Modular forms on $SL_2(\mathbb{Z})$

Archetypal example: $G_{2k}(\tau)$ $(k \geq 2)$

Fix $k \in \mathbb{Z}$ (k = weight). For $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{R})^+$ and $f: \mathcal{H} \to \mathbb{C}$ we let $\boxed{(f|_k \alpha)(\tau) = \det(\alpha)^{k/2} (c\tau+d)^{-k} f\left(\frac{a\tau+b}{c\tau+d}\right)}$.

Def. $A_k = A_k(SL_2(\mathbb{Z})) := \{ f: \mathcal{H} \to \mathbb{C}$ meromorphic such that

(1) $\forall \alpha \in SL_2(\mathbb{Z})$ $\quad f|_k \alpha = f$

(2) f is meromorphic at ∞

Explanation: [(1) for $\alpha = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$] $\iff f(\tau+1) = f(\tau)$ $\iff$ for $\tau \in \mathcal{H}$ with $\mathrm{Im}(\tau) >$ const. $f(\tau) = g(e^{2\pi i \tau})$, where g is meromorphic in $q = e^{2\pi i \tau}$ for $0 < |q| < (\text{const.}) < 1$. $\Rightarrow$ the condition (2) means that g extends to a meromorphic function on $\{ |q| < (\text{const.}) \}$, hence

$$f(\tau) = \sum_{n \geq n_0} a_n q^n \quad (\text{convergent if } \mathrm{Im}(\tau) > \text{const.})$$

Idea: $\tau = x+iy$, "$\tau \to i\infty$" $\iff y \to +\infty \iff q \to 0$.

Def. $M_k = M_k(SL_2(\mathbb{Z})) = \left\{ \begin{array}{l} f: \mathcal{H} \to \mathbb{C} \text{ holomorphic}, \forall \alpha \in SL_2(\mathbb{Z}) \; f|_k \alpha = f, \\ f \text{ holomorphic at } \infty: f(\tau) = \sum_{n \geq 0} a_n q^n \; (\tau \in \mathcal{H}) \end{array} \right\}$ $\quad q = e^{2\pi i \tau}$

"modular forms of weight k"

Def. Cusp forms of weight k on $SL_2(\mathbb{Z})$:

$$S_k = S_k(SL_2(\mathbb{Z})) = \{ f \in M_k \mid f(i\infty) = 0 \;\; (\iff f(\tau) = \sum_{n > 0} a_n q^n) \}$$

Ex: (1) $G_{2k}(\tau) \in M_{2k}$, $\quad G_{2k}(\tau) \notin S_{2k}$ $\quad (k \geq 2)$

(2) $J(\tau) = \dfrac{E_4(\tau)^3}{E_4(\tau)^3 - E_6(\tau)^2}$, $\quad j(\tau) = 12^3 J(\tau) \in A_0$

$E_{2k}(\tau) = G_{2k}(\tau) / G_{2k}(i\infty)$

Rmks: (1) $-I \in SL_2(\mathbb{Z}) \Longrightarrow (f|_k(-I))(\tau) = (-1)^k f \Longrightarrow A_k = 0$ if $2 \nmid k$.

(2) If $f \in A_0$ is holomorphic on $\mathcal{H}$ (e.g., if $f \in \mathbb{C}[j(\tau)]$), then f defines a function on $SL_2(\mathbb{Z}) \backslash \mathcal{H} \simeq \{\text{lattices } L \subset \mathbb{C}\} / \mathbb{C}^\times$

(3) $M_* = \sum_{k \in \mathbb{Z}} M_k = \bigoplus_{k \in \mathbb{Z}} M_k$ is a graded ring: $M_k \cdot M_\ell \subset M_{k+\ell}$

(idem for $A_* = \bigoplus_{k \in \mathbb{Z}} A_k = \sum_{k \in \mathbb{Z}} A_k$)

Goal: $\boxed{M_* = \mathbb{C}[G_4, G_6]}$

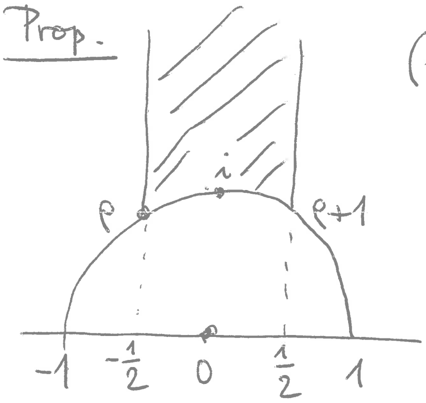
Action of $SL_2(\mathbb{Z})$ on $\mathcal{H}$

Notation: $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \Rightarrow S^2 = -I$, $ST = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$,
$(ST)^2 = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$, $(ST)^3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$.

Exercise: the stabilisers $SL_2(\mathbb{Z})_i = \{\pm I, \pm S\}$
$SL_2(\mathbb{Z})_\rho = \{\pm I, \pm ST, \pm (ST)^2\}$ $\quad (\rho = e^{2\pi i/3}, \quad G_x = \{g \in G \mid g(x) = x\})$

Def. A fundamental domain for $SL_2(\mathbb{Z})$ in $\mathcal{H}$ is an open set $D \subset \mathcal{H}$ such that (1) $\mathcal{H} = \bigcup_{\gamma \in SL_2(\mathbb{Z})} \gamma \overline{D}$,
(2) $\forall \gamma \in SL_2(\mathbb{Z}) \setminus \{\pm I\}$ $\quad \gamma D \cap D = \emptyset$
(3) the hyperbolic area of the boundary: $\mathrm{vol}(\overline{D} \setminus D) = 0$.

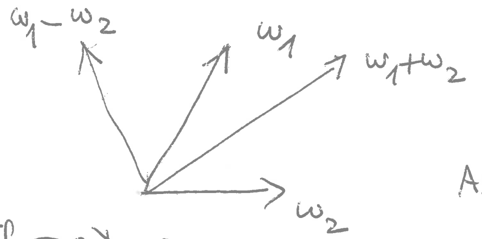
Prop.



(1) $D = \{\tau \in \mathcal{H} \mid |\mathrm{Re}(\tau)| < \frac{1}{2}, |\tau| > 1\}$ is a fundamental domain of $SL_2(\mathbb{Z})$ in $\mathcal{H}$.

(2) the set $D \cup (\partial D \cap \{\mathrm{Re}(\tau) \leq 0\})$ is a set of representatives of $SL_2(\mathbb{Z}) \backslash \mathcal{H}$.

Pf. (1) Need to show: (a) given a lattice $L \subset \mathbb{C}$ there is a positive basis $\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 = L$ such that $\frac{\omega_1}{\omega_2} \in \overline{D}$:
choose $\omega_2 \in L \setminus \{0\}$ s.t. $|\omega_2| = \min_{0 \neq u \in L} |u|$. then
$L \cap \mathbb{Q}\omega_2 = \mathbb{Z}\omega_2 \Rightarrow \omega_2$ can be completed to a basis of L.
Choose any $\omega_1' \in L \setminus \mathbb{Z}\omega_2$ s.t. $\mathbb{Z}\omega_1' \oplus \mathbb{Z}\omega_2$ is a positive basis and let $\omega_1 := \omega_1' + m\omega_2$ $(m \in \mathbb{Z})$, where m is chosen so that $|\omega_1| = \min_{u \in \omega_1' + \mathbb{Z}\omega_2} |u|$. Let $\tau := \frac{\omega_1}{\omega_2} \in \mathcal{H}$.



As $|\omega_1| \geq |\omega_2| \Rightarrow |\tau| \geq 1$.
As $|\omega_1 \pm \omega_2| \geq |\omega_1| \Rightarrow |\tau \pm 1| \geq |\tau|$
$\Downarrow$
$\boxed{|\mathrm{Re}(\tau)| \leq \frac{1}{2}}$
$\Rightarrow \tau \in \overline{D}$

(b) If $\tau \in D \Rightarrow \omega_1$ is unique (for given ω_2).
$\Downarrow$
$|m\tau \pm n|^2 = m^2|\tau|^2 \pm 2mn\,\mathrm{Re}(\tau) + n^2 > m^2 \pm mn + n^2 \geq 1$ if $m, n \in \mathbb{Z}, m \neq 0$
$\Rightarrow \omega_2$ is unique up to a sign.

(2) When is $\tau \in \overline{D} \cap \gamma\overline{D}$ for some $\gamma \in SL_2(\mathbb{Z})$, $\gamma \neq \pm I$?

(a) If $|\tau|>1 \Rightarrow \begin{cases} \omega_2 \text{ is unique up to a sign.} \\ \omega_1 \text{ is unique unless } |\omega_1+\omega_2|=|\omega_1| \text{ or } |\omega_1-\omega_2|=|\omega_1| \end{cases}$

$\Leftrightarrow \operatorname{Re}(\tau)=\pm\frac{1}{2}$, $\gamma = \pm\begin{pmatrix} 1 & \pm1 \\ 0 & 1\end{pmatrix} = \pm T^{\pm 1}$

(b) If $|\tau|=1$, $|\operatorname{Re}(\tau)| \neq \frac{1}{2}$ $\Rightarrow$ we can replace ω_2 by $\pm\omega_1$ and ω_1 by $\mp\omega_2$

$(\Leftrightarrow \gamma = \pm S = \pm\begin{pmatrix} 0 & -1 \\ 1 & 0\end{pmatrix})$.

(c) If $|\tau|=1$, $|\operatorname{Re}(\tau)|=\frac{1}{2}$: $\tau=\rho$ or $\rho+1$ $\Rightarrow \gamma = T^{\pm1}\gamma'$, $\gamma' \in SL_2(\mathbb{Z})_\rho$ or $\rho+1$.

Cor. $SL_2(\mathbb{Z})$ is generated by $S=\begin{pmatrix} 0 & -1 \\ 1 & 0\end{pmatrix}$ and $T=\begin{pmatrix} 1 & 1 \\ 0 & 1\end{pmatrix}$.

Pf. Let $\Gamma = \langle S, T\rangle \subseteq SL_2(\mathbb{Z})$. Note: $-I = S^2 \in \Gamma$.

Claim: $\mathcal{H} = \bigcup_{\gamma\in\Gamma} \gamma\overline{D}$.

Pf: Fix $\tau\in\mathcal{H}$. As $\{|c\tau+d| : \begin{pmatrix} a & b \\ c & d\end{pmatrix}\in\Gamma\}$ is bounded below

$\Rightarrow \{\operatorname{Im}(\gamma(\tau))\}_{\gamma\in\Gamma}$ attains maximum at some $\gamma\in\Gamma$.

Applying T^n $(n\in\mathbb{Z})$ we can assume $z=T^n\gamma(\tau)$ satisfies $|\operatorname{Re}(z)| \le \frac{1}{2}$. As $\operatorname{Im}(z) \ge \operatorname{Im}(S(z)) = \operatorname{Im}(z)/|z|^2 \Rightarrow |z|\ge 1$

$\Rightarrow z\in\overline{D} \Rightarrow \tau = \gamma^{-1}T^{-n}(z) \in \Gamma\overline{D}$.

Claim $\Rightarrow$ Cor. Let $\tau_0 \in D$, $g\in SL_2(\mathbb{Z})\setminus\{\pm I\}$. Then $\exists \gamma\in\Gamma\ \exists\tau\in\overline{D}$

$g(\tau_0) = \gamma(\tau) \Rightarrow \gamma^{-1}g(\tau_0) = \tau \in \overline{D} \Rightarrow \gamma^{-1}g(\tau_0)\in D$

$\Rightarrow \gamma^{-1}g = \pm I \Rightarrow g\in\Gamma$.

Def. An elliptic (resp. parabolic) point of $SL_2(\mathbb{Z})$ is $\tau\in\mathcal{H}$ (resp. $z\in\mathbb{P}^1(\mathbb{R})$) such that $\exists$ elliptic (resp. parabolic) $\gamma\in SL_2(\mathbb{Z})$ such that $\gamma(\tau)=\tau$ (resp. $\gamma(z)=z$).

Prop. $\{\text{elliptic points of } SL_2(\mathbb{Z})\} = \{\gamma(i), \gamma(\rho) \mid \gamma\in SL_2(\mathbb{Z})\}$.

Pf. $S(i)=i$, $ST(\rho)=\rho$, $TS(\rho+1)=\rho+1$. Let $\tau\in\overline{D}$ be elliptic:

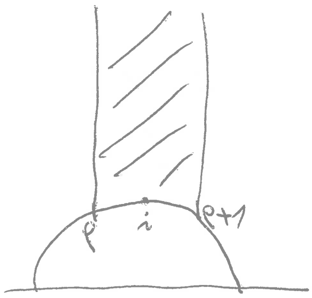
$\exists g=\begin{pmatrix} a & b \\ c & d\end{pmatrix}\in SL_2(\mathbb{Z})$ $\begin{pmatrix} a & b \\ c & d\end{pmatrix}\begin{pmatrix} \tau \\ 1\end{pmatrix} = (c\tau+d)\begin{pmatrix} \tau \\ 1\end{pmatrix} \Rightarrow c\tau+d$ is a root of

$X^2 - \operatorname{Tr}(g)X + \det(g) = X^2 \pm X+1$ or X^2+1 (since $|\operatorname{Tr}(g)|<2$)

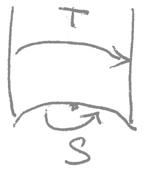
$\Rightarrow c\tau+d \in \{i, \rho, \rho+1\}$. As $\tau\in\overline{D} \Rightarrow \tau\in\{i,\rho,\rho+1\}$.

Fundamental domains for $\Gamma \subset \Gamma(1) = SL_2(\mathbb{Z})$ of finite index

Recall: $D = \{\tau \in \mathcal{H} \mid |\tau| > 1,\ |\mathrm{Re}(\tau)| < \frac{1}{2}\}$ is a fundamental domain of $\Gamma(1) = SL_2(\mathbb{Z})$ in $\mathcal{H}$



$\Gamma(1)\backslash\mathcal{H}$ is obtained from $\overline{D} = D \cup \partial D$ by glueing



$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} : \tau \longmapsto \tau + 1$

$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} : \tau \longmapsto -1/\tau$

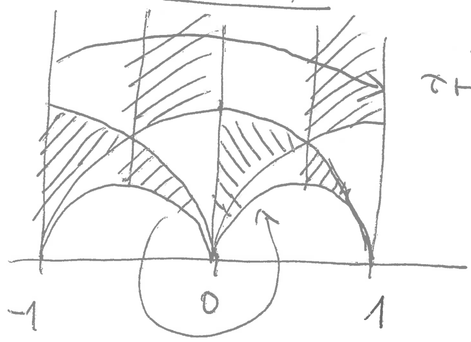
If $\Gamma \subset \Gamma(1)$, $(\Gamma(1) : \Gamma) < \infty$: fix coset representatives

$\overline{\Gamma(1)} = \bigsqcup_{\alpha_j} \overline{\Gamma}\alpha_j \qquad (\overline{\Gamma} = \mathrm{Im}(\Gamma \longrightarrow SL_2(\mathbb{R})/\{\pm I\}))$

$\Longrightarrow \bigcup_j \alpha_j D$ (disjoint union) is a fundamental domain of Γ.

One can also cut and move around pieces by applying suitable $\alpha \in SL_2(\mathbb{R})/\{\pm I\}$.

Ex 1. $\Gamma = \Gamma(2)$: $\quad (\Gamma(1) : \Gamma(2)) = (\overline{\Gamma(1)} : \overline{\Gamma(2)}) = |SL_2(\mathbb{Z}/2\mathbb{Z})| = 6$

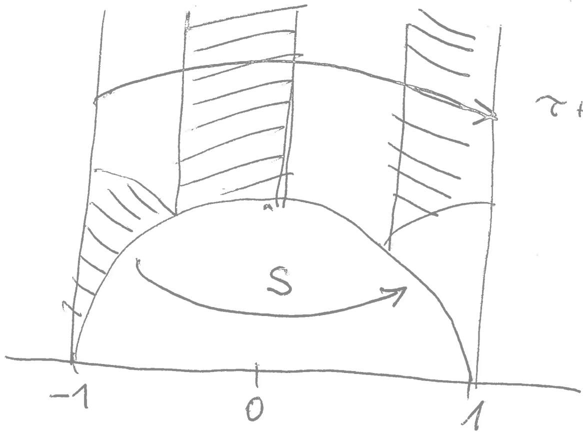


$\tau \xrightarrow{T^2} \tau + 2 \qquad \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$

$\overline{\Gamma(2)}$ = free on the images of T^2, T'^2 in $SL_2(\mathbb{R})/\{\pm I\}$

$\tau \longmapsto \frac{\tau}{2\tau+1} \qquad \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} = T'^2$

Ex 2. $\Gamma = \Gamma_\theta = \Gamma(2) \cup S\Gamma(2)$, $\quad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $\quad (\overline{\Gamma(1)} : \overline{\Gamma_\theta}) = 3$



$\tau \xrightarrow{T^2} \tau + 2$

$\Gamma_\theta = \langle S, T^2 \rangle$

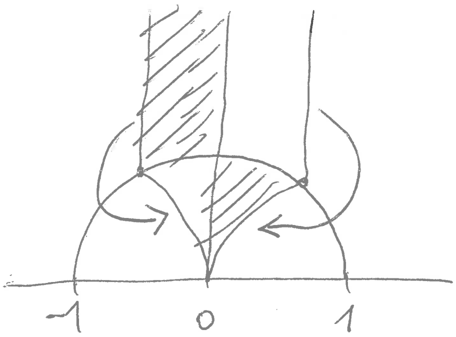
Ex 3: Γ = the unique subgroup of $\Gamma(1) = SL_2(\mathbb{Z})$ of index 2:

$$\Gamma = \mathrm{Ker}\left(\Gamma(1) \longrightarrow \Gamma(1)/\Gamma(2) = SL_2(\mathbb{Z}/2\mathbb{Z}) \xrightarrow{\sim} S_3 \xrightarrow{\mathrm{sgn}} \{\pm 1\}\right)$$

$$= \left\{ \alpha \in \Gamma(1) \;\middle|\; \alpha \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \pmod 2 \right\}$$

$$\overline{\Gamma(1)} = \overline{\Gamma} \cup \overline{\Gamma} S$$

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$



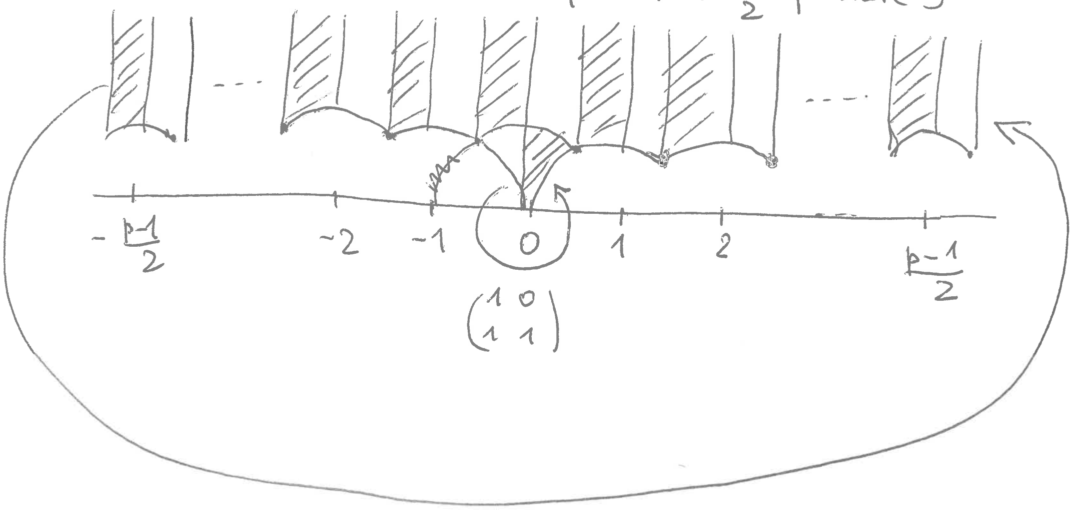
Ex 4: $\Gamma = \Gamma^0(p) = \left\{ \alpha \in \Gamma(1) \;\middle|\; \alpha \equiv \begin{pmatrix} * & 0 \\ * & * \end{pmatrix} \pmod p \right\}$, $p > 2$ prime

$$(\Gamma(1) : \Gamma^0(p)) = (\overline{\Gamma(1)} : \overline{\Gamma^0(p)}) = \left(SL_2(\mathbb{Z}/p\mathbb{Z}) : \left\{ \begin{pmatrix} * & 0 \\ * & * \end{pmatrix} \right\} \right)$$

$(-I \in \Gamma^0(p))$ $$= |\mathbb{P}^1(\mathbb{Z}/p\mathbb{Z})| = p+1$$

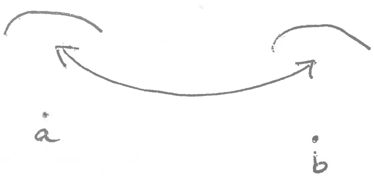
Representatives of $\Gamma^0(p) \backslash \Gamma$: $\left\{ T^a = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \;\middle|\; a \in \mathbb{Z}/p\mathbb{Z} \right\}, S$

(can take $\{ a \in \mathbb{Z} \mid |a| \le \frac{p-1}{2} \}$ here)



$T^p : \tau \longmapsto \tau + p$

Glueing of



$$T^b S T^{-a} = \begin{pmatrix} b & -1-ab \\ 1 & -a \end{pmatrix}$$

$$ab + 1 \equiv 0 \pmod p$$

Zeroes of modular forms

Def. For $\tau \in \mathcal{H} \cup \{\infty\}$ let $e_\tau = \begin{cases} 2 & \tau \in SL_2(\mathbb{Z})\cdot i \\ 3 & \tau \in SL_2(\mathbb{Z})\cdot \rho \\ 1 & \text{otherwise} \end{cases}$

Def. For $0 \neq f \in A_k(SL_2(\mathbb{Z}))$ we have

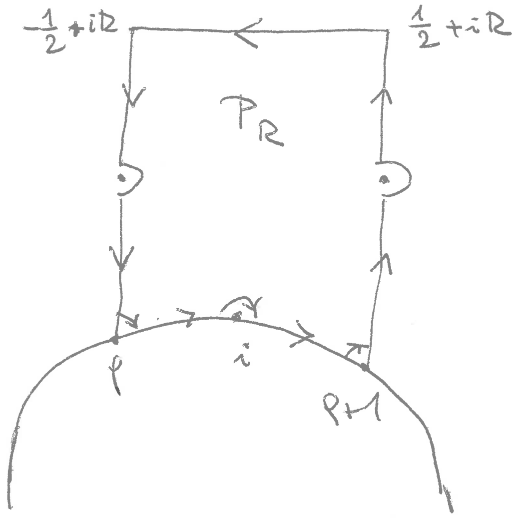
$f(\tau) = \sum_{n \geq n_0} a_n q^n$ with $a_{n_0} \neq 0$ $(q = e^{2\pi i \tau})$.

Let $\mathrm{ord}_\infty(f) := n_0$.

Thm. If $0 \neq f \in A_k(SL_2(\mathbb{Z}))$, then

$$\boxed{\sum_{\tau \in (SL_2(\mathbb{Z})\backslash\mathcal{H}) \cup \{\infty\}} \frac{\mathrm{ord}_\tau(f)}{e_\tau} = \frac{k}{12}}$$

Pf. We want to compute "$\frac{1}{2\pi i} \int_{\partial D} \frac{f'(\tau)}{f(\tau)} d\tau$" ($D$ as above)



$(R \gg 0)$ Make small circular cuts at ρ, i, $\rho+1$ and at singular points of f on ∂P_R, so that equivalent points have congruent cuts. Then let the radii of these circles $\longrightarrow 0$. Integrate along ∂P_R:

$$\sum_{\tau \in P_R} \mathrm{ord}_\tau(f) = \frac{1}{2\pi i} \int_{\partial P_R} \frac{f'(\tau)}{f(\tau)} d\tau$$

① horizontal integral from $\frac{1}{2}+iR$ to $-\frac{1}{2}+iR$:

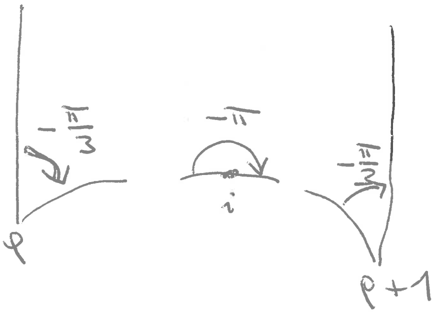
$f(\tau) = \sum_{n \geq n_0} a_n q^n$, $f'(\tau) = (2\pi i) \sum_{n \geq n_0} n a_n q^n$, $a_{n_0} \neq 0$, $n_0 = \mathrm{ord}_\infty(f)$

$\frac{1}{2\pi i} \frac{f'(\tau)}{f(\tau)} = n_0 + \sum_{n \geq 1} b_n e^{2\pi i n \tau} \Longrightarrow$ contribution of $-\frac{1}{2}+iR \longleftarrow \frac{1}{2}+iR$

is $-n_0 = -\mathrm{ord}_\infty(f)$.

(2) <u>vertical integrals</u> cancel each other ($f(\tau+1)=f(\tau)$)

(3) <u>around $\tau_0 \in \{\rho, i, \rho+1\}$</u> : $\frac{f'(\tau)}{f(\tau)} = \frac{\mathrm{ord}_{\tau_0}(f)}{\tau-\tau_0} + (\text{holomorphic at } \tau_0)$

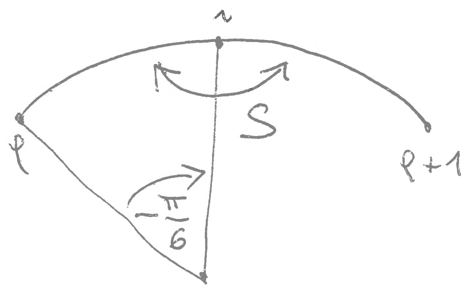


As radii $\longrightarrow 0$, the contribution is

$$\frac{1}{2\pi}\left(-\frac{\pi}{3}\,\mathrm{ord}_\rho(f) - \pi\,\mathrm{ord}_i(f) - \frac{\pi}{3}\,\mathrm{ord}_{\rho+1}(f)\right) =$$

$$= -\frac{1}{3}\,\mathrm{ord}_\rho(f) - \frac{1}{2}\,\mathrm{ord}_i(f).$$

(4)



$$S(\tau) = -\frac{1}{\tau}, \quad f(S(\tau)) = \tau^k f(\tau)$$

$$\tau^{-2} f'(S(\tau)) = k\tau^{k-1} f(\tau) + \tau^k f'(\tau)$$

$$\tau^{-2}\frac{f'(S(\tau))}{f(S(\tau))} = \frac{k}{\tau} + \frac{f'}{f}(\tau)$$

$$\frac{1}{2\pi i}\int_\rho^{\rho+1} \frac{f'}{f}(\tau)\,d\tau = \frac{1}{2\pi i}\int_\rho^i \frac{f'}{f}(\tau)\,d\tau - \frac{1}{2\pi i}\int_\rho^i \underbrace{\frac{f'}{f}(S(\tau))\,dS(\tau)}_{\left(\frac{k}{\tau}+\frac{f'}{f}(\tau)\right)d\tau} =$$

$$= \frac{1}{2\pi i}\int_\rho^i -\frac{k}{\tau}\,d\tau = \frac{1}{2\pi}\left(-\frac{\pi}{6}\right)(-k) = \frac{k}{12}.$$

Consequences of $\displaystyle\sum_{\tau \in \{\infty\} \cup (SL_2(\mathbb{Z})\backslash\mathcal{H})} \frac{\mathrm{ord}_\tau(f)}{e_\tau} = \frac{k}{12}$:

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(1) <u>$k<0$</u> : RHS $<0 \implies M_k = 0$.

(2) <u>$k<12$</u> : RHS $<1 \implies S_k = 0$.

(0) <u>$k=0$</u> : $S_0 = 0$. $\Downarrow$ $M_0 = \mathbb{C}$.

(3) <u>$k=2$</u> : $a + \frac{b}{2} + \frac{c}{3} \neq \frac{1}{6}$ $(a,b,c \in \mathbb{Z}_{\geq 0}) \implies M_2 = 0$.

(4) <u>$k=12$</u> : $\underbrace{E_4^3 - E_6^2}_{12^3\Delta(\tau)} \in S_{12}$, $\mathrm{ord}_\infty(\ ) = 1 \implies \forall \tau \in \mathcal{H}$ $(E_4^3 - E_6^2)(\tau) \neq 0$.

<u>Prop</u>. $\forall k \in \mathbb{Z}$ $M_k \xrightarrow{\cdot\Delta} S_{k+12}$, $f \longmapsto \Delta\cdot f$ <u>is a bijection.</u>

<u>Pf</u>. this is well-defined and so is its inverse : $g \longmapsto \frac{g}{\Delta}$.

Cor. $j: \underbrace{SL_2(\mathbb{Z})\backslash\mathcal{H}}_{\simeq\{\text{lattices } L\subset\mathbb{C}\}/\mathbb{C}^\times} \longrightarrow \mathbb{C}$ and

$j: (SL_2(\mathbb{Z})\backslash\mathcal{H}) \cup \{\infty\} \longrightarrow \mathbb{P}^1(\mathbb{C})$ are bijective.

Pf. $\forall a\in\mathbb{C}$ $\quad f(\tau)=j(\tau)-a\in A_0$ is holomorphic on $\mathcal{H}$ and $\mathrm{ord}_\infty(f)=-1 \implies \displaystyle\sum_{\tau\in SL_2(\mathbb{Z})\backslash\mathcal{H}} \frac{\mathrm{ord}_\tau(f)}{e_\tau}=1$

$\implies \exists!\ \tau\in\mathcal{H}$ $\quad f(\tau)=0$, and $\quad e_\tau=\mathrm{ord}_\tau(f)$.

Rmk: $\mathrm{ord}_\rho(j)=3$, $\quad \mathrm{ord}_i(j-12^3)=2$; the functions $J^{1/3}$,
$(J=j/12^3)$ $\qquad\qquad = \mathrm{ord}_i(J-1)$ $\qquad (J-1)^{1/2}$ are of interest

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Prop. $A_0=\mathbb{C}(j)$

Pf. If $0\neq f\in A_0$, f has only finitely many singularities in $\overline{D} \implies$ some $\quad g(\tau)=\displaystyle\prod_{k=1}^{n}(j(\tau)-j(\tau_k))^{n_k} f(\tau)\in A_0$ is holomorphic on $\mathcal{H}$. Write $\quad g(\tau)=\displaystyle\sum_{n=-N}^{\infty} a_n q^n = a_n j^{-N}+\dots$

Recursively we find a polynomial $P(j)\in\mathbb{C}[j]$ such that
$g-P(j)\in S_0=\{0\} \implies f=P(j)/Q(j)$.

Prop. If $f=\displaystyle\sum_{n=0}^{\infty}a_n q^n$, $g=\displaystyle\sum_{n=0}^{\infty}b_n q^n$, $f,g\in M_k$ and $\forall n\le\left[\frac{k}{12}\right]+1$, then $f=g$.

Pf. If $f\neq g \implies \dfrac{k}{12}=\displaystyle\sum\frac{\mathrm{ord}_\tau(f-g)}{e_\tau}\ge \mathrm{ord}_\infty(f-g)\ge\left[\frac{k}{12}\right]+1>\frac{k}{12}$
contradiction

Prop: $M_0 = \mathbb{C} = \mathbb{C}\cdot 1 \oplus S_0$, $\quad M_2 = S_2 = 0$

$\forall k \geq 2 \quad M_{2k} = \mathbb{C}\cdot G_{2k} \oplus S_{2k}$.

Pf. $G_{2k} \in M_{2k}$, $G_{2k} \notin S_{2k} \Longrightarrow \mathbb{C}\cdot G_{2k} \cap S_{2k} = \{0\}$. $\quad (k \geq 2)$

If $f(\tau) = \sum_{n=0}^{\infty} a_n q^n \in M_{2k} \Longrightarrow f - a_0 E_{2k} \in S_{2k}$

$\Longrightarrow M_{2k} = \mathbb{C}G_{2k} \oplus S_{2k}$

Table: basis of M_k, S_k (k even)

k	0	2	4	6	8	10	12	14	16	18	20	22	24	26
S_k	0	0	0	0	0	0	Δ	0	ΔG_4	ΔG_6	ΔG_8	ΔG_{10}	ΔG_{12}, Δ^2	ΔG_{14}
M_k	1	0	G_4	G_6	G_8	G_{10}	G_{12}, Δ	G_{14}	ΔG_4, G_{16}	ΔG_6, G_{18}	ΔG_8, G_{20}	ΔG_{10}, G_{22}	ΔG_{12}, Δ^2, G_{24}	ΔG_{14}, G_{26}

Prop. Let $k \geq 0$ be even. (1) $\dim(M_k) = \dim(S_k) + \begin{cases} 1 & k \neq 2 \\ 0 & k = 2. \end{cases}$

(2) $\dim(M_{k+12}) = \dim(M_k) + 1$, $\dim(S_{k+12}) = \dim(S_k) + (\text{— '' —})$.

(3) $\dim(M_k) = \left[\frac{k}{12}\right] + \begin{cases} 1, & k \not\equiv 2 \pmod{12} \\ 0, & k \equiv 2 \pmod{12} \end{cases}$

$\dim(S_k) = \left[\frac{k}{12}\right] + \begin{cases} 0, & k \not\equiv 2 \pmod{12} \text{ or } k = 2 \\ -1, & k \equiv 2 \pmod{12} \text{ but } k \neq 2 \end{cases}$

Pf: (1) Prop. above. (2) Use (1) and $\Delta : M_k \xrightarrow{\sim} S_{k+12}$.

(3) Use (2) and induction.

Prop. $M_* = \mathbb{C}[G_4, G_6]$, with G_4, G_6 algebraically independent over $\mathbb{C}$.

Cor. Basis of M_k: $G_4^a G_6^b$, $a, b \in \mathbb{Z}_{\geq 0}$, $4a + 6b = k$.

Pf. $\mathbb{C}[G_4, G_6] \subseteq M_*$.

If $\exists\, 0 \neq P \in \mathbb{C}[X, Y]$ $P(G_4, G_6) = 0$, take P of smallest degree d: $P = \sum_{4k+6l=d} a_{kl} X^k Y^l$. As $\left.\begin{array}{l} G_4(\rho) = 0 \neq G_6(\rho) \\ G_4(i) \neq 0 = G_6(i) \end{array}\right\} \Longrightarrow \begin{array}{l} a_{k,0} = 0 \\ a_{0,l} = 0 \end{array}$

$\Longrightarrow P$ is divisible by XY — contradiction.

As $M_{2k+12} = \mathbb{C}\, G_{2k+12} \oplus \Delta M_{2k} \xRightarrow{\text{induction}} M_{2k} \subset \mathbb{C}[G_4, G_6]$,

since Δ, $G_{2k} \in \mathbb{C}[G_4, G_6]$.

Alternatively: $\dim \mathbb{C}[G_4, G_6]_{\text{weight}=k} = \sum_{4a+6b=k} 1 = \sum_{\substack{b=0 \\ b \equiv \frac{k}{2} \pmod 2}}^{\left[\frac{k}{6}\right]} 1 =: d(k)$

satisfies $d(k+12) = d(k) + 1$ and

$d(2) = 0$, $d(k) = 1$ for $k = 0, 4, 6, 8, 10 \Longrightarrow \dim(M_k) = d(k)$ $\forall k \geq 0$ even.

$\Longrightarrow M_* = \mathbb{C}[G_4, G_6]$.

The differential operator $q\frac{d}{dq}$

Notation: $D = q\frac{d}{dq} = \frac{1}{2\pi i}\frac{d}{d\tau}$ $\quad (q = e^{2\pi i \tau})$

$$D\Big(\sum_{n\geq 0} a_n q^n\Big) = \sum_{n\geq 1} n a_n q^n$$

Recall: $(f|_k\alpha)(\tau) = (\det(\alpha))^{k/2}(c\tau+d)^{-k} f(\alpha(\tau))$, $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{R})^+$

$\frac{d}{d\tau}\alpha(\tau) = \frac{\det(\alpha)}{(c\tau+d)^2} = \alpha'(\tau)$ $\quad (k \in \mathbb{Z})$

Prop. $f: \mathcal{H} \to \mathbb{C}$ holomorphic $\Rightarrow D(f|_k\alpha) - (Df)|_{k+2}\alpha = -\frac{kc}{2\pi i(c\tau+d)} f|_k\alpha$

Pf: $(2\pi i)\big((Df)|_{k+2}\alpha\big)(\tau) = (f'|_{k+2}\alpha)(\tau) = \frac{\det(\alpha)^{(k+2)/2}}{(c\tau+d)^{k+2}} f'\Big(\frac{a\tau+b}{c\tau+d}\Big)$

$2\pi i\big(D(f|_k\alpha)\big)(\tau) = \det(\alpha)^{k/2}\Big(\frac{-kc}{(c\tau+d)^{k+1}} f\Big(\frac{a\tau+b}{c\tau+d}\Big) + \frac{\det(\alpha)}{(c\tau+d)^{k+2}} f'\Big(\frac{a\tau+b}{c\tau+d}\Big)\Big)$

Cor. If $f, g: \mathcal{H} \to \mathbb{C}$ holomorphic, $k, l \in \mathbb{Z}$, then:

(1) $l(g|_l\alpha)\big(D(f|_k\alpha) - (Df)|_{k+2}\alpha\big) = k(f|_k\alpha)\big(D(g|_l\alpha) - (Dg)|_{l+2}\alpha\big)$

(2) If $f|_k\alpha = f$ and $g|_l\alpha = g$ $\Rightarrow$ $h := lgDf - kfDg$ satisfies $h|_{k+l+2}\alpha = h$

(3) $f \in M_k$, $g \in M_l \Rightarrow lgDf - kfDg \in S_{k+l+2}$.

Ex: $\Delta \in S_{12}$, $\quad D(\Delta)/\Delta = E_2 = 1 - 24\sum_{n=1}^{\infty}\sigma_1(n)q^n$

therefore $\forall f \in M_k$ $\quad 12\Delta Df - kfE_2\Delta \in S_{k+14} = \Delta M_{k+2}$ $\quad (k \geq 0)$

$$\Rightarrow \boxed{\Big(D - \frac{k}{12}E_2\Big)f \in M_{k+2}}$$

Rmk: the bilinear map

$$M_k \times M_l \longrightarrow S_{k+l+2}$$
$$(f, g) \longmapsto lgDf - kfDg$$

is the simplest Rankin–Cohen bracket.

Prop. the Eisenstein series $E_{2k}(\tau) = G_{2k}(\tau)/G_{2k}(i\infty) = 1 - \frac{4k}{B_{2k}} \sum_{n\geq 1} \sigma_{2k-1}(n) q^n$
$(k \geq 1)$ satisfy

$\underbrace{(D - \tfrac{1}{3}E_2)E_4 = -\tfrac{1}{3}E_6}$, $\underbrace{(D - \tfrac{1}{2}E_2)E_6 = -\tfrac{1}{2}E_8}$, $\underbrace{(D - E_2)\Delta = 0}$

$\underbrace{(D - \tfrac{2}{3}E_2)E_8 = -\tfrac{2}{3}E_{10}}$, $\underbrace{(D - E_2)E_{12} = -E_{14}}$, $\underbrace{(D - \tfrac{1}{12}E_2)E_2 = -\tfrac{1}{12}E_4}$

$$(D - \tfrac{5}{6}E_2)E_{10} = -\frac{5}{6}E_{12} + \frac{2^7 \cdot 3^3 \cdot 11}{691}\Delta$$

Pf. $\forall k \geq 4$ even $\quad (D - \frac{k}{12}E_2)E_k = -\frac{k}{12} + c_1 q + \cdots \in M_{k+2}$

$$= -\frac{k}{12}E_{k+2} + (\underbrace{\text{sth.} \in S_{k+2}}_{=0 \text{ if } k=4,6,8,12})$$

$k=10$: must compare the coefficients at q

$k=2$: $\quad E_2|_2 \begin{pmatrix} a & b \\ c & d \end{pmatrix} = E_2 + \frac{12c}{2\pi i(c\tau+d)}$

the same calculation as before $\Rightarrow (D - \frac{1}{12}E_2)E_2 \in M_4$, etc.

Elementary congruences for $\tau_k(n)$

Notation: for $k \in \{12, 16, 18, 20, 22, 26\}$

$S_k = \mathbb{C} \cdot \Delta_k$, $\quad \Delta_k(\tau) = \sum_{n=1}^{\infty} \tau_k(n) q^n$, $\quad \tau_k(1) = 1 \quad (\Delta_k = \Delta E_{k-12})$.

Prop. $\quad \Delta = \Delta_{12} \equiv \sum_{n=1}^{\infty} \sigma_{11}(n) q^n \pmod{691}$

$(\Longleftrightarrow \tau(n) \equiv \sigma_{11}(n) \pmod{691} \quad \forall n \geq 1)$

Pf. $\quad E_{12} = 1 + \frac{a}{691} \sum_{n\geq 1} \sigma_{11}(n) q^n$, $\quad 691 \nmid a = 65520 = 2^4 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13$

$S_{12} \ni E_{12} - E_6^2 = (\frac{a}{691} + 1008)q + \cdots = (\frac{a}{691} + 1008)\Delta$

$$\Rightarrow \quad 691 E_{12} - 691 \underbrace{E_6^2}_{\text{coefficients in } \mathbb{Z}} = (a + 691 \cdot 1008)\Delta$$

$$\Rightarrow \quad a \sum_{n\geq 1} \sigma_{11}(n) q^n \equiv a\Delta \pmod{691}$$

<u>Recall</u>: $E_4 = 1 + 240 \sum_{n\geq 1} \sigma_3(n) q^n$, $E_2 = 1 - 24 \sum_{n\geq 1} \sigma_1(n) q^n$

$E_6 = 1 - 504 \sum_{n\geq 1} \sigma_5(n) q^n$, $E_8 = 1 + 480 \sum_{n\geq 1} \sigma_7(n) q^n = E_4^2$

$E_{10} = 1 - 264 \sum_{n\geq 1} \sigma_9(n) q^n = E_4 E_6$, $E_{14} = 1 - 24 \sum_{n\geq 1} \sigma_{13}(n) q^n = E_4^2 E_6$

$$E_{12} = \frac{441 E_4^3 + 250 E_6^2}{691} \quad (\in M_{12} = \mathbb{C}\cdot E_4^3 \oplus \mathbb{C}\cdot E_6^2 \text{; compare coeff. at } 1, q)$$

<u>Lemma</u>. p prime, $i, j \geq 1$, $i \equiv j \pmod{(p-1)}$ $\Rightarrow$ $\forall n \geq 1$ $\sigma_i(n) \equiv \sigma_j(n) \pmod p$

<u>Pf</u>: $\forall d \mid n$ $d^i \equiv d^j \pmod p$

Congruences (mod 5^n)

$E_4 \equiv 1 \pmod 5$, $E_2 \equiv E_6 \pmod 5$, $\Delta = \frac{E_4^3 - E_6^2}{12^3} \equiv 2(1 - E_6^2) \pmod 5$

$$D E_6 = \tfrac{1}{2}(E_2 E_6 - E_8) \equiv \tfrac{1}{2}(E_6^2 - 1) \equiv \Delta \equiv \underbrace{\Delta_{16}}_{\Delta E_4} \equiv \underbrace{\Delta_{20}}_{\Delta E_4^2} \pmod 5$$

$$D^2 E_6 \equiv -2 D(E_6^2) \equiv -4 E_6 \, D E_6 \equiv \underbrace{E_6 \Delta}_{\Delta_{18}} \equiv \underbrace{\Delta_{22}}_{E_4 \Delta_{18}} \equiv \underbrace{\Delta_{26}}_{E_4 \Delta_{22}} \pmod 5$$

<u>Cor</u>. $\tau_k(n) \equiv \begin{cases} n\,\sigma_5(n) \pmod 5, & k = 12, 16, 20 \\ n^2 \sigma_5(n) \pmod 5, & k = 18, 22, 26 \end{cases}$

<u>Prop</u>. $D E_{10} \equiv 11 \Delta \pmod{5^2}$

$\forall n \geq 1$ $n \sigma_9(n) \equiv \tau(n) \pmod{5^2}$

<u>Pf</u>: $$\left(D - \tfrac{5}{6} E_2\right) E_{10} = -\tfrac{5}{6} E_{12} + \underbrace{\frac{2^7 \cdot 3^3 \cdot 11}{691}}_{\equiv 1 \pmod{5^2}} \Delta$$

$E_{12} \equiv 1 \pmod 5$, $E_2 E_{10} \equiv E_6 E_6 E_4 \equiv E_6^2 \pmod 5$

$$D E_{10} \equiv 5 \underbrace{(E_6^2 - 1)}_{\equiv 2\Delta \pmod 5} + \Delta \pmod{5^2} \equiv 11 \Delta \pmod{5^2}$$

$$E_{10} \equiv 1 + 11 \sum_{n\geq 1} \sigma_9(n) q^n \pmod{5^2} \Rightarrow D E_{10} \equiv 11 \sum_{n\geq 1} \sigma_9(n) q^n \pmod{5^2}$$

<u>Exercise</u>: <u>(mod 7)</u> $E_6 \equiv 1 \pmod 7$, $E_2 \equiv E_4^2 \pmod 7$, $\Delta \equiv 1 - E_4^3 \pmod 7$

$$\tau_k(n) \equiv \begin{cases} n\sigma_3(n) \\ n\,\sigma_7(n) \\ n^2\sigma_3(n) \end{cases} \pmod 7, \qquad k = \begin{cases} 12, 18 \\ 16, 22 \\ 20, 26 \end{cases}$$

<u>(mod 11)</u>

$$\tau_{16}(n) \equiv \tau_{26}(n) \equiv n\,\sigma_3(n)$$
$$\tau_{18}(n) \equiv n\sigma_5(n) \qquad \pmod{11}$$
$$\tau_{20}(n) \equiv n\,\sigma_7(n)$$

<u>(mod 13)</u>

$$\tau_{18}(n) \equiv n\sigma_3(n)$$
$$\tau_{20}(n) \equiv n\,\sigma_5(n) \qquad \pmod{13}$$
$$\tau_{22}(n) \equiv n\,\sigma_7(n)$$

(mod 17)

$$\tau_{22}(n) \equiv n\,\sigma_3(n) \qquad \pmod{17}$$
$$\tau_{26}(n) \equiv n\,\sigma_7(n)$$

(mod 19) $\quad \tau_{26}(n) \equiv n\,\sigma_5(n) \pmod{19}$

<u>Facts</u>: (1) $\forall p \geq 2$ prime $\quad \underline{E_{p-1} \equiv 1 \pmod p}$

(2) All congruences (mod p) between elements of M_* with coefficients in $\mathbb{Z}$ are consequences of (1)

(3) There are <u>no other</u> congruences of the form

$$\tau_k(n) \equiv n^i\,\sigma_j(n) \pmod p \qquad (p > 3)$$

(4) $\forall p > 2$ prime $\quad E_2 \pmod p \equiv E_{p+1} \pmod p$ is a "(mod p) modular form"

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(5) In fact, $\quad E_i \equiv E_{i+(p-1)} \pmod p$

Ramanujan's congruences (mod ℓ) ($\ell = 5, 7, 11$) for $p(n)$

Differential operators: $\left(D - \frac{k}{12} E_2\right) : M_k \longrightarrow S_{k+2}$, $\quad D = q\frac{d}{dq}$

$$E_4 = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n, \quad E_6 = 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n, \quad E_2 = \frac{D\Delta}{\Delta} = 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n) q^n$$

$$E_8 = E_4^2, \quad E_{10} = E_4 E_6, \quad E_{14} = E_4^2 E_6$$

$$DE_2 = \frac{1}{12}(E_2^2 - E_4), \quad DE_4 - \frac{1}{3} E_2 E_4 = -\frac{1}{3} E_6, \quad DE_6 - \frac{1}{2} E_2 E_6 = -\frac{1}{2} E_4^2$$

$$E_4^3 - E_6^2 = 12^3 \Delta, \quad \Delta = q P(q)^{24} = \sum \tau(n) q^n, \quad \frac{1}{P(q)} = \prod_{n=1}^{\infty} (1-q^n)^{-1} = \sum_{n=0}^{\infty} p(n) q^n$$

<u>Thm.</u> For $\ell \in \{5, 7, 11\}$ and $n \geq 1$, $\quad p\left(\ell n - \frac{\ell^2 - 1}{24}\right) \equiv 0 \pmod{\ell}$.

$$p\left(13n - \frac{13^2 - 1}{24}\right) \equiv 11\, \tau(n) \pmod{13}$$

<u>Pf</u>: <u>$\ell = 5$</u>: $E_4 \equiv 1\ (5)$, $E_6 \equiv E_2\ (5)$ $\Rightarrow$ $E_4^3 - E_6^2 \equiv 1 - E_2^2 \equiv 3(DE_2) \pmod 5$

On the other hand, $\quad E_4^3 - E_6^2 \equiv 3\Delta = 3q P(q)^{24} \equiv 3q \frac{P(q^5)^5}{P(q)} \pmod 5$

$$\Rightarrow \quad q P(q^5)^5 \left(\sum_{n \geq 0} p(n) q^n\right) \equiv DE_2 \equiv \sum_{m \geq 1} m \sigma_1(m) q^m \pmod 5$$

$$\Rightarrow \quad P(q^5)^5 \sum_{m \geq 0} p(5m+4) q^{5m+5} \equiv 0 \pmod 5.$$

<u>$\ell = 7$</u>: $E_6 \equiv 1\ (7)$, $E_4^2 = E_8 \equiv E_2\ (7)$ $\Longrightarrow$ $(E_4^3 - E_6^2)^2 \equiv (E_2 E_4 - 1)^2 \equiv$

$$\equiv E_2^2 E_4^2 - 2E_2 E_4 + 1 \equiv E_2^3 - E_2 E_4 + E_6 - E_2 E_4 \equiv E_2(E_2^2 - E_4) + (E_6 - E_2 E_4) \equiv$$

$$\equiv 5 E_2 (DE_2) - 3(DE_4) \equiv \underbrace{D(6E_2^2 - 3E_4)}_{\text{the coefficients at } q^{7m} \text{ are all } \equiv 0 \pmod 7.} \ (7)$$

But $$(E_4^3 - E_6^2)^2 \equiv (12^3 \Delta)^2 \equiv \Delta^2 = q^2 \frac{P(q)^{49}}{P(q)} \equiv q^2 \frac{P(q^7)^7}{P(q)} = P(q^7)^7 \sum_{n=0}^{\infty} p(n) q^{n+2} \pmod 7$$

<u>$\ell = 11$</u>: $691 E_{12} = 441 E_4^3 + 250 E_6^2$, $\quad E_4 E_6 = E_{10} \equiv 1\ (11)$, $\quad E_{12} \equiv E_2\ (11)$

$$\underbrace{\Delta^5 \equiv (E_4^3 - E_6^2)^5}_{} \equiv \cdots \equiv \underbrace{D(3E_2^4 - 4E_2^2 E_4 + 5E_2 E_6)}_{\text{the coefficients at } q^{11m} \text{ are all } \equiv 0 \pmod{11}} \ (11)$$

$$\equiv q^5 \frac{P(q^{11})^{11}}{P(q)} \pmod{11} = P(q^{11})^{11} \sum_{n=0}^{\infty} p(n) q^{n+5} \pmod{11}$$

<u>$\ell = 13$</u>: exercise.

Structure of $M_k, S_k, M_k^!$

$k \in 2\mathbb{Z}$

$$M_k^! = \Big\{ f : \mathcal{H} \to \mathbb{C} \text{ holomorphic}, \ \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \ \ f\Big(\frac{a\tau+b}{c\tau+d}\Big) = (c\tau+d)^k f(\tau), \quad f(\tau) = \sum_{n \ge n_0} a_n e^{2\pi i n \tau} \ \ (\text{for some } n_0 \in \mathbb{Z}) \Big\}$$

$\cup$

$M_k = \{ \text{ —"— }, \ n_0 = 0 \}$

$\cup$

$S_k = \{ \text{ —"— }, \ n_0 = 1 \}$ $\qquad q = e^{2\pi i \tau}$

Note: (1) $\eta^{24}(\tau) = q \prod_{n=1}^{\infty} (1-q^n)^{24} \in S_{12}$ has no zeroes in $\mathcal{H}$

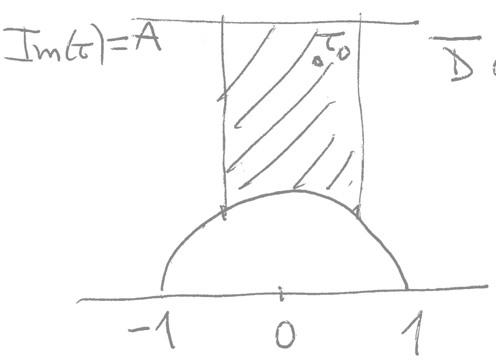
(2) $M_k^! = \bigcup_{r \ge 0} M_{k+12r} \, (\eta^{24}(\tau))^{-r}$

(3) $S_k = M_{k-12} (\eta^{24}(\tau))$

(4) $k > 2 \Rightarrow M_k = \mathbb{C} G_k \oplus S_k = \mathbb{C} E_k \oplus S_k$.

(5) $S_0 = 0$. Pf: If $0 \ne f \in S_0$, $\exists \tau_0 \in \overline{D}$ $f(\tau_0) \ne 0$

$\exists A \gg 0$ $\quad |f(\tau)| < \frac{1}{2} |f(\tau_0)|$ if $\mathrm{Im}(\tau) \ge A$.



$\overline{D} \cap \{\mathrm{Im}(\tau) \le A\}$ is compact $\Rightarrow$ $|f|$ attains its sup here at some τ_1

$\Rightarrow \forall \tau \in \overline{D} \quad |f(\tau)| \le |f(\tau_1)|$

$\Rightarrow \forall \tau \in \bigcup_{\gamma \in SL_2(\mathbb{Z})} \gamma \overline{D} = \mathcal{H} \quad |f(\tau)| \le |f(\tau_1)|$

$\Rightarrow f = \text{constant} = a_0$ (maximum principle) $\xRightarrow{f \in S_0} f = 0$.

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(6) $M_0 = \mathbb{C} \Rightarrow S_{12} = \mathbb{C} \cdot \eta^{24}(\tau) \Longrightarrow \boxed{\eta^{24}(\tau) = (E_4^3 - E_6^2)/12^3 = \Delta(\tau)}$

(7) $k < 0 \Rightarrow M_k = 0$: if $f \in M_k$, $f^{12} (\eta^{24}(\tau))^{-k} \in M_0 = \mathbb{C}$

$\Rightarrow f = (\text{const}) \, \eta^{2k}(\tau) = (\text{const.}) \, q^{\overbrace{k/12}} \underset{<0}{+} \cdots \Rightarrow \text{const.} = 0$.

(8) $k < 12 \Rightarrow S_k = 0$

(9) $k \in \{4, 6, 8, 10\} \Rightarrow M_k = \mathbb{C} E_k \Rightarrow E_8 = E_4^2, \ E_{10} = E_4 E_6$

(10) $M_2 = 0$: if $0 \ne f \in M_2 \Rightarrow f^2 \in M_4 = \mathbb{C} E_4, \ f^3 \in M_6 = \mathbb{C} E_6$,

$\underset{a, b \ne 0}{f^2 = a E_4, \ f^3 = b E_6}$, $\quad a^3 \underbrace{E_4^3}_{1+\cdots} = b^2 \underbrace{E_6^2}_{1+\cdots} \Rightarrow a^3 = b^2, \ E_4^3 = E_6^2$ – false.

(11) $S_{14} = 0 \Rightarrow M_{14} = \mathbb{C} E_{14} \Rightarrow E_{14} = E_4^2 E_6$

Summary: $k \in \mathbb{Z}$, $2 | k$.

k	0	2	4	6	8	10	12	14
S_k	0	0	0	0	0	0	$\mathbb{C}\Delta$	0
M_k	$\mathbb{C}$	0	$\mathbb{C}E_4$	$\mathbb{C}E_6$	$\mathbb{C}E_8$	$\mathbb{C}E_{10}$	$\mathbb{C}E_{12} \oplus \mathbb{C}\Delta$	$\mathbb{C}E_{14}$

Consequences: (1) $E_8 = E_4^2$, $E_{10} = E_4 E_6$, $E_{14} = E_4^2 E_6$

(2) For $0 \le k < 12$, $\dim S_k = 0$

$\dim M_k = 1 - \delta_{k,2}$

(3) For $k = 12r + \ell$, $0 \le \ell < 12$, $r \ge 0$:

$$\dim S_{k+12} = \dim M_k$$

$$\dim M_{k+12} = 1 + \dim S_{k+12} = 1 + \dim M_k$$

$$\Rightarrow \quad \dim M_{12r+\ell} = r + 1 - \delta_{k,2}$$

$$\dim S_{12r+\ell} = r$$

Differential operators: $D = q \frac{d}{dq} = \frac{1}{2\pi i} \frac{d}{d\tau}$

If $f\left(\frac{a\tau+b}{c\tau+d}\right) = f(\tau)$ $\Rightarrow$ $\frac{df}{d\tau}\left(\frac{a\tau+b}{c\tau+d}\right)(c\tau+d)^{-2} = \frac{df}{d\tau}$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$

$\Rightarrow \quad A_0 \xrightarrow{D} A_2$

Cor: If $f \in A_k$, then $f^{12}/\Delta^k \in A_0 \Rightarrow D(f^{12}/\Delta^k) \in A_2$.

But $D\Delta/\Delta = E_2 = 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n) q^n$, and

$$D(f^{12}/\Delta^k) = \underbrace{\frac{12 f^{11}}{\Delta^k}}_{\in A_{-k}} \left(Df - \frac{k}{12}\left(\frac{D\Delta}{\Delta}\right) f \right)$$

Cor: If $f \in A_k$, then $\underbrace{D_k f}_{\in A_{k+2}} := Df - \frac{k}{12}\left(\frac{D\Delta}{\Delta}\right) f = \Delta^{k/12} D(f/\Delta^{k/12})$.

Cor: $\Delta \cdot (Dj) = -1 + \cdots \in M_{14} = \mathbb{C} E_4^2 E_6 \Rightarrow \boxed{Dj = -\frac{E_4^2 E_6}{\Delta}}$

Vanishing of the constant term

$M_k^! \xrightarrow{CT} \mathbb{C}$, $\sum_{n\ge n_0} a_n q^n \longmapsto a_0$ | $D = q\frac{d}{dq}$

$D : \sum_{n\ge n_0} a_n q^n \mapsto \sum_{\substack{n\ge n_0 \\ n\neq 0}} n a_n q^n$

Prop 1 If $r\ge 0$, then the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{C}\cdot\Delta^r \arrow[r] \arrow[d, "\Delta^{-r}"] & M_{12r} \arrow[r, "D_{12r}"] \arrow[d, "\Delta^{-r}"] & M_{12r+2} \arrow[d, "\Delta^{-r}"] & \\
0 \arrow[r] & \mathbb{C} \arrow[r] & M_0^! \arrow[r, "D"] \arrow[rr, bend right, "0"'] & M_2^! \arrow[r, "CT"] & \mathbb{C}
\end{tikzcd}$$

commutes.

(a) D_{12r} is surjective

(b) $M_0^! \xrightarrow{D} M_2^!$ is surjective

(c) $CT(M_2^!) = 0$

Interpretation: for $f\in M_2^!$, $\omega = (2\pi i) f(\tau)\,d\tau \in \Omega^1_{mer}(\underbrace{SL_2(\mathbb{Z})\backslash(\mathcal{H}\cup\{\infty\})}_{\mathbb{C}\cup\{\infty\}})$

$= \sum_{n\ge n_0} a_n q^n \frac{dq}{q}$

$\underbrace{\mathrm{res}_{q=0}(\omega)}_{\mathrm{res}_\infty(\omega)} = a_0 = CT(f)$

But $\underbrace{\sum_{x\in\mathbb{C}\cup\{\infty\}} \mathrm{res}_x(\omega)}_{\mathrm{res}_\infty(\omega)} = 0$, $\omega\in\Omega^1(\mathbb{C})$.

Pf: (a) $\dim(M_{12r}) = r+1$, $\dim(M_{12r+2}) = r$.

(b) $\forall f\in M_2^!\ \exists r\ge 0$ $f\Delta^r \in M_{12r+2}$; apply (a).

(c) $CT\circ D = 0$.

Prop 2 Let $0\neq f\in M_k$, $k>0$ ($\Rightarrow k>2$). Let $m\ge 0$ ($m\neq 2$) be minimal such that $fE_m \in M_{12r+2}$ (convention: $E_0 = 1$).

(a) If $k = 12t+l$, $0\le l<12$, then

l	0	2	4	6	8	10
m	14	0	10	8	6	4
r	$t+1$	t	$t+1$	$t+1$	$t+1$	$t+1$
E_m	$E_4^2E_6$	1	E_4E_6	E_4^2	E_6	E_4
$\dim M_k$	$t+1$	t	$t+1$	$t+1$	$t+1$	$t+1$

$\Rightarrow \dim M_k = r$

(b) $CT(fE_m/\Delta^r) = 0$. Pf: Prop. 1 for fE_m.

Thm (Siegel) Let $0 \neq f(\tau) = \sum_{n=0}^{\infty} a(n) q^n \in \underbrace{M_k(SL_2(\mathbb{Z}))}_{\dim = r}$, $k > 0$.

Then $a(0) \in \sum_{n=1}^{r} \mathbb{Q}\, a(n)$.

Pf: Let m be as above: $f E_m \in M_{12r+2}$, $E_m = E_4^a E_6^b$ $\quad 0 \le a \le 2$, $0 \le b \le 1$

Write $E_4^a E_6^b / \Delta^r = \sum_{n \ge -r} c_n^r q^n$. Then $c_n^r \in \mathbb{Z}$ and

$$0 = CT(f E_4^a E_6^b / \Delta^r) = a(0)\, CT(E_4^a E_6^b / \Delta^r) + \sum_{n=1}^{r} a(n) \underbrace{c_{-n}^r}_{\Uparrow\ \mathbb{Z}}$$

So it is enough to prove:

Lemma. $CT(E_4^a E_6^b / \Delta^r) \neq 0$.

Pf: all coefficients of $1/\Delta^r$ (resp. of E_4^a) are > 0, which proves Lemma if $b = 0$. Enough to prove:

Sublemma. If $4 \mid k$ and $k + 4a \equiv 8 \pmod{12}$ $\quad (0 \le a \le 2)$,

then $\underline{CT(E_4^a E_6 / \Delta^r) < 0}$, unless $a = 2$ and $r = 1$

$(r = [k/12] + 1)$ $\quad (\Leftrightarrow k = 0)$.

Pf: $D E_4 - \frac{4}{12} \frac{D\Delta}{\Delta} E_4 = -\frac{4}{12} E_6 = -\frac{1}{3} E_6$

$$\Rightarrow \quad \frac{E_6 E_4^a}{\Delta^r} + \underbrace{D\left(\frac{1}{r} \frac{E_4^{a+1}}{\Delta^r}\right)}_{CT = 0} = \left(\frac{a+1}{r} - 3\right) \underbrace{\frac{E_4^a (D E_4)}{\Delta^r}}_{\text{all coeff.} > 0}$$

$\forall a \ge 0$

$\Rightarrow CT > 0$

If $0 \le a \le 2$ and $r \ge 1$, then $\frac{a+1}{r} - 3 < 0$, unless $a = 2$ and $r = 1$ $(\Leftrightarrow k = 0)$.
