MODULAR FORMS – INTRODUCTION

JAN NEKOVÁŘ

Books on modular forms:

[Se] J.-P. Serre, A course in arithmetic, ch. VII.
[Za] D. Zagier, in: The 1-2-3 of Modular Forms
[Iw] H. Iwaniec, Topics in Classical Automorphic Forms
[La 1] S. Lang, Elliptic Functions
[Mi] T. Miyake, Modular Forms
[DiSh] F. Diamond, J. Shurman, A First Course in Modular Forms
[CoSt] H. Cohen, F. Strömberg, Modular Forms, A Classical Approach
[La 2] S. Lang, Introduction to Modular Forms
[Ko] N. Koblitz, Introduction to Elliptic Curves and Modular Forms
[Sh 1] G. Shimura, Elementary Dirichlet Series and Modular Forms

---

[Sch] B. Schoeneberg, Elliptic Modular Functions
[Og] A. Ogg, Modular Forms and Dirichlet Series
[Le 1] J. Lehner, Discontinuous groups and automorphic functions
[Le 2] J. Lehner, A short course in automorphic functions
[Le 3] J. Lehner, Lectures on modular forms

[FrKl] R. Fricke, F. Klein, Vorlesungen über die Theorie der Modulfunktionen I, II
[We] H. Weber, Lehrbuch der Algebra III

More advanced:

[EZ] M. Eichler, D. Zagier, Jacobi Forms
[Mu Θ] D. Mumford, Tata Lectures on Theta I, II, III
[Sh 2] G. Shimura, Modular forms: basics and beyond
[Bu] D. Bump, Automorphic forms and representations
[MF] Modular Forms of One Variable I–VI
[LiVe] Lion, M. Vergne, The Maslov Index and the Weil Representation
[Ma] H. Maass, Lectures on modular functions of one complex variable
[Fr 1] E. Freitag, Siegelsche Modulfunktionen
[Fr 2] E. Freitag, Hilbert modular forms
[BFOR] K. Bringmann, A. Folsom, K. Ono, L. Rolen, Harmonic Maass Forms and Mock Modular Forms
[Mu AV] D. Mumford, Abelian Varieties

Related topics :

[Ra] H. Rademacher, Topics in Analytic Number Theory

[Si 1] C.L. Siegel, lectures on Advanced Analytic Number Theory

[Si 2] C.L. Siegel, lectures on Quadratic Forms

[We] A. Weil, Elliptic functions according to Eisenstein and Kronecker

Geometry :

[Be] A. Beardon, the geometry of discrete groups

[Ms] B. Maskit, Kleinian groups

[OS] A.L. Onishchik, R. Sulanke, Projective and Cayley–Klein geometries

History :

J. Lehner, [Le 1], ch. 1

[Gr] J. Gray, Linear differential equations and group theory, from Riemann to Poincaré

[Le] S. Levy (ed.), The Eightfold Way

[SG] H.P. de Saint-Gervais, Uniformisation des surfaces de Riemann, Retour sur un théorème centenaire

Special functions :

[WW] E.T. Whittaker, G.N. Watson, A course of modern analysis

[BW] R. Beals, R. Wong, Special functions and orthogonal polynomials

[AAR] G.E. Andrews, R. Askey, R. Roy, Special functions

[HTF] A. Erdelyi, W. Magnus, F. Oberhettinger, F.G. Tricomi, Higher Transcendental Functions I–III

[Lb] N.N. Lebedev, Special functions and their applications

Background in analysis

[Ah] L.V. Ahlfors, Complex Analysis

[Fo] O. Forster, Riemann Surfaces

[DM] H. Dym, H.P. McKean, Fourier series and integrals

[Ru] W. Rudin, Real and Complex Analysis

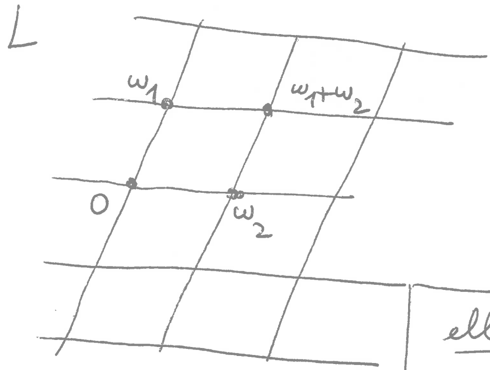
[DK] J.J. Duistermaat, J.A.C. Kolk, Distributions

Origins of the theory of modular forms

Elliptic functions: meromorphic $f: \mathbb{C} \to \mathbb{C}$ with two periods

$$\forall z \in \mathbb{C} \quad f(z+\omega_1) = f(z) = f(z+\omega_2)$$

Non-trivial case: $\omega_1/\omega_2 \notin \mathbb{R} \Leftrightarrow L := \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 \subset \mathbb{C}$ lattice



$$\forall z \in \mathbb{C} \quad \forall u \in L \quad f(z+u) = f(z)$$

Constructions of f: often work for all L

$\Longrightarrow$ functions of two variables $f(z, L)$

elliptic fns:	L fixed, z variable
modular forms:	z fixed, L variable

Space of lattices: $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 = \omega_2 (\underbrace{\mathbb{Z}\frac{\omega_1}{\omega_2} + \mathbb{Z}}_{\mathbb{Z}\tau + \mathbb{Z}})$, $\tau = \frac{\omega_1}{\omega_2} \in \mathbb{C} \setminus \mathbb{R}$

normalised lattice

change of basis of L: $L = \mathbb{Z}\omega_1' + \mathbb{Z}\omega_2' = \omega_2'(\mathbb{Z}\frac{\omega_1'}{\omega_2'} + \mathbb{Z})$, $\tau' = \frac{\omega_1'}{\omega_2'}$

$$\begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix} = \underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{\gamma} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Z}) := M_2(\mathbb{Z})^\times$$

$(a, b, c, d \in \mathbb{Z},\ \det(\gamma) \in \mathbb{Z}^\times = \{\pm 1\})$

$$\boxed{\tau' = \frac{a\omega_1 + b\omega_2}{c\omega_1 + d\omega_2} = \frac{a\tau + b}{c\tau + d}}$$

$SL_2(\mathbb{Z}) \backslash \underbrace{\mathcal{H}}_{\{\mathrm{Im}(\tau) > 0\}}$

$$\boxed{\mathbb{C}^\times \backslash \{\text{lattices } L \subset \mathbb{C}\} \xleftrightarrow{\sim} GL_2(\mathbb{Z}) \backslash (\mathbb{C} \setminus \mathbb{R}) = SL_2(\mathbb{Z}) \backslash \mathcal{H} \quad \text{bijection}}$$

$$\mathbb{C}^\times (\mathbb{Z}\tau + \mathbb{Z}) \longleftrightarrow GL_2(\mathbb{Z})\tau$$

$$\mathbb{C}^\times (\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2) \longleftrightarrow GL_2(\mathbb{Z}) \frac{\omega_1}{\omega_2}$$

$GL_2(\mathbb{C})$ acts on $\mathbb{C} \cup \{\infty\} = \mathbb{P}^1(\mathbb{C})$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}(z) = \frac{az+b}{cz+d} \qquad (*)$$

Analogy: (A) L acts on $\mathbb{C}$ by translations $z \longmapsto z+u$

(B) $GL_2(\mathbb{Z})$ acts on $\mathbb{C} \setminus \mathbb{R}$ by $(*)$

Classical constructions of elliptic functions

Abel: inverse function to $\int \frac{dx}{\sqrt{f(x)}}$ $\quad f\in\mathbb{C}[x]$, $\deg(f)=3,4$ (with distinct roots)

($f(x)=1-x^2$: inverse fn is $\sin(x)$)

($f(x)=1-x^4$: obtain the lemniscate sin (Gauss – unpublished his results))

Jacobi: theta-functions: holomorphic $\varphi:\mathbb{C}\to\mathbb{C}$

$$\forall u\in L\ \exists a(u),b(u)\in\mathbb{C}\ \forall z\in L\quad \varphi(z+u)=e^{a(u)z+b(u)}\varphi(z)$$

quotient of two such θ-functions φ_1/φ_2 is an elliptic fn

Eisenstein: Eisenstein series

$k\in\mathbb{Z},\ k\geq 1$
$$G_k(z;L):=\sum_{u\in L}\frac{1}{(z+u)^k} \qquad G_k(L):=\sum_{0\neq u\in L}\frac{1}{u^k}$$

(requires regularisation if $k=1,2$). This will be an elliptic function if $k\geq 2$.

Weierstrass: regularised version of "$z\prod_{0\neq u\in L}\left(1-\frac{z}{u}\right)$":

$$\boxed{\sigma(z;L)=z\prod_{0\neq u\in L}\left(1-\frac{z}{u}\right)e^{\frac{z}{u}+\frac{1}{2}\left(\frac{z}{u}\right)^2}}$$

not quite periodic, but it is a θ-function

$$\Rightarrow \wp(z;L):=-\left(\frac{d}{dz}\right)^2\log\sigma(z)=\frac{1}{z^2}+\sum_{0\neq u\in L}\left(\frac{1}{(z-u)^2}-\frac{1}{u^2}\right)$$ is an elliptic fn

($\forall u\in L\quad \wp(z+u)=\wp(z)$)

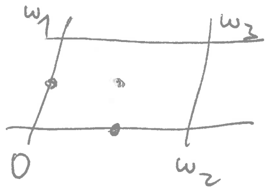
$$\Rightarrow \zeta(z;L)=\frac{\sigma'(z)}{\sigma(z)}=\frac{d}{dz}\log(z)=\frac{1}{z}+\sum_{0\neq u\in L}\left(\frac{1}{z-u}+\frac{1}{u}+\frac{z}{u^2}\right)=\frac{1}{z}+\sum_{m=2}^{\infty}G_{2m}(L)z^{2m-1}$$

satisfies $\forall u\in L\ \exists \eta(u)\in\mathbb{C}\ \forall z\in\mathbb{C}\quad \zeta(z+u)=\zeta(z)+\eta(u)$ $\quad(\zeta'(z)=-\wp(z))$

Properties: (1) $\wp(-z)=\wp(z)$, $\quad \wp'(-z)=-\wp'(z)$

(2) $\wp'(z)^2=4\wp(z)^3-g_2\wp(z)-g_3=4(\wp(z)-e_1)(\wp(z)-e_2)(\wp(z)-e_3)$

$e_k=\wp\left(\frac{\omega_k}{2}\right)$, $\quad \omega_3=\omega_1+\omega_2 \qquad g_2=60\,G_4(L),\quad g_3=140\,G_6(L)$



$\Rightarrow \wp(z)$ is the inverse function of $\int_{\infty}^{\wp}\frac{dx}{\sqrt{f(x)}}$, $\quad f(x)=4x^3-g_2x-g_3$

(3) f has distinct roots $\Rightarrow \Delta(L):=16\prod_{j<k}(e_j-e_k)^2=g_2^3-27g_3^2\neq 0$

(4) $\mathcal{M}(\mathbb{C}/L):=\{$meromorphic functions periodic w.r.t. $L\}$
$=\mathbb{C}(\wp(z),\wp'(z))=\{h_1(z)+\wp'(z)h_2(z)\mid h_k\in\mathbb{C}(\wp(z))\}$

(5) $\prod_{j=1}^{N}\sigma(z-a_j)^{n_j}\in\mathcal{M}(\mathbb{C}/L)\iff \sum n_j=0\in\mathbb{Z},\ \sum n_ja_j\in L\subset\mathbb{C}$
$(n_j\in\mathbb{Z})$

Modular forms – Introduction

What is a classical modular form?

holomorphic function $\quad f : \underbrace{\mathcal{H} = \{\tau = x+iy \in \mathbb{C} \mid y > 0\}}_{\text{cplx upper half plane}} \longrightarrow \mathbb{C}$

with many symmetries, such as

$$\forall \tau \in \mathcal{H} \quad f(\tau+1) = f(\tau), \quad f\left(-\frac{1}{\tau}\right) = \tau^k f(\tau) \qquad (1)$$

$(k \in \mathbb{Z})$

$$\Updownarrow$$

$$\forall \tau \ \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \quad f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k f(\tau) \qquad (1')$$

(and a growth condition "at infinity" when $\mathrm{Im}(\tau) \longrightarrow +\infty$).

Ex: Eisenstein series (holomorphic, of level 1, of weight k)

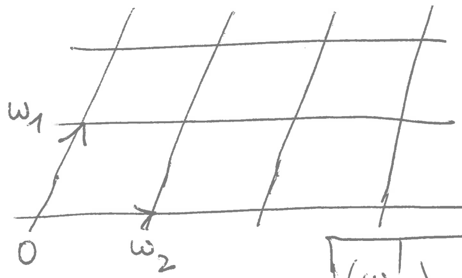
$$G_k(\tau) := \sum_{m,n\in\mathbb{Z}}{}' \frac{1}{(m\tau+n)^k} \qquad (k \in \mathbb{Z},\ k > 2) \qquad (\tau \in \mathbb{C} \setminus \mathbb{R})$$

homogeneous version: $\quad G_k(\omega_1, \omega_2) := \sum_{m,n\in\mathbb{Z}}{}' \frac{1}{(m\omega_1+n\omega_2)^k} = \sum_{0 \neq u \in L} \frac{1}{u^k} = G_k(L)$

$(\omega_1, \omega_2 \in \mathbb{C}$, linearly independent over $\mathbb{R})$

depends ~~onf~~ on the lattice $\quad L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 \subset \mathbb{C}$, and

$$G_k(tL) = t^{-k} G_k(L) \qquad (t > 0)$$



Change of basis of L:

$$\boxed{L = \mathbb{Z}\omega_1' + \mathbb{Z}\omega_2' \ \Updownarrow\ \begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Z}) := M_2(\mathbb{Z})^\times}$$

$$M_2(\mathbb{Z})^\times = \{\gamma \in M_2(\mathbb{Z}) \mid \det(\gamma) \in \underbrace{\mathbb{Z}^\times}_{\{\pm 1\}}\}$$

$$G_k(\tau) = G_k(\tau, 1) = G_k(\mathbb{Z}\tau + \mathbb{Z})$$

$$G_k(\omega_1, \omega_2) = \omega_2^{-k} G_k\left(\frac{\omega_1}{\omega_2}, 1\right)$$

$$L = \omega_2\Big(\mathbb{Z}\underbrace{\frac{\omega_1}{\omega_2}}_{\tau} + \mathbb{Z}\Big) = \omega_2'\left(\mathbb{Z}\frac{\omega_1'}{\omega_2'} + \mathbb{Z}\right) = (c\omega_1 + d\omega_2)\underbrace{\left(\mathbb{Z}\frac{a\omega_1+b\omega_2}{c\omega_1+d\omega_2} + \mathbb{Z}\right)}_{\left(\mathbb{Z}\frac{a\tau+b}{c\tau+d} + \mathbb{Z}\right)}$$

Summary: $\quad \forall \tau \in \mathbb{C} \setminus \mathbb{R}, \quad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Z})$

$$G_k(\tau) = G_k(\mathbb{Z}\tau + \mathbb{Z}) = G_k\Big(\underbrace{\mathbb{Z}(a\tau+b) + \mathbb{Z}(c\tau+d)}_{(c\tau+d)\left(\mathbb{Z}\frac{a\tau+b}{c\tau+d} + \mathbb{Z}\right)}\Big) = (c\tau+d)^{-k} G_k\left(\frac{a\tau+b}{c\tau+d}\right).$$

Notation above:

- $\sum_{m,n\in\mathbb{Z}}{}'$ — the term $m = n = 0$ is omitted
- $GL_n(R)$ $\quad$ (R ring) $:=$ invertible $n \times n$ matrices with coefficients in R

Convergence: **Exercise**: (1) For $\sigma>0$, $\left(\sum_{0\neq x\in\mathbb{Z}^n}\frac{1}{\|x\|^\sigma}<\infty \iff \sigma>n\right)$.

(2) For $\tau\in\mathbb{C}\setminus\mathbb{R}$ and $\sigma>0$, $\left(\sum_{m,n\in\mathbb{Z}}{}'\ \frac{1}{|m\tau+n|^\sigma}<\infty \iff \sigma>2\right)$.

Variants: (a) non-holomorphic Eisenstein series (of level 1):

$$G_{k,s}(\tau):=\sum_{m,n\in\mathbb{Z}}{}'\ \frac{1}{(m\tau+n)^k|m\tau+n|^s}\qquad \begin{pmatrix}k\in\mathbb{Z},\ s\in\mathbb{C},\ k+\mathrm{Re}(s)>2\\ \tau\in\mathbb{C}\setminus\mathbb{R}\end{pmatrix}$$

$\forall \begin{pmatrix}a&b\\c&d\end{pmatrix}\in GL_2(\mathbb{Z})$ $\quad G_{k,s}\left(\frac{a\tau+b}{c\tau+d}\right)=(c\tau+d)^k|c\tau+d|^s\,G_{k,s}(\tau)$.

Exercise: $\{$continuous group homomorphisms $\mathbb{C}^\times\longrightarrow\mathbb{C}^\times\}=$
$=\{z\longmapsto z^k|z|^s \mid k\in\mathbb{Z},\ s\in\mathbb{C}\}$

(a') $$F_{k,s}(\tau):=\sum_{m,n\in\mathbb{Z}}{}'\ \frac{|\mathrm{Im}(\tau)|^{s/2}}{(m\tau+n)^k|m\tau+n|^s},\qquad F_{k,s}\left(\frac{a\tau+b}{c\tau+d}\right)=(c\tau+d)^kF_{k,s}(\tau)$$
$$\left(=|\mathrm{Im}(\tau)|^{s/2}G_{k,s}(\tau)\right)\qquad \forall\begin{pmatrix}a&b\\c&d\end{pmatrix}\in GL_2(\mathbb{Z})$$

Fact: $\forall\begin{pmatrix}a&b\\c&d\end{pmatrix}\in GL_2(\mathbb{R})$, $\forall z,z'\in\mathbb{C}\setminus\mathbb{R}$

$$\frac{az+b}{cz+d}-\frac{az'+b}{cz'+d}=\frac{(ad-bc)(z-z')}{(cz+d)(cz'+d)}\overset{z':=\bar z}{\Longrightarrow}\mathrm{Im}\left(\frac{az+b}{cz+d}\right)=\frac{\left(\det\begin{pmatrix}a&b\\c&d\end{pmatrix}\right)\mathrm{Im}(z)}{|cz+d|^2}$$

(b) holomorphic Eisenstein series of level N (and weight k)

($k,N\in\mathbb{Z}$, $k>2$, $N\geq1$) fix $\phi:((\mathbb{Z}/N\mathbb{Z})^2)^*\longrightarrow\mathbb{C}$

$$G_k(\tau,\phi):=\sum_{m,n\in\mathbb{Z}}{}'\ \frac{\phi((m,n))}{(m\tau+n)^k}=\sum_{a,b\in\mathbb{Z}/N\mathbb{Z}}\phi(a,b)\sum_{\substack{m\equiv a\,(N)\\ n\equiv b\,(N)}}{}'\ \frac{1}{(m\tau+n)^k}$$
($\tau\in\mathbb{C}\setminus\mathbb{R}$)

$\forall\gamma=\begin{pmatrix}a&b\\c&d\end{pmatrix}\in GL_2(\mathbb{Z})$

$$m\tau+n=(m\ \ n)\begin{pmatrix}\tau\\1\end{pmatrix}=\underbrace{(m\ \ n)\begin{pmatrix}a&b\\c&d\end{pmatrix}^{-1}}_{(m'\ \ n')}\underbrace{\begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}\tau\\1\end{pmatrix}}_{\begin{pmatrix}a\tau+b\\c\tau+d\end{pmatrix}}=\underbrace{(m'\ \ n')\begin{pmatrix}\tau'\\1\end{pmatrix}}_{m'\tau'+n'}(c\tau+d)\qquad \tau'=\frac{a\tau+b}{c\tau+d}$$

$$\Longrightarrow G_k(\tau,\phi)=\sum_{m',n'\in\mathbb{Z}}{}'\ \frac{\phi\left((m'\ n')\begin{pmatrix}a&b\\c&d\end{pmatrix}\right)}{((m'\tau'+n')(c\tau+d))^k}=(c\tau+d)^{-k}\underbrace{\sum_{m',n'\in\mathbb{Z}}{}'\ \frac{\left(\begin{pmatrix}a&b\\c&d\end{pmatrix}*\phi\right)((m'\ n'))}{(m'\tau'+n')^k}}_{\boxed{G_k\left(\tau,\begin{pmatrix}a&b\\c&d\end{pmatrix}*\phi\right)}}$$

$$\left(\begin{pmatrix}a&b\\c&d\end{pmatrix}*\phi\right)((m'\ n')):=\phi\left((m'\ n')\begin{pmatrix}a&b\\c&d\end{pmatrix}\right)$$

Cor: $\forall\begin{pmatrix}a&b\\c&d\end{pmatrix}\in\Gamma(N):=\left\{\gamma\in GL_2(\mathbb{Z})\ \middle|\ \gamma\equiv\begin{pmatrix}1&0\\0&1\end{pmatrix}(\mathrm{mod}\ N)\right\}$

$$G_k\left(\frac{a\tau+b}{c\tau+d},\phi\right)=(c\tau+d)^k\,G_k(\tau,\phi)$$

Ex: Jacobi's θ-function $\quad \boxed{\theta(z,\tau) := \sum_{n\in\mathbb{Z}} e^{\pi i n^2\tau + 2\pi i n z}} \quad (\tau\in\mathcal{H},\ z\in\mathbb{C})$

$$\theta(\tau) := \theta(0,\tau) = \sum_{n\in\mathbb{Z}} e^{\pi i n^2 \tau} = \sum_{n\in\mathbb{Z}} q^{n^2/2} \qquad (q := e^{2\pi i\tau},\ q^{1/N} := e^{2\pi i \tau/N})$$

(1) $\theta(\tau+2) = \theta(\tau)$ (obvious)

(2) $\theta(-\frac{1}{\tau}) = \underbrace{\left(\frac{\tau}{i}\right)^{1/2}}_{} \theta(\tau)$

the branch on $\mathcal{H}$ equal to 1 at $\tau = i$

$(= e^{-2\pi i/8}\tau^{1/2},\ 0 < \arg(\tau^{1/2}) < \pi/2)$

Where does (2) come from? Poisson's summation formula + fact that $e^{-\pi x^2}$ = its Fourier transform.

Ultimate reason: rigidity of representations of Heisenberg's commutation relation $PQ - QP = 2\pi i$.

Combination of (1) and (2)

$\Downarrow$

$$\forall\, \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_\theta \quad \underline{\theta\left(\frac{a\tau+b}{c\tau+d}\right) = \varepsilon_\gamma (c\tau+d)^{1/2}\theta(\tau)}, \quad \varepsilon_\gamma^8 = 1, \qquad (3)$$

$$\Gamma_\theta = \left\{\gamma\in SL_2(\mathbb{Z}) \mid \gamma \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \pmod 2\right\}.$$

θ = holomorphic modular form of weight $\frac{1}{2}$ and level 2.

Variants: (a) $\theta(2\tau) = \sum_{n\in\mathbb{Z}} e^{2\pi i n^2\tau} = \sum_{n\in\mathbb{Z}} q^{n^2}$ satisfies a version of (3)

for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in g^{-1}\Gamma_\theta g$, $g = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$. Note: $g^{-1}\Gamma_\theta g \not\subset M_2(\mathbb{Z})$, but

$\underline{g^{-1}\Gamma_\theta g \cap M_2(\mathbb{Z}) = \Gamma_0(4)}$, $\quad \underline{\Gamma_0(N) := \left\{\gamma\in SL_2(\mathbb{Z}) \mid \gamma \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod N\right\}}$

(b) $\theta(2\tau)^k = \sum_{n_1,\dots,n_k\in\mathbb{Z}} q^{n_1^2+\dots+n_k^2} = 1 + \sum_{m=1}^{\infty} r_k(m) q^m \qquad \underline{\text{weight } \frac{k}{2}}$

$r_k(m) = |\{(n_1,\dots,n_k)\in\mathbb{Z}^k \mid n_1^2+\dots+n_k^2 = m\}|$

(c) $S[x] := \sum_{a,b=1}^{k} s_{ab} x_a x_b$ quadratic form ($\underline{s_{ab} = s_{ba} \in \mathbb{Z}}$) in k variables, $\underline{\text{positive definite}}$

$$\boxed{\theta_S(\tau) := \sum_{n\in\mathbb{Z}^k} e^{\pi i S[n]\tau}}$$

$$\theta_S(\tau) = 1 + \sum_{m=1}^{\infty} r_S(m) q^{m/2}, \quad r_S(m) = |\{n\in\mathbb{Z}^k \mid S[n] = m\}|$$

Rmk: By definition, $\left(f\big|_k \underbrace{\begin{pmatrix}a&b\\c&d\end{pmatrix}}_{\alpha}\right)(\tau) := \dfrac{\det(\alpha)^{k/2}}{(c\tau+d)^k}\, f\left(\dfrac{a\tau+b}{c\tau+d}\right)$.
$(\det(\alpha)>0)$

As $(f|_k\alpha)|_k\beta = f|_k(\alpha\beta)$, a function $f: \mathcal{H}\longrightarrow V$ satisfies

$$(\forall \alpha\in \text{set } \Sigma)\quad f|_k\alpha = f$$

$$\Updownarrow$$

$$\left(\begin{array}{c}\forall\alpha\in \text{the group } \langle\Sigma\rangle\subset GL_2^+(\mathbb{R})\\ \text{generated by } \Sigma\end{array}\right)\quad f|_k\alpha = f.$$

Notation: $S=\begin{pmatrix}0&-1\\1&0\end{pmatrix}$, $T=\begin{pmatrix}1&1\\0&1\end{pmatrix}$ $\quad S(\tau)=-\frac{1}{\tau}$, $T^k(\tau)=\tau+k$ $(k\in\mathbb{Z})$

$S^2=-I$

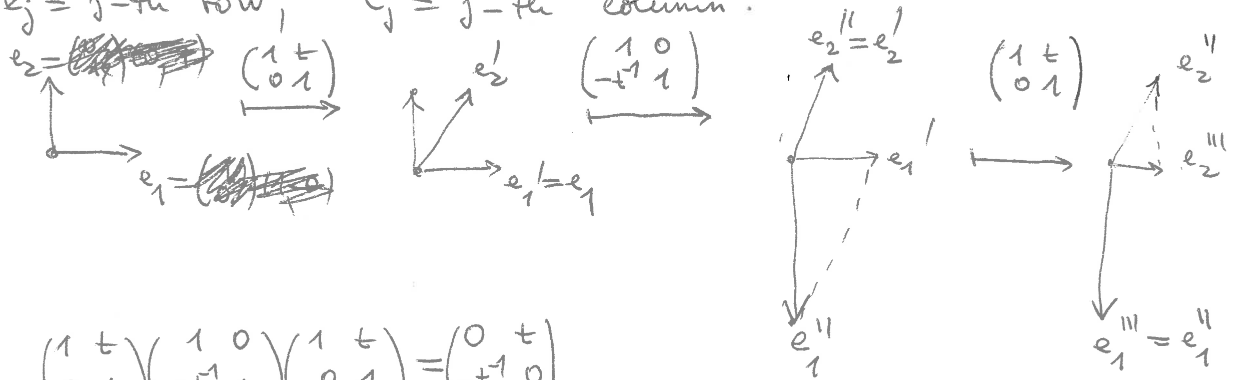
$\left(\Longrightarrow STS^{-1}=\begin{pmatrix}1&0\\-1&1\end{pmatrix}\right)$, $ST=\begin{pmatrix}0&-1\\1&1\end{pmatrix}$, $(ST)^2=\begin{pmatrix}-1&-1\\1&0\end{pmatrix}$, $(ST)^3=\begin{pmatrix}-1&0\\0&-1\end{pmatrix}=-I$

Prop. $\langle S,T\rangle = SL_2(\mathbb{Z})$ $\qquad T(STS^{-1})T=-S$

Special case of Gauss elimination (= elementary operations) over Euclidean domains.

Ex: $\begin{pmatrix}1&t\\0&1\end{pmatrix}\begin{pmatrix}R_1\\ \hline R_2\end{pmatrix}=\begin{pmatrix}R_1+tR_2\\ \hline R_2\end{pmatrix}$, $\left(C_1\,\middle|\,C_2\right)\begin{pmatrix}1&t\\0&1\end{pmatrix}=\left(C_1\,\middle|\,C_2+tC_1\right)$

R_j = j-th row, C_j = j-th column.



$$\begin{pmatrix}1&t\\0&1\end{pmatrix}\begin{pmatrix}1&0\\-t^{-1}&1\end{pmatrix}\begin{pmatrix}1&t\\0&1\end{pmatrix}=\begin{pmatrix}0&t\\-t^{-1}&0\end{pmatrix}$$

Thm. Let A be a Euclidean domain. Let $E_n(A)\subset GL_n(A)$ be the subgroup generated by elementary matrices $e_{ij}(t)=I_n+\begin{array}{c}\\ \\ \end{array}$ $\begin{array}{cc} & j \\ & \\ i & \end{array}\begin{pmatrix}0 & \vdots & 0\\ \cdots & t & \cdots\\ 0 & \vdots & 0\end{pmatrix}$ $(i\neq j)$.
For each $m\times n$ matrix $M\in M_{m,n}(A)$ $\exists\, g\in E_m(A)$, $h\in E_n(A)$

$$gMh=\begin{pmatrix}d_1 & & 0 & 0\\ & \ddots & & \\ 0 & & d_r & \\ 0 & & & 0\end{pmatrix},\quad r\geq 0,\ d_1,\dots,d_r\neq 0,\ d_1|d_2|\cdots|d_r.$$

By changing g,h, we can replace

$$d_i\longmapsto u_id_i,\quad u_i\in A^\times,\quad u_1\cdots u_r=1.$$

Pf: Euclid's algorithm + elementary operations on rows ($\Longrightarrow g$) and columns ($\Longrightarrow h$).

Cor: For a Euclidean domain A, $E_n(A)=SL_n(A)$ $\forall n\geq 1$.

Cor. of $\langle S, T\rangle = SL_2(\mathbb{Z})$: it is equivalent (for $f: \mathcal{H} \longrightarrow V$):

$$\left\{\begin{array}{l} f(\tau+1) = f(\tau) \\ f(-\frac{1}{\tau}) = \tau^k f(\tau) \end{array}\right\} \overset{(k\in\mathbb{Z})}{\iff} \left\{ \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \qquad f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k f(\tau) \right\}$$

Exercise: $\langle S, T^2\rangle = \Gamma_\theta := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \,\middle|\, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \pmod 2 \right\}$

Cor. it is equivalent (for $f: \mathcal{H} \longrightarrow V$)

$$\left\{\begin{array}{l} f(\tau+2) = f(\tau) \\ f(-\frac{1}{\tau}) = \tau^k f(\tau) \end{array}\right\} \overset{(k\in\mathbb{Z})}{\iff} \left\{ \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_\theta \qquad \text{—\,''\,—} \right\}$$

Exercise: What is $\langle T^2, ST^{-2}S^{-1}\rangle = \left\langle \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \right\rangle$ equal to?

Fact: for $k \geq 6$, the smallest normal subgroup of $SL_2(\mathbb{Z})$ containing $T^k = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$ (it also contains $\begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix} = ST^{-k}S^{-1}$) has *infinite* index in $SL_2(\mathbb{Z})$.

Back to Eisenstein series

$k \in \mathbb{Z},\ k > 2,\qquad \tau \in \mathcal{H},\qquad G_k(\tau) := \sum_{m,n\in\mathbb{Z}}' \frac{1}{(m\tau+n)^k}$

$G_k(-\tau) = (-1)^k G_k(\tau) \implies \boxed{G_k(\tau) = 0 \text{ if } 2 \nmid k}$

At infinity: $G_k(i\infty) := \lim_{\operatorname{Im}(\tau)\to+\infty} G_k(\tau) = \sum_{n\in\mathbb{Z}}' \frac{1}{n^k} = 2\sum_{n=1}^{\infty}\frac{1}{n^k} = 2\zeta(k)$

normalised Eisenstein series: $\boxed{E_k(\tau) := G_k(\tau)/G_k(i\infty)}$ $\quad E_k(i\infty) = 1$

$G_k(\tau+1) = G_k(\tau) \implies G_k(\tau)$ is a holomorphic function of

$q = e^{2\pi i\tau},\quad 0 < |q| = e^{-2\pi\operatorname{Im}(\tau)} < 1$

$$\implies G_k(\tau) = \sum_{m\in\mathbb{Z}} a_{k,m}\, e^{2\pi i m\tau} = \sum_{m\in\mathbb{Z}} a_{k,m}\, q^m \qquad \forall \tau \in \mathcal{H}$$

$\boxed{G_k(i\infty) \neq \infty \implies a_{k,m} = 0 \text{ for } m < 0}$

Calculation (pf - later): ($k \in \mathbb{Z}$, $k > 2$, $2 \mid k$):

$$E_k(\tau) = 1 - \frac{2k}{B_k}\sum_{m=1}^{\infty}\frac{m^{k-1}q^m}{1-q^m} = 1 - \frac{2k}{B_k}\sum_{M=1}^{\infty}\sigma_{k-1}(M)\,q^M \qquad (q = e^{2\pi i\tau},\ \tau\in\mathcal{H})$$

$\sigma_r(M) := \sum_{d\mid M} d^r,\qquad \frac{t}{e^t-1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}\qquad (\{B_n\} = \text{Bernoulli numbers})$

(above, $-\frac{B_k}{k} = \zeta(1-k)$)

Variants: (a) for Eisenstein series of level N:

$\sum_{d\mid n} d^{k-1}$ is modified by terms depending on $d \pmod N$ and $n \pmod N$

(b) for non-holomorphic Eisenstein series: the term $e^{-2\pi n y}$ in $q^n = e^{2\pi i n x}e^{-2\pi n y}$ $(\tau = x+iy)$ needs to be replaced by $W(2\pi n y)$ for a suitable Whittaker function W.

Ex: $E_4(\tau) = 1 + 240\sum_{n=1}^{\infty}\sigma_3(n)q^n,\quad E_6(\tau) = 1 - 504\sum_{n=1}^{\infty}\sigma_5(n)q^n$

Dirichlet series attached to $G_k(\tau)$:

$$\sum_{n=1}^{\infty}\frac{\sigma_{k-1}(n)}{n^s} = \underbrace{\left(\sum_{k=1}^{\infty}\frac{d^{k-1}}{d^s}\right)}_{\prod_p\left(1-\frac{1}{p^{s-k+1}}\right)^{-1}}\underbrace{\left(\sum_{m=1}^{\infty}\frac{1}{m^s}\right)}_{\prod_p\left(1-\frac{1}{p^s}\right)^{-1}} = \zeta(s-k+1)\,\zeta(s)$$

Ex: Discriminant $\Delta(\tau)$, Dedekind η-function $\eta(\tau)$

$$\Delta(\tau) := \frac{E_4(\tau)^3 - E_6(\tau)^2}{12^3} = q - 24q^2 + \cdots \qquad (q = e^{2\pi i \tau})$$

Jacobi's product formula for $\Delta(\tau)$: $\boxed{\Delta(\tau) = q\prod_{n=1}^{\infty}(1-q^n)^{24}}$ $(\tau \in \mathcal{H})$

$$\Delta(\tau) = \eta(\tau)^{24}, \quad \eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$$

$$\forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \qquad \Delta\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^{12}\Delta(\tau) \qquad \left(\begin{matrix}\text{since true for}\\ E_4^3 \text{ and } E_6^2\end{matrix}\right)$$

Note: $$\frac{\left(q\frac{d}{dq}\right)\Delta(\tau)}{\Delta(\tau)} = q\frac{d}{dq}\left(\log \Delta(\tau)\right) = q\frac{d}{dq}\Big(\log(q) + 24\sum_{n=1}^{\infty}\underbrace{\log(1-q^n)}_{-\sum_{k\geq 1}\frac{q^{nk}}{k}}\Big) =$$

$$= 1 - 24\sum_{n,k=1}^{\infty} n q^{nk} = 1 - 24\sum_{m=1}^{\infty}\sigma_1(m)q^m =: E_2(\tau)$$

Previous formula $\Longrightarrow$ (assuming the validity of the product formula for $\Delta(\tau)$)

$$\boxed{\begin{gathered}\forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \\ E_2\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^2 E_2(\tau) + \underbrace{\frac{12c(c\tau+d)}{2\pi i}}_{\text{extra term}}\end{gathered}}$$

Exercise / Question: can one modify the function $E_2(\tau)$ in order to get rid of the term $\frac{12c(c\tau+d)}{2\pi i}$?

As we shall see,

$$2\zeta(2)\left(E_2(\tau) - 1\right) = 2\sum_{m=1}^{\infty}\left(\sum_{n\in\mathbb{Z}}\frac{1}{(m\tau+n)^2}\right)$$

Coefficients of $\Delta(\tau) = \sum_{n=1}^{\infty}\tau(n)q^n = q\,P(q)^{24}$, $\quad P(q) = \prod_{n=1}^{\infty}(1-q^n)$

---

recursive relation: $$\left(q\frac{d}{dq}\right)\Delta = E_2\Delta \iff \sum_{n\geq 1} n\,\tau(n)q^n = \left(\sum_{k\geq 1}\tau(k)q^k\right)E_2(\tau)$$

$$\iff \forall n \geq 1 \quad (1-n)\,\tau(n) = 24\sum_{k=1}^{n-1}\tau(k)\,\sigma_1(n-k)$$

Values of $\tau(n)$ ("Ramanujan's function")

n	$\tau(n)$		n	$\tau(n)$	
1	1		6	-6048	$= -2^5\cdot 3^3\cdot 7$
2	-24	$= -2^3\cdot 3$	7	-16744	$= -2\cdot 3\cdot 7\cdot 13\cdot 23$
3	252	$= 2^2\cdot 3^2\cdot 7$	8	84480	$= 2^{8}\cdot 3\cdot 5\cdot 11$
4	-1472	$= -2^6\cdot 23$	9	-113643	$= -3^4\cdot 23\cdot 61$
5	4830	$= 2\cdot 3\cdot 5\cdot 7\cdot 23$	10	-115920	$= -2^4\cdot 3^2\cdot 5\cdot 7\cdot 23$

What is going on? $\quad \tau(6) = \tau(2)\tau(3)$ (!)

$\tau(10) = \tau(2)\tau(5)$ (!)

$\tau(4) \neq \tau(2)\tau(2)$, but $\tau(4) - \tau(2)\tau(2) = -2048 = -2^{11}$ (!!)

$\tau(8) \neq \tau(2)\tau(4)$, but $\tau(8) - \tau(2)\tau(4) = 2^{14}\cdot 3 = -2^{11}\tau(2)$ (!!)

$\tau(9) \neq \tau(3)\tau(3)$, but $\tau(9) - \tau(3)\tau(3) = -3^{11}$ (!!)

Ramanujan conjectured: $\quad (m,n) = 1 \Longrightarrow \tau(mn) = \tau(m)\tau(n)$

p prime, $n \geq 1 \Longrightarrow \tau(p^{n+1}) - \tau(p)\tau(p^n) + p^{11}\tau(p^{n-1}) = 0$

$$\left(\Longleftrightarrow \quad L(\Delta, s) := \sum_{n=1}^{\infty} \frac{\tau(n)}{n^s} = \prod_p \left(1 - \frac{\tau(p)}{p^s} + \frac{p^{11}}{p^{2s}}\right)^{-1}\right)$$

this was proved by Mordell. General theory: Hecke (but "Hecke operators" appeared already in Mordell's work).

Ramanujan's conjecture: $|\tau(p)|$ is "small":

$$1 - \tau(p)X + p^{11}X^2 = (1-\alpha_p X)(1-\beta_p X) \qquad \text{with } \beta_p = \overline{\alpha_p}$$

$(\Longleftrightarrow \quad |\alpha_p| = |\beta_p| = p^{11/2}) \qquad (\Longrightarrow |\tau(p)| \leq 2p^{11/2})$.

Proved by Deligne.

Congruences for $\tau(n)$: p prime

$\tau(p) \equiv 1 + p^{11} \pmod{2^6} \quad (p \neq 2)$

$\tau(p) \equiv p^{135} + p^{362} \pmod{3^6} \quad (p \neq 3)$

$\tau(p) \equiv p^{41} + p^{70} \pmod{5^3} \quad (p \neq 5)$

$\tau(p) \equiv p + p^{10} \pmod{7} \quad (p \neq 7)$

$\tau(p) \equiv 1 + p^{11} \pmod{691} \quad (p \neq 691)$ (Ramanujan)

n	1	2	3	4	5	6	7	8	9	10
$\tau(n) \pmod{23}$	1	-1	-1	0	0	1	0	1	0	0

$p \neq 23$

$$\tau(p) \equiv \begin{Bmatrix} 0 \\ 2 \\ -1 \end{Bmatrix} \pmod{23}, \quad \begin{matrix} p \not\equiv x^2 \pmod{23} \\ p = u^2 + 23v^2 \\ p \neq u^2 + 23v^2, \ p \equiv x^2 \pmod{23} \end{matrix}$$

Modern interpretation: Galois representations

Representation of integers by sums of squares

$$\theta(2\tau)^k = 1 + \sum_{n=1}^{\infty} r_k(n) q^n, \qquad r_k(n) = \left| \{(n_1, \dots, n_k) \in \mathbb{Z}^k;\ n_1^2 + \dots + n_k^2 = n\} \right|$$

<u>classical formulas</u>: $r_2(n) = 4 \sum_{2 \nmid d \mid n} (-1)^{\frac{d-1}{2}}$, $\quad r_4(n) = 8 \sum_{4 \nmid d \mid n} d$

$$r_6(n) = 4 \sum_{2 \nmid d \mid n} (-1)^{\frac{d-1}{2}} \left((2n/d)^2 - d^2 \right), \qquad r_8(n) = 16 \sum_{d \mid n} (-1)^{d+n} d^3$$

$$\Updownarrow$$

$$\theta(2\tau)^2 = 1 + 4 \sum_{m=1}^{\infty} \frac{(-1)^m q^{2m-1}}{1 - q^{2m-1}}, \qquad \theta(2\tau)^4 = 1 + 8 \sum_{m=1}^{\infty} \frac{m q^m}{1 + (-q)^m} \qquad \text{(Jacobi)}$$

Eisenstein series of level 2

$$\theta(2\tau)^6 = 1 + 16 \sum_{m=1}^{\infty} \frac{m^2 q^m}{1 + q^{2m}} + 4 \sum_{m=1}^{\infty} (-1)^m \frac{(2m-1)^2 q^{2m-1}}{1 - q^{2m-1}}$$

$$\theta(2\tau)^8 = 1 + 16 \sum_{m=1}^{\infty} \frac{m^3 q^m}{1 - (-q)^m}$$

<u>Fact</u>: $k > 8$ even $\Longrightarrow \theta(2\tau)^k \neq$ Eisenstein series.

<u>Liouville</u>: (a) $r_{10}(n) = \frac{4}{5} \sum_{2 \nmid d \mid n} (-1)^{\frac{d-1}{2}} \left((2n/d)^4 + d^4 \right) + \frac{8}{5} \sum_{\substack{a, b \in \mathbb{Z} \\ a^2 + b^2 = n}} (a + bi)^4$

(improved formulation, due to Glashier)

$(\Longleftrightarrow \underbrace{\theta(2k)^{10}}_{(\theta\text{-series of } x_1^2 + \dots + x_{10}^2)} = (\text{Eisenstein series}) + \begin{pmatrix} \theta\text{-series of } x_1^2 + x_2^2 \text{ with} \\ \text{harmonic polynomial } (x_1 + i x_2)^4 \end{pmatrix})$

(b) $\quad r_{12}(2m) = 8 \sum_{d \mid 2m} (-1)^{d + (2m/d) - 1} d^5$

<u>Glashier</u> (among other things): general formula for $r_{12}(n)$ in terms of quaternions $a + bi + cj + dk$ instead of $a + bi$

<u>Ramanujan</u>: $\theta(2\tau)^{24} = \underbrace{1 + \frac{16}{691} \sum_{m=1}^{\infty} \frac{m^{11} q^m}{1 - (-q)^m}}_{\text{Eisenstein series}} + \frac{(33152)\Delta(\tau + \frac{1}{2}) - (65536)\Delta(2\tau)}{691}$

$$\Updownarrow$$

$$\frac{691}{16} r_{24}(n) = \sum_{d \mid n} (-1)^{d+n} d^{11} + 2072 (-1)^{n-1} \tau(n) - 4096\, \tau(n/2)$$

<u>If $n = p$ prime</u>: $\quad \underbrace{\sum_{d \mid n} (-1)^{d+n} d^{11}}_{\substack{1 + p^{11} \\ \text{elementary term}}} \qquad \underbrace{2072 (-1)^{n-1} \tau(n) - 4096\, \tau(n/2)}_{\substack{\leq C \cdot p^{11/2} \\ \text{error term}}}$

Modular invariant $j(\tau)$

$$j(\tau) := \frac{E_4^3(\tau)}{\Delta(\tau)} = \frac{1}{q} + 744 + \sum_{n=1}^{\infty} c(n) q^n \qquad (q = e^{2\pi i \tau})$$

$c(1) = d_1 + d_2$, $c(2) = d_1 + d_2 + d_3$, $c(3) = 2d_1 + 2d_2 + d_3 + d_4$, ...

$$\underbrace{d_1 = 1,\ d_2 = 196883,\ d_3 = 21296876,\ d_4 = 842609326, \ldots}_{\text{dimensions of irreducible representations of the Monster simple group}}$$

Congruences for $c(n)$:

$\forall m \geq 1$ $\quad c(2m) \equiv 0 \pmod{2^{11}}$, $\quad c(3m) \equiv 0 \pmod{3^5}$,

$\quad c(5m) \equiv 0 \pmod{5^2}$, $\quad c(7m) \equiv 0 \pmod{7}$

$\forall n \geq 1\ \forall k \geq 1$

$$c(2^k n) \equiv 0 \pmod{2^{3k+8}}$$
$$c(3^k n) \equiv 0 \pmod{3^{2k+3}}$$
$$c(5^k n) \equiv 0 \pmod{5^{k+1}}$$
$$c(7^k n) \equiv 0 \pmod{7^k}$$

(not very interesting)

First interesting case: $\pmod{13}$

(a) $\forall k \geq 1\ \exists \alpha_k \not\equiv 0 \pmod{13}$ $\quad \forall n \geq 1 \quad c(13^{k+1} n) \equiv \alpha_k\, c(13^k n) \pmod{13^k}$

(b) $\forall p \neq 13$ prime and $t_k(n) = t(n) := c(13^k n)/c(13^k) \pmod{13^k}$,

$$t(np) - t(n)t(p) + p^{-1} t\left(\tfrac{n}{p}\right) \equiv 0 \pmod{13^k} \qquad (k, n \geq 1)$$

Modern interpretation:

p-adic properties of the operator

$$U_p : \sum_{n=0}^{\infty} a(n) q^n \longmapsto \sum_{n=0}^{\infty} a(np) q^n$$

on certain spaces of p-adic modular forms.

$$\boxed{\forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \qquad j\left(\frac{a\tau + b}{c\tau + d}\right) = j(\tau)}$$

Fractional linear maps $\frac{ax+b}{cx+d}$ (algebra and geometry)

K – field, $\quad g=\begin{pmatrix}a&b\\c&d\end{pmatrix}\in GL_2(K)$, $\quad \begin{pmatrix}x_1\\x_2\end{pmatrix}\in K^2$

$$\begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}=\begin{pmatrix}ax_1+bx_2\\cx_1+dx_2\end{pmatrix}=\begin{pmatrix}x_1'\\y_1'\end{pmatrix}$$

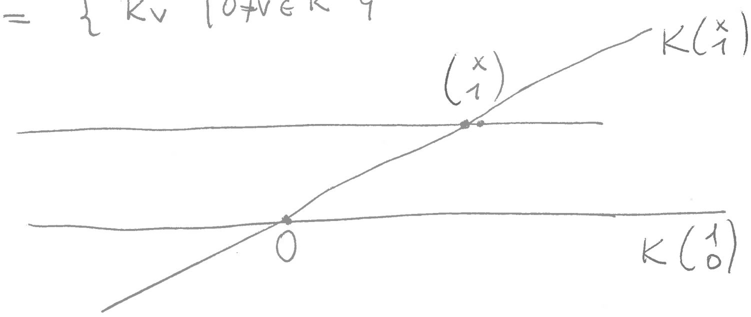
Dehomogeneisation: $x=\frac{x_1}{x_2}$, $\quad x'=\frac{x_1'}{\cancel{x_2'}}$

$$\boxed{x'=\frac{ax_1+bx_2}{cx_1+dx_2}=\frac{ax+b}{cx+d}}$$

$x, x'\in K\cup\{\infty\}$

Projective line: $\mathbb{P}^1(K):=\mathbb{P}(K^2)=\{W\subset K^2 \mid \text{vector subspace}, \dim(W)=1\}$
$= \{Kv \mid 0\neq v\in K^2\}$

$$\boxed{\begin{array}{rcl} K\cup\{\infty\} & \longleftrightarrow & \mathbb{P}^1(K)\\ x & \longleftrightarrow & K\begin{pmatrix}x\\1\end{pmatrix}\\ \infty & \longleftrightarrow & K\begin{pmatrix}1\\0\end{pmatrix}\end{array}}$$



$$K\underbrace{\begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}x\\1\end{pmatrix}}_{\begin{pmatrix}ax+b\\cx+d\end{pmatrix}}=\begin{cases}K\begin{pmatrix}\frac{ax+b}{cx+d}\\1\end{pmatrix} & (\text{if } cx+d\neq 0)\\ K\begin{pmatrix}1\\0\end{pmatrix} & (\text{if } cx+d=0)\end{cases}$$

$$K\begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}=K\begin{pmatrix}a\\c\end{pmatrix}=\begin{cases}K\begin{pmatrix}a/c\\1\end{pmatrix} & (\text{if } c\neq 0)\\ K\begin{pmatrix}1\\0\end{pmatrix} & (\text{if } c=0)\end{cases}$$

Summary: the standard linear action $\begin{pmatrix}x_1\\x_2\end{pmatrix}\longmapsto\begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}$ of $GL_2(K)$ on K^2 defines an action of $GL_2(K)$ on $\mathbb{P}^1(K)\simeq K\cup\{\infty\}$ given by

$$g=\begin{pmatrix}a&b\\c&d\end{pmatrix}:\quad \begin{array}{l} x\longmapsto \frac{ax+b}{cx+d}=g(x) \quad (=\infty \text{ if } cx+d)\\ \infty\longmapsto \frac{a}{c} \quad (=\infty \text{ if } c=0).\end{array}$$

Formulas: (1) $g\begin{pmatrix}x\\1\end{pmatrix}=\begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}x\\1\end{pmatrix}=\begin{pmatrix}ax+b\\cx+d\end{pmatrix}=\begin{pmatrix}\frac{ax+b}{cx+d}\\1\end{pmatrix}(cx+d)=\begin{pmatrix}g(x)\\1\end{pmatrix}J(g,x)$

$$\boxed{J\left(\begin{pmatrix}a&b\\c&d\end{pmatrix},x\right)=cx+d \qquad \text{automorphy factor}}$$

(2) $g_1(g_2(x))=(g_1g_2)(x)$

(3) $g=\begin{pmatrix}t&\\&t\end{pmatrix}=t\cdot I \iff \forall x \quad g(x)=x$

($\Longrightarrow$ action of $GL_2(K)/K^\times\cdot I=PGL_2(K)$ (faithful, transitive) on $\mathbb{P}^1(K)$)

(4) 1-cocycle identity

$$\boxed{J(g_1g_2,x)=J(g_1,g_2(x))J(g_2,x)}$$

(provided $g_2(x)\neq\infty$)

Additional Formulas: (1) $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(K)$, $x, y \in K$

$$\boxed{g(x) - g(y) = \frac{ax+b}{cx+d} - \frac{ay+b}{cy+d} = \frac{(ad-bc)(x-y)}{(cx+d)(cy+d)} = \frac{\det(g)}{J(g,x)J(g,y)}(x-y)}$$

(1') **Abstract version**: $B: K^2 \times K^2 \longrightarrow K$ skew-symmetric bilinear non-degenerate

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \longmapsto \begin{vmatrix} u_1 & v_1 \\ u_2 & v_2 \end{vmatrix}$$

$\forall g \in GL_2(K)$ $\forall u, v \in K^2$ $\underline{B(gu, gv) = \det(g)\, B(u,v)}$

$$\left.\begin{array}{l} u = \begin{pmatrix} x \\ 1 \end{pmatrix}, \quad v = \begin{pmatrix} y \\ 1 \end{pmatrix}, \quad gu = \begin{pmatrix} g(x) \\ 1 \end{pmatrix} J(g,x), \quad gv = \begin{pmatrix} g(y) \\ 1 \end{pmatrix} J(g,y) \\ B(u,v) = x-y, \quad B\left(\begin{pmatrix} g(x) \\ 1 \end{pmatrix}, \begin{pmatrix} g(y) \\ 1 \end{pmatrix}\right) = g(x) - g(y) \end{array}\right\} \Longrightarrow (1)$$

Question: Is there a higher-dimensional version of (1') ?

(2) Limit $y \longrightarrow x$: $\boxed{d(g(x)) = \frac{\det(g)}{J(g,x)^2}\, dx}$

(3) Over $K = \mathbb{C}$: $g \in GL_2(\mathbb{R})$, $x = z$, $y = \bar{z}$:

$$\boxed{\operatorname{Im}(g(z)) = \frac{\det(g)}{|J(g,z)|^2} \operatorname{Im}(z)} \qquad \boxed{\begin{array}{c} g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ J(g,z) = cz+d \end{array}}$$

$$g^*(dz) \wedge g^*(d\bar{z}) = d(g(z)) \wedge d(g(\bar{z})) = \frac{\det(g)^2}{|J(g,z)|^4}\, dz \wedge d\bar{z}$$

$$\Longrightarrow g^*\left(\frac{i}{2} \frac{dz \wedge d\bar{z}}{\operatorname{Im}(z)^2}\right) = \underbrace{\frac{i}{2} \frac{dz \wedge d\bar{z}}{\operatorname{Im}(z)^2}}_{\frac{dx \wedge dy}{y^2}} \qquad \forall g \in GL_2(\mathbb{R})$$

$\boxed{z = x+iy}$

$\Longrightarrow$ (a) $PGL_2(\mathbb{R})$-invariant measure $\mu = \frac{dx\,dy}{y^2}$ on $\mathbb{C} \setminus \mathbb{R}$

(b) $g^*(\cancel{\ } |dz|^2) = d(g(z))\, d(g(\bar{z})) = \frac{\det(g)^2}{|J(g,z)|^4} |dz|^2$

$$\Longrightarrow g^*\left(\frac{|dz|^2}{\operatorname{Im}(z)^2}\right) = \frac{|dz|^2}{\operatorname{Im}(z)^2} = \frac{dx^2 + dy^2}{y^2}$$

$PGL_2(\mathbb{R})$-invariant Riemannian metric on $\mathbb{C} \setminus \mathbb{R}$

(Poincaré metric)

Question: Is there a higher-dimensional analogue of (2), (3) ?

Action on differentials:

$$g^*\left(f(z)(dz)^{\otimes m}\right) = \underbrace{\frac{\det(g)^m}{(cz+d)^{2m}} f\left(\frac{az+b}{cz+d}\right)}_{\left(f|_{2m}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)(z)} (dz)^{\otimes m}$$

($g \in GL_2(\mathbb{R})^+$
($\det(g) > 0$))

<u>Cross ratio</u>: given <u>distinct</u> $x_1, x_2, x_3, x_4 \in \mathbb{P}^1(K)$ there is a <u>unique</u> $g \in PGL_2(\mathbb{R})$ such that

$$(x_2, x_3, x_4) \overset{g}{\mapsto} (1, 0, \infty)$$

$$\boxed{g(x) = \frac{x - x_3}{x - x_4} : \frac{x_2 - x_3}{x_2 - x_4}}$$

$$\boxed{\begin{array}{l} r(x_1, x_2, x_3, x_4) := g(x_1) = \dfrac{x_1 - x_3}{x_1 - x_4} : \dfrac{x_2 - x_3}{x_2 - x_4} \\ \text{cross ratio of } x_1, \dots, x_4 \end{array}}$$

x_1	x_2	x_3	x_4	x
$r(x_1, \dots, x_4)$	1	0	∞	$g(x)$

$\forall h \in PGL_2(K)$ $\quad$ <u>$r(x, 1, 0, \infty) = x$</u>

<u>$r(h(x_1), \dots, h(x_4)) = r(x_1, \dots, x_4)$</u>

$$r : PGL_2(K) \backslash \{\text{ordered } (x_1, \dots, x_4) \mid x_i \in \mathbb{P}^1(K) \text{ distinct}\} \xrightarrow{\sim} \underbrace{\mathbb{P}^1(K) \smallsetminus \{0, 1, \infty\}}_{K \smallsetminus \{0, 1\}}$$

<u>Permutations</u>: if $r(x_1, \dots, x_4) = \lambda$, then

$$\{ r(x_{\sigma(1)}, \dots, x_{\sigma(4)}) \mid \sigma \in S_4 \} = \left\{ \lambda, \frac{1}{\lambda}, 1 - \lambda, \frac{1}{1 - \lambda}, \frac{\lambda}{\lambda - 1}, \frac{\lambda - 1}{\lambda} \right\}$$

<u>Abstract version</u>: V K-vector space, $\dim(V) = 2$

$\mathbb{P}(V)$ = abstract projective line $= \{ W \subset V \mid \dim(W) = 1 \}$

$P_1, \dots, P_4 \in \mathbb{P}(V)$ distinct. What is $r(P_1, \dots, P_4)$?

Fix a linear isomorphism $f : V \xrightarrow{\sim} K^2 \implies \mathbb{P}(f) : \mathbb{P}(V) \xrightarrow{\sim} \mathbb{P}^1(K)$

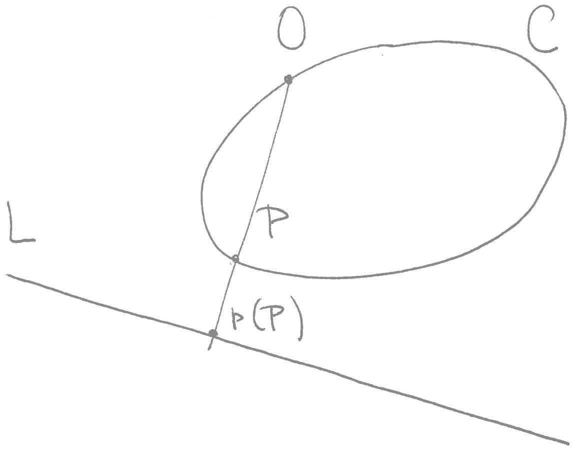
and define $r(P_1, \dots, P_4) := r(\mathbb{P}(f)(P_1), \dots, \mathbb{P}(f)(P_4)) \in K \smallsetminus \{0, 1\}$

<u>Exercise</u>: this value does not depend on the choice of f.

<u>Projective generation of conics</u>

<u>Given</u>: $C \subset \mathbb{P}^2(K) = \mathbb{P}(K^3)$ non-singular projective conic

$O \in C(K)$ K-rational point, $L \subset \mathbb{P}^2(K)$ projective line, $O \notin L$



projection with centre O onto L:

$$p : C(K) \xrightarrow{\sim} L(K) \quad \text{bijection}$$

$$P \longmapsto L \cap \overleftrightarrow{OP}$$

$(\overleftrightarrow{OO} :=$ the tangent to C at $O)$

<u>Thm</u>: For another pair (O', L') as above, and a choice of $L \xrightarrow{\sim} \mathbb{P}^1(K) \xleftarrow{\sim} L'$, there exists $g \in PGL_2(K)$ such that

$$\begin{array}{ccccc} C(K) & \xrightarrow{p} & L & \xrightarrow{\sim} & \mathbb{P}^1(K) \\ & \searrow^{p'} & & & \downarrow g \\ & & L' & \xrightarrow{\sim} & \mathbb{P}^1(K) \end{array}$$

<u>Cor</u>: $\forall P_1, \dots, P_4 \in C(K)$ distinct, $r(p(P_1), \dots, p(P_4)) \in K \smallsetminus \{0, 1\}$ is well-defined.

SL_n and PGL_n

$K = $ field, $g \in GL_n(K)$. If $\det(g) = t^n$ with $t \in K^\times$, then $t^{-1} g \in SL_n(K)$ has the same image in $PGL_n(K) = GL_n(K)/K^\times I_n$ as g does.

Summary: exact sequence $1 \longrightarrow SL_n(K)/\underbrace{\mu_n(K)}_{\{\zeta \in K \mid \zeta^n = 1\}} \cdot I_n \longrightarrow PGL_n(K) \xrightarrow{\det} K^\times / K^{\times n} \to 1$

Exercise: (a) $PGL_2(\mathbb{R}) = PGL_2(\mathbb{C})^{\sigma = id}$ $\quad$ ($\sigma(z) := \bar{z}$, cplx conjugation), $X^{\sigma = id} := \{x \in X \mid \sigma(x) = x\}$

(b) $SL_2(\mathbb{R})/\{\pm I\} \neq (SL_2(\mathbb{C})/\{\pm I\})^{\sigma = id}$

(c) Generalise (a).

Action of $PGL_2(\mathbb{C}) = SL_2(\mathbb{C})/\{\pm I\}$ on $\mathbb{P}^1(\mathbb{C})$: fixed pts, orbits

Given: $g \in SL_2(\mathbb{C})$ with eigenvalues $\lambda_1, \lambda_2 \in \mathbb{C}^\times$, $\lambda_1 \lambda_2 = 1$.

① $\{$fixed points of g on $\mathbb{P}^1(\mathbb{C})\} = \{\mathbb{C}v \mid v \in \mathbb{C}^2$ eigenvector of $g\}$

② Assume: $g \neq \pm I$ $\quad$ ($\Rightarrow g$ has 1 or 2 eigenvectors)

(a) $\lambda_1 = \lambda_2$ ($\Rightarrow \lambda_1 = \lambda_2 = \pm 1$) $\Longrightarrow$ g has 1 eigenvector

$\exists h \in SL_2(\mathbb{C})$ $\quad hgh^{-1} = \pm \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ $\quad$: $z \longmapsto z+1$

$\{$fixed pts of $hgh^{-1}\} = \{\infty\} \Longrightarrow \{$fixed pts of $g\} = \{h^{-1}(\infty)\}$

terminology: g is parabolic

(b) $\lambda_1 \neq \lambda_2 \ (= \lambda_1^{-1}) \Longrightarrow g$ has 2 eigenvectors,

$\exists h \in SL_2(\mathbb{C})$ $\quad hgh^{-1} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1^{-1} \end{pmatrix}$ $\quad$: $z \longmapsto \lambda_1^2 z$ $\qquad$ $(\lambda_1^2 = \frac{\lambda_1}{\lambda_2} \neq 1)$

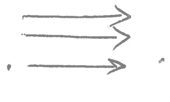
$\{$fixed pts of $hgh^{-1}\} = \{0, \infty\}$, $\quad \{$fixed pts of $g\} = \{h^{-1}(0), h^{-1}(\infty)\}$

terminology: g is hyperbolic if $\lambda_1 \neq \lambda_2$ and $\lambda_1, \lambda_2 \in \mathbb{R}$

g is loxodromic if $\lambda_1 \neq \lambda_2$, $\lambda_1, \lambda_2 \notin \mathbb{R}$ and $|\lambda_1|, |\lambda_2| \neq 1$

g is elliptic if $\lambda_1 \neq \lambda_2$, $\lambda_1, \lambda_2 \notin \mathbb{R}$, $|\lambda_1| = |\lambda_2| = 1$

Ex: $\exp\left(t \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$: $z \longmapsto z+t$



parabolic

$\exp\left(t \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right) = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$: $z \longmapsto e^{2t} z$

hyperbolic

Q: What about $\exp\left(t \cdot (1+i) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right) = \begin{pmatrix} e^t(\cos(t) + i\sin(t)) & 0 \\ 0 & e^{-t}(\cos(t) - i\sin(t)) \end{pmatrix}$?

(loxodromic)

Geometry of the action of $PGL_2(\mathbb{C}) \simeq SL_2(\mathbb{C})/\{\pm I\}$ on $\underbrace{\mathbb{P}^1(\mathbb{C})}_{\mathbb{C}\cup\{\infty\}}$

Ex: $\begin{pmatrix}1 & a\\ 0 & 1\end{pmatrix} : z \mapsto z+a$ translation, $\begin{pmatrix}a & 0\\ 0 & b\end{pmatrix} : z \mapsto \frac{a}{b} z$ complex homothety

$t \in \mathbb{R}^*,\ t>0$: $\begin{pmatrix}t & 0\\ 0 & 1\end{pmatrix} : z \mapsto tz$ homothety with factor t } centre $=0$

$\lambda \in \mathbb{C},\ |\lambda| = 1$: $\begin{pmatrix}\lambda & 0\\ 0 & 1\end{pmatrix} : z \mapsto \lambda z$ rotation }

$\left\{\text{fixed points of } \begin{pmatrix}1 & a\\ 0 & 1\end{pmatrix}\right\} = \{\infty\}$ $\quad (a \neq 0)$

$\left\{\quad \text{—— '' ——} \quad \begin{pmatrix}a & 0\\ 0 & b\end{pmatrix}\right\} = \{0,\infty\}$ $\quad (a \neq b)$

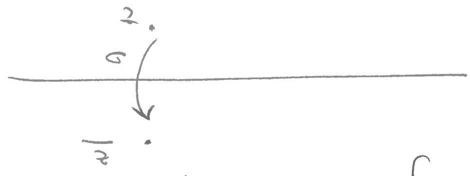
Question: describe $z \mapsto \frac{1}{z} = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}(z)$ geometrically

$\boxed{\text{Complex conjugation } \sigma(z) = \overline{z}}$:

$$\overline{\left(\frac{az+b}{cz+d}\right)} = \frac{\overline{a}\,\overline{z}+\overline{b}}{\overline{c}\,\overline{z}+\overline{d}} \iff \sigma(g(z)) = \overline{g}(\sigma(z)) = ({}^{\sigma}g)(\sigma(z))$$

Action of $PGL_2(\mathbb{C}) \rtimes \underbrace{\{1,\sigma\}}_{\langle\sigma\rangle}$ on $\mathbb{P}^1(\mathbb{C}) = \mathbb{C}\cup\{\infty\}$

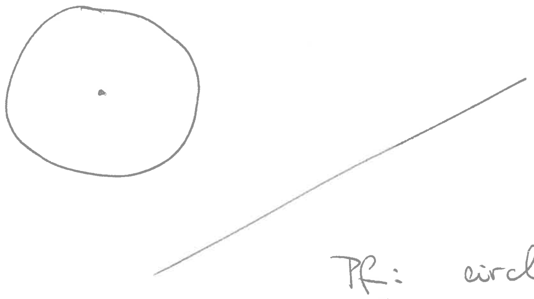
Geometry:



$\mathbb{P}^1(\mathbb{R}) = \mathbb{R}\cup\{\infty\}$

σ = symmetry w.r.t. $\mathbb{R}$

Def: circle in $\mathbb{P}^1(\mathbb{C})$ $:= \begin{cases} \text{Euclidean circle } \{|z-a| = r\} \\ \quad\text{or} \\ \{\infty\} \cup \text{ affine line} \end{cases}$



$\boxed{\text{Prop: } C \text{ circle},\ g \in PGL_2(\mathbb{C}) \Rightarrow \sigma(C),\ g(C) \text{ are circles}}$

Pf: circles are given by equations

$$A z\overline{z} + \overline{B} z + B\overline{z} + C = 0 \qquad (A, C \in \mathbb{R}),\ B \in \mathbb{C}$$

substitute $z = \frac{aw+b}{cw+d}$

Cor: $\{\text{circles } C \text{ in } \mathbb{P}^1(\mathbb{C})\} = \{g(\mathbb{P}^1(\mathbb{R})) \mid g \in PGL_2(\mathbb{C})\}$

Pf: map 3 pts of C to $1, 0, \infty \in \mathbb{P}^1(\mathbb{R})$ by $h \Rightarrow g = h^{-1}$

Cor: $z_1, \dots, z_4 \in \mathbb{P}^1(\mathbb{C})$ distinct lie on a circle $\iff r(z_1,\dots,z_4) \in \mathbb{R}$.

Rmk: above, we have used: $\forall z_1, z_2, z_3$ distinct $\exists$ unique circle C containing z_1, z_2, z_3.

Rmk: $z \mapsto g(z)$ is holomorphic $\Rightarrow$ it preserves angles between curves
($z \mapsto \overline{z}$ changes the sign of angles)

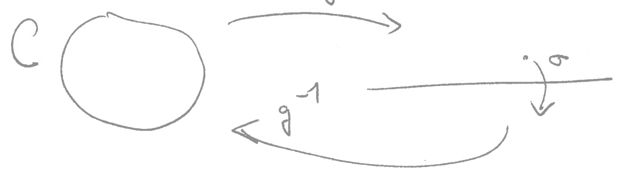
Symmetry s_C with respect to a circle C:

(a) $C = \mathbb{R}\cup\{\infty\}$:

$s_C(z) = \bar z = \sigma(z)$

$\forall\ a_1,a_2,a_3 \in \mathbb{R}\cup\{\infty\}$ distinct $\quad \overline{r(a_1,a_2,a_3,z)} = r(a_1,a_2,a_3,\bar z)$

(b) C any circle: $\exists\ g\in PGL_2(\mathbb{C})$ $\quad g(C) = \mathbb{R}\cup\{\infty\}$



Def: $\boxed{s_C := g^{-1}\circ\sigma\circ g}$

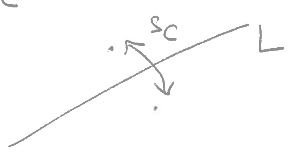
$s_C(z) = g^{-1}(\overline{g(z)}) = (g^{-1}\bar g)(\bar z)$

this is well-defined: if $h(C) = \mathbb{R}\cup\{\infty\}$, then $gh^{-1}(\mathbb{R}\cup\{\infty\}) = \mathbb{R}\cup\{\infty\}$

$\Longrightarrow gh^{-1} \in PGL_2(\mathbb{R}) \Longrightarrow \overline{gh^{-1}} = gh^{-1} \Longrightarrow g^{-1}\bar g = h^{-1}\bar h$.

Formulas: (1) $\boxed{\forall\ z_1,z_2,z_3\in C \text{ distinct} \quad \overline{r(z,z_1,z_2,z_3)} = r(s_C(z),z_1,z_2,z_3)}$

(2) If $C = \{\infty\}\cup(\text{line } L) \Longrightarrow s_C$ usual symmetry



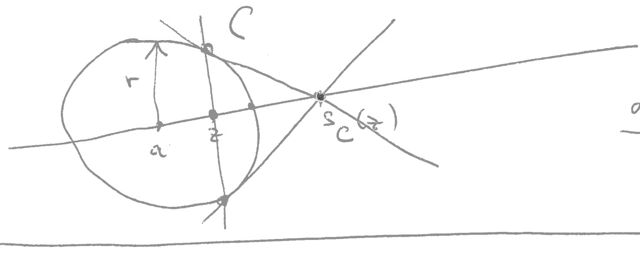
(3) $C = \{\text{fixed points of } s_C\}$

(4) If $C =$ usual circle $\{|z-a| = r\}$ and $z_1,z_2,z_3\in C$, then

$r(s_C(z),z_1,z_2,z_3) = \overline{r(z,z_1,z_2,z_3)} = \overline{r(z-a, z_1-r, z_2-r, z_3-r)} = r\left(\overline{z-a}, \frac{r^2}{z_1-a}, \frac{r^2}{z_2-a}, \frac{r^2}{z_3-a}\right)$

$= r\left(\frac{r^2}{\overline{z-a}}, z_1-a, z_2-a, z_3-a\right) = r\left(\frac{r^2}{\overline{z-a}}+a, z_1,z_2,z_3\right)$

$\Longrightarrow \boxed{(s_C(z)-a)\overline{(z-a)} = r^2} \qquad s_C(a) = \infty,\quad s_C(\infty) = a$



$\text{dist}(a,z)\ \text{dist}(a, s_C(z)) = r^2$

$s_C(z) \in$ half line $\overrightarrow{az}$

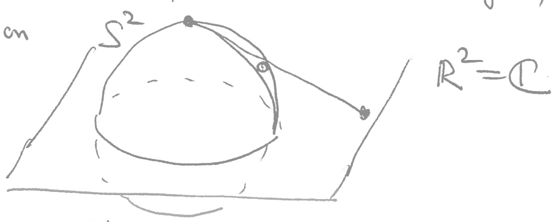
(5) $C =$ unit circle $= \{|z| = 1\} \Longrightarrow s_C(z) = \frac{1}{\bar z}$

Exercise: (i) $\{s_C \mid C \text{ circle}\}$ generate $PGL_2(\mathbb{C}) \rtimes \langle\sigma\rangle$.

(ii) $\forall$ circle C $\forall\ g\in PGL_2(\mathbb{C})$ $\forall z\in\mathbb{P}^1(\mathbb{C})$ $\quad g(s_C(z)) = s_{g(C)}(g(z))$

(iii) Under the stereographic projection unit sphere $S^2 \xrightarrow{\sim} \mathbb{R}^2\cup\{\infty\} = \mathbb{C}\cup\{\infty\}$,

circles $\longleftrightarrow$ circles



(iv) Which subgroup of $PGL_2(\mathbb{C})\rtimes\langle\sigma\rangle$ corresponds to $\{A\in GL_3(\mathbb{R}) \mid A(S^2) = S^2\} = O(3)$?

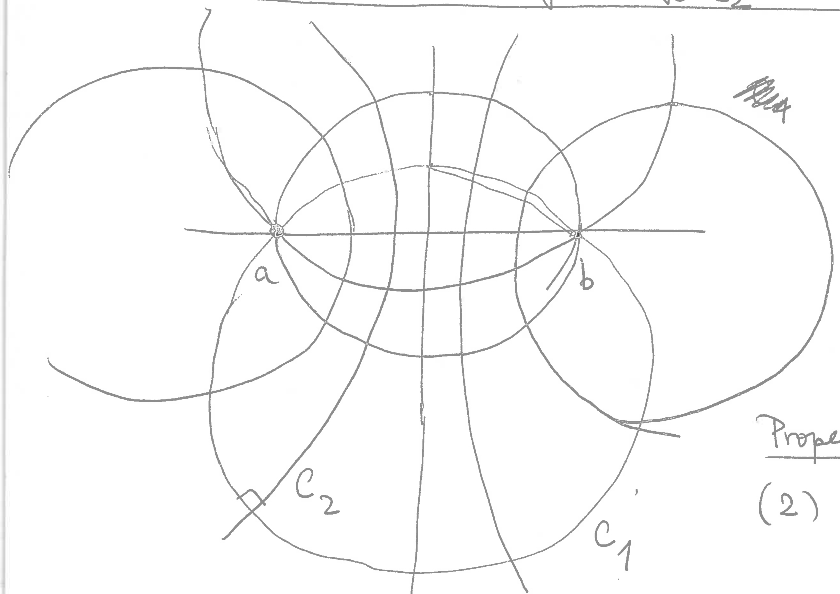
Apollonius circles

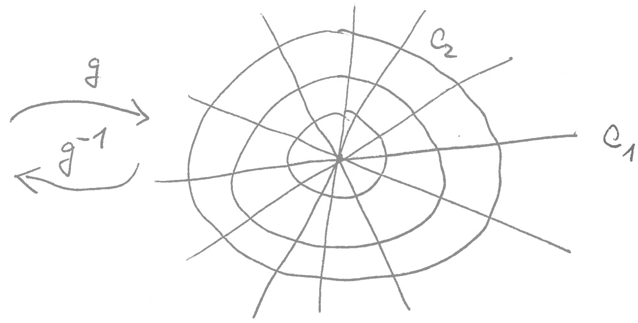
Given $a, b \in \mathbb{C}$ distinct ; fix $g \in GL_2(\mathbb{C})$ $\quad g : \begin{array}{l} a \longmapsto 0 \\ b \longmapsto \infty \end{array}$, $g(z) = k \dfrac{z-a}{z-b}$

$$g : \{\text{circles } C_1 \text{ containing } a, b\} \xrightarrow{\sim} \{\text{lines containing } 0 \ (\cup\{\infty\})\}$$

$$\{\text{— " —} \ C_2 : \left|\frac{z-a}{z-b}\right| = \frac{r}{|k|}\} \xrightarrow{\sim} \{\text{circles with centre } 0 \text{ (radius } r)\}$$

each C_1 is orthogonal to C_2 $\Longleftarrow$ orthogonal





Properties : (1) $C_1 \perp C_2$

(2) $\forall z \in \mathbb{C} \setminus \{a, b\}$ $\exists$ unique $C_1 \ni z$
$\exists$ unique $C_2 \ni z$

(3) $s_{C_1}(C_2) = C_2$, $s_{C_1}(C_1') = C_1''$
$s_{C_2}(C_1) = C_1$, $s_{C_2}(C_2') = C_2''$

(4) $\left(s_C(a) = b \iff C = C_2\right)$
C circle

$$\boxed{\text{Exercise: } C, C' \text{ circles} \quad s_C(C') = C' \iff C \perp C'}$$

Special case : a, b above $=$ the fixed points of $h \in PGL_2(\mathbb{C})$

$$\frac{h(z)-a}{h(z)-b} = \rho \, \frac{z-a}{z-b} \qquad (\rho \neq 1)$$

($\Rightarrow h$ has distinct eig. values)

Properties : (1) $h(C_1) = C_1'$, $h(C_2) = C_2'$

(2) angle $(C_1, h(C_1)) = \arg(\rho)$, $\quad \dfrac{|(z-a)/(z-b)| \text{ on } h(C_2)}{\text{— " — on } C_2} = |\rho|$

Special cases (h is Hyperbolic): $\rho > 0 \iff h(C_1) = C_1$ (orientation preserving)

$\underbrace{\text{as } \rho \text{ varies from } 0 \text{ to } +\infty}_{h \text{ is replaced by } h^\sigma, \ \sigma \in \mathbb{R}}$, $z \in \mathbb{C} \setminus \{a, b\}$ will flow from a to b along C_1

(h is Elliptic): $|\rho| = 1 \iff h(C_2) = C_2$

$\underbrace{\text{as } \rho \text{ varies in } \{|\rho| = 1\}}_{h \text{ is replaced by } h^\sigma, \ \sigma \in \mathbb{R}}$, $z \in \mathbb{C} \setminus \{a, b\}$ will flow around C_2

h is Loxodromic : $\rho \notin \mathbb{R}$, $|\rho| \neq 1$

Exercise: Let $h \in GL_2(\mathbb{C})$, $h \neq \lambda \cdot I$. Show that:

(a) $\exists$ circle C $h(C) = C \iff \exists g \in GL_2(\mathbb{C})$ $g h g^{-1} \in GL_2(\mathbb{R})$

(b) $\left\{ \begin{array}{l} \text{— " —} \\ \text{and } h \text{ preserves each connected} \\ \text{component of } \mathbb{P}^1(\mathbb{C}) \setminus C \end{array} \right\} \iff$ — " — $\in \underbrace{GL_2(\mathbb{R})^+}_{\det > 0}$

$\iff h \neq$ loxodromic

(c) Describe the trajectories $\sigma \mapsto h^\sigma(z)$ $(\sigma \in \mathbb{R})$ for $h \in SL_2(\mathbb{C})$ loxodromic

---

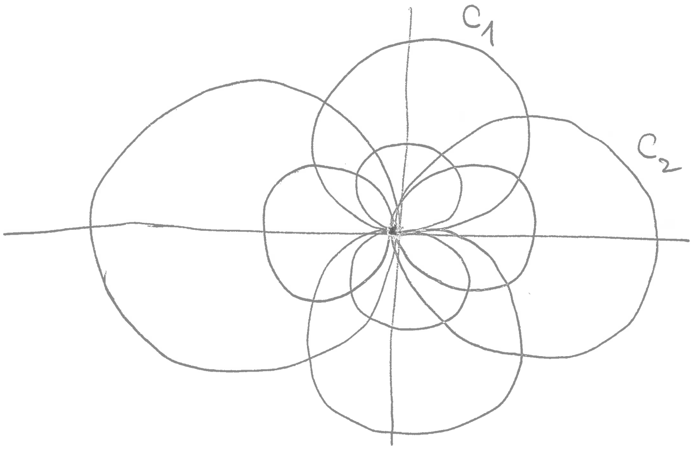
Degenerate situation $a = b$ (**h Parabolic**) **$h(a) = a$ unique fixed point**

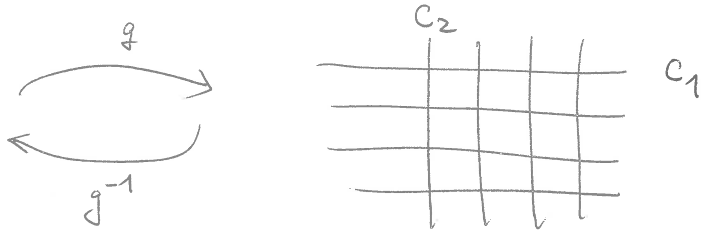
$w = g(z) = \dfrac{k}{z-a} + k'$ $\qquad g : \{\text{circles } C \text{ through } a\} \xrightarrow{\sim} \{\text{lines } (\cup\{\infty\})\}$

$g : a \mapsto \infty$ $\qquad$ tangent circles $\longleftrightarrow$ parallel lines

$C_1 \longleftrightarrow \{\mathrm{Im}(w) = \text{const.}\}$

$C_2 \longleftrightarrow \{\mathrm{Re}(w) = \text{const.}\}$





z $\qquad$ w

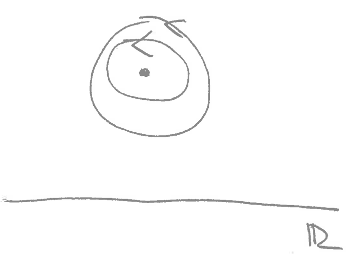
$\mathbb{C}$
$\downarrow$

$g h g^{-1} = t \cdot \begin{pmatrix} 1 & t' \\ 0 & 1 \end{pmatrix}$ $\quad t, t' \neq 0$

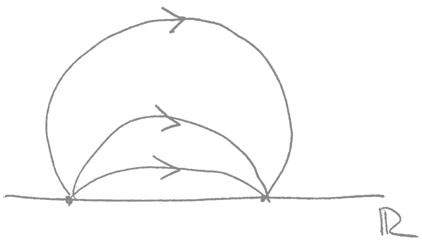
(choose k, k' such that $t' \in \mathbb{R}$)

$\Rightarrow$ $\sigma \mapsto h^\sigma$ $(\sigma \in \mathbb{R})$ will correspond to flow along C_1 $\qquad$ $w \mapsto w + t'$ / $w \mapsto w + \sigma t'$

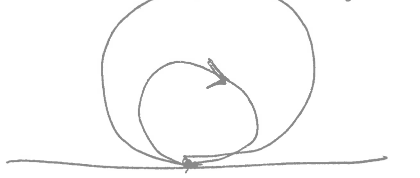
Summary : If $h \in SL_2(\mathbb{R})$, $h \neq \pm I$: flow along $\sigma \mapsto h^\sigma$



h elliptic



h hyperbolic



h parabolic

Cayley transform

$g = \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$, $\quad g(\tau) = \frac{\tau - i}{\tau + i} = w$

$g(\mathbb{P}^1(\mathbb{R})) = \{|w| = 1\}$

$g(\mathcal{H}) = \{|w| < 1\} = D$ unit disc

$g(i) = 0$

$SL_2(\mathbb{R})$ preserves $\mathcal{H}$ $\Rightarrow$ $g\, SL_2(\mathbb{R}) g^{-1}$ preserves D.

<u>Questions</u>: (a) Describe $g\, SL_2(\mathbb{R}) g^{-1}$.

(b) Describe all holomorphic automorphisms of D.

(c) ——— " ——— $\mathcal{H}$.

<u>Hint for (b)</u>: use <u>Schwartz's lemma</u>: If $f : D \to D$ is holomorphic and $f(0) = 0$, then: (1) $\forall z \in D \quad |f(z)| \leq |z|$.

(2) If $\exists z_0 \in D \quad |f(z_0)| = |z_0|$, then $\exists u \in \mathbb{C}, |u| = 1$ $f(z) = uz$.

(<u>Pf</u>: maximum principle for $f(z)/z$)

(d) For $a, b \in D$, give an explicit element $h \in g\, SL_2(\mathbb{R}) g^{-1}$ such that $h(a) = b$.

<u>Rmk</u>. For $\tau = x + iy \in \mathcal{H}$, $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : z \mapsto z + x$, $\begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix} : z \mapsto yz$

$\Rightarrow \underbrace{\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix}}_{g_\tau} \cancel{\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}} : i \longmapsto x + iy = \tau.$

<u>Cor</u>: $G = SL_2(\mathbb{R})$ acts transitively on $\mathcal{H}$.

<u>Stabiliser of i</u>: $G_i = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R}) \;\middle|\; \frac{ai + b}{ci + d} = i \right\} = \left\{ \begin{pmatrix} a & -c \\ c & a \end{pmatrix} \;\middle|\; a^2 + c^2 = 1 \right\} = SO(2)$

<u>Cor</u> $SL_2(\mathbb{R}) / SO(2) \xrightarrow{\sim} \mathcal{H}$ bijection

$h\, SO(2) \longmapsto h(i)$

<u>Question</u>: Do the same for D: write it as G'/K'.

(!) <u>Question</u>: What are possible higher-dimensional analogues of $\mathcal{H}$ and D (and of $G = SL_2(\mathbb{R})$ and G')?