Poincaré series

General principle: if $\phi : \mathcal{H} \to \mathbb{C}$ is holomorphic, $k \in \mathbb{Z}$, $\Gamma \subset SL_2(\mathbb{R})$ discrete subgroup, $\Gamma' := \{\gamma \in \Gamma \mid \phi|_k \gamma = \phi\}$, then

$$\mathbb{P}_k(\phi) := \sum_{\gamma \in \Gamma' \backslash \Gamma} \underbrace{\phi|_k \gamma}_{\text{depends only on } \Gamma'\gamma.}$$

(if convergent) satisfies $\forall \gamma \in \Gamma$ $\mathbb{P}_k(\phi)|_k \gamma = \mathbb{P}_k(\phi)$

The case $\Gamma = SL_2(\mathbb{Z})$, $\Gamma_\infty = \pm \begin{pmatrix} 1 & \mathbb{Z} \\ 0 & 1 \end{pmatrix} = \pm \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\rangle$

Motivation: $\underset{M_k}{E_k} = 1 + \cdots$, $\underset{M_0^!}{j} = \frac{1}{q} + \cdots$, $\underset{S_{12}}{\Delta} = q + \cdots$, $\underset{M_2^!}{Dj} = -\frac{1}{q} + \cdots$

General question: given $k \in 2\mathbb{Z}$ and $m \in \mathbb{Z}$, can one define

$f = q^m + \cdots \in M_k^!$ ($\in M_k$ if $m \geq 0$, resp. $\in S_k$ if $m > 0$)

in a suitably canonical way? $(q = e^{2\pi i \tau})$

Idea: for $\tau \to i\infty$, f is approximated by q^m;

but $f|_k \gamma = f$ $\forall \gamma \in \Gamma$ $\Rightarrow$ we need to consider

$$P_{m,k}(\tau) := \mathbb{P}_k(q^m) = \mathbb{P}_k(e^{2\pi i m \tau}) = \sum_{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_\infty \backslash \Gamma} (c\tau + d)^{-k} e^{2\pi i m \left(\frac{a\tau + b}{c\tau + d}\right)}$$

($k > 2 \Rightarrow$ convergence)

Note: (1) $\Gamma_\infty \backslash \Gamma \xrightarrow{\sim} \{(c,d) \in \mathbb{Z}^2 \mid \gcd(c,d) = 1\} / \{\pm 1\} = \underbrace{\{(0,1)\}}_{c=0} \cup \{(c,d) \in \mathbb{Z}^2 \mid c \geq 1, \gcd(c,d) = 1\}$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto (0,1)\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (c, d)$

(2) $\Gamma_\infty \backslash \Gamma / \Gamma_\infty \xrightarrow{\sim} \{(0,1)\} \cup \{(c, \bar{d}) \mid c \in \mathbb{Z}, c \geq 1, \bar{d} \in (\mathbb{Z}/c\mathbb{Z})^*\}$

$\Gamma_\infty \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Gamma_\infty \longmapsto (0,1)$ if $c = 0$

$\Gamma_\infty \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Gamma_\infty \longmapsto (c, d \pmod c)$ (if $c \neq 0$)

(= integral version of the Bruhat decomposition for SL_2)

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(3) If $k > 2$, then $P_{m,k}$ converges uniformly on compact subsets of $\mathcal{H}$ (and is holomorphic on $\mathcal{H}$).

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(4) $P_{m,k}(\tau + 1) = P_{m,k}(\tau) \Longrightarrow P_{m,k}(\tau) = \sum_{n \in \mathbb{Z}} p_n e^{2\pi i n \tau}$

Goal: compute p_n.

$$\frac{a\tau+b}{c\tau+d} = \frac{a}{c} - \frac{1}{c(c\tau+d)} \qquad (\text{if } c\neq 0) \qquad ad-bc=1 \implies ad\equiv 1 \pmod{c}$$

$$P_n = \int_{\mathbb{R}/\mathbb{Z}} e^{-2\pi i n\tau}\Big(\underbrace{e^{2\pi i m\tau}}_{c=0} + \sum_{c\geq 1}\sum_{\substack{d\in\mathbb{Z}\\(c,d)=1}} (c\tau+d)^{-k} e^{2\pi i am/c - \frac{2\pi i m}{c(c\tau+d)}}\Big)\,d\tau$$

$$= \delta_{mn} + \int_{\mathbb{R}} \Big(\sum_{c\geq 1}\sum_{d\in(\mathbb{Z}/c\mathbb{Z})^*} (cz)^{-k} e^{2\pi i am/c} e^{-\frac{2\pi i m}{c^2 z} - 2\pi i n(z-\frac{d}{c})}\Big)\,dx$$

$(z=x+iy,\ y>0)$

$$\boxed{\begin{matrix} d=\bar{d}+cl,\ l\in\mathbb{Z} \\ c\tau+d = \underbrace{c(\tau+l)+\bar{d}}_{c(z)} \end{matrix}}$$

$$= \delta_{mn} + \sum_{c\geq 1} c^{-k} \underbrace{\sum_{d\in(\mathbb{Z}/c\mathbb{Z})^*} e^{2\pi i(md^*+nd)/c}}_{S(c,m,n)} F_k\Big(\frac{m}{c^2},n\Big)$$

$(dd^*\equiv 1 \pmod{c})$

Kloosterman sum

$$F_k\Big(\frac{m}{c^2},n\Big) = \int_{\mathbb{R}} \underbrace{(x+iy)^{-k}}_{z^{-k}} e^{-\frac{2\pi i m}{c^2 z} - 2\pi i n z}\,dx \qquad (z=x+iy,\ y>0)$$

Case $m=0$: $P_{0,k}$ = Eisenstein series $E_k(\tau)$

Case $m\neq 0$: integral representation of Bessel functions:

For $r,y,\nu>0$:

$$\frac{1}{2\pi}\int_{\mathrm{Im}(z)=y} e^{-ir(z\pm z^{-1})/2} \frac{dz}{\underbrace{(z/i)^{\nu+1}}_{\text{real if } z/i\in\mathbb{R}_{>0}}} = \begin{cases} J_\nu(r) = \sum_{l\geq 0} \frac{(x/2)^{\nu+2l}(-1)^l}{l!\,\Gamma(\nu+l+1)} \\ I_\nu(r) = \sum_{l\geq 0}\frac{(x/2)^{\nu+2l}}{l!\,\Gamma(\nu+l+1)} \end{cases}$$

Case $m>0$: $F_k\Big(\frac{m}{c^2},n\Big) = \Big(c\sqrt{\frac{n}{m}}\Big)^{k-1}\frac{1}{i^k}\,2\pi\, J_{k-1}\Big(\frac{4\pi\sqrt{mn}}{c}\Big)$

Case $m<0$: $F_k\Big(\frac{m}{c^2},n\Big) = \Big(c\sqrt{\frac{n}{|m|}}\Big)^{k-1}\frac{1}{i^k}\,2\pi\, I_{k-1}\Big(\frac{4\pi\sqrt{|m|n}}{c}\Big)$

$(k>2)$

Thm. If $m\neq 0$, then

$$P_{m,k}(\tau) = q^m + \sum_{n\geq 1}\frac{2\pi}{i^k}\Big(\frac{n}{|m|}\Big)^{\frac{k-1}{2}}\Big(\sum_{c\geq 1}\frac{S(c,m,n)}{c}\begin{cases} J_{k-1}\big(\frac{4\pi\sqrt{mn}}{c}\big), & m>0 \\ I_{k-1}\big(\frac{4\pi\sqrt{|m|n}}{c}\big), & m<0\end{cases}\Big) q^n$$

$P_{m,k}(\tau) \in M_k^!(SL_2(\mathbb{Z}))$

$\boxed{\text{Cor. } P_{1,12} = \Delta(\tau)}$

Poincaré series with multiplier systems

$\Gamma = SL_2(\mathbb{Z})$, $\eta(\tau) = e^{2\pi i\tau/24} \prod_{n\geq 1}(1-e^{2\pi i n\tau}) = q^{1/24}\prod_{n\geq 1}(1-q^n)$

$\forall \gamma = \begin{pmatrix} a & b \\ c & d\end{pmatrix} \in \Gamma$ $\qquad \eta\left(\frac{a\tau+b}{c\tau+d}\right) = \underset{\in\, \mu_{24}}{v_\eta(\gamma)}\, \underbrace{(c\tau+d)^{1/2}}_{\exp\left(\frac{1}{2}\underbrace{\log(c\tau+d)}_{-\pi\leq \arg <\pi}\right)}\, \eta(\tau)$

$\underline{v_\eta(\gamma)^{24} = 1}$

Define: for $k \in \frac{1}{2}\mathbb{Z}$, let

$$(f|_{k,v}\gamma)(\tau) = v(\gamma)^{-1}\left((c\tau+d)^{1/2}\right)^{-2k} f(\gamma(\tau))$$

Goal: construct Poincaré series of weight $k = \frac{\ell}{2} > 2$ and multiplier system v_η^ℓ $(\ell\in\mathbb{Z},\ \ell>4)$

If $m \in \frac{1}{24}\mathbb{Z}$ and $k-12m \in 2\mathbb{Z}$, then

$q^m = e^{2\pi i m\tau}$ is invariant under $|_{k,v_\eta^{24m}} \begin{pmatrix} 1 & 1 \\ 0 & 1\end{pmatrix}$

— " — $\begin{pmatrix} -1 & 0 \\ 0 & -1\end{pmatrix}$

(since $v_\eta\begin{pmatrix} 1 & 1 \\ 0 & 1\end{pmatrix} = e^{2\pi i/24}$)

$\Longrightarrow$ $P_{k,m}(\tau) := \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} q^m |_{k,v_\eta^{24m}}\gamma$ is well-defined.

In this case

$$P_{k,m}(\tau) = q^m + \sum_{0<n\in(\mathbb{Z}+m)} \frac{2\pi}{i^k}\left(\frac{n}{|m|}\right)^{\frac{k-1}{2}} \sum_{c\geq 1}\frac{S^*(c,m,n)}{c}\begin{Bmatrix} J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right), & m>0 \\ I_{k-1}\left(\frac{4\pi\sqrt{|m|n}}{c}\right), & m<0\end{Bmatrix} q^n$$

$$S^*(c,m,n) = \sum_{d\in(\mathbb{Z}/c\mathbb{Z})^*} v_\eta\left(\begin{pmatrix} * & * \\ c & d\end{pmatrix}\right)^{-24m} e^{2\pi i(md^*+nd)/c}, \quad dd^*\equiv 1 \pmod c$$

Recall: $M_0^! \xrightarrow{D} M_2^!$ is surjective

$f = \sum_{n\geq n_0} a_n q^n \in M_2^! \Longrightarrow \omega = \sum_{n\geq n_0} a_n q^n \frac{dq}{q} \in \Omega^1_{mer}(\Gamma(1)\backslash\mathfrak{H}^*) \cap \Omega^1(\Gamma(1)\backslash\mathfrak{H})$

$\Longrightarrow$ $0 = \operatorname{Res}_{q=0}(\omega) = a_0$.

Cor. If $1\leq r\leq 24$, $\sum_{n\geq 0} p_r(n) q^n = \prod_{n\geq 1}(1-q^n)^{-r} = q^{r/24}\eta(\tau)^{-r}$.

$\forall n\geq 1$ $\quad P_{2+r/2,\,-n+r/24}(\tau)\cdot\eta(\tau)^{-r} \in M_2^! \Longrightarrow$ its constant term is 0

$\Longrightarrow$ $\underline{p_r(n) = -\left(\text{coefficient of } q^{r/24} \text{ in } P_{2+r/2,\,-n+r/24}\right)}$ | $\underline{r=1}$: Rademacher's formula

Coefficients of $\prod_{n=1}^{\infty}(1-q^n)^{-1}$

Partition function: $\quad p(n) =$ number of partitions of $n \in \mathbb{N}$

$$\sum_{n=0}^{\infty} p(n)\, q^n = \prod_{n=1}^{\infty}(1-q^n)^{-1} = q^{1/24}\underbrace{\eta(\tau)^{-1}}\ , \quad q = e^{2\pi i \tau}$$

"element of $M^!_{-1/2}(SL_2(\mathbb{Z}), v_\eta^{-1})$"

Rademacher:
~~Hardy – Ramanujan: asymptotic series for $p(n)$ ($\alpha > 0$ fixed)~~

$$p(n) = \frac{1}{\pi\sqrt{2}} \sum_{k=1}^{\infty} k^{1/2} A_k(n)\, \frac{d}{dn}\left(\frac{\sinh\left(\frac{\pi}{k}\sqrt{\frac{2}{3}}\sqrt{n-\frac{1}{24}}\right)}{\sqrt{n-\frac{1}{24}}} \right) \cancel{+\, O(n^{-1/4})}$$

$$\Updownarrow$$

$$p(n) = 2\pi\left(\frac{\pi}{12}\right)^{3/2} \sum_{k=1}^{\infty} k^{-5/2} A_k(n) \left(\frac{\pi}{2k}\sqrt{\frac{2}{3}\left(n-\frac{1}{24}\right)}\right)^{-3/2} I_{3/2}\left(\frac{\pi}{k}\sqrt{\frac{2}{3}\left(n-\frac{1}{24}\right)}\right)$$

$$A_k(n) = \sum_{h \in (\mathbb{Z}/k\mathbb{Z})^\times} \omega_{hk}\, e^{-2\pi i h n/k} \qquad \omega_{hk}^{24} = 1$$

comes from the transformation formula for $\eta(\tau)$ under $\begin{pmatrix} h & * \\ k & * \end{pmatrix}$

Truncation at $k \le [\alpha\sqrt{n}]$ $\quad$ ($\alpha > 0$ fixed): $\qquad \sinh(x) = \frac{e^x - e^{-x}}{2}$; keep $\frac{e^x}{2}$ only

$$p(n) = \frac{1}{2\pi\sqrt{2}} \sum_{k=1}^{[\alpha n^{1/2}]} k^{1/2} A_k(n)\, \frac{d}{dn}\left(\frac{\exp\left(\frac{\pi}{k}\sqrt{\frac{2}{3}}\sqrt{n-\frac{1}{24}}\right)}{\sqrt{n-\frac{1}{24}}} \right) + O(n^{-1/4})$$

Hardy – Ramanujan

keep only $k=1$: $\qquad p(n) = \frac{1}{2\pi\sqrt{2}}\, \frac{d}{dn}\left(\frac{\exp(C\lambda_n)}{\lambda_n}\right) + O\left(\frac{\exp\left(\frac{C}{2}\lambda_n\right)}{n}\right)$

$C = \pi\sqrt{\frac{2}{3}}\ , \quad \lambda_n = \sqrt{n - \frac{1}{24}}$

$$p(n) \sim \frac{1}{4\sqrt{3}}\, \frac{e^{C\lambda_n}}{n}$$

Interpretation of Rademacher's formula:

(1) RHS = coeff. of a "Poincaré series of weight $-1/2$"

(2) "duality" between $M^!_{-1/2}(v_\eta^{-1})$ and $M_{5/2}(v_\eta^{5/2})$

$\qquad\qquad\qquad\qquad \Psi$ true (convergent) Poincaré series

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Analogous formula for $c(n)$, $\quad j(\tau) = q^{-1} + 744 + \sum_{n \ge 1} c(n) q^n$,

involves $I_1\left(\frac{4\pi}{k}\sqrt{n}\right)$ $\qquad$ (Rademacher, Petersson)

Convergence of Poincaré series

Prop. If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ $(a_n \in \mathbb{C})$ has radius of convergence $> R$, then

$$\int_{|z|\le R} |f(z)|^2 \, dx\,dy \ \ge\ |f(0)|^2 \int_{|z|\le R} dx\,dy = \pi R^2 |f(0)|^2$$

Pf: $|f(z)|^2 = \sum_{m,n\ge 0} a_m \overline{a_n}\, r^{m+n} e^{i(m-n)\theta}$, $z = re^{i\theta}$

$$\Rightarrow \int_{|z|\le R} |f|^2 dx\,dy = \sum_{m,n\ge 0} a_m \overline{a_n} \int_0^R r^{m+n+1} \underbrace{\Big(\int_0^{2\pi} e^{i(m-n)\theta}\, d\theta\Big)}_{2\pi\delta_{m,n}} dr = \underbrace{\pi \sum_{n\ge 0} \frac{|a_n|^2 r^{2n+2}}{n+1}}_{\ge \pi r^2 |a_0|^2}$$

Thm (Poincaré). Assume that a group Γ acts <u>discontinuously</u> by holomorphic maps on a <u>bounded</u> open set $U \subset \mathbb{C}^n$. For $\gamma \in \Gamma$, denote by $j_\gamma : U \longrightarrow \mathbb{C}$ the Jacobian of $\gamma : U \longrightarrow U$ ($j_\gamma(z) =$ determinant of $(D\gamma)(z) \in M_n(\mathbb{C})$). Then

$$\sum_{\gamma\in\Gamma} |j_\gamma|^2 \quad \text{converges uniformly on compact}$$

subsets of U.

Ex: $\Gamma \subset SU(1,1)$ discrete subgroup, $U = D = \{|z|<1\}$

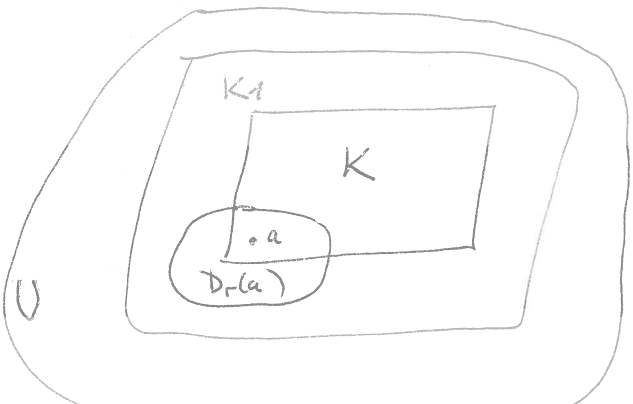
$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\quad j_\gamma(z) = \dfrac{1}{|cz+d|^2}$

Pf. Let $K \subset U$ be compact, $2r := d_\infty(K, \partial U)$ ($d_\infty =$ ~~Euclidean~~ distance in $\max(|z_j|)_{1\le j\le n}$ metric)

$K_1 := \{z \in \mathbb{C}^n \mid d_\infty(z,K) \le r\} \subset U$ compact

$\forall a \in K \quad D_r(a) := \{z\in\mathbb{C}^n \mid d_\infty(z,a) \le r\} \subset K_1$

polydisc



If $\gamma_1, \gamma_2 \in \Gamma$ and $\gamma_1(D_r(a)) \cap \gamma_2(D_r(a)) \ne \emptyset$

$$\Rightarrow \gamma_2^{-1}\gamma_1 \in \underbrace{\{\gamma\in\Gamma \mid \gamma(K_1)\cap K_1 \ne \emptyset\}}_{\text{finite set, with } N \text{ elements}}$$

If $\lambda =$ Lebesgue measure on $\mathbb{C}^n$, then $\forall a \in K$

$$(\pi r^2)^n \sum_{\gamma\in\Gamma} |j_\gamma(a)|^2 \le \sum_{\gamma\in\Gamma} \int_{D_r(a)} |j_\gamma(z)|^2 d\lambda = \sum_{\gamma\in\Gamma} \int_{\gamma(D_r(a))} d\lambda \le N \int_{\bigcup_{\gamma\in\Gamma}\gamma(D_r(a))} d\lambda \le N\lambda(U) < \infty$$

$\Rightarrow$ [redacted] convergence on K.

Uniform convergence: for given $\varepsilon > 0$, fix $K_2 \subset U$ compact such that $K_1 \subset K_2$ and $N \cdot \lambda(U \setminus K_2) < \varepsilon$.

the set $\Delta := \{\gamma \in \Gamma \mid \gamma(K_1) \cap K_2 \neq \emptyset\}$ is finite and $\forall a \in K$

$$(\pi r^2)^n \sum_{\gamma \in \Gamma \setminus \Delta} |j_\gamma(a)|^2 \leq \sum_{\gamma \in \Gamma \setminus \Delta} \int_{D_r(a)} |j_\gamma(z)|^2 \, d\lambda = \sum_{\gamma \in \Gamma \setminus \Delta} \int_{\gamma(D_r(a))} d\lambda \leq$$

$$\leq N \cdot \lambda\Big(\underbrace{\bigcup_{\gamma \in \Gamma \setminus \Delta} \gamma(D_r(a))}_{\subseteq \bigcup_{\gamma \in \Gamma \setminus \Delta} \gamma(K_1) \subseteq U \setminus K_2} \Big) \leq N \cdot \lambda(U \setminus K_2) < \varepsilon .$$

Cor. Let $\Gamma \subset SU(1,1)$ be a discrete subgroup, $k \in \mathbb{Z}$, $k \geq 4$,

~~$f(z) = \frac{g(z)}{h(z)}$, $g, h \in \mathbb{C}[z]$ polynomials~~

$f : D = \{|z| < 1\} \longrightarrow \mathbb{C}$ holomorphic and bounded; then

$$F(z) := \sum_{\gamma \in \Gamma} (f|_k \gamma)(z) = \sum_{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma} (cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right)$$

converges uniformly (and normally) on compact subsets of D to a holomorphic function F satisfying

$$\forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \qquad F\left(\frac{az+b}{cz+d}\right) = (cz+d)^k F(z) \qquad (z \in D)$$

Thm. Let $\Gamma \subset SL_2(\mathbb{R})$ be a discrete subgroup such that $\Gamma\backslash\mathcal{H}^*_\Gamma$ is compact, $\infty \in \mathrm{Cusps}_\Gamma$ and $-I \in \Gamma$. Then, $\forall r > 2$,

$$\sum_{\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_\infty \backslash \Gamma} |c\tau + d|^{-r}$$

converges uniformly on each set

$$\{ \operatorname{Im}(\tau) \geq C_1, \ |\operatorname{Re}(\tau)| \leq C_2 \} \qquad (C_1, C_2 > 0).$$

Proof. $\exists\, h > 0$ $\pm \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}^{\mathbb{Z}} = \Gamma_\infty$. Choose a set of representatives $\Delta \subset \Gamma$ of $\Gamma_\infty \backslash \Gamma$ such that $\forall \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Delta \smallsetminus \{I\}$ $\left|\frac{a}{c}\right| \leq \frac{h}{2}$.

We know that $|c| \geq \frac{1}{h}$ (if $c \neq 0$).

There exists a fundamental domain D_Γ (for Γ acting on $\mathcal{H}$) contained in the closed set

$$F = \left\{ |\operatorname{Re}(\tau)| \leq \tfrac{h}{2} \right\} \cap \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \smallsetminus \Gamma_\infty \ \middle|\ \underbrace{|c\tau + d| \geq 1}_{\iff \operatorname{Im}(\gamma(\tau)) \leq \operatorname{Im}(\tau)} \right\}$$

$(\mathcal{H} = \Gamma \cdot F)$

$\Longrightarrow$ $\forall \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Delta \smallsetminus \{I\}$ $\forall \tau \in F$

$|c\gamma(\tau) - a| \leq 1 \Longrightarrow |\operatorname{Re}(\gamma(\tau))|$

satisfies $|\operatorname{Re}(\gamma(\tau))| \leq \left|\frac{a}{c}\right| + \frac{1}{|c|} \leq \frac{h}{2} + h = \frac{3}{2}h$.

Fix a compact disc $K \subset D_\Gamma$. Then the discs $\gamma(K)$ $(\gamma \in \Delta)$ are disjoint and lie in

$$\{ |\operatorname{Re}| \leq \underbrace{\tfrac{3}{2}h}_{A} \} \cap \{ |\operatorname{Im}| \leq \underbrace{\sup_{\tau \in K} \operatorname{Im}(\tau)}_{B} \}$$

So

$$\infty > \int_{\substack{|\operatorname{Re}| \leq A \\ \operatorname{Im} \leq B}} y^{r/2} \frac{dx\,dy}{y^2} \geq \sum_{\gamma \in \Delta} \int_{\gamma(K)} y^{r/2}\, d\mu =$$

$$= \sum_{\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \gamma \in \Delta} \int_K \left(\frac{\operatorname{Im}(\tau)}{|c\tau + d|^2} \right)^{r/2} d\mu \geq \mathrm{const}(K) \sum_{\gamma \in \Delta} \frac{1}{|c\tau_0 + d|^r} \qquad (\tau_0 \in K \text{ fixed})$$

However, if $\operatorname{Im}(\tau) \geq C_1$ and $|\operatorname{Re}(\tau)| \leq C_2$ $(C_1, C_2 > 0$ fixed$)$, then $\forall\, c, d \in \mathbb{R}$

$$|c\tau + d| \geq \underbrace{\mathrm{const}(\tau_0, C_1, C_2)}_{>0} |c\tau_0 + d|.$$

Bessel functions

Laplace–Beltrami operator: metric $ds^2 = g_{ij}\, dx^i dx^j$

$$g_{ij} = (\partial_i, \partial_j) \qquad \partial_i = \frac{\partial}{\partial x^i},\ \partial_j = \frac{\partial}{\partial x^j}$$

dual: $g^{ij} = (dx^i, dx^j)$, $\quad g^{ij} g_{jk} = \delta^i_k = (dx^i, g_{jk}\, dx^j)$

$I \subset [1,n] = \{1, \dots, n\}$, $\quad vol = \sqrt{g}\ dx^{[1,n]}$, $\quad g = \det(g_{ij})$

Hodge $*$-operator: $\alpha \wedge * \beta = (\alpha, \beta)\, vol$

$$*1 = vol = \sqrt{g}\ dx^{[1,n]}, \qquad * dx^{[1,n]} = \frac{1}{\sqrt{g}}$$

$$dx^k \wedge * dx^l = \delta^k_j\, dx^j \wedge * dx^l = \sqrt{g}\, g^{kl}\ dx^{[1,n]}$$

$$df \wedge * dx^l = \sqrt{g}\ g^{kl} (\partial_k f)\ dx^{[1,n]}$$

$$\Delta f = * d * df = \frac{1}{\sqrt{g}}\, \partial_k \left(\sqrt{g}\, g^{kl}\, \partial_l f \right)$$

Orthogonal coordinates: $g_{ij} = h_i^2 \delta_{ij}$, $\quad \sqrt{g} = h_1 \cdots h_n = h$

$$\Delta f = \frac{1}{h}\, \partial_k \left(\frac{h}{h_k^2}\, \partial_k f \right)$$

Polar coordinates: $n = 2$, $x = r\cos(\varphi)$, $y = r \sin(\varphi)$, $ds^2 = (dr)^2 + r^2 (d\varphi)^2$

$$h_r = 1,\ h_\varphi = r,\ h = r, \quad \Delta = \frac{1}{r}\left(\frac{\partial}{\partial r}\left(r \frac{\partial}{\partial r}\right) + \frac{\partial}{\partial \varphi}\left(\frac{1}{r} \frac{\partial}{\partial \varphi}\right)\right) = \partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r}\partial_\varphi^2$$

Eigenfunctions of Δ: $\quad (\Delta + |k|^2)(e^{i(k, \vec{r})}) = 0$

$\vec{r} = (r\cos\varphi, r\sin\varphi) \qquad (\Delta + \lambda) f = 0, \quad f = a(r)\, e^{in\varphi}$

$$\Longleftrightarrow \quad a'' + \frac{1}{r} a' + \left(\frac{(in)^2}{r^2} + \lambda \right) a = 0$$

Plane wave:

$$(*) \quad r^2 a'' + r a' + (r^2 - n^2)\, a = 0 \qquad \lambda = 1$$

For $\overset{\uparrow}{0}\ k = (0,1)$

$$f = e^{iy} = e^{ir\sin(\varphi)} = \sum_{n \in \mathbb{Z}} \underbrace{c_n(r)}_{\text{a solution of } (*)} e^{in\varphi}$$

$$c_n(r) = c_{-n}(r)$$

Residue thm:

$$2\pi i\, c_n(r) = \int_{|z|=1} z^{-n}\, \underbrace{e^{r(z - z^{-1})/2}}_{\sum_{k=0}^{\infty} \frac{(r/2)^k}{k!} \sum_{l=0}^{k} \binom{k}{l} (-1)^l z^{l-(k-l)}} \frac{dz}{z}$$

$$\Rightarrow c_n(r) = \sum_{\substack{0 \le l \le k \\ 2l - k = n}} (-1)^l \left(\frac{r}{2}\right)^k \binom{k}{l} \underset{\text{if } n \ge 0}{=} \sum_{l=0}^{\infty} \left(\frac{r}{2}\right)^{n+2l} \frac{(-1)^l \binom{n+2l}{l}}{(n+2l)!} = \underbrace{\sum_{l=0}^{\infty} \frac{(r/2)^{n+2l} (-1)^l}{l!\, \Gamma(n+l+1)}}_{J_n(r)}$$

$$\boxed{e^{ir\sin(\varphi)} = \sum_{n \in \mathbb{Z}} J_n(r)\, e^{in\varphi}} \qquad J_{-n}(r) = (-1)^n J_n(r)$$

In general: $\quad J_\nu(x) = \sum_{\ell \geq 0} \frac{(x/2)^{\nu+2\ell}(-1)^\ell}{\ell!\,\Gamma(\nu+\ell+1)}$

$$I_\nu(x)\left(= e^{-\pi i \nu/2} J_\nu(ix)\right) = \sum_{\ell\geq 0} \frac{(x/2)^{\nu+2\ell}}{\ell!\,\Gamma(\nu+\ell+1)} \qquad (x>0,\ \nu \notin \mathbb{Z}_{<0})$$

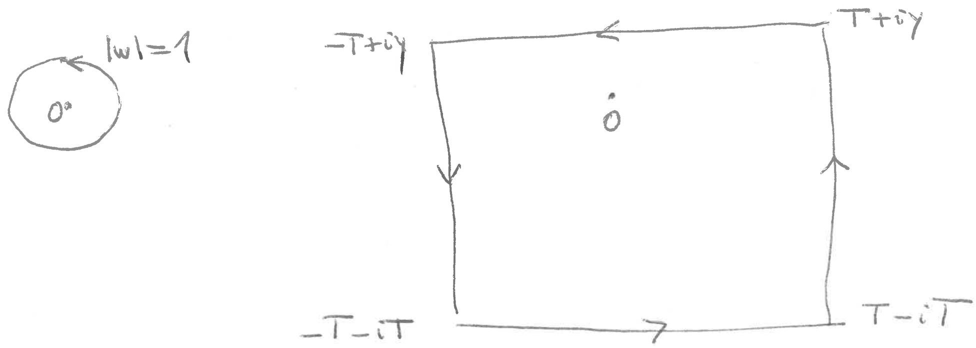
$$J_\nu' = \frac{1}{2}(J_{\nu-1} - J_{\nu+1}), \quad \nu J_\nu = \frac{x}{2}(J_{\nu-1}+J_{\nu+1}), \quad J_\nu'' + \frac{1}{x}J_\nu' + \left(1-\frac{\nu^2}{x^2}\right)J_\nu = 0$$

$$I_\nu' = \frac{1}{2}(I_{\nu-1}+I_{\nu+1}), \quad \nu I_\nu = \frac{x}{2}(I_{\nu-1}-I_{\nu+1}), \quad I_\nu'' + \frac{1}{x}I_\nu' - \left(1+\frac{\nu^2}{x^2}\right)I_\nu = 0$$

Integral representation: $\quad n \in \mathbb{Z}_{>0}$, $r>0$

$$2\pi i\, J_n(r) = \int_{|z|=1} e^{r(z-z^{-1})/2}\frac{dz}{z^{n+1}} = i^n \int_{|w|=1} e^{-ir(w+w^{-1})/2}\frac{dw}{w^{n+1}} \qquad (z=-iw)$$

deformation of the contour of integration: $\quad$ fix $y>0$, let $T \to +\infty$:



For fixed $r, y > 0$: $\quad \left| \int_{\sqcup} \right| \leq \frac{\mathrm{const.}(r,y)}{T^n}$; let $T \to +\infty$:

$$2\pi i\, J_n(r) = -i^n \int_{iy-\infty}^{iy+\infty} e^{-ir(w+w^{-1})/2}\frac{dw}{w^{n+1}} \qquad (r,y>0;\ n\geq 1)$$

$$\Rightarrow \quad 2\pi i\, I_n(r) = (-i)^n \int_{|z|=1} e^{ir(z-z^{-1})/2}\frac{dz}{z^{n+1}} = i^n \int_{|w|=1} e^{-ir(w-w^{-1})/2}\frac{dw}{w^{n+1}} \qquad (w=-z)$$

as above

$$(r,y>0;\ n\geq 1) \qquad = -i^n \int_{\mathrm{Im}(w)=y} e^{-ir(w-w^{-1})/2}\frac{dw}{w^{n+1}}$$

General formulas: $\quad \forall\, r,y>0 \qquad \mathrm{Re}(\nu)>0$

$$\underbrace{\int_{\mathrm{Im}(w)=y}}_{\int_{-\infty+iy}^{+\infty+iy}} e^{-ir(w\pm w^{-1})/2}\frac{dw}{\underbrace{(w/i)^{\nu+1}}_{\substack{\text{real if}\\ w/i\in\mathbb{R}_{>0}\\ \text{and } \nu\in\mathbb{R}_{>0}}}} = 2\pi \left\{ \begin{matrix} J_\nu(r) \\ I_\nu(r) \end{matrix} \right\}$$

Fourier coefficients of Poincaré series – abstract approach

$G = SL_2(\mathbb{R})$ acts on $\mathcal{H}$, $K = G_i = SO(2)$

automorphy factor: $\rho(g,\tau) = J(g,\tau)^k$ $(k\in\mathbb{Z})$, $J(\begin{pmatrix} a & b \\ c & d\end{pmatrix},\tau) = c\tau+d$

usual dictionary: $\rho_K = \rho|_{K\times\{i\}} : K \longrightarrow \mathbb{C}^\times$ $\quad \rho_K(h_\theta) = e^{ik\theta}$

$\{ f : G/K = \mathcal{H} \longrightarrow \mathbb{C} \} \longleftrightarrow \mathrm{Ind}_K^G(\rho_K)$ $\quad h_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

with ρ-twisted (right) G-action $\quad \{F : G \to \mathbb{C} \mid F(g h_\theta) = \rho_K(h_\theta)^{-1} F(g)\}$

$(f|_\rho g)(\tau) = \rho(g,\tau)^{-1} f(g(\tau))$ $\quad (F|g)(g') = F(gg')$

$f(g(i)) = \rho(g,i)\, F(g)$

Iwasawa decomposition: $G = NAK$, $K = \{h_\theta \mid \theta\in\mathbb{R}/2\pi\mathbb{Z}\}$

$N = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1\end{pmatrix} \mid x\in\mathbb{R}\right\}$, $A = \left\{ \begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2}\end{pmatrix} \mid y>0\right\}$

$\Gamma = SL_2(\mathbb{Z})$, $N_\Gamma = N\cap\Gamma = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1\end{pmatrix} \mid x\in\mathbb{Z}\right\}$

Fix: $\psi_0 : \mathbb{R}/\mathbb{Z} \cong N/N_\Gamma \longrightarrow U(1)$ character $(\neq 1)$

$\psi_0(x) = e^{2\pi i m x}$ $(m\in\mathbb{Z}\setminus\{0\})$

let $F : G = NAK \longrightarrow \mathbb{C}$ be a function of the form

$$F\left(\begin{pmatrix} 1 & x \\ 0 & 1\end{pmatrix}\begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2}\end{pmatrix} h_\theta\right) = e^{2\pi i m x}\, W(y)\, e^{-ik\theta}$$

satisfying $(k\in 2\mathbb{Z})$

$\Delta F \;(= -\frac{\Omega}{2}F) = \lambda F$ (and W of rapid decay as $y\to+\infty$)

Whittaker function

then $F \in \mathrm{Ind}_K^G(\rho_K)^{\pm N_\Gamma}$.

the corresponding Poincaré series is

$$\mathbb{P}(F) = \sum_{\gamma\in\pm N_\Gamma\backslash\Gamma} F|\gamma.$$

It satisfies $\mathbb{P}(F)|u = \mathbb{P}(F)$ $\forall u\in N_\Gamma$

Fourier coefficients of $\mathbb{P}(F)$: fix $\psi_1 : \mathbb{R}/\mathbb{Z} \cong N/N_\Gamma \longrightarrow U(1)$

$\psi_1(x) = e^{2\pi i n x}$ $(n\in\mathbb{Z})$

$$a_{\psi_1}(\mathbb{P}(F))(g) = \int_{N_\Gamma\backslash N} \psi_1(u)^{-1} \underbrace{\mathbb{P}(F)(ug)}_{\sum_{\gamma\in\pm N_\Gamma\backslash\Gamma} F(\gamma u g)}\, du$$

Bruhat decomposition: $G = \underbrace{\pm NA}_{B = \text{Borel subgroup of } G} \cup (\pm NwNA)$ $\quad w = \begin{pmatrix} 0 & -1 \\ 1 & 0\end{pmatrix}$

Note: the right action of N on $N\backslash NwNA$ is free

$$a_{\psi_1}(\mathbb{P}(F))(I) = \int_{N_\Gamma \backslash N} \psi_1(u)^{-1} \underbrace{\sum_{\gamma \in \pm N_\Gamma \backslash \Gamma}}_{\sum_{\gamma = I} + \sum_{\gamma \neq I}} F(\gamma u)\, du = \underbrace{\int_{N_\Gamma \backslash N} (\psi_0 \psi_1^{-1})(u)\, du}_{\substack{(=0 \text{ if } \psi_0 \neq \psi_1) \\ = 1 \text{ if } \psi_0 = \psi_1}} +$$

$$+ \sum_{\pm I \neq \gamma \in \pm N_\Gamma \backslash \Gamma / N_\Gamma} \int_N \psi_1(u)^{-1} F(\gamma u)\, du$$

We need to investigate the following function:

$\tilde{F} : G \longrightarrow \mathbb{C}$, $\quad \tilde{F}(g) := \int_N \psi_1(u)^{-1} F(gu)\, du$

Properties of $\tilde{F}$: (1) $\forall u_0, u_1 \in N \quad \tilde{F}(u_0 g u_1) = \psi_0(u_0)\psi_1(u_1)\tilde{F}(g)$

(2) $\Delta \tilde{F} = \lambda \tilde{F}$

Then: $a_{\psi_1}(\mathbb{P}(F))(I) = \delta_{\psi_0, \psi_1} + \sum_{\pm I \neq \gamma \in \pm N_\Gamma \backslash \Gamma / N_\Gamma} \tilde{F}(\gamma)$

$\tilde{F}$ — abstract version of a Bessel function.

F — ——— " ——————— Whittaker function.

Bessel functions on the big cell of $SL_2(\mathbb{R})$

$$G = SL_2 = B \cup \underline{BwB}, \quad w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} = TU$$

$$UwB = UwTU \qquad T = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}, \quad U = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$$

<u>Formulas:</u> $(c \neq 0)$ $(\det(g) = 1)$

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix} \begin{pmatrix} 1 & dc^{-1} \\ 0 & 1 \end{pmatrix}$$

$$\underline{g = n_1 w h n_2}, \quad n_j = \begin{pmatrix} 1 & u_j \\ 0 & 1 \end{pmatrix}, \quad h = \begin{pmatrix} y & 0 \\ 0 & y^{-1} \end{pmatrix} \qquad u_1 = \frac{a}{c}, \quad u_2 = \frac{d}{c}, \quad y = c$$

<u>Lie algebra action:</u> $\underline{X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}$, $e^{tX} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$

$$\boxed{g = \underbrace{\begin{pmatrix} 1 & u_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y & 0 \\ 0 & y^{-1} \end{pmatrix} \begin{pmatrix} 1 & u_2 \\ 0 & 1 \end{pmatrix}}_{[u_1, u_2, y]}}$$

$$R(e^{tX}): g = [u_1, u_2, y] \longmapsto [u_1, u_2 + t, y] = g e^{tX}$$

$$\underline{H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}, \quad e^{tH} = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}, \quad R(e^{tH}): g = [u_1, u_2, y] \longmapsto g e^{tH} = [u_1, e^{-2t} u_2, e^t y]$$

$$\underline{Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}, \quad e^{tY} = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}, \quad R(e^{tY}): g = [u_1, u_2, y] = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto g e^{tY} = [u_1', u_2', y']$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} = \begin{pmatrix} a + tb & b \\ c + td & d \end{pmatrix}, \quad u_1' = \frac{a + tb}{c + td} = \frac{a}{c} - \frac{t}{c(c+td)} = \frac{a}{c} - \frac{t}{y^2} + O(t^2)$$

$$u_2' = \frac{d}{c + td} = \frac{d}{c}\left(1 - \frac{td}{c}\right) + O(t^2) = u_2 - u_2^2 t + O(t^2)$$

$$y' = c + td = c(1 + tu_2) = y + u_2 y t$$

$$X \longmapsto \frac{\partial}{\partial u_2}, \quad Y \longmapsto -y^{-2} \frac{\partial}{\partial u_1} - u_2^2 \frac{\partial}{\partial u_2} + u_2 y \frac{\partial}{\partial y}, \quad H \longmapsto -2u_2 \frac{\partial}{\partial u_2} + y \frac{\partial}{\partial y}$$

$$\Omega = XY + YX + \frac{H^2}{2} = 2YX + \left(\frac{H^2}{2} + H\right) = 2XY + \left(\frac{H^2}{2} - H\right)$$ *Casimir element*

<u>Special functions:</u> $\boxed{f(u_1, u_2, y) = e^{\alpha_1 u_1 + \alpha_2 u_2} F(y)} = E(u) F(y)$

$$U_{,\psi_1} \backslash G / U_{,\psi_2} \qquad \underline{\boxed{\psi_j\left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\right) = e^{\alpha_j u}}} \quad E(u)^{-1} \frac{\partial}{\partial u_j} E(u) F = \alpha_j, \quad \frac{\partial}{\partial u_j} f = \alpha_j f$$

$$Xf = \alpha_2 f, \quad YXf = \alpha_2 Yf, \quad Yf = -\frac{\alpha_1}{y^2} f - \alpha_2 u_2^2 f + u_2 E(u) y \frac{dF}{dy}$$

$$Hf = -2\alpha_2 u_2 f + E(u) y \frac{dF}{dy}, \quad H^2 f = 4\alpha_2 u_2 (1 + \alpha_2 u_2) f - 2\alpha_2 u_2 E(u) y \frac{\partial F}{\partial y} +$$

$$+ 2\alpha_2 u_2 E(u) y \frac{\partial F}{\partial y} + E(u)\left(y^2 \frac{d^2 F}{dy^2} + y \frac{dF}{dy}\right)$$

$$E(u)^{-1} \frac{\Omega^2}{2} (E(u) F(y)) = \Big(-\frac{\alpha_1 \alpha_2}{y^2} - \cancel{\alpha_2^2 u_2^2} + \cancel{\alpha_2 u_2 y \frac{d}{dy}} - \cancel{\alpha_2 u_2} + \frac{1}{2} y \frac{d}{dy} + \cancel{\alpha_2 u_2 (1 + \alpha_2 u_2)} -$$

$$- \cancel{\alpha_2 u_2 y \frac{d}{dy}} + \frac{1}{4}\left(y^2 \left(\frac{d}{dy}\right)^2 + y \frac{d}{dy}\right) \Big) (F)$$

$$= \frac{1}{4}\left(y^2 \left(\frac{d}{dy}\right)^2 + 3y\left(\frac{d}{dy}\right) - \frac{4\alpha_1 \alpha_2}{y^2}\right)$$

Bessel's equation: $x^2u'' + xu' + (x^2-\nu^2)u = 0$.

Lommel's equation: $v(x) = x^a u(bx^c)$

$$v'' + \frac{1-2a}{x}v' + \left[(bcx^{c-1})^2 + \frac{a^2-\nu^2c^2}{x^2}\right]v = 0.$$

Want: $\frac{\Omega}{2}f = \lambda f$, $\quad \lambda = \frac{k^2-2k}{4} = \frac{(k-1)^2-1}{4}$

$$\Longleftrightarrow \quad F''(y) + \frac{3}{y}F'(y) + \left(\frac{4\alpha_1\alpha_2}{y^4} + \frac{1-(k-1)^2}{\cancel{4}\,y^2}\right)F(y) = 0$$

$1-2a = 3$, $\quad a^2-\nu^2c^2 = 1-(k-1)^2$, $\quad (bc\cancel{x}y^{c-1})^2 = -\frac{4\alpha_1\alpha_2}{y^4}$

$a = -1$, $\quad c = -1$, $\quad \nu = \pm(k-1)$, $\quad \left(\frac{b}{2}\right)^2 = -\alpha_1\alpha_2$

So: $F(y) = y^{-1}u\left(\frac{2\sqrt{-\alpha_1\alpha_2}}{y}\right)$, $\quad \underbrace{x^2u'' + xu' + (x^2-(k-1)^2)u = 0}_{\text{one solution: } J_{k-1}(x)}$.

Our case: $\alpha_1 = 2\pi i m$, $\alpha_2 = 2\pi i n$ $\quad (m, n \in \mathbb{Z})$

$$F(y) = \frac{1}{y}J_{k-1}\left(\frac{4\pi\sqrt{mn}}{y}\right) \qquad \text{(one solution)}$$

This function appears in the n-th Fourier coefficient of the corresponding Poincaré series $\sum_{\gamma \in \pm\left(\begin{smallmatrix}1 & \mathbb{Z}\\ 0 & 1\end{smallmatrix}\right)\backslash SL_2(\mathbb{Z})} e^{2\pi i m\tau}\big|_k\gamma$
