Modular forms on Γ

$\Gamma \subset SL_2(\mathbb{R})$ discrete subgroup such that $\Gamma \backslash \mathcal{H}^*_\Gamma$ is compact

Notation: for $k \in \mathbb{Z}$, $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{R})^+$, $f : \mathcal{H} \longrightarrow \mathbb{C} \cup \{\infty\}$

$$(f|_k \alpha)(\tau) = \det(\alpha)^{k/2} (c\tau+d)^{-k} f\left(\frac{a\tau+b}{c\tau+d}\right)$$

Facts: (a) $f|_k \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} = \mathrm{sgn}(a)^k f$ $\|$ (b) $(f|_k\alpha)|_k \beta = f|_k \alpha\beta$

Def. Fix $k \in \mathbb{Z}$.

$$A_k(\Gamma) = \left\{ \begin{array}{l} f : \mathcal{H} \longrightarrow \mathbb{C} \cup \{\infty\} \text{ meromorphic, } \forall \alpha \in \Gamma \ f|_k \alpha = f \\ \qquad f \text{ is meromorphic at all cusps} \end{array} \right\}$$

$\cup$

$M_k(\Gamma) = \{ f \in A_k(\Gamma) \mid f \text{ holomorphic on } \mathcal{H} \text{ and at all cusps} \}$

$\cup$

$S_k(\Gamma) = \{ f \in M_k(\Gamma) \mid f \text{ vanishes at all cusps} \}$

Terminology: $M_k(\Gamma) = \{$modular forms of weight k on $\Gamma\}$

$S_k(\Gamma) = \{$ cusp —— " —— $\}$

Meromorphy at cusps: assume that $f \in \mathcal{M}(\mathcal{H})$ satisfies $\forall \alpha \in \Gamma$ $f|_k \alpha = f$. For $x \in \mathrm{cusps}(\Gamma)$, fix $\sigma \in SL_2(\mathbb{R})$, $\sigma(x) = \infty$.

$x \xrightarrow{\sigma} \infty$

$$\sigma \Gamma_x \sigma^{-1} \cdot \{\pm 1\} = \pm \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}^{\mathbb{Z}}, \quad h > 0.$$

Put $g = f|_k \sigma^{-1}$; then $\forall \alpha \in \Gamma$ $g|_k (\sigma\alpha\sigma^{-1}) = g$

$\exists\, \varepsilon \in \{\pm 1\}$ $\varepsilon \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \in \sigma \Gamma_x \sigma^{-1}$.

(1) k even: $g(\tau) = \left(g|_k \varepsilon \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}\right)(\tau) = g(\tau+h)$

$$\Longrightarrow (f|_k \sigma^{-1})(\tau) = \sum_{n=-\infty}^{\infty} a_n q_h^n \quad \text{if } \mathrm{Im}(\tau) > r \quad (q_h = e^{2\pi i \tau/h})$$

(2) k odd: if $-I \in \Gamma \Longrightarrow f = f|_k(-I) = -f \Longrightarrow f = 0$.

Assume $-I \notin \Gamma$.

If x is a regular cusp of Γ: $\begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \in \sigma\Gamma\sigma^{-1} \Longrightarrow g(\tau) = g(\tau+h)$

$$\Longrightarrow (f|_k \sigma^{-1})(\tau) = \sum_{n \in \mathbb{Z}} a_n q_h^n$$

If x is an irregular cusp of Γ: $-\begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \in \sigma\Gamma\sigma^{-1} \Longrightarrow g(\tau) = -g(\tau+h)$

$$\Longrightarrow (f|_k \sigma^{-1})(\tau) = \sum_{m \in \mathbb{Z}} b_m q_{2h}^{2m+1} = \sum_{n \in \mathbb{Z}+\frac{1}{2}} a_n q_h^n$$

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f being meromorphic at x: by definition, $\exists\, n_0$ $a_m = 0$ if $m < n_0$.

Def. f is $\begin{Bmatrix}\text{meromorphic}\\ \text{holomorphic}\\ \text{zero}\end{Bmatrix}$ at $x \iff \begin{Bmatrix}(\exists n_0)\ \forall n<n_0 & a_n=0\\ \forall n<0 & a_n=0\\ \forall n\le 0 & a_n=0\end{Bmatrix}$

Def. For such $f\neq 0$, $\mathrm{ord}_x(f) := \min\{n \mid a_n \neq 0\}$.

Generalisation: given $\chi:\Gamma\longrightarrow\mathbb{C}^\times$ homomorphism with finite image ("character of finite order"), put $\Gamma_\chi := \mathrm{Ker}(\chi)$.

Def.
$$A_k(\Gamma,\chi) := \{f\in A_k(\Gamma_\chi) \mid \forall\gamma\in\Gamma\quad f|_k\gamma=\chi(\gamma)f\}$$
$$\cup$$
$$M_k(\Gamma,\chi) := A_k(\Gamma,\chi)\cap M_k(\Gamma_\chi)$$
$$\cup$$
$$S_k(\Gamma,\chi) := A_k(\Gamma,\chi)\cap S_k(\Gamma_\chi)$$

Examples: (0) $A_0(\Gamma)=\mathcal{M}(\Gamma\backslash\mathcal{H}^*)$

$M_0(\Gamma)=\mathcal{O}(\Gamma\backslash\mathcal{H}^*)=\mathbb{C}$

$S_0(\Gamma)=\begin{cases}\mathbb{C} & \text{if } \mathrm{cusps}(\Gamma)=\emptyset\\ 0 & \text{otherwise}\end{cases}$

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(1) $f(\tau)=\theta^2(\tau)$, $\theta(\tau)=\sum_{n\in\mathbb{Z}}e^{\pi i n^2\tau}$; $\theta(\underbrace{\tau+2}_{T^2(\tau)})=\theta(\tau)$, $\theta(\underbrace{-\tfrac{1}{\tau}}_{S(\tau)})=\sqrt{\tfrac{\tau}{i}}\,\theta(\tau)$

$S=\begin{pmatrix}0&-1\\1&0\end{pmatrix}$, $T=\begin{pmatrix}1&1\\0&1\end{pmatrix}$

We know: $\Gamma(2)=\langle -I, T^2, ST^{-2}S^{-1}=\begin{pmatrix}1&0\\2&1\end{pmatrix}\rangle$

$\Gamma_\theta=\langle S,T^2\rangle=\langle S,\Gamma(2)\rangle$

$$SL_2(\mathbb{Z})\underbrace{\supset}_{\text{index } 3}\Gamma_\theta=\Gamma(2)\cup S\Gamma(2)\underbrace{\supset}_{\text{index } 2}\Gamma(2)$$

Prop. $\theta^2(\tau)\in M_1(\Gamma_\theta,\chi)$, $\chi:\Gamma_\theta\longrightarrow\{\pm1,\pm i\}=\mu_4$

$\chi(S)=-i$, $\chi(T^2)=1$

$$\mathrm{Ker}(\chi)=\left\{\begin{pmatrix}a&b\\c&d\end{pmatrix}\in SL_2(\mathbb{Z}) \;\middle|\; a\equiv d\equiv 1 \pmod 4,\ b\equiv c\equiv 0 \pmod 2\right\}$$

Pf. $\theta^2(\tau)$ is holomorphic at $\infty \Longrightarrow$ also at 0 (app. of S) $\Longrightarrow$ also at 1 (app. of T). We know: $\mathrm{cusps}(\Gamma(2))=\Gamma(2)\cdot\{0,1,\infty\}\Longrightarrow \mathrm{cusps}(\Gamma_\theta)=\Gamma_\theta\cdot\{1,\infty\}$.

Cor. $\theta^4(\tau)\in M_2(\Gamma_\theta,\chi^2)$, $\chi^2:\Gamma_\theta/\Gamma(2)\xrightarrow{\sim}\{\pm1\}$

$\theta^8(\tau)\in M_4(\Gamma_\theta)$

Ex (2). $\alpha \in GL_2(\mathbb{R})^+$, $f \in M_k(\Gamma) \Longrightarrow f|_k\alpha \in M_k(\alpha^{-1}\Gamma\alpha)$ (idem for S_k)

Interesting case: $\alpha^{-1}\Gamma\alpha \cap \Gamma$ has finite index in Γ

$\Longrightarrow$ $M_k(\alpha^{-1}\Gamma\alpha \cap \Gamma)$ is defined (and contains $M_k(\alpha^{-1}\Gamma\alpha)$)

e.g., for $\Gamma = \Gamma_0(M)$, $\alpha = \begin{pmatrix} N & 0 \\ 0 & 1 \end{pmatrix}$ $\qquad (f|_k\alpha)(\tau) = N^{k/2} f(N\tau)$

$$\alpha^{-1}\begin{pmatrix} a & b \\ Mc & d \end{pmatrix}\alpha = \begin{pmatrix} a & b/N \\ MNc & d \end{pmatrix} \Longrightarrow \alpha^{-1}\Gamma_0(M)\alpha \cap \Gamma_0(M) = \Gamma_0(MN)$$

Conclusion: $f(\tau) \in M_k(\Gamma_0(M)) \Longrightarrow f(N\tau) \in M_k(\Gamma_0(MN))$

Ex (3). Eisenstein series on $\Gamma(N)$

fix $\phi : (\mathbb{Z}^2)^*/N \longrightarrow \mathbb{C}$ and an integer $k > 2$

$$G_k(\tau, \phi) := \sum_{m,n \in \mathbb{Z}}{}' \frac{\phi((m,n))}{(m\tau+n)^k}$$

For $\alpha \in GL_2(\mathbb{Z})$, let $(\alpha^*\phi)((m,n)) := \phi((m,n)\alpha)$; then

$\forall \alpha \in SL_2(\mathbb{Z})$ $\quad G_k(\cdot, \phi)|_k\alpha = G_k(\cdot, \alpha^*\phi) \Longrightarrow$

$\forall \alpha \in \Gamma(N)$ $\quad G_k(\cdot, \phi)|_k\alpha = G_k(\cdot, \phi)$.

As $\displaystyle\lim_{\mathrm{Im}(\tau) = +\infty} G_k(\tau, \phi) = \sum_{0 \neq n \in \mathbb{Z}} \frac{\phi((0,n))}{n^k}$ exists,

$G_k(\cdot, \phi)$ is holomorphic at ∞. The same holds for $G_k(\cdot, \alpha^*\phi) = G_k(\cdot, \phi)|_k\alpha$ $\quad \forall \alpha \in SL_2(\mathbb{Z})$

$\Longrightarrow$ $G_k(\cdot, \phi)$ is holomorphic at all cusps.

$\Longrightarrow$ $G_k(\cdot, \phi) \in M_k(\Gamma(N))$.

Observe: If $\Gamma' \triangleleft \Gamma$ is a normal subgroup of finite index, $A := \Gamma/\Gamma'$ acts on the right on $M_k(\Gamma')$ by $f \mapsto f|_k\alpha$.
If A is abelian, then $\displaystyle M_k(\Gamma') = \bigoplus_{\chi : A \to \mathbb{C}^\times} M_k(\Gamma, \chi)$ (idem for S_k)

Ex: (1) $\psi : \Gamma_\theta/\Gamma(2) \xrightarrow{\sim} \{\pm 1\}$ $\qquad (\psi = \chi^2)$

$$M_k(\Gamma(2)) = M_k(\Gamma_\theta) \oplus M_k(\Gamma_\theta, \psi)$$

(2) $\Gamma_1(N) \triangleleft \Gamma_0(N)$, $\quad \Gamma_0(N)/\Gamma_1(N) \xrightarrow{\sim} (\mathbb{Z}/N\mathbb{Z})^\times$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\Gamma_1(N) \longmapsto d \pmod N$$

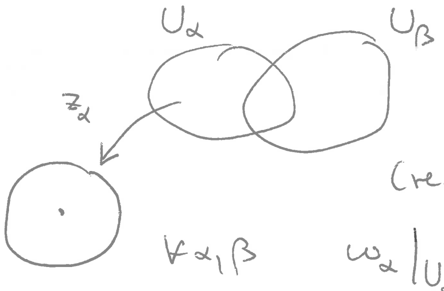
$$M_k(\Gamma_1(N)) = \bigoplus_{\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times} M_k(\Gamma_0(N), \chi)$$

Differentials on $\Gamma\backslash\mathcal{H}^*_\Gamma$ and modular forms of even weight

$\Gamma \subset SL_2(\mathbb{R})$ discrete subgroup such that $X_\Gamma = \Gamma\backslash\mathcal{H}^*_\Gamma$ is compact.

Assume: X = Riemann surface, $m \in \mathbb{Z}$

Def. A holomorphic (resp. meromorphic) m-differential on X is a collection $\omega = (\omega_\alpha = f_\alpha(z_\alpha)\,(dz_\alpha)^m)$, where



$\{z_\alpha : U_\alpha \longrightarrow \mathbb{C}\}$ are some coordinate charts of X, f_α are holomorphic (resp. meromorphic) and

$\forall \alpha, \beta \quad \omega_\alpha|_{U_\alpha \cap U_\beta} = \omega_\beta|_{U_\alpha\cap U_\beta}$ in the sense that

$$f_\alpha(z_\alpha) = f_\beta(z_\beta)\left(\frac{dz_\beta}{dz_\alpha}\right)^m \quad \text{on } U_\alpha \cap U_\beta \quad \left(\text{note: } \frac{dz_\beta}{dz_\alpha} \neq 0 \text{ there}\right)$$

Notation: $\Omega^m(X)$ (resp. $\Omega^m_{mer}(X)$) the set of such m-differentials.

From now on: X = compact

Def: For $0 \neq \omega \in \Omega^m_{mer}(X)$, $\quad \mathrm{div}(\omega) := \sum_{x\in X} \underbrace{\mathrm{ord}_x(f_\alpha)}_{\text{for any } \alpha \text{ s.t. } x \in U_\alpha}\cdot(x)$

Rmks: (0) $\Omega^0_{mer}(X) = \mathcal{M}(X)$, $\Omega^0(X) = \mathcal{O}(X)$ ($=\mathbb{C}$ since X is compact)

(1) $\dim_{\mathcal{M}(X)} \Omega^m_{mer}(X) = 1$ [Pf: $f \in \mathcal{M}(X)\smallsetminus\mathbb{C}$, $(df)^m \in \Omega^m_{mer}(X)\smallsetminus\{0\}$
$0 \neq \omega_1, \omega_2 \in \Omega^m_{mer}(X) \Longrightarrow \omega_1/\omega_2 \in \mathcal{M}(X)$]

(2) Every holomorphic map $f: Y \longrightarrow X$ induces
$f^* : \Omega^m(X) \longrightarrow \Omega^m(Y)$ (injective if f is non-constant)
and, if f is non-constant, $f^*: \Omega^m_{mer}(X) \longrightarrow \Omega^m_{mer}(Y)$ (injective)

Our case:

$$\begin{array}{ccc} \mathcal{H} & & \\ \downarrow \pi & & \\ Y_\Gamma = \Gamma\backslash\mathcal{H} & \overset{j}{\hookrightarrow} & \Gamma\backslash\mathcal{H}^*_\Gamma = X_\Gamma \end{array}$$

X_Γ compact,
$X_\Gamma \smallsetminus Y_\Gamma$ finite

Action of $\gamma = \begin{pmatrix} a & b \\ c & d\end{pmatrix} \in GL_2(\mathbb{R})^+$ on $\Omega^m_{mer}(\mathcal{H})$ by $\omega \longmapsto \gamma^*(\omega)$

[$(\gamma_1\gamma_2)^* = \gamma_2^*\gamma_1^*$: right action] $\qquad \omega = f(\tau)(d\tau)^m$

$$\gamma^*(f(\tau)(d\tau)^m) = f\left(\frac{a\tau+b}{c\tau+d}\right)\left(d\left(\frac{a\tau+b}{c\tau+d}\right)\right)^m = \underbrace{(ad-bc)^m(c\tau+d)^{-2m} f\left(\frac{a\tau+b}{c\tau+d}\right)}_{(f|_{2m}\gamma)(\tau)}(d\tau)^m$$

Prop : (1) $\mathcal{H} \xrightarrow{\pi} \Gamma\backslash\mathcal{H} = Y_\Gamma \xhookrightarrow{j} \Gamma\backslash\mathcal{H}^*_\Gamma = X_\Gamma$ induces a bijection

$$\pi^*: \Omega^m_{mer}(Y_\Gamma) \xrightarrow{\sim} \Omega^m_{mer}(\mathcal{H})^\Gamma .$$

(2) For $\omega = f(\tau)(d\tau)^m \in \Omega^m_{mer}(\mathcal{H})$ $\quad (f \in \mathcal{M}(\mathcal{H}))$ and $\gamma \in GL_2(\mathbb{R})^+$,
$\gamma^*(\omega) = ((f|_{2m}\gamma)(\tau))(d\tau)^m$. therefore π^* gives a bijection

$$\Omega^m_{mer}(Y_\Gamma) \xrightarrow{\sim} \{ f \in \mathcal{M}(\mathcal{H}) \mid \forall \gamma \in \Gamma \quad f|_{2m}\gamma = f \}.$$

(3) the image of $j^*\Omega^m_{mer}(X_\Gamma)$ under (2) is $A_{2m}(\Gamma)$, i.e.,

$$\pi^* j^* = (j\pi)^* : \Omega^m_{mer}(X_\Gamma) \xrightarrow{\sim} \{ f(\tau)(d\tau)^m \mid f \in A_{2m}(\Gamma) \} \subset \Omega^m_{mer}(\mathcal{H})^\Gamma$$

Pf : (1) $\forall \gamma \in \Gamma \quad \pi \circ \gamma = \pi \implies \mathrm{Im}(\pi^*) \subset \Omega^m_{mer}(\mathcal{H})^\Gamma$.

We need to check surjectivity locally on $\mathcal{H}$. this reduces to the corresponding statement for the map

$$\begin{array}{ccc} z \in D & \longrightarrow & \langle \zeta \rangle \backslash D \\ & \searrow^{\pi'} & \wr\!\!\downarrow \\ & & z^e \in D \end{array}$$

, where $D =$ unit disc, $\zeta = \exp(2\pi i/e)$
$e \geq 1$ ramification index

$$\Omega^m_{mer}(D) = \{ \omega = f(z)(dz)^m \mid f(z) = \sum_{n \geq n_0} a_n z^n \in \mathcal{M}(D) \}$$

$$\Omega^m_{mer}(D)^{\langle \zeta \rangle} = \{ \omega = f(z)(dz)^m \mid \underbrace{f(z)(dz)^m = f(\zeta z)\, d(\zeta z)^m}_{\boxed{\iff f(\zeta z) = \zeta^{-m} f(z) \iff a_n = 0 \text{ if } n \not\equiv -m \ (\mathrm{mod}\ e)}}$$

But $(\pi')^* \Omega^m_{mer}(D) = \{ \underbrace{g(z^e)\,(d(z^e))^m}_{(e z^{e-1})^m \left(\sum_{n \geq n_1} b_n z^{en}\right)(dz)^m} \mid g(z) = \sum_{n \geq n_1} b_n z^n \in \mathcal{M}(D) \}$

$$\implies (\pi')^* \Omega^m_{mer}(D) = \Omega^m_{mer}(D)^{\langle \zeta \rangle}.$$

(2) We have already checked the formula for $\gamma^*(\omega)$.

(3) At $x \in \mathrm{cusps}(\Gamma)$: we can assume $x = \infty$, local coordinate $q_h = e^{2\pi i \tau/h}$

If $f(\tau) = \sum_{n \in \mathbb{Z}} a_n q_h^n$ (if $\mathrm{Im}(\tau) > (\mathrm{const.})$), then $\quad \dfrac{dq_h}{q_h} = \dfrac{2\pi i}{h} d\tau$

$$\omega = f(\tau)(d\tau)^m = \sum_{n \in \mathbb{Z}} a_n q_h^n \left(\frac{2\pi i}{h}\right)^{-m} \left(\frac{dq_h}{q_h}\right)^{-m}, \text{ hence}$$

ω is meromorphic at $\pi(\infty) \iff f$ is meromorphic at $q_h = 0$.
$\Updownarrow$
$\iff \exists n_0 \ \forall n < n_0 \ a_n = 0$

Def. For $0 \neq f \in A_{2m}(\Gamma)$ and $P \in X_\Gamma$, we define

$$\mathrm{div}(f) := \sum_{P \in X_\Gamma} \mathrm{ord}_P(f) \cdot (P) \in \mathrm{Div}(X_\Gamma) \otimes \mathbb{Q}, \quad \text{where}$$

$$\mathrm{ord}_P(f) := \begin{cases} \dfrac{\mathrm{ord}_\tau(f)}{e_\tau}, & \text{if } P = \pi(\tau),\ \tau \in \mathcal{H} \\ \text{as above } (\in \mathbb{Z}), & \text{if } P \in \mathrm{cusps}(X_\Gamma) = X_\Gamma \setminus Y_\Gamma \end{cases}$$

(recall: $e_P = e_\tau = |\overline{\Gamma}_\tau| \cancel{\ } =$ the ramification index of $\mathcal{H} \xrightarrow{\pi} Y_\Gamma$ at τ)

Prop. Let $m \in \mathbb{Z}$. If $0 \neq \omega \in \Omega^m_{mer}(X_\Gamma)$ corresponds to $0 \neq f \in A_{2m}(\Gamma)$ (i.e., $\omega = f(\tau)(d\tau)^m$), then

$$\mathrm{div}(f) = \mathrm{div}(\omega) + m \sum_{P \in Y_\Gamma} \left(1 - \frac{1}{e_P}\right)(P) + m \sum_{P \in \underbrace{X_\Gamma \setminus Y_\Gamma}_{\mathrm{cusps}(X_\Gamma)}} (P)$$

[Rmk: $e_P \neq 1 \iff P =$ elliptic point of Γ]

Cor. $f \in S_2(\Gamma) \iff \omega \in \Omega^1(X_\Gamma)$

$f \in M_2(\Gamma) \iff \omega \in \underbrace{\Omega^1(X_\Gamma)(\text{cusps})}_{\substack{\text{differentials holomorphic on } Y_\Gamma \\ \text{with at most simple poles at cusps}}}$

Pf. We need to check this locally.

(a) at a cusp x: (say, ∞) local coordinate $q_h = e^{2\pi i \tau / h}$

$$f(\tau) = \sum_{n \geq n_0} a_n q_h^n, \quad a_{n_0} \neq 0, \quad \mathrm{ord}_x(f) = n_0.$$

$$\omega = f(\tau)(d\tau)^m = \left(\frac{h}{2\pi i}\right)^m \sum_{n \geq n_0} a_n q_h^n \left(\frac{dq_h}{q_h}\right)^m \implies \mathrm{ord}_x(\omega) = n_0 - m.$$

(b) at $P = \pi(\tau)$, $\tau \in \mathcal{H}$: in local coordinates around τ_0:

$\pi : \mathcal{H} \longrightarrow \Gamma \backslash \mathcal{H} = Y_\Gamma$ is isomorphic to $\pi : D \longrightarrow D$, $z \longmapsto z^e$; $e = e_P$

If $\omega = \sum_{n \geq n_1} b_n z^n (dz)^m$, $b_{n_1} \neq 0$,

$$\pi^*(\omega) = \sum_{n \geq n_1} b_n (z^e)^n (d(z^e))^m = \sum_{n \geq n_1} b_n (e z^{e-1})^m z^{ne} (dz)^m$$

$$\implies \underbrace{\mathrm{ord}_{z=0}\, \pi^*(\omega)}_{\mathrm{ord}_\tau(f) = e_P\, \mathrm{ord}_P(f)} = e n_1 + m(e-1) = e\left(\mathrm{ord}_{z=0}(\omega) + m\left(1 - \frac{1}{e}\right)\right)$$

Cor. For $0 \neq f \in A_{2m}(\Gamma)$,

$$\deg(\mathrm{div}(f)) = m(2g-2) + m \sum_{P \in Y_\Gamma} \left(1 - \frac{1}{e_P}\right) + m \sum_{P \in \mathrm{cusps}(X_\Gamma)} 1.$$

(g = genus of X_Γ)

Pf. $0 \neq \omega = f(\tau)(d\tau)^m \in \Omega^m_{mer}(X_\Gamma)$ has degree $\deg(\mathrm{div}(\omega)) = m(2g-2)$

Cor. If $f, g \in M_{2m}(\Gamma)$ and if $\exists\, x \in \mathrm{cusps}(X_\Gamma)$ such that

$$\mathrm{ord}_x(f-g) > m(2g-2) + m \sum_{P \in X_\Gamma} \left(1 - \frac{1}{e_P}\right) + m\,|\mathrm{cusps}(X_\Gamma)|,$$

then $f = g$.

Notation: (1) X = compact Riemann surface, $D \in \mathrm{Div}(X)$

$$\Omega^m(D) := \{\omega \in \Omega^m_{mer}(X) \mid \omega = 0 \text{ or } \mathrm{div}(\omega) + D \geq 0\}$$

(2) For $D = \sum n_i \underbrace{(P_i)}_{\text{distinct}} \in \mathrm{Div}(X) \otimes \mathbb{Q}$ $\quad (n_i \in \mathbb{Q})$,

$$[D] := \sum [n_i](P_i) \in \mathrm{Div}(X) \quad ([n_i] = \text{the integral part of } n_i)$$

Def. For Γ as above, put

$$(\mathrm{cusps}) := \sum_{P \in X_\Gamma \setminus Y_\Gamma} (P) \in \mathrm{Div}(X_\Gamma), \qquad E := \sum_{P \in Y_\Gamma} \left(1 - \frac{1}{e_P}\right)(P) + (\mathrm{cusps})$$

$E \in \mathrm{Div}(X_\Gamma) \otimes \mathbb{Q}$

Reformulation of Prop. above: under the bijection

$$A_{2m}(\Gamma) \xleftrightarrow{\sim} \Omega^m_{mer}(X_\Gamma)$$

$$f \longmapsto f(\tau)(d\tau)^m = \omega$$

$$\boxed{\mathrm{div}(f) = \mathrm{div}(\omega) + mE}$$

Cor. Under this bijection,

$$\begin{array}{ccc} M_{2m}(\Gamma) & \xleftrightarrow{\sim} & \Omega^m([mE]) \\ \cup & & \cup \\ S_{2m}(\Gamma) & \xleftrightarrow{\sim} & \Omega^m([mE] - (\mathrm{cusps})) \end{array}$$

Special case $m=1$: $[mE] = [E] = (\mathrm{cusps})$

$$\begin{array}{ccc} M_2(\Gamma) & \xleftrightarrow{\sim} & \Omega^1(X_\Gamma)(\mathrm{cusps}) \\ \cup & & \cup \\ S_2(\Gamma) & \xleftrightarrow{\sim} & \Omega^1(X_\Gamma) \end{array}$$

Riemann – Roch Theorem

$X =$ compact Riemann surface of genus g

Def. For $D \in \mathrm{Div}(X)$, $\mathcal{L}(D) := \{0\} \cup \{f \in \mathcal{M}(X) \setminus \{0\} \mid \mathrm{div}(f) + D \geq 0\}$

$\ell(D) := \dim_{\mathbb{C}} \mathcal{L}(D)$.

Note: (1) $\deg(D) < 0 \implies \mathcal{L}(D) = 0$

(2) If $D = D' + \mathrm{div}(g) \implies \mathcal{L}(D) \xrightarrow{\sim} \mathcal{L}(D')$ bijection, $f \longmapsto fg$

$\implies \ell(D)$ depends only on the class $cl(D)$ of D in $Cl(X)$

$$Cl(X) = \mathrm{Div}(X) / \{\mathrm{div}(g) \mid g \in \mathcal{M}(X) \setminus \{0\}\}$$

Def. the canonical class $K \in Cl(X)$ is the class of $\mathrm{div}(\omega)$, for any $0 \neq \omega \in \Omega^1_{mer}(X)$.

Riemann – Roch Theorem: $\boxed{\ell(D) - \ell(K-D) = \deg(D) - g + 1}$

Note: (3) $\forall m \in \mathbb{Z}$ $\ell(mK+D) = \dim \underbrace{\{\omega \in \Omega^m_{mer}(X) \mid \mathrm{div}(\omega) + D \geq 0\}}_{\Omega^m(D)}$

(4) $D = 0$: $\dim \Omega^1(X) = \ell(K) = \ell(0) - 1 + g = g$

(5) Symmetry under $D \longleftrightarrow K - D \implies \deg(K) = 2g-2$.

(6) If $\deg(D) > 2g-2 \implies \deg(K-D) < 0 \implies \ell(K-D) = 0$

$\implies \ell(D) = \deg(D) - g + 1$.

Back to $X_\Gamma = \Gamma \backslash \mathcal{H}^*_\Gamma$: we have defined

$Y_\Gamma = \Gamma \backslash \mathcal{H}$

$$E := \sum_{P \in Y_\Gamma} \left(1 - \frac{1}{e_P}\right)(P) + \underbrace{\sum_{P \in X_\Gamma \setminus Y_\Gamma}}_{\text{(cusps)}} (P) \in \mathrm{Div}(X_\Gamma) \otimes \mathbb{Q}.$$

Proposition. $d := \deg(K+E) = 2g - 2 + \sum_{P \in Y_\Gamma} \left(1 - \frac{1}{e_P}\right) + \underbrace{|\text{cusps}|}_{\nu_\infty} > 0$.

Pf (only if $(SL_2(\mathbb{Z}) : \Gamma) < \infty$):

$$2g - 2 = -2\mu + \frac{2}{3}(\mu - \nu_3) + \frac{1}{2}(\mu - \nu_2) + (\mu - \nu_\infty),$$

$$d = (2g-2) + \frac{2}{3}\nu_3 + \frac{1}{2}\nu_2 + \nu_\infty = \frac{\mu}{6} > 0.$$

General case: $d = \frac{1}{2\pi} \mathrm{vol}(\Gamma \backslash \mathcal{H})$.

Theorem. Let $m \in \mathbb{Z}$, $g = \text{genus}(X_\Gamma)$, $\nu_\infty = |\text{cusps}(X_\Gamma)|$. Then:

(1)

$$\dim S_{2m}(\Gamma) = \begin{cases} (2m-1)(g-1) + \sum_{P \in Y_\Gamma} \left[m\left(1 - \frac{1}{e_P}\right)\right] + (m-1)\nu_\infty, & m > 1 \\ g, & m = 1 \\ 1, & m = 0, \nu_\infty = 0 \\ 0, & m = 0, \nu_\infty > 0 \\ 0, & m < 0 \end{cases}$$

(2)

$$\dim M_{2m}(\Gamma) = \begin{cases} \dim S_{2m}(\Gamma) + \nu_\infty & m > 1 \\ \dim S_2(\Gamma) + \nu_\infty - 1 \quad = g + \nu_\infty - 1 & m = 1, \nu_\infty > 0 \\ \dim S_2(\Gamma) \quad = g & m = 1, \nu_\infty = 0 \\ 1 & m = 0 \\ 0 & m < 0 \end{cases}$$

Pf. For $E = \sum_{P \in Y_\Gamma} \left(1 - \frac{1}{e_P}\right)(P) + (\text{cusps})$,

$\dim S_{2m}(\Gamma) = \ell([mE] - (\text{cusps}) + mK)$

$\dim M_{2m}(\Gamma) = \ell([mE] + mK)$

We know that $\deg(K + E) > 0$.

$\underline{m < 0}$: $\deg([mE] + mK) \leq m \deg(K+E) < 0 \Rightarrow M_{2m}(\Gamma) = 0$.

$\underline{m = 0}$: $M_0(\Gamma) = \mathcal{O}(X_\Gamma) = \mathbb{C}$, $S_0(\Gamma) = \begin{cases} 0 & \nu_\infty > 0 \\ \mathbb{C} & \nu_\infty = 0 \end{cases}$

$\underline{m = 1}$: $S_2(\Gamma) \simeq \Omega^1(X_\Gamma)$, $\dim = \ell(K) = g$

$M_2(\Gamma) \simeq \Omega^1(X_\Gamma)(\text{cusps})$,

$\dim M_2(\Gamma) = \ell(K + (\text{cusps})) = g - 1 + \nu_\infty + \underbrace{\ell(-\text{cusps})}_{\begin{cases} 0, & \nu_\infty > 0 \\ 1, & \nu_\infty = 0 \end{cases}}$

$\underline{m > 1}$: $\deg(mK + [mE] - (\text{cusps})) = (2g-2)m + \sum_{P \in Y_\Gamma} \underbrace{\left[m\left(1 - \frac{1}{e_P}\right)\right]}_{\geq (m-1)\left(1 - \frac{1}{e_P}\right)} + (m-1)\nu_\infty$

$\geq 2g - 2 + (m-1)\deg(K+E) > 2g - 2$

$\Rightarrow \underbrace{\ell(mK + [mE] - (\text{cusps}))}_{\dim S_{2m}(\Gamma)} = \deg(\text{—"—}) - g + 1$.

Similarly for $\dim M_{2m}(\Gamma)$.

Examples : (1) $\Gamma = \Gamma(2)$: $g = 0$, {elliptic points} $= \emptyset$, $\nu_\infty = 3$

$$\dim S_{2m}(\Gamma(2)) = \begin{cases} (2m-1)\cdot(-1) + (m-1)\cdot 3 = m-2 & m \geq 2 \\ 0 & m \leq 1 \end{cases}$$

$$\dim M_{2m}(\Gamma(2)) = \begin{cases} m+1 & m \geq 2 \\ 2 & m = 1 \\ 1 & m = 0 \\ 0 & m < 0 \end{cases}$$

(2) $\Gamma = \Gamma_\theta$: $g = 0$, {elliptic points} = {one at i}, $\nu_\infty = 2$

$\deg(K+E) = \frac{1}{2}$ $\qquad E = \frac{1}{2}(i) + (\text{cusps})$

$$\dim S_{2m}(\Gamma_\theta) = \begin{cases} (2m-1)\cdot(-1) + \left[\frac{m}{2}\right] + (m-1)\cdot 2 = \left[\frac{m}{2}\right] - 1, & m \geq 2 \\ 0 & m \leq 1 \end{cases}$$

$$\dim M_{2m}(\Gamma_\theta) = \begin{cases} \left[\frac{m}{2}\right] + 1 & m \geq 2 \\ 1 & m = 0, 1 \\ 0 & m < 0 \end{cases}$$

(3) $\psi = \chi^2 : \Gamma_\theta / \Gamma(2) \xrightarrow{\sim} \{\pm 1\}$

$M_k(\Gamma(2)) = M_k(\Gamma_\theta) \oplus M_k(\Gamma_\theta, \psi)$ (idem for S_k)

$$\dim S_{2m}(\Gamma_\theta, \psi) = \begin{cases} m - 2 - \left(\left[\frac{m}{2}\right] - 1\right) = \left[\frac{m-1}{2}\right] & m \geq 2 \\ 0 & m \leq 1 \end{cases}$$

$$\dim M_{2m}(\Gamma_\theta, \psi) = \begin{cases} m + 1 - \left(\left[\frac{m}{2}\right] + 1\right) = \left[\frac{m+1}{2}\right] & m \geq 2 \\ 1 & m = 1 \\ 0 & m \leq 0 \end{cases}$$

Application : sums of squares

$$\theta(\tau)^k = \sum_{n=0}^{\infty} r_k(n) q^{n/2}, \quad r_k(n) = \left| \left\{ (x_1, \dots, x_k) \;\middle|\; \begin{array}{l} x_j \in \mathbb{Z} \\ x_1^2 + \dots + x_k^2 = n \end{array} \right\} \right| \quad (k \geq 1)$$

We know : $\theta^2 \in M_1(\Gamma_\theta, \chi)$, $\chi : \Gamma_\theta \longrightarrow \{\pm 1, \pm i\}$

$\theta^4 \in M_2(\Gamma_\theta, \psi) \longleftarrow \dim = 1$, $S_2(\Gamma_\theta, \psi) = 0$

$\theta^8 \in M_4(\Gamma_\theta) \longleftarrow \dim = 2$, $S_4(\Gamma_\theta) = 0$

General principle: if $SL_2(\mathbb{Z}) \supset \Gamma \supset \Gamma(N)$, then

$\forall k \geq 2 \qquad M_k(\Gamma) = S_k(\Gamma) \oplus$ (Eisenstein series)

Let us identify θ^4 and θ^8 with appropriate Eisenstein series.

Recall: $\forall k \geq 2 \quad \sum_{n \in \mathbb{Z}} \frac{1}{(z+n)^k} = \frac{(-2\pi i \varepsilon)^k}{(k-1)!} \sum_{n=1}^{\infty} n^{k-1} e^{2\pi i n z \varepsilon} \qquad (\varepsilon z \in \mathcal{H})$
$\varepsilon = \pm 1$

Fix $N \geq 1$, $a, b \in \mathbb{Z}$.

$$G_{k,N,a,b}(\tau) := \sum_{\substack{m,n \in \mathbb{Z} \\ m \equiv a \,(\mathrm{mod}\, N) \\ n \equiv b \,(\mathrm{mod}\, N)}}' \frac{1}{(m\tau+n)^k}$$

(absolute convergence if $k \geq 3$)

For $k=2$: one can still make sense of $G_{2,a,b} - G_{2,c,d}$.

Fourier expansion of $G_{k,N,a,b}(\tau)$:

$$G_{k,N,a,b}(\tau) = \delta\left(\frac{a}{N}\right) \sum_{\substack{0 \neq n \in \mathbb{Z} \\ n \equiv b \,(\mathrm{mod}\, N)}} n^{-k} + \frac{(-2\pi i)^k}{N^k (k-1)!} \sum_{\substack{0 \neq m \in \mathbb{Z} \\ m \equiv a \,(\mathrm{mod}\, N)}} \sum_{\substack{n \in \mathbb{Z} \\ mn > 0}} \mathrm{sgn}(n) n^{k-1} \left(\zeta_N^b q_N^m\right)^n$$

$\zeta_N = e^{2\pi i/N}$

$q_N = e^{2\pi i \tau}$, $\delta\left(\frac{a}{N}\right) = \begin{cases} 1, & \frac{a}{N} \in \mathbb{Z} \\ 0, & \frac{a}{N} \notin \mathbb{Z} \end{cases}$

$M_2(\Gamma_\theta, \psi)$: $\dim = 1$

Consider
$$g(\tau) = \text{"} G_{2,0,1}(\tau) - G_{2,1,0}(\tau) \text{"}$$
$$= \sum_{m,n \in \mathbb{Z}} \left[\frac{1}{(2m\tau + (2n+1))^2} - \frac{1}{((2m+1)\tau + 2n)^2} \right]$$

$g \in M_2(\Gamma(2))$, $\qquad g|_2 \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -g \implies g \in M_2(\Gamma_\theta, \psi)$
$\implies g = c \cdot \theta^4(\tau)$

Need to make explicit the Fourier expansion of $g(\tau)$.

In terms of $q_2 = e^{2\pi i \tau/2} = e^{\pi i \tau}$:

$$g(\tau) = \underbrace{\sum_{\substack{n\in\mathbb{Z} \\ n\equiv 1 \pmod 2}} \frac{1}{n^2}}_{2\left(1-\frac{1}{2^2}\right)\zeta(2) = \frac{\pi^2}{4}} - \pi^2 \sum_{0\neq m\in 2\mathbb{Z}} \sum_{\substack{n\in\mathbb{Z} \\ mn>0}} \operatorname{sgn}(n)\, n\, (-1)^n q_2^{mn} + $$

$$+ \pi^2 \sum_{m\in 2\mathbb{Z}+1} \sum_{\substack{n\in\mathbb{Z} \\ mn>0}} \operatorname{sgn}(n)\, n\, q_2^{mn}$$

$$\frac{4}{\pi^2} g(\tau) = 1 + 4 \sum_{n=1}^{\infty} \Big(2 \sum_{\substack{d\mid n \\ 2\nmid n/d}} d - 2 \sum_{\substack{d \mid n \\ 2\mid (n/d)}} (-1)^d d \Big) q_2^n \qquad (d>0)$$

$$\Rightarrow \frac{4}{\pi^2} g(\tau) = \theta(\tau)^4,$$

$$\forall n \geq 1 \qquad r_4(n) = 8 \Big(\sum_{\substack{d\mid n \\ 2\nmid n/d}} d - \sum_{\substack{d\mid n \\ 2\mid (n/d)}} (-1)^d d \Big) = 8 \sum_{4\nmid d \mid n} d$$

$M_4(\Gamma_\theta)$: $\dim = 2$, contains $G_4(\tau) \in M_4(SL_2(\mathbb{Z}))$ and $G_{4,2,1,1}(\tau)$

$$G_4(\tau) = \underbrace{2\zeta(4)}_{\pi^4/45} + \frac{(2\pi)^4}{6} \sum_{\substack{m,n\in\mathbb{Z} \\ mn>0}} |n|^3 q_2^{2mn} = \frac{\pi^4}{45}\left(1 + 240 q_2^2 + \cdots\right)$$

$$G_{4,2,1,1}(\tau) = \sum_{\substack{m,n\in\mathbb{Z} \\ m\equiv n\equiv 1 \pmod 2}} \frac{1}{(m\tau+n)^4} = \frac{\pi^4}{6} \sum_{\substack{m\in\mathbb{Z} \\ 2\nmid m}} \sum_{\substack{n\in\mathbb{Z} \\ mn>0}} \operatorname{sgn}(n)\, n^3 (-1)^n q_2^{mn}$$

$$= \frac{\pi^4}{6}\left(0 - 2\cdot q_2 + 16 q_2^2 + \cdots\right)$$

$$\Rightarrow \underbrace{\theta(\tau)^8}_{1+16 q_2 + \cdots} = \frac{45}{\pi^4} G_4(\tau) - 8 \cdot \frac{6}{\pi^4} G_{4,2,1,1}(\tau)$$

$$\Rightarrow \forall n\geq 1 \qquad r_8(n) = 240 \sum_{\substack{d \mid \frac{n}{2} \\ (=0 \text{ if } 2\nmid n)}} d^3 - 8\cdot 2 \sum_{\substack{d\mid n \\ 2\nmid \frac{n}{d}}} (-1)^d d^3$$

$$= 16 \sum_{d\mid n} (-1)^{d+n} d^3 \qquad (d>0)$$

Similar calculations:

$\theta^2 \in M_1(\Gamma_\theta, \chi)$, $\quad S_1(\Gamma_\theta, \chi) = 0$

$\theta^6 \in M_3(\Gamma_\theta, \chi^{-1})$, $\quad S_3(\Gamma_\theta, \chi^{-1}) = 0$

One can write explicitly θ^2, θ^6 as suitable Eisenstein series.

What happens for $\theta(\tau)^{2m} \in M_m(\Gamma_\theta, \chi^m)$ is $m > 4$?

In this case $\theta(\tau)^{2m} = (\text{Eisenstein series}) + \underbrace{(\text{cusp form})}_{\neq 0}$

Ex (Ramanujan) $\quad \theta(\tau)^{24} = 1 + \frac{16}{691} \sum_{m=1}^{\infty} \frac{m^{11} q^m}{1-(-q)^m} + a\Delta(\tau + \tfrac{1}{2}) + b\Delta(2\tau)$

$691a = 33152$, $\quad 691b = -65536$

However: $\theta(\tau)^{2m}$ can be written as a linear combination of products of two Eisenstein series (of level 2):

$$\theta(2\tau)^{24} = \frac{1}{9}\left(1 + 16\sum_{m=1}^{\infty} \frac{m^3 q^m}{1-(-q)^m}\right)\left(17 + 32\sum_{m=1}^{\infty} \frac{m^7 q^m}{1-(-q)^m}\right) -$$

$$- \frac{8}{9}\left(1 - 8\sum_{m=1}^{\infty} \frac{m^5 q^m}{1-(-q)^m}\right)^2$$

What about $\theta(\tau)^k$, $2 \nmid k$?

In this case the weight $\frac{k}{2} \in \mathbb{Z} + \frac{1}{2}$ and the coefficients $r_k(n)$ are much more complicated.

Gauss ($k=3$): showed that

$$r_3^*(n) := \{(n_1, n_2, n_3) \mid n_j \in \mathbb{Z},\ \gcd(n_1, n_2, n_3) = 1,\ n_1^2 + n_2^2 + n_3^2 = n\}$$

is equal to $r_3^*(n) = (\text{elementary term}) \cdot (h(-n) \text{ or } h(-4n))$, where $h(D)$ ($D \in \mathbb{Z}$, $D < 0$, $D \equiv 0, 1 \pmod 4$) is the class number of primitive positive definite quadratic forms $Ax^2 + Bxy + Cy^2$ ($A, B, C \in \mathbb{Z}$, $(A, B, C) = 1$) of discriminant $D = B^2 - 4AC$. This is much stronger than Legendre's theorem saying that

$$r_3(n) > 0 \iff n \neq 4^k(8m-1).$$