d, mapsto] \\ {[y]_{\Gamma'}} \arrow[d, mapsto] \\ {[y]_\Gamma} \end{tikzcd}$$

$$p^{-1}([y]_\Gamma) = \{[\alpha_i y]_{\Gamma'}\}$$

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(2) $\alpha \in G$

$$\begin{tikzcd} Y \arrow[r, "\alpha", "\sim"'] \arrow[d, "p_\Gamma"'] & Y \arrow[d, "p_{\alpha\Gamma\alpha^{-1}}"] \\ Y_\Gamma \arrow[r, "\alpha", "\sim"'] & Y_{\alpha\Gamma\alpha^{-1}} \end{tikzcd} \qquad \begin{tikzcd} y \arrow[r, mapsto] \arrow[d, mapsto] & \alpha y \arrow[d, mapsto] \\ {[y]_\Gamma} \arrow[r, mapsto] & {[\alpha y]_{\alpha\Gamma\alpha^{-1}}} \end{tikzcd}$$

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(3) $\alpha \in G$, $\quad \Gamma, \Gamma' < G$ subgroups

$$\Gamma \overset{(1)}{=} \coprod_i (\Gamma \cap \alpha^{-1}\Gamma'\alpha)\alpha_i$$

$$\begin{tikzcd} Y \arrow[rr, "\alpha", "\sim"'] \arrow[d] \arrow[dd, bend right=60, "p_\Gamma"'] & & Y \arrow[d] \arrow[dd, bend left=60, "p_{\Gamma'}"] \\ Y_{(\Gamma\cap\alpha^{-1}\Gamma'\alpha)} \arrow[rr, "\alpha", "\sim"'] \arrow[d, "p"] & & Y_{\alpha\Gamma\alpha^{-1}\cap\Gamma'} \arrow[d, "p'"] \\ Y_\Gamma \arrow[rr, dashed, "T(\alpha)_{\Gamma,\Gamma'}"] & & Y_{\Gamma'} \end{tikzcd}$$

$$\begin{tikzcd} \{[\alpha_i y]_{\Gamma\cap\alpha^{-1}\Gamma'\alpha}\} \arrow[r, mapsto, "\alpha"] \arrow[d, mapsto] & \{[\alpha\alpha_i y]_{\alpha\Gamma\alpha^{-1}\cap\Gamma'}\} \arrow[d, mapsto] \\ {[y]_\Gamma} \arrow[r, dashed] & \{[\alpha\alpha_i y]_{\Gamma'}\} \end{tikzcd}$$

"multivalued map"
(correspondence)

$T(\alpha)_{\Gamma,\Gamma'}$ is given by

$$\begin{tikzcd} & Y_{\Gamma\cap\alpha^{-1}\Gamma'\alpha} \arrow[dl, "p"'] \arrow[dr, "p'\circ\alpha"] & \\ Y_\Gamma \arrow[rr, dashed] & & Y_{\Gamma'} \end{tikzcd}$$

Equivalent disjoint decompositions:

$$\boxed{\Gamma = \coprod_i (\Gamma\cap\alpha^{-1}\Gamma'\alpha)\alpha_i \iff \Gamma'\alpha\Gamma = \coprod_i \Gamma'\alpha\alpha_i}$$ (exercise!)

Summary: if $\Gamma'\alpha\Gamma = \coprod_i \Gamma'\beta_i$, then $T(\alpha)_{\Gamma,\Gamma'} : [y]_\Gamma \longmapsto \{[\beta_i y]_{\Gamma'}\}$

Ex: $Y = \mathcal{H}$, $G = SL_2(\mathbb{R})^+$, $\Gamma = \Gamma' = SL_2(\mathbb{Z})$, $\alpha = \begin{pmatrix} n & 0 \\ 0 & 1 \end{pmatrix}$

$$\Gamma \cap \alpha^{-1}\Gamma'\alpha = \Gamma_0(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \;\middle|\; c \equiv 0 \pmod n \right\}$$

$$\alpha\Gamma\alpha^{-1} \cap \Gamma' = \Gamma^0(n) = \left\{ \text{———} \;\; '' \;\; \text{———} \;\middle|\; b \equiv 0 \pmod n \right\}$$

Thm on elementary divisors:

$$\Gamma\alpha\Gamma' = SL_2(\mathbb{Z}) \begin{pmatrix} n & 0 \\ 0 & 1 \end{pmatrix} SL_2(\mathbb{Z}) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in M_2(\mathbb{Z}) \;\middle|\; \begin{array}{l} \det = n, \\ \mathbb{Z}(A,B) + \mathbb{Z}(C,D) \\ \text{primitive sublattice} \\ \text{of } \mathbb{Z}\oplus\mathbb{Z} \text{ of index } n \end{array} \right\}$$

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = \begin{pmatrix} A\omega_1 + B\omega_2 \\ C\omega_1 + D\omega_2 \end{pmatrix}$$

So $T(\alpha)_{\Gamma,\Gamma'}$: $[\tau]_\Gamma \longmapsto \left\{ [\tau']_\Gamma \;\middle|\; \begin{array}{l} \mathbb{Z}\tau' + \mathbb{Z} \subset \mathbb{Z}\tau + \mathbb{Z} \\ (\mathbb{Z}\tau + \mathbb{Z})/(\mathbb{Z}\tau' + \mathbb{Z}) \simeq \mathbb{Z}/n\mathbb{Z} \end{array} \right\}$

is given by $T(n)_{\mathrm{prim}}$.

Action on functions, differentials, ...:

Need to assume: (a) $(\Gamma : (\Gamma \cap \alpha^{-1}\Gamma'\alpha)) < \infty$, $(\Gamma' : (\alpha\Gamma\alpha^{-1} \cap \Gamma')) < \infty$

(b) Trace maps associated to p, p' are defined

$$\begin{tikzcd} & Y_{\Gamma \cap \alpha^{-1}\Gamma'\alpha} \arrow[dl, "p"'] \arrow[dr, "p'\circ\alpha"] & \\ Y_\Gamma \arrow[rr, dashed, "T(\alpha)_{\Gamma,\Gamma'}"] & & Y_{\Gamma'} \end{tikzcd}$$

Covariant action: $(p'\circ\alpha)_* \circ p^* = (T(\alpha)_{\Gamma,\Gamma'})_*$

Contravariant action: $p_* \circ (p'\circ\alpha)^* = (T(\alpha)_{\Gamma,\Gamma'})^*$

Depend only on the double coset $\Gamma'\alpha\Gamma$ (and Γ, Γ')

$\Gamma(N)$ – level structures

Notation : K field , $K^n = \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = x \mid x_i \in K \right\}$

$(K^n)^*$ = the dual space $= \{ (a_1, \dots, a_n) = a \mid a_i \in K \}$

duality pairing $(K^n)^* \times K^n \xrightarrow{\langle , \rangle} K$

$$a , \quad x \longmapsto ax = \sum a_i x_i$$

dual map to $X \xrightarrow{f} Y$ (linear) : $X^* \xleftarrow{f^*} Y^*$

$$\langle f^*(y^*), x \rangle_X = \langle y^*, f(x) \rangle_Y \qquad (g \circ f)^* = f^* \circ g^*$$

matrix notation : $X = K^m$, $Y = K^n$, $f(x) = Ax$, $A \in M_{n,m}(K)$

$$\langle y^*, Ax \rangle = y^*(Ax) = (y^*A)x \qquad (x \in K^m, \ y^* \in (K^n)^*)$$

$$\Longrightarrow \quad A^*(y^*) = y^*A$$

Lattices in $\mathbb{C}$: $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 = \left\{ (m_1 \ m_2) \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} \middle| (m_1 \ m_2) \in (\mathbb{Z}^2)^* \right\}$

$L_\tau = \mathbb{Z}\tau + \mathbb{Z} = \left\{ (m_1 \ m_2) \begin{pmatrix} \tau \\ 1 \end{pmatrix} \middle| (m_1 \ m_2) \in (\mathbb{Z}^2)^* \right\} \qquad (\tau \in \mathbb{C} \setminus \mathbb{R})$

$\forall \alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Z}) \qquad \alpha \begin{pmatrix} \tau \\ 1 \end{pmatrix} = \begin{pmatrix} a\tau + b \\ c\tau + d \end{pmatrix} = \begin{pmatrix} \alpha(\tau) \\ 1 \end{pmatrix} (c\tau + d)$

$\lambda_\tau : (\mathbb{Z}^2)^* \xleftarrow{\sim} L_\tau$, $\boxed{(m_1 \ m_2) \overset{\lambda_\tau}{\longleftrightarrow} (m_1 \ m_2) \begin{pmatrix} \tau \\ 1 \end{pmatrix} = m_1\tau + m_2}$

$$\begin{tikzcd}
L_\tau \arrow[r, "\lambda_\tau", "\sim"'] & (\mathbb{Z}^2)^* \\
L_{\alpha(\tau)} \arrow[u, "(c\tau+d)", "\wr"'] \arrow[r, "\lambda_{\alpha(\tau)}", "\sim"'] & (\mathbb{Z}^2)^* \arrow[u, "\alpha^*"', "\wr"]
\end{tikzcd}
\qquad
\begin{tikzcd}
(m \ n) \begin{pmatrix} a\tau + b \\ c\tau + d \end{pmatrix} \arrow[r, leftrightarrow] & (m \ n)\alpha \\
(m \ n) \begin{pmatrix} \alpha(\tau) \\ 1 \end{pmatrix} \arrow[u] \arrow[r, leftrightarrow] & (m \ n) \arrow[u, "\alpha^*"']
\end{tikzcd}$$

Def. A full level N structure on a lattice $L \subset \mathbb{C}$ is a group isomorphism $\lambda : (L/NL) \xrightarrow{\sim} (\mathbb{Z}^2)^*/N$

(another terminology : $\Gamma(N)$ – level structure)

Ex. $\forall \tau \in \mathbb{C} \setminus \mathbb{R}$ $L_\tau = \mathbb{Z}\tau + \mathbb{Z}$ has a canonical $\Gamma(N)$ – level structure $\lambda_{\tau,N} = \lambda_\tau \pmod N$: $\underset{(\mathrm{mod}\ nL)}{m\tau + n} \longmapsto \underset{(\mathrm{mod}\ N)}{(m, n)}$

Action of $\beta \in Gl_2(\mathbb{Z}/N\mathbb{Z})$: $\beta : (\mathbb{Z}/N\mathbb{Z})^2 \longrightarrow (\mathbb{Z}/N\mathbb{Z})^2$

$$\begin{tikzcd}
L/NL \arrow[r, "\lambda", "\sim"'] & (\mathbb{Z}^2)^*/N \arrow[d, "\beta"] \\
& (\mathbb{Z}^2)^*/N
\end{tikzcd}
\qquad
\begin{aligned}
\beta^* : (\mathbb{Z}^2)^*/N &\longrightarrow (\mathbb{Z}^2)^*/N \\
(x \ y) &\longmapsto (x \ y)\beta
\end{aligned}$$

Prop. $V = \mathbb{C}$-vector space, $k \in \mathbb{Z}$

there are natural bijections between

(1) Functions $\{(L, \lambda) \mid L \subset \mathbb{C}$ lattice
$\lambda : L/NL \xrightarrow{\sim} (\mathbb{Z}^2)^*/N$ full level N structure$\}$

such that $\forall t \in \mathbb{C}^\times \quad F((tL, \lambda \circ t^{-1})) = t^{-k} F(L, \lambda)$

(2) Functions $f : (\mathbb{C} \setminus \mathbb{R}) \times GL_2(\mathbb{Z}/N\mathbb{Z}) \longrightarrow V$ such that

$\forall \alpha \in GL_2(\mathbb{Z}), \alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\quad f\Big(\underbrace{\frac{a\tau + b}{c\tau + d}}_{\alpha(\tau)}, \alpha\beta\Big) = (c\tau + d)^k f(\tau, \beta)$

$\forall \beta \in GL_2(\mathbb{Z}/N\mathbb{Z})$

(3) Collections of functions $f_u : \mathcal{H} \longrightarrow V$

(u representatives in $\mathbb{Z}$ of $(\mathbb{Z}/N\mathbb{Z})^\times \subset \mathbb{Z}/N\mathbb{Z}$)

such that $\forall \alpha \in \sigma_u \Gamma(N) \sigma_u^{-1}$ $\qquad \sigma_u = \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}$

$$f_u\Big(\frac{a\tau + b}{c\tau + d}\Big) = (c\tau + d)^k f_u(\tau)$$

Formulas: (1) $\Rightarrow$ (2) Given F, define

$f(\tau, \beta) := F((L_\tau = \mathbb{Z}\tau + \mathbb{Z}, \underbrace{\beta^*}_{\text{right multiplication by } \beta} \circ \lambda_{\tau, N}))$

(2) $\Rightarrow$ (1) Given f: for L and $\lambda : L/NL \xrightarrow{\sim} (\mathbb{Z}^2)^*/N$,

choose a basis $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 = \omega_2(\mathbb{Z}\tau + \mathbb{Z})$, $\quad \tau = \frac{\omega_1}{\omega_2} \in \mathbb{C} \setminus \mathbb{R}$

$\exists!\ \beta \in GL_2(\mathbb{Z}/N\mathbb{Z})$ such that $\lambda = \beta^* \circ \lambda_{\tau, N}$

let $F((L, \lambda)) := \omega_2^{-k} f\big(\frac{\omega_1}{\omega_2}, \beta\big)$

---

(2) $\Rightarrow$ (3) $\qquad f_u(\tau) = f(\tau, \sigma_u)$

Rmks: (A) $SL_2(\mathbb{Z}) \backslash GL_2(\mathbb{Z}/N\mathbb{Z}) \xrightarrow[\det]{\sim} (\mathbb{Z}/N\mathbb{Z})^\times$

(B) $f_1(\tau)$ satisfies $\forall \alpha \in \Gamma(N) \quad f_1|_k \alpha = f_1$

Corresponds to symplectic $\Gamma(N)$-level structures λ

(compatible with natural positive generators of $\Lambda^2_{\mathbb{Z}} L$, $\Lambda^2_{\mathbb{Z}}(\mathbb{Z}^2)^*$)

So: $\Gamma(N)\backslash\mathcal{H} = \{(L\subset\mathbb{C} \text{ lattice}, \text{symplectic } \Gamma(N)\text{-level structure})\}/\mathbb{C}^\times$

$$\Gamma(N)\tau \longmapsto \left(L_\tau = \mathbb{Z}\tau+\mathbb{Z},\ \begin{array}{ccc} L_\tau/NL_\tau & \xrightarrow{\sim} & (\mathbb{Z}^2)^*/N \\ m\tau+n & \longmapsto & (m,n) \bmod N \\ \bmod NL_\tau & & \end{array}\right)$$

$$\Gamma_1(N) = \left\{\alpha = \begin{pmatrix} a & b \\ c & d\end{pmatrix} \in SL_2(\mathbb{Z}) \,\middle|\, \underset{\substack{\uparrow \\ (\mathbb{Z}^2)^*/N}}{(0\ 1)(\bmod N)} = (0\ 1)\alpha \ (\bmod N)\right\}$$

$\Gamma_0(N) = \{$ —— " —— $|$ the cyclic subgroup $(0,*)\subset(\mathbb{Z}^2)^*/N$ is stable under α $\}$

$\Rightarrow$ $\Gamma_1(N)\backslash\mathcal{H} = \{(L\subset\mathbb{C} \text{ lattice}, P\in\mathbb{C}/L \text{ of exact order } N)\}/\mathbb{C}^\times$

$\Gamma_1(N)\tau \longleftrightarrow (L_\tau = \mathbb{Z}\tau+\mathbb{Z},\ P = \frac{1}{N} (\bmod L_\tau))$

$\Gamma_0(N)\backslash\mathcal{H} = \{(L\subset\mathbb{C} \text{ lattice}, \underset{C\simeq\mathbb{Z}/N\mathbb{Z}}{C\subset(\mathbb{C}/L)[N]} \text{ subgroup}\}/\mathbb{C}^\times$

$\Gamma_0(N)\tau \longleftrightarrow (L_\tau = \mathbb{Z}\tau+\mathbb{Z},\ C = \{a\cdot\frac{1}{N} \mid a = 1,\dots,N\} (\bmod L_\tau))$

Hecke operators – general definition

$\Gamma_0 = SL_2(\mathbb{Z})$, $\Gamma_1, \Gamma_2 \subset \Gamma_0$ congruence subgroups, $\alpha \in GL_2(\mathbb{Q})^+$
$(\exists N \ \Gamma(N) \subset \Gamma_1, \Gamma_2)$

$\Rightarrow$ $\exists$ finite set of representatives

$$\Gamma_1 \alpha \Gamma_2 = \coprod_i \Gamma_1 \alpha_i \qquad (\alpha_i \in GL_2(\mathbb{Q})^+)$$

Ex: $\Gamma_1 = \Gamma_2 = SL_2(\mathbb{Z})$, $\alpha = \begin{pmatrix} n & 0 \\ 0 & 1 \end{pmatrix}$, $\alpha_i = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \,\middle|\, \begin{array}{l} a, d > 0,\ ad = n \\ (a,d) = 1 \\ 0 \le b < d \end{array} \right\}$

primitive Hecke operator $T(n)_{prim}$

Def. For $f \in A_k(\Gamma_1)$, $f|\Gamma_1 \alpha \Gamma_2 := \det(\alpha)^{\frac{k}{2}-1} \sum_i f|_k \alpha_i$

is well-defined and $f|\Gamma_1\alpha\Gamma_2 \in A_k(\Gamma_2)$

Prop. For $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ as above, let $\alpha' = \det(\alpha)\,\alpha^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

then $\forall f \in S_k(\Gamma_1)$, $\forall g \in S_k(\Gamma_2)$

(1) $(f|_k\alpha, g) = (f, g|_k \alpha')$ (normalised Petersson scalar product)

(2) ~~For~~ If $\Gamma_1 = \Gamma_2$, then

$$(f|\Gamma_1\alpha\Gamma_1, g) = (f, g|\Gamma_1\alpha'\Gamma_1).$$

Pf: (1) $(f|_k\alpha, h|_k\alpha) = (f, h) \Rightarrow$ result

(2) $|\Gamma_1 \backslash \Gamma_1\alpha\Gamma_1| = |(\Gamma_1 \cap \alpha^{-1}\Gamma_1\alpha)\backslash\Gamma_1| = |\Gamma_1/(\alpha\Gamma_1\alpha^{-1} \cap \Gamma_1)| = |\Gamma_1\alpha\Gamma_1/\Gamma_1|$

compare their covolumes

$\xRightarrow{\text{exercise}}$ $\exists$ common representatives $\{\alpha_i\}$ of

$\Gamma_1\backslash\Gamma_1\alpha\Gamma_1$ and $\Gamma_1\alpha\Gamma_1/\Gamma_1 \Rightarrow$
$$\Gamma_1\alpha\Gamma_1 = \coprod \Gamma_1\alpha_i$$
$$\Gamma_1\alpha^{-1}\Gamma_1 = \coprod \Gamma_1\alpha_i^{-1}$$
$$\Gamma_1\alpha'\Gamma_1 = \coprod \Gamma_1\alpha_i'$$

Apply (1)

Easy Hecke operators : assume $\Gamma(1) = SL_2(\mathbb{Z}) \supset \Gamma \supset \Gamma(N)$

If $l \mid m$, $lm = n$, $(n, N) = 1$: For simplicity: $\boxed{\Gamma = \Gamma_0(N)}$

$\Gamma \begin{pmatrix} l & 0 \\ 0 & m \end{pmatrix} \Gamma$ act by the same formulas as for $SL_2(\mathbb{Z})$

$T(n)_{prim} \longleftrightarrow \Gamma \begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix} \Gamma = \Gamma \begin{pmatrix} n & 0 \\ 0 & 1 \end{pmatrix} \Gamma$

$T(l,l)\, T\left(\frac{m}{l}\right)_{prim} = T(l,l)\, T\left(\frac{n}{l^2}\right)_{prim} \longleftrightarrow \Gamma \begin{pmatrix} l & 0 \\ 0 & m \end{pmatrix} \Gamma = \Gamma \begin{pmatrix} m & 0 \\ 0 & l \end{pmatrix} \Gamma$

Generate a commutative algebra acting on $M_k(\Gamma)$, $S_k(\Gamma)$

$$T(n) = \sum_{d \mid n} T(d,d)\, T\left(\frac{n}{d^2}\right)_{prim} \longleftrightarrow \coprod_{d^2 \mid n} \Gamma \underbrace{\begin{pmatrix} n/d & 0 \\ 0 & d \end{pmatrix}}_{\alpha = \alpha'} \Gamma$$

Cor. If $k \geq 2$, $(f \mid T(n), g) = (f, g \mid T(n))$ $\forall f, g \in S_k(\Gamma)$.

Prop. $S_k(\Gamma)$ has a basis of eigenforms of all $T(n,n)$, $T(n)_{prim}$, $T(n)$ for $(n, N) = 1$.

Cor. For $\Gamma = SL_2(\mathbb{Z})$, S_k has a basis of eigenforms of all $T(n)$.

Difficult Hecke operators $(\Gamma(1) \supset \Gamma \supset \Gamma(N))$

Correspond to $\Gamma \begin{pmatrix} l & 0 \\ 0 & m \end{pmatrix} \Gamma$ with $(lm, N) \neq 1$.

Ex: $\Gamma = \Gamma_0(N)$, , $\alpha = \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix}$, q prime

$\Gamma \cap \alpha^{-1} \Gamma \alpha = \Gamma_0(Nq)$, $|(\Gamma \cap \alpha^{-1}\Gamma\alpha) \backslash \Gamma| = \begin{cases} q & \text{if } q \mid N \\ q+1 & \text{if } q \nmid N \end{cases}$

In terms of lattices:

$F : \{(L, C) \mid L \subset \mathbb{C} \text{ lattice}, \; C \subset \frac{1}{N}L/L \text{ subgroup}, \; C \simeq \mathbb{Z}/N\mathbb{Z}\} \longrightarrow \mathbb{C}$

$F(tL, tC) = t^{-k} F(L, C)$

$$(T(q)F)(L, C) = q \sum_{\substack{L' \subset L \\ (L : L') = q \\ L' \text{ transversal to } C}} F(L', \text{"} L' \cap C \text{"})$$

(if $q \mid N$)

Atkin–Lehner Theory:

$$S_k(\Gamma_0(N)) \xrightarrow{\delta_d} S_k(\Gamma_0(N'))$$ if $dN \mid N'$

$$f(\tau) \longmapsto f(d\tau)$$ compatible with $T(m)$, $(m, dN) = 1$

Old forms: subspace of $S_k(\Gamma_0(M))$ generated by

$$\mathrm{Im}\,(\delta_d : S_k(\Gamma_0(M')) \longrightarrow S_k(\Gamma_0(M)))$$ $dM' \mid M$

New forms: orthogonal complement of old forms

Main thm: $S_k(\Gamma_0(M))^{new}$ has a basis of eigenforms not just for all $T(m)$, $T(m,m)$ for $(m, N) = 1$, but also of $T\left(\begin{pmatrix} m & 0 \\ 0 & 1 \end{pmatrix}\right)$, W_M (and more general involutions defined by Atkin–Lehner).

The eigenvalues of $\{T(m), T(m,m)$ for $(m,N)=1\}$ occur in $S_k(\Gamma_0(M))^{new}$ with multiplicity 1 (and ~~these~~ eigenvalues do not occur in $S_k(\Gamma_0(M))^{old}$.

Atkin–Li: the same for $S_k(\Gamma_0(M), \chi)$.
