Growth of coefficients of cusp forms

Prop. Let $f(\tau) = \sum_{n=1}^{\infty} a_n q^n \in S_k(SL_2(\mathbb{Z}))$ $\quad (q = e^{2\pi i \tau})$.

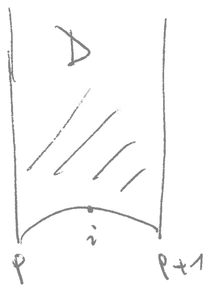
(1) the function $g(\tau) := \mathrm{Im}(\tau)^{k/2} |f(\tau)|$ is $SL_2(\mathbb{Z})$-invariant.

(2) $\exists\, C > 0$ $\quad \forall \tau = x+iy \in \mathcal{H}$ $\quad |f(\tau)| \leq \dfrac{C}{y^{k/2}}$

(3) $\exists\, c > 0$ $\quad |a_n| \leq c n^{k/2}$.

Pf. (1) $\forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ $\quad \underbrace{\mathrm{Im}\left(\frac{a\tau+b}{c\tau+d}\right)^{k/2}}_{\frac{\mathrm{Im}(\tau)^{k/2}}{|c\tau+d|^k}} \left|f\left(\frac{a\tau+b}{c\tau+d}\right)\right| = \mathrm{Im}(\tau)^{k/2} |f(\tau)|$

(2) Let $\mathcal{D}$ be the standard fundamental domain of $SL_2(\mathbb{Z})$.



Fix $R > 0$ such that

$$|f(\tau)| \leq 2|a_k q^k| = 2|a_k e^{-2\pi k y}| \qquad (\tau = x+iy)$$

if $y = \mathrm{Im}(\tau) \geq R$

$(k = \min\{n \geq 1 \mid a_n \neq 0\})$.

then: (a) g is bounded in $\underbrace{\overline{\mathcal{D}} \cap \{\mathrm{Im}(\tau) \leq R\}}_{\text{compact}}$

(b) $|f(\tau)|$ is bounded in $\{\mathrm{Im}(\tau) \geq R\}$ by

$C_1 e^{-2\pi y} \Rightarrow g$ is bounded there

$\overset{(1)}{\Longrightarrow}$ g is bounded on $\mathcal{H}$.

(3) $$f(x+iy) = \sum_{n=1}^{\infty} a_n e^{2\pi i n x} e^{-2\pi n y}$$

$$\int_0^1 f(x+iy)\, e^{-2\pi i n x}\, dx = a_n e^{-2\pi n y}$$

Take $y = \frac{1}{n}$: $\quad |RHS| = |a_n| e^{-2\pi}$, $\quad |LHS| \leq \dfrac{C}{y^{k/2}} = C n^{k/2}$.

Rmks : (a) $G_k(\tau) = a_k + b_k \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n$, $\sigma_{k-1}(n) \geq n^{k-1}$

(b) Prop. above (and its proof) is valid for all $\mathcal{C}^\infty$ functions $f : \mathcal{H} \longrightarrow \mathbb{C}$ satisfying $\forall \alpha \in SL_2(\mathbb{Z}) \quad f|_k \alpha = f$ and

$\forall l > 0 \quad \lim_{y \to +\infty} y^l f(x+iy) = 0$ (rapidly decreasing f).

(c) the same argument works if $SL_2(\mathbb{Z})$ is replaced by any discrete subgroup $\Gamma \subset SL_2(\mathbb{R})$ for which X_Γ is compact, if one considers Fourier coefficients at <u>every cusp</u> x of Γ: $\exists \sigma \in SL_2(\mathbb{R}) \quad \sigma(x) = \infty$,

$\pm \sigma \Gamma_x \sigma^{-1} = \pm \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}^{\mathbb{Z}}$, $(f|_k \sigma^{-1})(\tau) = \sum_n a_n q_h^n$

$q_h = e^{2\pi i \tau / h}$,

(d) For $f \in S_k(\Gamma, \chi)$, $SL_2(\mathbb{Z}) \supset \Gamma \supset \Gamma(N)$ for some N, Deligne's proof of Ramanujan's conjecture implies that $|a_n| \leq C_\varepsilon n^{\frac{k-1}{2} + \varepsilon}$, for any $\varepsilon > 0$.

Mellin transform

$f \longmapsto Mf \qquad (Mf)(s) = \int_0^\infty f(y)\, y^s \, \frac{dy}{y}$

Assumptions: (a) $f : (0,\infty) \longrightarrow \mathbb{C}$ continuous

(b) $\exists\ \sigma_1 < \sigma_2 \quad \exists\ c_1, c_2 > 0$

(b1) $|f(y)| \le \frac{c_1}{y^{\sigma_1}}$ for $y \longrightarrow 0+$

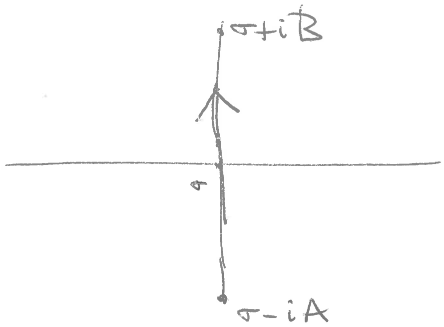
(b2) $|f(y)| \le \frac{c_2}{y^{\sigma_2}}$ for $y \longrightarrow +\infty$

Prop. (1) $\boxed{(Mf)(s) := \int_0^\infty f(y)\, y^s \, \frac{dy}{y}}$

is absolutely convergent for $\sigma_1 < \mathrm{Re}(s) < \sigma_2$.

(2) For $\sigma_1 < \sigma < \sigma_2$ and $y > 0$,

$$\underbrace{\boxed{f(y) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} (Mf)(s)\, y^{-s}\, ds}}_{\lim\limits_{A,B \to +\infty} \int_{\sigma - iA}^{\sigma + iB}}$$



Pf: (1) $\mathrm{Re}(s) > \sigma_1 \Longrightarrow \int_0^1$ absolutely convergent

$\mathrm{Re}(s) < \sigma_2 \Longrightarrow \int_1^\infty$ (— " —).

(2) $g_\sigma(x) := f(e^x)\, e^{\sigma x} \quad (x \in \mathbb{R})$ has exponential decay at $\pm\infty$,

$$(Mf)(\underbrace{\sigma + it}_{s}) = \int_{\mathbb{R}} \underbrace{f(e^x)\, e^{(\sigma+it)x}}_{g_\sigma(x)\, e^{itx}} \, dx = (\mathcal{F} g_\sigma)\left(-\frac{t}{2\pi}\right)$$

Fourier inversion: $\frac{1}{2\pi} \int_{-\infty}^{\infty} (Mf)(\sigma + it)\, e^{-itx}\, dt = g_\sigma(x)$

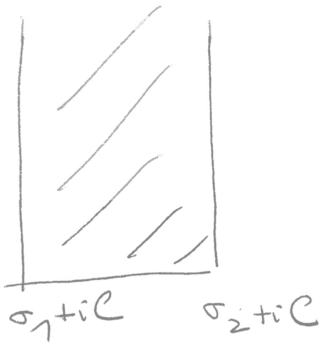
$$\Longrightarrow \quad f(e^x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} (Mf)(\sigma + it)\, e^{-(\sigma+it)x}\, dt$$

$$f(y) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} (Mf)(s)\, y^{-s}\, ds \qquad \square$$

Ex: $f(y) = e^{-ay}$ $(a>0)$, $\sigma_1 = 0$, $\sigma_2 = +\infty$

$$(Mf)(s) = \int_0^\infty e^{-ay} y^s \frac{dy}{y} = \frac{1}{a^s} \int_0^\infty e^{-y} y^s \frac{dy}{y} = \frac{\Gamma(s)}{a^s} \quad (\mathrm{Re}(s) > 0)$$

Phragmen – Lindelöf Principle:



Assume: (a) $F(s)$ holomorphic in the region $\sigma_1 \le \mathrm{Re}(s) \le \sigma_2$, $\mathrm{Im}(s) \ge C$

(b) $F(\sigma + it) = O(e^{(t^\beta)})$ for some $\beta > 0$ if $\sigma_1 \le \sigma \le \sigma_2$, $t \longrightarrow +\infty$

(c) $F(\sigma + it) = O(t^M)$ if $\sigma = \sigma_1, \sigma_2$, $t \longrightarrow +\infty$.

Then $F(\sigma + it) = O(t^M)$ uniformly for $\sigma \in [\sigma_1, \sigma_2]$, $t \longrightarrow +\infty$.

Functional equation of L-series

Notation: $\lambda > 0$

$$f(\tau) = \sum_{n=0}^{\infty} a_n e^{2\pi i n\tau/\lambda}, \quad g(\tau) = \sum_{n=0}^{\infty} b_n e^{2\pi i n \tau/\lambda}, \quad (\tau \in \mathcal{H})$$

$a_n, b_n = O(n^C)$ as $n \to +\infty$

$$L(f,s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}, \quad L(g,s) = \sum_{n=1}^{\infty} \frac{b_n}{n^s} \quad \left(\text{abs. convergent if } \mathrm{Re}(s) > C+1\right).$$

$$\Lambda(f,s) = \left(\frac{2\pi}{\lambda}\right)^{-s} \Gamma(s) L(f,s), \quad \Lambda(g,s) = \left(\frac{2\pi}{\lambda}\right)^{-s}\Gamma(s) L(g,s).$$

Thm (Hecke) Let $k > 0$. The following statements are equivalent.

(1) $\forall \tau \in \mathcal{H}$ $\quad \boxed{f\left(-\frac{1}{\tau}\right) = \underbrace{\left(\frac{\tau}{i}\right)^k}_{\text{branch equal to } 1 \text{ at } \tau = i} g(\tau)}$

(2) $\Lambda(f,s) + \frac{a_0}{s} + \frac{b_0}{k-s}$ has holomorphic continuation to $\mathbb{C}$, it is bounded in every vertical strip $\{\sigma_1 < \mathrm{Re}(s) < \sigma_2\}$, and

$$\Lambda(f,s) = \Lambda(g, k-s).$$

Pf. As $\displaystyle\int_0^\infty e^{-ay} y^s \frac{dy}{y} = \frac{\Gamma(s)}{a^s}$ $\quad (a, \mathrm{Re}(s) > 0)$,

$$\Lambda(f,s) = \sum_{n=1}^{\infty} a_n \left(\frac{2\pi n}{\lambda}\right)^{-s} \Gamma(s) = \sum_{n=1}^{\infty} a_n \int_0^\infty e^{-\frac{2\pi n y}{\lambda}} y^s \frac{dy}{y} =$$

$$= \int_0^\infty \underbrace{\left(\sum_{n=1}^{\infty} a_n e^{-\frac{2\pi n y}{\lambda}}\right)}_{f(iy) - a_0 \text{ exponential decay for } y \to +\infty} y^s \frac{dy}{y}$$

$\Downarrow$

$\displaystyle\int_1^\infty (f(iy) - a_0) y^s \frac{dy}{y}$ has analytic continuation to $\mathbb{C}$ and is bounded in vertical strips.

$$\int_0^1 (f(iy)-a_0)\, y^s \frac{dy}{y} = -a_0 \int_0^1 y^{s-1}\,dy + \int_0^1 f(iy)\, y^s \frac{dy}{y} \overset{y=\frac{1}{y}}{=\!=}$$

$$= -\frac{a_0}{s} + \int_1^\infty f\left(\frac{i}{y}\right) y^{-s} \frac{dy}{y}$$

Pf of (1) $\Longrightarrow$ (2):

$$\int_0^1 (f(iy)-a_0)\, y^s \frac{dy}{y} = -\frac{a_0}{s} + \int_1^\infty (g(iy)-b_0+b_0)\, y^{k-s} \frac{dy}{y} \qquad (\mathrm{Re}(s)<k)$$

$$= -\frac{a_0}{s} - \frac{b_0}{k-s} + \underbrace{\int_1^\infty (g(iy)-b_0)\, y^{k-s}\frac{dy}{y}}_{\text{has analytic continuation to } \mathbb{C},\ \text{bounded in vertical strips}}$$

$$\Longrightarrow \Lambda(f,s) + \frac{a_0}{s} + \frac{b_0}{k-s} = \int_1^\infty \left[y^s(f(iy)-a_0) + y^{k-s}(g(iy)-b_0)\right] \frac{dy}{y}$$

$$= \Lambda(g,k-s) + \frac{b_0}{k-s} + \frac{a_0}{s}$$ has analytic continuation to $\mathbb{C}$, bounded in vertical strips.

Pf of (2) $\Longrightarrow$ (1): $L(f,s)$, $L(g,s)$ abs. convergent for $\mathrm{Re}(s)>C$

Stirling formula: $|\Gamma(\sigma+it)| \approx \sqrt{2\pi}\, |t|^{\sigma-\frac{1}{2}}\, e^{-\pi|t|/2}$ for $t\to\pm\infty$ ($\sigma=$ const.)

$\Longrightarrow$ $\Lambda(f,\sigma+it)$, $\Lambda(g,\sigma+it)$ have exponential decay as $t\to\pm\infty$ (if $\sigma >$ const.) $\xrightarrow[\text{Lindelöf}]{\text{Phragmen}}$ $\Lambda(f,\sigma+it)$, $\Lambda(g,\sigma+it) \longrightarrow 0$ uniformly for $\sigma\in[\sigma_1,\sigma_2]$, $t\longrightarrow\pm\infty$.

$\Downarrow$ (2)

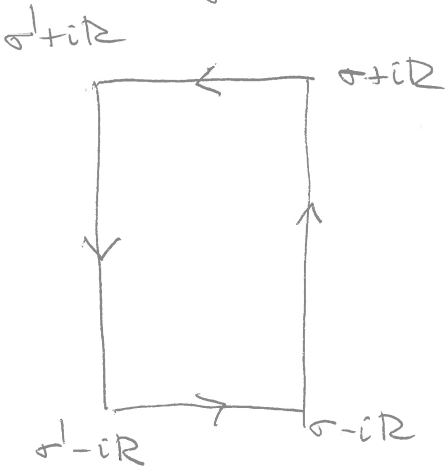
(also if $\sigma <$ const.!)

Mellin transform: $\Lambda(f,s) = M(f(iy)-a_0)$

$\Lambda(g,s) = M(g(iy)-b_0)$

$$\Longrightarrow f(iy) - a_0 = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \Lambda(f,s)\, y^{-s}\, ds \quad , \sigma > \text{const.}$$

Moving the line of integration:



Integrals of $\Lambda(f,s)\,y^{-s}\,ds$ along the horizontal paths $\longrightarrow 0$ as $R\to+\infty$

$$f(iy)-a_0=\frac{1}{2\pi i}\int_{\sigma'-i\infty}^{\sigma'+i\infty}\Lambda(f,s)\,y^{-s}\,ds+\underbrace{\sum_{\sigma'<\mathrm{Re}(s_0)<\sigma}\mathrm{Res}_{s=s_0}\,y^{-s}\Lambda(f,s)}_{-a_0+y^{-k}b_0}$$

Take $\sigma\gg 0$, $\sigma'\ll 0$:

$$f(iy)=\frac{1}{2\pi i}\int_{\sigma'-i\infty}^{\sigma'+i\infty}y^{-s}\Lambda(f,s)\,ds+y^{-k}b_0 \qquad (\sigma'\ll 0)$$

$$\overset{s\leftrightarrow k-s}{=}\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}y^{s-k}\Lambda(g,s)\,ds+y^{-k}b_0 \qquad (\sigma\gg 0)$$

$$=y^{-k}g\left(\frac{i}{y}\right)\quad\Longrightarrow\quad (1)\text{ for }\tau=iy\ (y>0).$$

By analytic continuation, (1) holds for all $\tau\in\mathcal{H}$.

Ex: (1) $f=g=\frac{1}{2}\theta(\tau)=\frac{1}{2}+\sum_{n=1}^{\infty}e^{\pi i n^2\tau}$, $\lambda=2$, $k=\frac{1}{2}$

$L(f,s)=L(g,s)=\zeta(2s)$ $\qquad a_0=b_0=\frac{1}{2}$

thm $\Rightarrow$ $\Lambda(s)=\pi^{-s}\Gamma(s)\zeta(2s)$ is holomorphic in $\mathbb{C}\setminus\{0,\frac{1}{2}\}$,

has simple poles at $s=0,\frac{1}{2}$ with residues $-\frac{1}{2},\frac{1}{2}$,

$\Lambda(s)=\Lambda(\frac{1}{2}-s)$

($\Leftrightarrow$ $\pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s)$ is invariant under $s\leftrightarrow 1-s$).

$\Longrightarrow$ $\forall k\in\mathbb{Z}_{>0}$ $\quad\zeta(-2k)=0$

$$\zeta(1-2k)=-\frac{B_{2k}}{2k}.$$

(2) $f(\tau) = \sum_{n=1}^{\infty} a_n q^n$ with $a_n = O(n^c)$ $(q = e^{2\pi i \tau})$

lies in $S_k(SL_2(\mathbb{Z}))$ $(k \geq 2$ even$)$

$\Updownarrow$

$\Lambda(f,s) := (2\pi)^{-s}\Gamma(s) L(f,s) = (2\pi)^{-s}\Gamma(s) \sum_{n=1}^{\infty} \frac{a_n}{n^s}$

has holomorphic continuation to $\mathbb{C}$, is bounded in every vertical strip, and $\Lambda(f,s) = (-1)^{k/2}\Lambda(f,k-s)$

(take $\lambda = 1$, $g = (-1)^{k/2} f = i^k f$)

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(3) $\underline{k \geq 2}$: $f = \widetilde{G}_{2k} := \underbrace{-\frac{B_{2k}}{4k}} + \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n = c_{2k} E_{2k}$

$(\lambda = 1)$ $g = (-1)^k f$ $(\frac{1}{2}\zeta(1-2k)) = c_{2k}$

$L(f,s) = \zeta(s)\zeta(s-2k+1)$, $\Lambda(f,s) = (2\pi)^{-s}\Gamma(s)L(f,s)$

$\Lambda(f, 2k-s) = (-1)^k \Lambda(f,s)$

$\Lambda(f,s) + c_{2k}\left(\frac{1}{s} + \frac{(-1)^k}{2k-s}\right)$ has holomorphic continuation to $\mathbb{C}$

As $s \to 0$, $(2\pi)^{-s}\Gamma(s) = \frac{1}{s}$

$\Rightarrow -c_{2k} = L(f,0) = \underbrace{\zeta(0)}_{-\frac{1}{2}}\zeta(1-2k)$

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(4) $f = -\frac{1}{24} E_2(\tau) = -\frac{1}{24} + \sum_{n \geq 1} \sigma_1(n) q^n$

$L(f,s) = \zeta(s)\zeta(s-1)$, $\Lambda(f,s) = (2\pi)^{-s}\Gamma(s)\zeta(s)\zeta(s-1)$

$2^{2s-1}\Gamma(s)\Gamma(s+\frac{1}{2}) = \sqrt{\pi}\,\Gamma(2s)$ + funct. equation for $\zeta(s)$

$\Rightarrow$ $\underline{\Lambda(f, 2-s) = -\Lambda(f,s)}$

We are going to apply the inverse Mellin transform to both sides of the above functional equation.

$\Lambda(f,s)$ has simple poles at $s = 0, 1, 2$

$$\operatorname{Res}_{s=2} \hat{\Lambda}(f,s) = (2\pi)^{-2}\Gamma(2)\zeta(2) = \frac{\zeta(2)}{4\pi^2} = \frac{1}{24} = \operatorname{Res}_{s=0} \Lambda(f,s)$$

$$\operatorname{Res}_{s=1} \Lambda(f,s) = (2\pi)^{-1}\Gamma(1)\underbrace{\zeta(0)}_{-1/2} = -\frac{1}{4\pi}$$

<u>So</u>: $$f(iy) = -\frac{1}{24} + \frac{1}{2\pi i}\int_{\mathrm{Re}(s)=\sigma} y^{-s}\Lambda(f,s)\,ds = \qquad (\sigma>2)$$

$$= -\frac{1}{24} + \frac{1}{24}y^{-0} - \frac{1}{4\pi}y^{-1} + \frac{1}{24}y^{-2} + \frac{1}{2\pi i}\int_{\mathrm{Re}(s')=\sigma'} y^{-s}\Lambda(f,s)\,ds \qquad (\sigma'<0)$$

$$\overset{s\leftrightarrow 2-s}{=} -\frac{1}{4\pi y} + \frac{1}{24y^2} - \frac{1}{2\pi i}\int_{\mathrm{Re}(s)=2-\sigma'>2} y^{s-2}\Lambda(f,s) = -\frac{1}{4\pi y} - y^{-2}f\left(\frac{i}{y}\right)$$

$$\Longrightarrow \quad f\left(\frac{i}{y}\right) = -y^2 f(iy) - \frac{y}{4\pi} \qquad \Big|\cdot(-24)$$

$$E_2\left(\frac{i}{y}\right) = -y^2 E_2(iy) + \frac{12y}{2\pi}$$

$$E_2\left(-\frac{1}{\tau}\right) = \tau^2 E_2(\tau) + \frac{12\tau}{2\pi i} \qquad \forall \tau\in\mathcal{H}$$

Petersson scalar product

Data: $\Gamma \subset G = SL_2(\mathbb{R})$ discrete subgroup such that X_Γ is compact

Ex: $f, g \in S_2(\Gamma) \iff \omega_f = f(\tau)\,d\tau, \quad \omega_g = g(\tau)\,d\tau \in \Omega^1(X_\Gamma)$.

$$\frac{i}{2}\int_{X_\Gamma} \omega_f \wedge \overline{\omega_g} = \int_{Y_\Gamma} f(\tau)\overline{g(\tau)}\,dx\,dy = \int_{Y_\Gamma} \underbrace{f(\tau)\overline{g(\tau)}\,y^2}_{\Gamma\text{-invariant}} \underbrace{\frac{dx\,dy}{y^2}}_{d\mu} \qquad \tau = x+iy$$

$d\mu$: G-invariant measure

Def: Assume that $k \geq 2$, $f, g \in M_k(\Gamma, \chi)$, at least one of f, g lies in $S_k(\Gamma, \chi)$. Then $(\tau = x+iy)$

$$(f, g) := \mu(Y_\Gamma)^{-1} \int_{Y_\Gamma} \underbrace{f(\tau)\overline{g(\tau)}\,y^k}_{\Gamma\text{-invariant}} \frac{dx\,dy}{y^2} \quad \text{is well-defined}$$

$$= \mu(D_\Gamma)^{-1} \int_{D_\Gamma} \text{— " —}$$

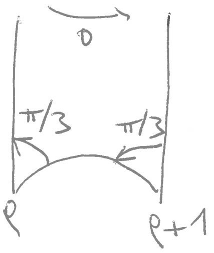
for any fundamental domain D_Γ of Γ.

Rmks: (a) X_Γ compact $\iff \mu(D_\Gamma) < \infty$

(b) If $(\Gamma : \Gamma') < \infty$, then $f, g \in M_k(\Gamma', \chi)$ and (f, g) does not change if we compute it on $Y_{\Gamma'}$.

Rmk on volumes: the Poincaré metric $\dfrac{(dx)^2 + (dy)^2}{y^2}$ has constant Gaussian curvature -1, which implies that the volume ($=$ area) of a geodesic triangle with angles α, β, γ is equal to $\pi - (\alpha+\beta+\gamma)$, by the Gauss–Bonnet formula.

Ex: (1)

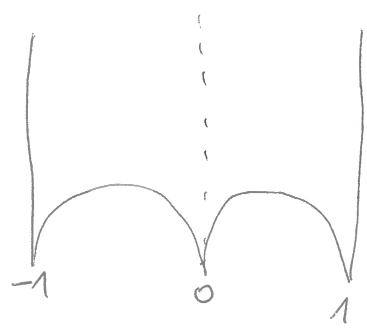


$D_{SL_2(\mathbb{Z})} =$ geodesic triangle with angles $0, \frac{\pi}{3}, \frac{\pi}{3} \implies \text{vol} = \frac{\pi}{3}$

Note: $(\overline{SL_2(\mathbb{Z})} : \overline{\Gamma(2)}) = 6$

$2\pi / \frac{\pi}{3} = 6$

(2)



$D_{\Gamma(2)} =$ union of two geodesic triangles with angles $0, 0, 0$
$\implies \text{vol} = 2 \cdot \pi$

Variant: (f,g) is defined if we merely assume that $f,g \in \mathcal{C}^\infty(\mathcal{H})$ satisfy: (i) $\forall \alpha \in \Gamma$ $\quad f|_k\alpha = \chi(\alpha) f$, $g|_k \alpha = \chi(\alpha) g$

(ii) $\forall$ cusp z of Γ fix $\sigma \in G$ such that $\sigma(z) = \infty$.

We require $(f|_k \sigma^{-1})(\tau)$, $(g|_k\sigma^{-1})(\tau) = O\left(\mathrm{Im}(\tau)^C\right)$ as $\mathrm{Im}(\tau) \to +\infty$ ("f,g are slowly increasing at ∞"),

and at least one among f,g is rapidly decreasing at ∞

$\left(= O\left(\mathrm{Im}(\tau)^{-m}\right) \quad \forall m > 0 \right)$

Examples for $\Gamma = SL_2(\mathbb{Z})$: let $\Gamma_\infty = \left\{ \pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \,\middle|\, n \in \mathbb{Z} \right\}$

(1) **Holomorphic Eisenstein series $E_k(\tau)$** $\quad (2|k,\ k>2)$

$$E_k(\tau) := \sum_{\alpha \in \Gamma_\infty \backslash \Gamma} \mathbb{1}|_k\alpha = \frac{1}{2} \sum_{\substack{c,d \in \mathbb{Z} \\ \gcd(c,d)=1}} \frac{1}{(c\tau+d)^k} = \frac{1}{2\zeta(k)} \underbrace{\sum_{c,d\in\mathbb{Z}}{}' \frac{1}{(c\tau+d)^k}}_{G_k(\tau)}$$

$$\Gamma_\infty\backslash\Gamma \xrightarrow{\sim} \left\{ (c,d) \in (\mathbb{Z}^2)^* \,\middle|\, \gcd(c,d) = 1 \right\} / \{\pm 1\}$$

$$\Gamma_\infty \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto (0\ 1)\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (c,d)$$

Given: $g \in M_\ell(SL_2(\mathbb{Z}))$, $f \in S_{k+\ell}(SL_2(\mathbb{Z}))$

Goal: Compute $(f, gE_k) = ?$

$$v(SL_2(\mathbb{Z})\backslash\mathcal{H})(f, gE_k) = \int_{\Gamma\backslash\mathcal{H}} \sum_{\alpha\in\Gamma_\infty\backslash\Gamma} \underbrace{f \cdot \overline{g} \cdot \overline{\mathbb{1}|_k\alpha}}_{f|_{k+\ell}\alpha\, \overline{(g|_\ell\alpha)\, \mathbb{1}|_k\alpha}} \, y^{k+\ell} d\mu$$

unfolding

$$= \int_{\Gamma_\infty\backslash\mathcal{H}} f\,\overline{g}\, y^{k+\ell-2} \, dx\,dy = \int_0^1\int_0^\infty \sum_{m,n} a_m \overline{b_n}\, e^{2\pi i(m-n)x} \cdot e^{-2\pi(m+n)y} y^{k+\ell-2} \, dx\,dy$$

$$f = \sum_{m=1}^\infty a_m e^{2\pi i m\tau}$$

$$g = \sum_{n=0}^\infty b_n e^{2\pi i n \tau}$$

$$\overset{m=n}{=} \sum_{n=1}^\infty a_n\overline{b_n} \, \frac{\Gamma(k+\ell-1)}{(4\pi n)^{k+\ell-1}}$$

$$= \frac{\Gamma(k+\ell-1)}{(4\pi)^{k+\ell-1}} \underbrace{\sum_{n=1}^\infty \frac{a_n\overline{b_n}}{n^{k+\ell-1}}}_{\text{Rankin-Selberg L-series}}$$

(2) Non-holomorphic Eisenstein series

$y : \mathcal{H} \longrightarrow \mathbb{R}_{>0}$, $\quad y(\tau) = \mathrm{Im}(\tau)$. $\quad$ Fix $s \in \mathbb{C}$, $\mathrm{Re}(s) > 1$.

$$\Big(\sum_{\alpha \in \Gamma_\infty \backslash \Gamma} y^s|_0 \alpha\Big)(\tau) = \frac{1}{2} \sum_{\substack{c,d \in \mathbb{Z} \\ \gcd(c,d)=1}}{}' \frac{\mathrm{Im}(\tau)^s}{|c\tau + d|^{2s}} =: E(s,\tau)$$

$$\boxed{\forall \alpha \in \Gamma = SL_2(\mathbb{Z}) \quad E(s,\alpha(\tau)) = E(s,\tau)}$$

Goal: for $f, g \in M_k(\Gamma)$, at least one of them cuspidal, compute $(f, g\,E(s,\cdot))$. As before,

$$\mathrm{vol}(\Gamma\backslash\mathcal{H})\,(f, g\,E(s,\cdot)) = \int_{\Gamma\backslash\mathcal{H}} \sum_{\alpha \in \Gamma_\infty\backslash\Gamma} f(\tau)\,\overline{g(\tau)}\,\overline{y^s(\alpha(\tau))}\,y^k\,d\mu$$

$$= \int_{\Gamma_\infty\backslash\mathcal{H}} f(\tau)\,\overline{g(\tau)}\,\mathrm{Im}(\tau)^{s+k}\,d\mu =$$

$f = \sum a_n q^n$

$g = \sum b_n q^n$

$$= \int_0^1 \int_0^\infty \sum_{m,n} a_m \overline{b_n}\, e^{2\pi i(m-n)x}\, e^{-2\pi(m+n)y}\, y^{s+k-2}\, dx\,dy$$

$$= \sum_{n=1}^\infty a_n \overline{b_n} \int_0^\infty e^{-4\pi n y}\, y^{s+k-2}\,dy = \frac{\Gamma(s+k-1)}{(4\pi)^{s+k-1}} \sum_{n=1}^\infty \frac{a_n \overline{b_n}}{n^{s+k-1}}$$

Exercise: Deduce from known analytic properties of the function $s \longmapsto E(s,\tau)$ analogous statements for $\displaystyle\sum_{n=1}^\infty \frac{a_n \overline{b_n}}{n^{s+k-1}}$.

Variant: one often omits the normalisation factor $\mu(\Gamma\backslash\mathcal{H})$ and defines

$$(f,g)_\Gamma := \int_{\Gamma\backslash\mathcal{H}} f(\tau)\,\overline{g(\tau)}\,y^k\,\frac{dx\,dy}{y^2} \qquad (\tau = x+iy)$$

Poincaré series

Poincaré: introduced series $\sum_{\alpha\in\Gamma} \varphi|_k\alpha$ for suitable rational functions φ and showed that they converge under appropriate assumptions.

Variant: let $\Gamma_\varphi := \{\alpha\in\Gamma \mid \varphi|_k\alpha = \alpha\}$; consider

$$F := \sum_{\alpha\in\Gamma_\varphi\backslash\Gamma} \varphi|_k\alpha. \quad \text{Formally, } \forall\beta\in\Gamma \;\; F|_k\beta = F.$$

---

this construction includes Eisenstein series:

(1) $\Gamma = SL_2(\mathbb{Z})$, $\varphi = \mathbb{1}$ (constant function) $\Rightarrow \Gamma_\varphi = \Gamma_\infty$ as above, (k>2 even)

$$\sum_{\alpha\in\Gamma_\varphi\backslash\Gamma} \varphi|_k\alpha = \frac{1}{2}\sum_{\substack{c,d\in\mathbb{Z}\\ \gcd(c,d)=1}}{}' \frac{1}{(c\tau+d)^k} = E_k(\tau)$$

---

(2) $\Gamma = SL_2(\mathbb{Z})$, $\varphi = y^s : \tau \longmapsto \mathrm{Im}(\tau)^s$, $\Gamma_\varphi = \Gamma_\infty = \pm\begin{pmatrix}1 & \mathbb{Z}\\ 0 & 1\end{pmatrix}$

~~k=0~~

$$\sum_{\alpha\in\Gamma_\varphi\backslash\Gamma} \varphi|_k\alpha = \frac{1}{2}\sum_{\substack{c,d\in\mathbb{Z}\\ \gcd(c,d)=1}}{}' \frac{1}{(c\tau+d)^k}\,\frac{\mathrm{Im}(\tau)^s}{|c\tau+d|^{2s}}$$

Poincaré series for $\varphi_m(\tau) = e^{2\pi i m\tau}$ (Petersson)

$\Gamma = SL_2(\mathbb{Z})$, $m\geq 1$,

$$P_m^{(k)}(\tau) = P_m(\tau) = \sum_{\alpha\in\Gamma_\infty\backslash\Gamma} (\varphi_m|_k\alpha)(\tau) = \frac{1}{2}\sum_{\Gamma_\infty\left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right)} (c\tau+d)^{-k} e^{2\pi i m\frac{a\tau+b}{c\tau+d}}$$

Unfolding: $\forall f\in S_k(\Gamma)$, $f(\tau) = \sum_{n=1}^{\infty} a_n e^{2\pi i n\tau}$

$$(f,P_m)_\Gamma = \int_{\Gamma_\infty\backslash\mathcal{H}} f(\tau)\overline{e^{2\pi i m\tau}}\, y^{k-2}\,dx\,dy = \int_0^\infty\int_0^1 \sum_{n=1}^{\infty} a_n e^{2\pi i(n-m)x} e^{-2\pi(m+n)y} y^{k-2}\,dx\,dy$$

$$= a_m \int_0^\infty e^{-4\pi m y} y^{k-2}\,dy = \frac{\Gamma(k-1)}{(4\pi m)^{k-1}}\, a_m$$

Summary: $f=\sum_{m=1}^{\infty} a_m q^m \in S_{k+\ell}(\Gamma)$, $g=\sum_{n=0}^{\infty} b_n q^n \in M_\ell(\Gamma)$, $\Gamma=SL_2(\mathbb{Z})$

(1) $$(f, gE_k)_\Gamma = \frac{\Gamma(k+\ell-1)}{(4\pi)^{k+\ell-1}} \sum_{m=1}^{\infty} \frac{a_m \overline{b_m}}{m^{k+\ell-1}}$$

(2) Special case $\ell=0$ of (1): $(f, E_k)_\Gamma = 0$ $(g=1)$

cusp forms are orthogonal to Eisenstein series

(3) If $k=0$: $f\in S_\ell(\Gamma)$, $g\in M_\ell(\Gamma)$

$$(f, g \underbrace{E(s,\cdot)}_{\text{non-holomorphic Eisenstein series}})_\Gamma = \frac{\Gamma(s+\ell-1)}{(4\pi)^{s+\ell-1}} \sum_{m=1}^{\infty} \frac{a_m \overline{b_m}}{m^{s+\ell-1}}$$

(4) If $f\in S_k(\Gamma)$: $$(f, \underbrace{P_m}_{\text{Poincaré series}})_\Gamma = \frac{\Gamma(k-1)}{(4\pi m)^{k-1}} a_m \qquad (m\geq 1)$$

Exercise. What happens if one replaces in (1) E_k by

$$E_{k,s} = \sum_{\alpha\in\Gamma_\infty\backslash\Gamma} y^s|_k \alpha = \frac{1}{2} \sum_{\substack{c,d\in\mathbb{Z} \\ \gcd(c,d)=1}}{}' \frac{1}{(c\tau+d)^k} \frac{\operatorname{Im}(\tau)^s}{|c\tau+d|^{2s}} \quad ?$$

Exercise. Determine the Fourier expansion of the Poincaré series

$$P_m(\tau) = \frac{1}{2} \sum_{\substack{c,d\in\mathbb{Z} \\ \gcd(c,d)=1}}{}' \frac{1}{(c\tau+d)^k} e^{2\pi i m \frac{a\tau+b}{c\tau+d}} = \sum_{n=0}^{\infty} a_n(P_m) e^{2\pi i n\tau} \qquad \begin{matrix}(m\geq 1) \\ \left(\begin{matrix}k>2, \\ \text{even}\end{matrix}\right)\end{matrix}$$

Additive twists

Data: (a) $N \geq 1$, $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \longrightarrow \mathbb{C}^\times$, $k \in \mathbb{Z}$

(b) $f : \mathcal{H} \longrightarrow \mathbb{C}$ such that $f(x+iy) = \sum_{n\in\mathbb{Z}} a_n(y) e^{2\pi i n x}$

and $\forall \alpha \in \Gamma_0(N)$ $\quad f|_k \alpha = \chi(\alpha) f$

$\left(\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \ \chi(\alpha) = \chi(d)\right)$

(c) $M \geq 1$, $\lambda : \mathbb{Z}/M\mathbb{Z} \longrightarrow \mathbb{C}$

(d) $\phi : \mathbb{Z}/M\mathbb{Z} \longrightarrow \mathbb{C}$

Define: $$(\mathrm{Tw}_\phi f)(x+iy) := \sum_{n\in\mathbb{Z}} a_n(y)\phi(n) e^{2\pi i n x}$$

$$f|_k R_\lambda := \sum_{u \in \mathbb{Z}/M\mathbb{Z}} \lambda(u)\, f|_k \begin{pmatrix} 1 & -u/M \\ 0 & 1 \end{pmatrix}$$

$$\sum_{n\in\mathbb{Z}} a_n(y) \underbrace{\left(\sum_{u\in\mathbb{Z}/M\mathbb{Z}} \lambda(u) e^{-\frac{2\pi i n u}{M}} \right)}_{\hat\lambda(-n)} e^{2\pi i n x}$$

$$\left(\hat\phi(v) = \sum_{u\in\mathbb{Z}/M\mathbb{Z}} \phi(u) e^{2\pi i u v/M} \right)$$

So: $f|_k R_\lambda = \mathrm{Tw}_{\hat\lambda_{-1}}$, where $\phi_k(u) := \phi(ku)$

Cor: As $\hat{\hat\phi} = M\phi_{-1}$, $\boxed{\mathrm{Tw}_\phi f = \frac{1}{M} f|_k R_{\hat\phi}}$

Prop. If χ factors through $(\mathbb{Z}/N_\chi\mathbb{Z})^\times \longrightarrow \mathbb{C}^\times$ $(N_\chi | N)$, let $N' := \mathrm{lcm}(M^2, MN_\chi, N)$. Then

$\forall \beta \in \Gamma_0(N')$ $\quad (\mathrm{Tw}_\phi f)|_k \beta = \chi(d)\, \mathrm{Tw}_{\phi_{d^2}} f$

$\beta = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Pf. $$\begin{pmatrix} 1 & -u/M \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \underbrace{\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}}_{\in\, \Gamma_0(N)}\begin{pmatrix} 1 & -ud^2/M \\ 0 & 1 \end{pmatrix}$$

$\Gamma_0(N)$, $d' \equiv d \pmod{N_\chi}$.

Special case: ϕ is a Dirichlet character $\phi : (\mathbb{Z}/M\mathbb{Z})^\times \to \mathbb{C}^\times$ extended by 0 on $\mathbb{Z}/M\mathbb{Z} \setminus (\mathbb{Z}/M\mathbb{Z})^\times$: $\phi_{d^2} = \phi(d)^2 \phi$

$$\Rightarrow \quad (Tw_\phi f)|_k \beta = (\chi\phi^2)(d)(Tw_\phi f) \quad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \beta \in \Gamma_0(N')$$

Gauss sums: if $\phi : (\mathbb{Z}/M\mathbb{Z})^\times \to \mathbb{C}^\times$ is a primitive Dirichlet character (extended by zero, as above), then

(a) $\hat{\phi}(1) = \sum_{u \in (\mathbb{Z}/M\mathbb{Z})^\times} \phi(u) e^{2\pi i u/M} = \tau(\phi)$ is the Gauss sum attached to ϕ

(b) $\hat{\phi}(u) = \begin{cases} 0, & (u,M) > 1 \\ \overline{\phi(u)}\tau(\phi), & (u,M) = 1 \end{cases}$

(c) $|\tau(\phi)|^2 = M$, $\overline{\tau(\phi)} = \phi(-1)\tau(\overline{\phi})$

Additive twists and Fricke involutions

Notation: $W_N = \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}$

Matrix identity: $\begin{pmatrix} 1 & -u/M \\ 0 & 1 \end{pmatrix} W_{NM^2} = M W_N \begin{pmatrix} M & v \\ uN & \frac{1+uvN}{M} \end{pmatrix} \begin{pmatrix} 1 & -v/M \\ 0 & 1 \end{pmatrix}$

Assume: $\left\{ \begin{array}{l} (M,N) = 1 \\ \phi : (\mathbb{Z}/M\mathbb{Z})^\times \to \mathbb{C}^\times \text{ primitive Dirichlet character (extended by zero to } \mathbb{Z}/M\mathbb{Z}) \end{array} \right\}$

Prop. Under these assumptions, if $\forall \alpha \in \Gamma_0(N)$ $f|_k \alpha = \chi(\alpha) f$, then

$$(Tw_\phi f)|_k W_{NM^2} = \chi(M)\underbrace{\phi(-N)\frac{\tau(\phi)}{\tau(\overline{\phi})}}_{\phi(N)\,\tau(\phi)^2/M} Tw_{\overline{\phi}}(f|_k W_N)$$

Pf. $\text{LHS} = \frac{1}{M} \sum_{u \in (\mathbb{Z}/M\mathbb{Z})^\times} \tau(\phi)\phi^{-1}(u) f|_k \begin{pmatrix} 1 & -u/M \\ 0 & 1 \end{pmatrix} W_{NM^2}$

choose $v \in (\mathbb{Z}/M\mathbb{Z})^\times$ such that $1 + uvN \equiv 0 \pmod M$ $\Rightarrow$ the sum is equal to

$$\frac{\tau(\phi)}{M} \underbrace{\chi(M)}_{\overline{\chi}\left(\frac{1+uvN}{M}\right)} \sum_{v \in (\mathbb{Z}/M\mathbb{Z})^\times} \underbrace{\phi(-vN)}_{\phi(-N)\frac{\hat{\phi}(v)}{\tau(\overline{\phi})}} f|_k W_N \begin{pmatrix} 1 & -v/M \\ 0 & 1 \end{pmatrix} = c\, Tw_{\overline{\phi}}(f|_k W_N),$$

$$c = \chi(M)\phi(-N)\frac{\tau(\phi)}{\tau(\overline{\phi})}$$

Note: $\boxed{(f|_k W_N)(\tau) = (N^{1/2}\tau)^{-k} f\left(-\frac{1}{N\tau}\right)}$

Functional equations for twists

$0 < k \in \mathbb{Z}$, $N \geq 0$

Variant of Hecke's thm: $f(\tau) = \sum_{n=0}^{\infty} a_n e^{2\pi i n \tau}$, $g(\tau) = \sum_{n=0}^{\infty} b_n e^{2\pi i n \tau}$

$$\Lambda_N(f,s) = \left(\frac{N^{1/2}}{2\pi}\right)^s \Gamma(s) \underbrace{L(f,s)}_{\sum_{n=1}^{\infty} \frac{a_n}{n^s}}, \quad \Lambda_N(g,s) = \left(\frac{N^{1/2}}{2\pi}\right)^s \Gamma(s) L(g,s)$$

$(a_n, b_n = O(n^c))$ $\qquad W_N = \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}$ $\qquad$ ☆

Equivalence between:

(A) $\quad (f|_k W_N)(\tau) = (N^{1/2}\tau)^{-k} f\left(-\frac{1}{N\tau}\right) = i^{-k} g(\tau)$

(B) $\quad \Lambda_N(f,s) + \frac{a_0}{s} + \frac{b_0}{k-s}$, $\Lambda_N(g,s) + \frac{b_0}{s} + \frac{a_0}{k-s}$ have holomorphic continuation to $\mathbb{C}$ (and are bounded in vertical strips) and $\Lambda_N(f,s) = \Lambda_N(g,k-s)$

Previous Propositions: If $f \in S_k(\Gamma_0(N), \chi)$, $f|_k W_N = i^{-k} g$, $(M,N) = 1$, $\phi : (\mathbb{Z}/M\mathbb{Z})^\times \longrightarrow \mathbb{C}^\times$ primitive character, then $Tw_\phi f \in S_k(\Gamma_0(NM^2), \chi\phi^2)$ and

$$(Tw_\phi f)|_k W_{NM^2} = \chi(M)\phi(-N) \frac{\tau(\phi)}{\tau(\bar\phi)} Tw_{\bar\phi} (\underbrace{f|_k W_N}_{i^{-k} g})$$

$$\Rightarrow \quad \Lambda_{NM^2}(Tw_\phi f, s) = \chi(M)\phi(-N)\frac{\tau(\phi)}{\tau(\bar\phi)} \Lambda_{NM^2}(Tw_{\bar\phi} g, s) \qquad (*)_\phi$$

Rmk: In many cases $f|_k W_N = c \cdot f_\rho$, where $f_\rho(\tau) = \overline{f(-\bar\tau)} = \sum_{n=0}^{\infty} \bar{a}_n e^{2\pi i n\tau}$

Thm (A. Weil). If f and g from Hecke's thm ☆ satisfy $(*)_\phi$ for "many" primitive characters $\phi : (\mathbb{Z}/p\mathbb{Z})^\times \longrightarrow \mathbb{C}^\times$ (sufficiently "many" of them), with $(p \nmid N$ primes $-$ $\Lambda_{Np^2}(Tw_\phi f, s)$ holomorphic and bounded in vertical strips$)$, then $f \in M_k(\Gamma_0(N), \chi)$, $g \in M_k(\Gamma_0(N), \bar\chi)$ and $f|_k W_N = i^{-k} g$.