Quotient spaces $\Gamma\backslash\mathcal{H}$

Recall: bijections

$$\tau \longmapsto L = \mathbb{Z}\tau + \mathbb{Z}$$

$$\begin{tikzcd} SL_2(\mathbb{Z})\backslash\mathcal{H} \arrow[r, leftrightarrow] \arrow[d, leftrightarrow, "j"] & \{L \subset \mathbb{C} \text{ lattice}\}/\mathbb{C}^\times \arrow[r, leftrightarrow] & \{\mathbb{C}/L\}/\mathrm{Isom} \arrow[d, leftrightarrow] \\ \mathbb{C} & & \left\{\begin{matrix}\text{elliptic curves} \\ E \text{ over } \mathbb{C}\end{matrix}\right\}/\mathrm{Isom} \end{tikzcd}$$

$$E \setminus \{0\} : y^2 = 4x^3 - g_2 x - g_3$$

$$g_2 = 60 G_4(L), \quad g_3 = 140 G_6(L), \quad j = (12 g_2)^3/(g_2^3 - 27 g_3^2).$$

Goals: (1) Define a structure of a Riemann surface on $SL_2(\mathbb{Z})\backslash\mathcal{H}$ and show that

$$\begin{tikzcd} j : SL_2(\mathbb{Z})\backslash\mathcal{H} \arrow[r] \arrow[d, phantom, "\cap" description] & \mathbb{C} \arrow[d, phantom, "\cap" description] \\ \{\infty\} \cup (SL_2(\mathbb{Z})\backslash\mathcal{H}) \arrow[r] & \mathbb{P}^1(\mathbb{C}) \end{tikzcd}$$

are isomorphisms of Riemann surfaces.

(2) Study $\Gamma\backslash\mathcal{H}$ for more general discrete subgroups $\Gamma \subset SL_2(\mathbb{R})$, e.g. for congruence subgroups of $SL_2(\mathbb{Z})$:

$$SL_2(\mathbb{Z}) \supset \Gamma \supset \underbrace{\Gamma(N)}_{} = \{\alpha \in SL_2(\mathbb{Z}) \mid \alpha \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{N}\}$$

principal congruence subgroup modulo N.

Ex : $\Gamma = \Gamma(2)$

$$\begin{tikzcd} (x,y) \arrow[d, mapsto] & \in E \arrow[d, "p"] \\ x & \in \mathbb{P}^1(\mathbb{C}) \end{tikzcd}$$

$$: y^2 = \underbrace{4x^3 - g_2 x - g_3}_{4(x-e_1)(x-e_2)(x-e_3)}, \quad g_2^3 - 27 g_3^2 \neq 0$$

p = 2-fold covering, ramified at $\{0, (e_1,0), (e_2,0), (e_3,0)\}$

$$\mathbb{C}/L \xrightarrow{\sim} E(\mathbb{C})$$

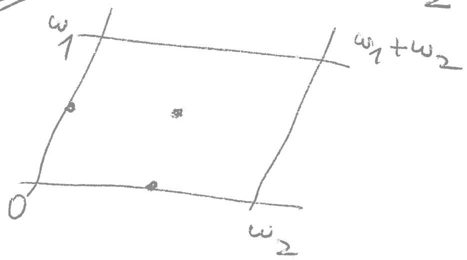
$$z \longmapsto (x,y) = (\wp(z), \wp'(z))$$

$\Updownarrow$

on $\mathbb{C}/L$: $\{0, \frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_1+\omega_2}{2}\}$ $=$ $\frac{1}{2}L/L$ $=$ $(\mathbb{C}/L)[2]$

$$L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$$

$$\mathrm{Im}(\omega_1/\omega_2) > 0$$



Ordering of the roots $\{e_1, e_2, e_3\}$

$\Updownarrow$

choice of an isomorphism $(\mathbb{Z}^2)^*/2 \xrightarrow{\sim} E(\mathbb{C})[2]$

(full level 2 structure on $\mathbb{C}/L$)

$(1,0) \longleftrightarrow \frac{\omega_1}{2} \pmod{L}$

$(0,1) \longleftrightarrow \frac{\omega_2}{2} \pmod{L}$

$\alpha \in GL_2(\mathbb{Z})$ preserves orientation <u>and</u> a full level 2 structure

$\Updownarrow$

$\alpha \in SL_2(\mathbb{Z})$ and $\alpha \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{2} \iff \alpha \in \Gamma(2)$.

Having fixed the order of e_1, e_2, e_3, we define

$$\lambda(E, \text{ordering of } \{e_1, e_2, e_3\}) := \frac{e_1 - e_3}{e_1 - e_2}$$

$$\Rightarrow \lambda(\tau) := \lambda\left(\mathbb{C}/(\mathbb{Z}\tau + \mathbb{Z}), \frac{\tau}{2}, \frac{1}{2}, \frac{\tau+1}{2}\right) = \frac{\wp(\tau/2) - \wp(\frac{\tau+1}{2})}{\wp(\frac{\tau}{2}) - \wp(\frac{1}{2})} = \left(\frac{\theta_{10}}{\theta_{00}}\right)^4 =$$

$(q = e^{2\pi i\tau})$

$$= 2^4 q^{1/2} \prod_{n=1}^{\infty} \left(\frac{1+q^n}{1+q^{n-1/2}}\right)^8 = 2^4 \left(\frac{\eta(2\tau)}{\eta(\frac{\tau+1}{2})}\right)^8$$

$$1 - \lambda(\tau) = \left(\frac{\theta_{01}}{\theta_{00}}\right)^{\cancel{8}4} = \prod_{n=1}^{\infty} \left(\frac{1-q^{n-1/2}}{1+q^{n-1/2}}\right)^8 = \left(\eta(\tau/2)/\eta(\tfrac{\tau+1}{2})\right)^8$$

Rmk: λ = cross ratio of the ramification points of $E \xrightarrow{p} \mathbb{P}^1(\mathbb{C})$ (in an order coming from the level 2 structure)

$\Rightarrow \exists!$ Möbius transformation mapping $(e_1, e_2, e_3, \infty) \mapsto (0, 1, \lambda, \infty)$, which transforms E into $(y/2)^2 = x(x-1)(x-\lambda)$ (Legendre's form)

Summary: (vertical maps are bijective), $\deg(\pi) = 6 = 3!$

$$\begin{tikzcd}
\Gamma(2)\backslash\mathcal{H} \arrow[r, "\pi"] \arrow[d, leftrightarrow] \arrow[ddddd, bend right, "\lambda"'] & \mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H} = \Gamma(1)\backslash\mathcal{H} \arrow[d, leftrightarrow] \arrow[dddd, bend right, "j"'] \\
\{(L \subset \mathbb{C} \text{ lattice}, (\mathbb{Z}^2)^*/2 \xrightarrow{\sim} \tfrac{1}{2}L/L)\}/\mathbb{C}^\times \arrow[r, "\pi"] \arrow[d, leftrightarrow] & \{L \subset \mathbb{C}\}/\mathbb{C}^\times \arrow[d, leftrightarrow] \\
\{(\mathbb{C}/L, (\mathbb{Z}^2)^*/2 \xrightarrow{\sim} (\mathbb{C}/L)[2])\}/\mathrm{Isom} \arrow[r, "\pi"] \arrow[d, leftrightarrow] & \{\mathbb{C}/L\}/\mathrm{Isom} \arrow[d, leftrightarrow] \\
\left\{\begin{array}{l}\text{elliptic curves } E \text{ over } \mathbb{C} \text{ with} \\ \text{full level 2 structure}\end{array}\right\}/\mathrm{Isom} \arrow[r, "\pi"] \arrow[d, leftrightarrow] & \{\text{elliptic curves over } \mathbb{C}\}/\mathrm{Isom} \\
\mathrm{PGL}_2(\mathbb{C})\backslash\backslash\left\{\begin{array}{l}4 \text{ distinct points in } \mathbb{P}^1(\mathbb{C}) \\ \text{(ordered)}\end{array}\right\} \arrow[d, leftrightarrow, "\text{cross ratio}"] & \mathbb{C} \\
\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\} = \mathbb{C} \setminus \{0, 1\} \arrow[ur, "\pi"] &
\end{tikzcd}$$

$\Gamma(1)/\Gamma(2) \xrightarrow{\sim} \mathrm{SL}_2(\mathbb{Z}/2\mathbb{Z}) \xrightarrow{\sim} S_3$ permutes all 6 full level 2 structures

Relation between λ and j:

$\underline{y^2 = 4x(x-1)(x-\lambda)}$: replace x by $x + \frac{\lambda+1}{3}$

get $y^2 = 4x^3 - g_2 x - g_3$, $g_2 = \frac{4}{3}(\lambda^2-\lambda+1)$, $g_3 = -\frac{8}{27}(\lambda+1)(\lambda-2)(\lambda-\frac{1}{2})$

$\Delta = 16\prod_{j<k}(e_j-e_k)^2 = 16\lambda^2(\lambda-1)^2$, $j = \frac{(12 g_2)^3}{\Delta} = 2^8 \frac{(\lambda^2-\lambda+1)^3}{\lambda^2(\lambda-1)^2}$

We know: λ and j induce bijections

$$\begin{tikzcd}
\Gamma(2)\backslash\mathcal{H} \arrow[r, "\pi"] \arrow[d, leftrightarrow, "\lambda"] & \Gamma(1)\backslash\mathcal{H} \arrow[d, leftrightarrow, "j"] \\
\mathbb{P}^1(\mathbb{C})\smallsetminus\{0,1,\infty\} \arrow[r, "\pi"] & \mathbb{P}^1(\mathbb{C})\smallsetminus\{\infty\} \\
\lambda \arrow[u, phantom, "\rotatebox{90}{$\in$}"] \arrow[r, mapsto] & 2^8\frac{(\lambda^2-\lambda+1)^3}{\lambda^2(\lambda-1)^2}
\end{tikzcd}$$

Relation to Gauss's differential ~~equat~~ equation

$\left(\lambda(\lambda-1)\left(\frac{d}{d\lambda}\right)^2 + (2\lambda-1)\frac{d}{d\lambda} + \frac{1}{4}\right)\omega_j = 0$, where $\lambda \in \mathbb{C}\smallsetminus\{0,1\}$

$E: y^2 = x(x-1)(x-\lambda)$, periods: $\omega_1 = 2\int_0^1 \omega$, $\omega_2 = 2\int_1^\lambda \omega$, $\omega = \frac{dx}{y}$

$\overset{\gamma_1(\lambda)}{\underset{0 \qquad 1}{\bullet\!\!-\!\!\!-\!\!\!-\!\!\bullet}}$ $\overset{\gamma_2(\lambda)}{\underset{1 \qquad \lambda}{\bullet\!\!-\!\!\!-\!\!\!-\!\!\bullet}}$ (multivalued functions of λ)

Monodromy: if λ goes aroud $\underset{0 \qquad 1}{\bullet\!\!-\!\!\!-\!\!\!-\!\!\bullet}$: $\begin{pmatrix}\gamma_1\\ \gamma_2\end{pmatrix} \longmapsto \begin{pmatrix}\gamma_1\\ \gamma_2+2\gamma_1\end{pmatrix} = \begin{pmatrix}1 & 0\\ 2 & 1\end{pmatrix}\begin{pmatrix}\gamma_1\\ \gamma_2\end{pmatrix}$

— " — — through $\underset{0 \qquad 1}{\bullet\!\!-\!\!\!-\!\!\!-\!\!\bullet}$: $\begin{pmatrix}\gamma_1\\ \gamma_2\end{pmatrix} \longmapsto \begin{pmatrix}\gamma_1+2\gamma_2\\ \gamma_2\end{pmatrix} = \begin{pmatrix}1 & 2\\ 0 & 1\end{pmatrix}\begin{pmatrix}\gamma_1\\ \gamma_2\end{pmatrix}$

$\tau = \frac{\omega_1}{\omega_2} \in \mathcal{H}$ multivalued function of $\lambda \in \mathbb{C}\smallsetminus\{0,1\}$

When λ goes around a loop in $\mathbb{C}\smallsetminus\{0,1\}$, τ is replaced by $\frac{a\tau+b}{c\tau+d}$, for some $\begin{pmatrix}a & b\\ c & d\end{pmatrix} \in \Gamma(2)$ (and, up to a sign, every element of $\Gamma(2)$ arises in this way).

Examples of discrete subgroups of $SL_2(\mathbb{R})$ and $SL_2(\mathbb{R})/\{\pm I\}$

Notation: $H \subset \underbrace{SL_2(\mathbb{R})}_{G}$ subgroup $\Rightarrow \overline{H} := \mathrm{Im}(H \longrightarrow \underbrace{SL_2(\mathbb{R})/\{\pm I\}}_{\mathrm{Aut}_{hol}(\mathcal{H}) = \overline{G}})$

Ex: (1) $SL_2(\mathbb{Z}) \subset G$ discrete

(2) $\Gamma \subset G$ discrete $\Rightarrow \forall\, g \in GL_2(\mathbb{R})$ $g\Gamma g^{-1} \subset G$ discrete

(3) $\Gamma \subset G$ discrete, $\underbrace{\Gamma' \text{ commensurable with } \Gamma}_{(\Gamma : \Gamma\cap\Gamma'),\ (\Gamma' : \Gamma\cap\Gamma') < \infty} \Rightarrow \Gamma' \subset G$ discrete

(4) Geometry: if X is a Riemann surface, then its universal covering $\widetilde{X}$ is holomorphically isomorphic either to

(A) $\mathbb{P}^1(\mathbb{C})$

(B) $\mathbb{C}$ (Uniformisation Theorem)

(C) (unit disc D) $\simeq \mathcal{H}$

The fundamental group $\Gamma = \pi_1(X, x_0)$ ($x_0 \in X$ fixed) acts freely discontinuously on $\widetilde{X}$ by holomorphic isomorphisms and $\Gamma \backslash \widetilde{X} = X$.

In case (A), $\mathrm{Aut}_{hol}(\mathbb{P}^1(\mathbb{C})) = PGL_2(\mathbb{C})$, but each $g \in PGL_2(\mathbb{C})$ has a fixed point in $\mathbb{P}^1(\mathbb{C}) \Rightarrow \Gamma = \{1\}$, $X \simeq \mathbb{P}^1(\mathbb{C})$

In case (B), $\Gamma \simeq \mathbb{Z}^a$ for $a \in \{0,1,2\}$ and $X \simeq \mathbb{C}$, $\mathbb{C}^\times$, $\mathbb{C}/(\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2)$, respectively.

In case (C), $\Gamma \subset \mathrm{Aut}_{hol}(\mathcal{H}) = SL_2(\mathbb{R})/\{\pm I\}$ acts freely discontinuously on $\mathcal{H} \Rightarrow \Gamma$ is a discrete subgroup of $\overline{G}$. (without fixed points in $\mathcal{H}$).

The case when X is of finite type: $X = \overline{X} \setminus S$,

$\overline{X}$ = compact Riemann surface of genus $g \geq 0$

$S \subset \overline{X}$ finite set, $|S| = s \geq 0$.

Structure of $\Gamma = \pi_1(X, x_0)$: generators $\underbrace{A_1, \ldots, A_g, B_1, \ldots, B_g}_{\text{come from } \overline{X}}, \underbrace{C_1, \ldots, C_s}_{\text{loops around pts in } S}$

one relation: $\boxed{[A_1, B_1] \cdots [A_g, B_g]\, C_1 \cdots C_s = 1}$

$[A, B] = ABA^{-1}B^{-1}$

Cor: $\boxed{\Gamma \text{ is free on } 2g+s-1 \text{ generators if } s > 0}$

Ex: $\Gamma = \overline{\Gamma(2)} = \left\langle \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \right\rangle$, $X = \Gamma(2)\backslash\mathcal{H} \xrightarrow[\sim]{\lambda} \mathbb{P}^1 \setminus \{0, 1, \infty\}$

$\overline{\Gamma(2)} = \pi_1(X)$ free on 2 generators $\quad g = 0,\ s = 3$

Discontinuous actions and discrete subgroups

Def. A topological group is a group G equipped with a topology such that

(A) The group law $G\times G \longrightarrow G$ and the inverse $G\longrightarrow G$ are continuous.
$(g,h)\longmapsto gh \qquad\qquad g\longmapsto g^{-1}$

(B) The one point set $\{e\}$ (where $e=$ the neutral el^t of G) is closed.

Rmks : (A) $\Rightarrow$ left/right translations $L(g): h\longmapsto gh$, $R(g): h\longmapsto hg$ are homeomophisms (so is $g\longmapsto g^{-1}$).

(A) $\Rightarrow$ $\forall$ open $U\ni e$ $\exists$ open $V\ni e$ $U\supset VV^{-1}:=\{gh^{-1}\mid g,h\in V\}$ (since $(g,h)\mapsto gh^{-1}$ is continuous

(A), (B) $\Rightarrow$ $\forall g\in G$ $\{g\}$ is closed

(A), (B) $\Rightarrow$ G is a Hausdorff topological space (if $g\neq h\in G$, then $g^{-1}h\neq e \Rightarrow \exists$ open $U\ni e$ $g^{-1}h\notin U \Rightarrow \exists$ $V\ni e$ open $g^{-1}h\notin VV^{-1} \Rightarrow \underbrace{gV}_{}\cap\underbrace{hV}_{\text{open}}=\emptyset$

Cor. $H\lhd G$ closed normal subgroup $\Rightarrow$ G/H with quotient topology is a topological group

---

Def. A subset $Y\subset X$ of a topological space X is discrete if the topology induced on Y from X is discrete, i.e., $\forall y\in Y$ $\exists U\subset X$ open $U\cap Y=\{y\}$.

Thm. A discrete subgroup Γ of a topological group G is closed.

Warning: this is false for subsets: $Y=\{1,\frac{1}{2},\frac{1}{3},\dots\}\subset\mathbb{R}$ discrete, not closed.

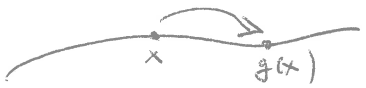
Pf. Assume $g\in G$ lies in the closure of Γ. By assumption, $\exists U\ni e$ open $U\cap\Gamma=\{e\}$, $\exists V\ni e$ open $VV^{-1}\subset U$. If $\gamma_1,\gamma_2\in\Gamma\cap Vg$
$\Rightarrow \gamma_1\gamma_2^{-1}\in\Gamma\cap VV^{-1}\subset\Gamma\cap U=\{e\} \Rightarrow \gamma_1=\gamma_2 \Rightarrow |\Gamma\cap V_g|\leq 1.$
But $g\in V_g$ is open $\Rightarrow$ g lies in the closure of $\Gamma\cap V_g \Rightarrow \Gamma\cap V_g=\{g\}$
$\Rightarrow g\in\Gamma.$

Cor. $\Gamma\subset G$ discrete subgroup, $K\subset G$ compact subgroup $\Big\}\Rightarrow$ $\Gamma\cap K\subset G$ finite subgroup.

Def. A left action of a topological group G on a topological space X is continuous if $\begin{array}{l} G\times X \longrightarrow \cancel{X} \\ (g,x)\longmapsto gx \end{array}$ is continuous

($\Longleftrightarrow$ the action map $\begin{array}{l} G\times X \longrightarrow X\times X \\ (g,x)\longmapsto (x,gx) \end{array}$ is continuous)

($\Longrightarrow$ $\forall x\in X$ the orbit map $\begin{array}{l} \mathrm{orb}_x : G \longrightarrow X \\ \qquad g \longmapsto g(x) \end{array}$ is continuous).



Exercise. Assume: G = locally compact top. group
X = — " — space (Hausdorff) } with countable topology
G acts continuously and transitively on X, $x\in X$
$\Longrightarrow$ orb_x induces a homeomorphism $\underbrace{G/G_x}_{\text{quotient topology}} \xrightarrow{\sim} X$ $(gG_x \longmapsto g(x))$

Def. A group Γ acting on a topological space X by homeomorphisms act freely discontinuously at $x\in X$ if $\exists U\ni x$ open such that $\{\gamma\in\Gamma \mid \gamma(U)\cap U\neq\emptyset\}=\{e\}$.
It acts freely discontinuously if this holds $\forall x\in X$

($\Longrightarrow$ the projection $p: X\longrightarrow \underbrace{\Gamma\backslash X}_{\text{quotient topology}}$ is a covering)

Prop. If a continuous action of a topological group Γ on a topological space X is free discontinuous at some $x\in X$, then the topology on Γ is discrete.

Pf. For $U\ni x$ as above, continuity of $\begin{array}{l} G\times X \longrightarrow X\times X \\ (g,x)\longmapsto (x,gx) \end{array}$ implies that $\{e\}\times U\subset\Gamma\times X$ is open $\Longrightarrow$ $\{e\}\subset\Gamma$ open.

Ex. The Bianchi group $\Gamma = SL_2(\mathbb{Z}[i])/\{\pm I\}\subset G = SL_2(\mathbb{C})/\{\pm I\}$ is discrete, but its action on $X=\mathbb{P}^1(\mathbb{C})$ is not freely discontinuous at any $x\in\mathbb{P}^1(\mathbb{C})$, since $\bigcup_{\gamma\in\Gamma\smallsetminus\{e\}}\{\text{fixed points of }\gamma\}$ is dense in X.

Explanation: $\mathbb{P}^1(\mathbb{C}) = SL_2(\mathbb{C})/B$ is too small, $B=\left\{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}\right\}$ is too big (not compact)

Weaker notion of discontinuity (fixed points allowed)

Assume: $X =$ locally compact Hausdorff topological space

Def. An action of a group Γ on X (by homeomorphisms) is discontinuous if $\forall K_1, K_2 \subset X$ compact $\{\gamma \in \Gamma \mid \gamma(K_1) \cap K_2 \neq \emptyset\}$ is finite.

Key example: If G is a topological group acting continuously on X with finitely many orbits, in such a way that $\forall x \in X$ $\mathrm{orb}_x : G \longrightarrow X$, $g \longmapsto g(x)$ is proper (i.e., orb_x^{-1}(compact set) is compact), then every discrete subgroup $\Gamma \subset G$ acts discontinuously on X.

[Note: enough to check the assumption on orb_x only for one point on each G-orbit in X.]

Pf. Fix representatives $x_1, \dots, x_n \in X$ of the G-orbits in X. Given compact sets $A, B \subset X$, then each

$A_j = \mathrm{orb}_{x_j}^{-1}(A)$, $B_j = \mathrm{orb}_{x_j}^{-1}(B) \subset G$ is compact

$$\Rightarrow \text{ so is } B_j A_j^{-1} = \mathrm{Im}\big(B_j \times A_j \hookrightarrow G \times G \longrightarrow G\big) \subset G.$$
$$(g, h) \longmapsto gh^{-1}$$

$$\Rightarrow \{\gamma \in \Gamma \mid \gamma(A) \cap B \neq \emptyset\} = \bigcup_{j=1}^{n} \big(\underbrace{\Gamma}_{\substack{\text{closed}\\ \text{discrete}}} \cap \underbrace{B_j A_j^{-1}}_{\text{compact}}\big) \text{ is finite.}$$

Note: If the action map $G \times X \longrightarrow X \times X$, $(g, x) \longmapsto (x, gx)$ is proper, so is $\mathrm{orb}_x : G \longrightarrow X$, $g \longmapsto g(x)$, $\forall x \in X$.

Example: $G = SL_2(\mathbb{R})$, $X = \mathcal{H} \simeq G/K$: the action is transitive and the orbit map $\mathrm{orb}_i : G \longrightarrow \mathcal{H}$, $g \longmapsto g(i)$ is a locally trivial fibration with compact fibre $K = G_i = SO(2)$. More precisely, the ~~action~~ orbit map orb_i is homeomorphic, via Iwasawa decomposition, to the projection

$$\begin{tikzcd} G \arrow[d, mapsto] & \cong & N \times A \times K \arrow[r, "pr"] & N \times A \\ nak & & (n, a, k) \arrow[ll, mapsto] \arrow[r, mapsto] & (n, a) \end{tikzcd}$$

$N = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \,\middle|\, x \in \mathbb{R} \right\}$

$A = \left\{ \begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix} \,\middle|\, y > 0 \right\}$

<u>Ex</u>: this suggests that in the case $G = SL_2(\mathbb{C})$ one should consider the action of G on G/K, for a maximal compact subgroup $K \subset SL_2(\mathbb{C})$. For $K = SU(2)$, the quotient $SL_2(\mathbb{C})/SU(2)$ can be identified with the 3-dimensional hyperbolic space $H^3 = \left\{ x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 \mid x_3 > 0 \right\}$, whose boundary is $\partial H^3 = \begin{pmatrix} \mathbb{R}^2 \\ 0 \end{pmatrix} \cup \{\infty\} = \mathbb{P}^1(\mathbb{C})$ (as in the case of the $SL_2(\mathbb{R})$ action on $\mathcal{H}$ and $\partial \mathcal{H} = \mathbb{P}^1(\mathbb{R})$). The action of $SL_2(\mathbb{C})/\{\pm I\} \simeq SO(3,1)^+$ on H^3 has a natural geometric description ("Poincaré extension").

<u>Exercise</u>: If a continuous action of a topological group Γ on X is discontinuous, then the topology of Γ is discrete.

<u>Prop.</u> Assume that a group Γ acts discontinuously on X (by homeomorphisms). Then:

(1) $\forall x, y \in X$ $\quad \{\gamma \in \Gamma \mid \gamma(x) = y\}$ is finite ($\xRightarrow{y=x}$ Γ_x is finite).

(2) $\forall x, y \in X$ $\quad \exists$ open $U \ni x, V \ni y$ $\quad \{\gamma \in \Gamma \mid \gamma U \cap V \neq \emptyset\}$ is finite.

(3) $\forall x, y \in X$ —— " —— $\{$ —"— $\} = \{\gamma \in \Gamma \mid \gamma(x) = y\}$

(4) $\Gamma \backslash X$ with quotient topology is Hausdorff.

(5) If $\forall x \in X$ $\Gamma_x = \{e\}$, then $X \xrightarrow{p} \Gamma \backslash X$ is a covering map (more precisely, if $\Gamma_x = \{e\}$, then Γ acts freely discontinuously at x)

<u>Pf</u>: (1) By definition: $\{x\}$, $\{y\}$ are compact.

(2) X locally compact $\Rightarrow \exists U \ni x, V \ni y$ open with $\overline{U}, \overline{V}$ compact.

(3) $\exists U_0 \ni x, V_0 \ni y$ open $\{\gamma \in \Gamma \mid \gamma(U_0) \cap V_0 \neq \emptyset\} = \{\gamma_1, \ldots, \gamma_m\} \cup \{\delta_1, \ldots, \delta_n\}$

$\exists U_j \ni \delta_j(x), V_j \ni y$ open $U_j \cap V_j = \emptyset$ $\qquad \forall i, j$ $\quad \underline{\gamma_i(x) = y}$ $\quad \underline{\delta_j(x) \neq y}$

Take $U = U_0 \cap \bigcap_{j=1}^{n} \delta_j^{-1}(U_j)$, $V = V_0 \cap \bigcap_{j=1}^{n} V_j$. <u>Note</u>: (3) $\xRightarrow{x=y}$ (5).

(4) Projection $X \xrightarrow{p} \Gamma \backslash X$. Given $x, y \in X$ such that $p(x) \neq p(y)$, $\exists U \ni x, V \ni y$ open $\{\gamma \in \Gamma \mid \gamma(U) \cap V \neq \emptyset\} = \{\gamma \in \Gamma \mid \gamma(x) = y\} \overset{\Downarrow}{=} \emptyset$.

Then $p^{-1}(p(U)) = \bigcup_{\gamma \in \Gamma} \gamma U$, $p^{-1}(p(V)) = \bigcup_{\gamma \in \Gamma} \gamma V$ are open in X and <u>disjoint</u>

$\Rightarrow p(U) \ni p(x)$, $p(V) \ni p(y)$ are open and disjoint.

Discontinuous action of subgroups of $SL_2(\mathbb{C})$

Notation: $\Gamma \subset SL_2(\mathbb{C})$ subgroup, $\overline{\Gamma} := \mathrm{Im}(\Gamma \longrightarrow SL_2(\mathbb{C})/\{\pm I\})$
$\simeq \Gamma/(\Gamma \cap \{\pm I\})$

Old-fashioned terminology (Poincaré):

$\Gamma \subset SL_2(\mathbb{C})$ is a $\left\{\begin{matrix}\text{Fuchsian}\\ \text{Kleinian}\end{matrix}\right\}$ subgroup if $\exists\, x \in \mathbb{P}^1(\mathbb{C})$ at which $\overline{\Gamma}$ acts freely discontinuously ($\Longrightarrow \Gamma$ is discrete in $SL_2(\mathbb{C})$)

and $\left\{\begin{matrix}\exists \text{ circle } C \text{ such that } \Gamma \text{ preserves the interior of } C\\ \text{no such } C \text{ exists.}\end{matrix}\right\}$

We are interested only in Fuchsian subgroups $\Gamma \subset SL_2(\mathbb{C})$:

(1) $\exists\, g \in SL_2(\mathbb{C})$ $g^{-1}(C)g = \mathbb{P}^1(\mathbb{R})$ (and $g^{-1}(\text{interior of } C)g = \mathcal{H}$)
$\Longrightarrow$ $\underline{g^{-1}\Gamma g \subset SL_2(\mathbb{R})}$ (discrete subgroup).

(2) Conversely, a discrete subgroup $\Gamma \subset SL_2(\mathbb{R})$ acts discontinously on $\mathcal{H} \Longrightarrow \forall \tau \in \mathcal{H}$ $\Gamma_\tau \subset SL_2(\mathbb{R})_\tau = g_\tau SO(2) g_\tau^{-1}$
is finite $\Longrightarrow$ finite cyclic $\Rightarrow$ with fixed points $\{\tau, \overline{\tau}\}$
$\Longrightarrow$ $\underbrace{\{\tau \in \mathcal{H} \mid \overline{\Gamma}_\tau = \{e\}\}}_{\Gamma \text{ acts freely discontinuously at } \tau.}$ is dense in $\mathcal{H} \Longrightarrow \Gamma$ is Fuchsian.

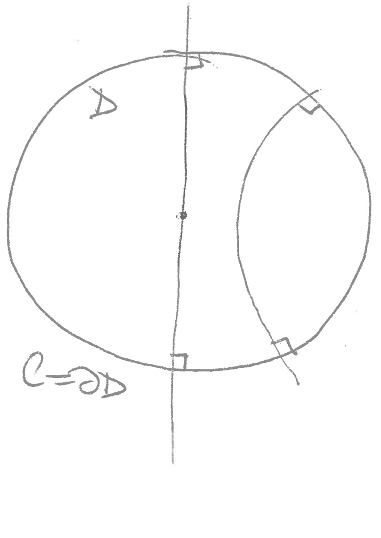
Summary: Fuchsian subgroups of $SL_2(\mathbb{R})$ = discrete subgroups of $SL_2(\mathbb{R})$

---

(3) It is often useful to pass from $(\mathcal{H}, \Gamma \subset SL_2(\mathbb{R}))$ to
$\left(\underbrace{g(\mathcal{H})}_{\text{interior of circle } C}, g\Gamma g^{-1} \subset g\, SL_2(\mathbb{R}) g^{-1}\right)$, for suitable $g \in SL_2(\mathbb{C})$.

Ex: $g = \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$ gives $(D = \{|w| < 1\}, \Gamma' \subset SU(1,1))$

Geometry:



$\{\text{geodesics in } D\} = \{D \cap (\text{circles} \perp \partial D)\}$

isometries: generated by symmetries with respect to the geodesics

Recall: given $C \subset \mathbb{P}^1(\mathbb{C})$ circle, D one of the two components of $\mathbb{P}^1(\mathbb{C}) \smallsetminus C$ (e.g., $C = \mathbb{P}^1(\mathbb{R})$, $D = \mathcal{H}$), elements of $G_C := \{ g \in SL_2(\mathbb{C}) \mid g(D) = D \}$ are of the following types:

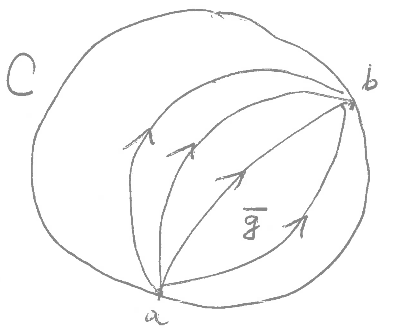
(0) $g = \pm I$

(1) g hyperbolic: g has real eigenvalues $\lambda_1 \neq \lambda_2 = \lambda_1^{-1}$,

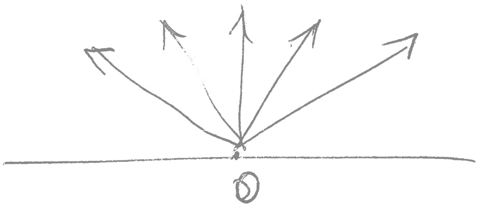
$\exists h \in SL_2(\mathbb{C})$ $\quad \pm h g h^{-1} = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$, $t > 0$, $t \neq 1$

~~orbits~~ $\mathrm{Fix}(g) = \{ x \in \mathbb{P}^1(\mathbb{C}) \mid g(x) = x \} = \{a, b\}$, $a, b \in C$, $a \neq b$

orbits of $\bar{g}^\sigma$ ($\sigma \in \mathbb{R}$, $\bar{g}$ = the image of g in $\bar{G}_C = G_C/\{\pm I\}$) are (the circles passing through $\mathrm{Fix}(g)$) $\cap D$:



If $C = \mathcal{H}$, $b = \infty$, $a = 0$: $\quad \pm g = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$, $t > 1$

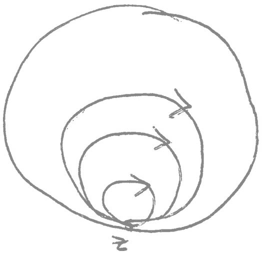


(2) g parabolic: $g \neq \pm I$ has double eigenvalue $\lambda_1 = \lambda_2 = \lambda_1^{-1}$:

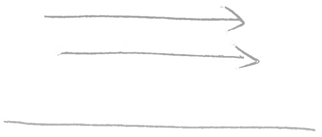
$\exists h \in SL_2(\mathbb{C})$ $\quad \pm h g h^{-1} = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$, $a \in \mathbb{R}$, $a \neq 0$ (can take $a=1$)

$\mathrm{Fix}(g) = \{z\}$, $z \in C$

orbits of $\bar{g}^\sigma$ are the horocycles at z: circles in D tangent to ∂D at z ($\Longleftrightarrow$ $\perp$ to geodesics from z)



If $C = \mathcal{H}$, $z = \infty$: $\quad g = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

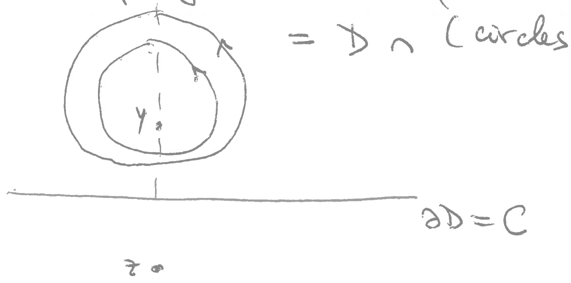


---

(3) g elliptic: g has non-real complex conjugate eigenvalues $\lambda_1 = \bar{\lambda}_2 = \bar{\lambda}_1^{-1}$ ($\Rightarrow |\lambda_1| = 1$), $\mathrm{Fix}(g) = \{y, z\}$ $\quad y \in D$, $z = s_C(y) \notin D$

orbits of $\bar{g}^\sigma$ $= D \cap$ (circles C' such that $s_{C'}(y) = z$)

$= D \cap$ (circles orthogonal to all geodesics through y)



Fact: A discrete subgroup $\Gamma \subset SL_2(\mathbb{R})$ consisting only of elliptic elements (and $\pm I$) has a common fixed point $\tau \in \mathcal{H}$
$\Longrightarrow$ is equal to the finite cyclic group Γ_τ.

Prop. If $g, h \in SL_2(\mathbb{C}) \setminus \{\pm I\}$, g hyperbolic, h shares with g precisely one fixed point $\Longrightarrow$ the subgroup $\langle g, h \rangle$ is not discrete.

Pf. Can assume $g = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}$, $\alpha \in \mathbb{R}$, $\cancel{\neq 0}$, $\alpha^2 \neq 1$, $\mathrm{Fix}(g) \cap \mathrm{Fix}(h) = \{\infty\}$

$\Longrightarrow h = \begin{pmatrix} \beta & b \\ 0 & \beta^{-1} \end{pmatrix}$, $b \neq 0$ $(\beta, b \in \mathbb{C})$. For $n \in \mathbb{Z}$,

$u_n := g^n h g^{-n} h^{-1} = \begin{pmatrix} 1 & (\alpha^{2n} - 1) b \\ 0 & 1 \end{pmatrix} \in \langle g, h \rangle$ are distinct, $\lim\limits_{n \to +\infty} (u_n)$ (if $|\alpha| < 1$)
resp. $\lim\limits_{n \to -\infty} (u_n)$ (if $|\alpha| > 1$) exists $\Longrightarrow \langle g, h \rangle$ is not discrete.

Def. Let $\Gamma \subset SL_2(\mathbb{R})$ be a discrete subgroup. A point
($\Longrightarrow \Gamma$ is countable)
$z \in \mathcal{H} \cup \mathbb{P}^1(\mathbb{R})$ is

$$\left\{ \begin{array}{l} \text{an elliptic point} \\ \text{a hyperbolic point} \\ \text{a parabolic point (or a cusp)} \end{array} \right\} \text{ of } \Gamma \text{ if } \exists \gamma \in \Gamma \left\{ \begin{array}{l} \gamma \text{ elliptic} \\ \gamma \text{ hyperbolic} \\ \gamma \text{ parabolic} \end{array} \right\} \gamma(z) = z$$

The corresponding sets Ell_Γ, Hyp_Γ, Cusps_Γ of all such points satisfy: (1) $\mathrm{Ell}_\Gamma \subset \mathcal{H}$; (2) Hyp_Γ, $\mathrm{Cusps}_\Gamma \subset \mathbb{P}^1(\mathbb{R}) = \partial \mathcal{H}$

(3) $\boxed{\mathrm{Hyp}_\Gamma \cap \mathrm{Cusps}_\Gamma = \emptyset}$ (by Prop. above)

(4) Ell_Γ is discrete and (at most) countable.

(5) $\tau \in \mathrm{Ell}_\Gamma \Longrightarrow \Gamma_\tau \subset SL_2(\mathbb{R})_\tau = g_\tau SO(2) g_\tau^{-1}$ is finite cyclic.

(6) $\boxed{x \in \mathrm{Hyp}_\Gamma \Longrightarrow \overline{\Gamma}_x \text{ is infinite cyclic.}}$

Pf: replace Γ by $g \Gamma g^{-1}$ for suitable $g \in SL_2(\mathbb{R}) \Longrightarrow$ can assume
$\gamma = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$, $t \in \mathbb{R}^\times$, $t^2 \neq 1$, $z = \infty$. By Prop. above, $\Gamma_\infty = \Gamma_\infty \cap \Gamma_0 =$
$= \Gamma \cap \left\{ \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \right\}$
$\mathrm{Fix}(\gamma) = \{\infty, 0\}$ $\Longrightarrow \overline{\Gamma_\infty}$ discrete subgroup of $\left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \middle| a > 0 \right\} \overset{\exp}{\xleftarrow{\sim}} \mathbb{R}$
(non-trivial)

(7) $\boxed{x \in \mathrm{Cusps}_\Gamma \Longrightarrow \overline{\Gamma}_x \text{ is infinite cyclic}}$

Pf: again, can assume $x = \infty$, $\pm\gamma = \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}$, $h \in \mathbb{R}$, $h \neq 0$
Prop. above $\Longrightarrow \overline{\Gamma}_x \subset \pm \begin{pmatrix} 1 & \mathbb{R} \\ 0 & 1 \end{pmatrix} \Longrightarrow \overline{\Gamma}_x = \pm \begin{pmatrix} 1 & A \\ 0 & 1 \end{pmatrix}$, $A \subset \mathbb{R}$ non-trivial discrete subgroup.

Fundamental domains

Datum: $\Gamma \subset SL_2(\mathbb{R})$ discrete subgroup
($\overline{\Gamma} :=$ image of Γ in $SL_2(\mathbb{R})/\{\pm I\}$)

Def. A Fundamental set of $\overline{\Gamma}$ acting on $\mathcal{H}$: a subset $F \subset \mathcal{H}$ such that

$$\mathcal{H} = \coprod_{\gamma \in \overline{\Gamma}} \gamma F \qquad \text{(disjoint union)}$$

Def. A Fundamental domain of $\overline{\Gamma}$ acting on $\mathcal{H}$: a subset $E \subset \mathcal{H}$ s.t.

(a) $E \subset \mathcal{H}$ is open

(b) $\exists$ fundamental set F of $\overline{\Gamma}$ such that $E \subset F \subset \underbrace{\widetilde{E}}_{\text{closure of } E \text{ in } \mathcal{H}}$

(c) $\mathrm{vol}(\partial E) = 0$ (hyperbolic area)

Small technical problem: $E \cup \partial E = \widetilde{E} \hookrightarrow \mathcal{H} \twoheadrightarrow \Gamma\backslash\mathcal{H}$ (composite p) is surjective

and injective when restricted to E, but the continuous bijection

$\underbrace{p(\widetilde{E})}_{\text{quotient topology}} \xrightarrow{\sim} \Gamma\backslash\mathcal{H}$ is not necessarily a homeomorphism

---

Remedy: **Def.** A fundamental domain E of $\overline{\Gamma}$ is locally finite
if $\forall K \subset \mathcal{H}$ compact $\{\gamma \in \overline{\Gamma} \mid K \cap \gamma(\widetilde{E})\}$ is finite.

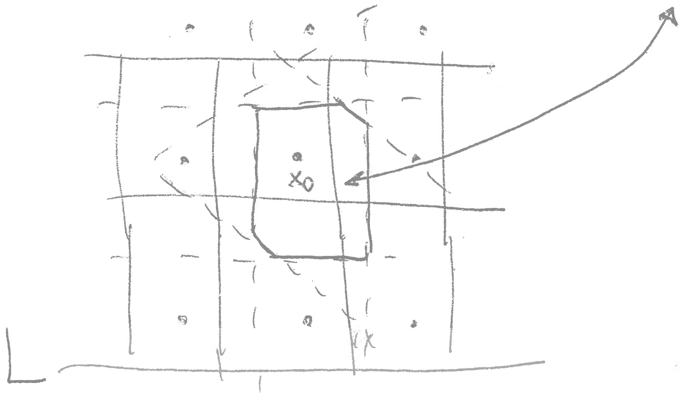
Warning: $\exists$ example when $E =$ interior of a convex hyperbolic pentagon, but E is not locally finite (see [Beardon], 9.2.5)

Prop. E is locally finite $\Longleftrightarrow$ $p(\widetilde{E}) \xrightarrow{\sim} \Gamma\backslash\mathcal{H}$ is a homeomorphism.
$\Downarrow$
the subset $\{\gamma \in \overline{\Gamma} \mid \gamma(\widetilde{E}) \cap \widetilde{E} \neq \emptyset\}$ generates $\overline{\Gamma}$. (see [Beardon])

Normal fundamental domain

Euclidean version:
Dirichlet (later revisited by Voronoi): given a lattice $L \subset \mathbb{R}^n$,

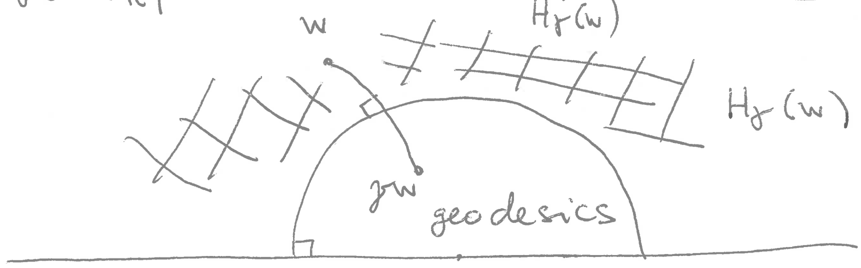
fix $x_0 \in \mathbb{R}^n$ and consider $E(x_0) := \{x \in \mathbb{R}^n \mid \forall u \in L \setminus \{0\}\ \|x - x_0\| < \|x - (x_0 + u)\|\}$



This is a fundamental domain for the action of L on $\mathbb{R}^n$.
It is the interior of a polygon with finitely many sides.

Hyperbolic version (Poincaré): $\Gamma \subset SL_2(\mathbb{R})$ discrete subgroup
$dist(x,y)$ = hyperbolic distance (in the Poincaré metric)
Fix $w \in \mathcal{H}$ such that $\Gamma_w = \{e\}$ and define

$$E(w) := \bigcap_{\gamma\in\Gamma\setminus\{e\}} \underbrace{\{z\in\mathcal{H} \mid dist(z,w) < dist(z,\gamma w)\}}_{H_\gamma(w)}$$



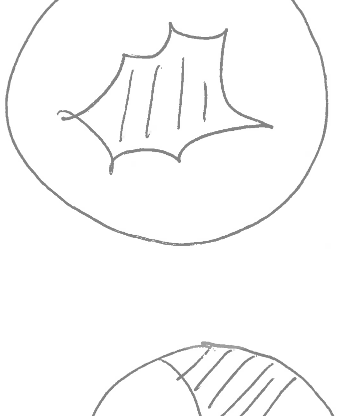
Prop. $E(w)$ is a locally finite fundamental domain of Γ. It is the interior of a convex hyperbolic polygon (possibly with infinitely many sides).

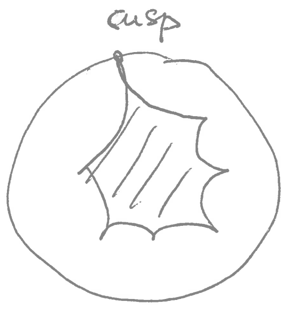
Def. $\Gamma \subset SL_2(\mathbb{R})$ (a discrete subgroup) is a Fuchsian group of the 1st kind if every $x \in \mathbb{P}^1(\mathbb{R}) = \partial\mathcal{H}$ is a limit point of Γ: $\exists\ \tau \in \mathcal{H}$ and (distinct) $\gamma_n \in \Gamma$ such that $\lim_{n\to+\infty} \gamma_n(\tau) = x$. If not, we say that Γ is of 2nd kind.

Ex of fundamental domains (transported to D by the Cayley map):

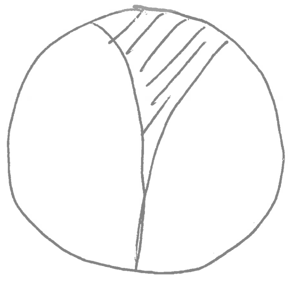
1st kind

(Γ purely hyperbolic)





2nd kind:



(or more complicated)

Adding points at infinity to $\Gamma\backslash\mathcal{H}$

Notation: $\Gamma\subset G=SL_2(\mathbb{R})$ discrete subgroup, $\overline{\Gamma}:=\Gamma/(\Gamma\cap\{\pm I\})$

Prop. For each $\tau\in\mathcal{H}$, $\overline{\Gamma_\tau}$ is a finite cyclic group. Put $e_\tau:=|\overline{\Gamma_\tau}|$.
If $-I\notin\Gamma$, then e_τ is odd.

Pf. $\Gamma_\tau=\Gamma\cap G_\tau$ is closed, discrete and compact $\Rightarrow$ it is a finite subgroup of $G_\tau=g\,SO(2)g^{-1}\simeq SO(2)$ $\quad(g(i)=\tau)$, hence cyclic.

If $-I\notin\Gamma\Rightarrow \underbrace{-I}_{\text{the only element of } G_\tau \text{ of order } 2}\notin\Gamma_\tau \Rightarrow \overline{\Gamma_\tau}=\Gamma_\tau$ has odd order.

Recall: $\tau\in\mathcal{H}$ is an elliptic point of $\Gamma\Leftrightarrow e_\tau>1$.

Recall: $x\in\mathbb{P}^1(\mathbb{R})$ is a cusp ($=$ a parabolic point) of Γ

$\Leftrightarrow \exists\,\gamma\in\Gamma$ parabolic such that $\gamma(x)=x$.

$$\exists\, g\in G\quad \gamma=\pm g\begin{pmatrix}1&a\\0&1\end{pmatrix}g^{-1},\ a\neq0$$

Lemma: $\{\text{cusps of } SL_2(\mathbb{Z})\}=\mathbb{P}^1(\mathbb{Q})=\mathbb{Q}\cup\{\infty\}$.

Pf. $\begin{pmatrix}1&1\\0&1\end{pmatrix}(\infty)=\infty \Rightarrow$ every $\underset{\uparrow\atop SL_2(\mathbb{Z})}{\begin{pmatrix}a&b\\c&d\end{pmatrix}}(\infty)=\frac{a}{c}$ is a cusp.

Converse: if $\gamma(x)=x$ for $\gamma=\begin{pmatrix}a&b\\c&d\end{pmatrix}\in SL_2(\mathbb{Z})$ parabolic

$\Rightarrow cx+d=(\text{eigen value of }\gamma)=\pm1\Rightarrow x=\frac{\pm1-d}{c}\in\mathbb{P}^1(\mathbb{Q})$.

Compactification of $SL_2(\mathbb{Z})\backslash\mathcal{H}$:

$$SL_2(\mathbb{Z})\backslash\mathcal{H}\hookrightarrow (SL_2(\mathbb{Z})\backslash\mathcal{H})\cup\{\infty\}=SL_2(\mathbb{Z})\backslash\big(\overbrace{\mathcal{H}\cup\underbrace{SL_2(\mathbb{Z})\cdot\infty}_{\text{cusps of } SL_2(\mathbb{Z})}}^{\mathcal{H}^*}\big)$$

Prop. If $x\in\mathbb{P}^1(\mathbb{R})$ is a cusp of Γ, then $\overline{\Gamma_x}\simeq\mathbb{Z}$ and for any $\sigma\in SL_2(\mathbb{R})$ such that $\sigma(x)=\infty$

$$(\sigma\Gamma_x\sigma^{-1})\cdot\{\pm1\}=\pm\begin{pmatrix}1&h\\0&1\end{pmatrix}^{\mathbb{Z}},\ h>0.$$

Pf. Replacing x,Γ by $\sigma(x),\sigma\Gamma\sigma^{-1}$, we can assume $x=\infty$.
$\exists\ \gamma=\pm\begin{pmatrix}1&\ell\\0&1\end{pmatrix}\in\Gamma_\infty$ $(\ell\neq0)$; taking γ^2 if necessarily we can assume $\gamma=\begin{pmatrix}1&\ell\\0&1\end{pmatrix}$. If $\exists\,\alpha\in\Gamma_\infty$, $\alpha\neq\pm I$, $\alpha\neq$ parabolic
$\Rightarrow\alpha=\begin{pmatrix}a&b\\0&a^{-1}\end{pmatrix}$, $a\neq\pm1$. Replacing $\alpha\mapsto\alpha^{-1}$, we can assume $|a|<1\Rightarrow$
$\forall n\geq1\quad \alpha^n\gamma\alpha^{-n}=\begin{pmatrix}1&a^{2n}\ell\\0&1\end{pmatrix}\in\Gamma$, $\quad\alpha^n\gamma\alpha^{-n}\xrightarrow{n\to+\infty}\begin{pmatrix}1&0\\0&1\end{pmatrix}$ – contradicts the discreteness of Γ. Therefore $\Gamma_\infty\subseteq\{\pm1\}\begin{pmatrix}1&\mathbb{R}\\0&1\end{pmatrix}\Rightarrow\overline{\Gamma_\infty}\subset N=\begin{pmatrix}1&\mathbb{R}\\0&1\end{pmatrix}\xrightarrow{\sim}(\mathbb{R},+)$

Non-trivial discrete subgroups of $\mathbb{R}$ are $h\mathbb{Z}$, $h>0$.

Cor. If $\Gamma' \subset \Gamma$ is a subgroup of finite index, then
$\{\text{cusps of } \Gamma'\} = \{\text{cusps of } \Gamma\}$.

Pf. $\boxed{\subseteq}$ trivial.

$\boxed{\supseteq}$ let $x \in \text{cusp}(\Gamma)$. Again, we can assume $x = \infty$. As

$$(\Gamma_x : \underbrace{\Gamma' \cap \Gamma_x}_{\Gamma'_x}) \le (\Gamma : \Gamma') < \infty \Longrightarrow \Gamma_x \cdot \{\pm 1\} = \pm\begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}^{\mathbb{Z}}, \quad \Gamma'_x \cdot \{\pm 1\} = \pm \begin{pmatrix} 1 & nh \\ 0 & 1 \end{pmatrix} \ (n \ge 1)$$

Regular and irregular cusps

For $x \in \text{cusps}(\Gamma)$, fix $\sigma \in G$, $\sigma(x) = \infty$. Then

$$(\sigma \Gamma_x \sigma^{-1}) \cdot \{\pm 1\} = \pm \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}^{\mathbb{Z}} \qquad (h > 0).$$

3 possibilities for Γ_x:

- $-I \in \Gamma$: $\sigma \Gamma_x \sigma^{-1} = \pm \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}^{\mathbb{Z}}$ } "regular cusp"
- $-I \notin \Gamma$: $\sigma \Gamma_x \sigma^{-1} = \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}^{\mathbb{Z}}$ } "regular cusp"
  - $\sigma \Gamma_x \sigma^{-1} = \begin{pmatrix} -1 & -h \\ 0 & -1 \end{pmatrix}^{\mathbb{Z}}$ } "irregular cusp"

Exercise: "regularity" of $x \in \text{cusps}(\Gamma)$ is well-defined (does not depend on the choice of σ).

The space $\mathcal{H}^*_\Gamma$

Def. $\mathcal{H}^*_\Gamma := \mathcal{H} \cup \text{cusps}(\Gamma)$

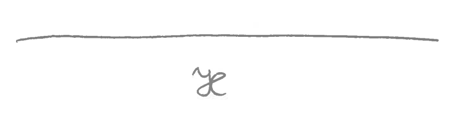
Topology on $\mathcal{H}^*_\Gamma$: for $r > 0$, put $U_r := \{\tau \in \mathcal{H} \mid \text{Im}(\tau) > r\}$

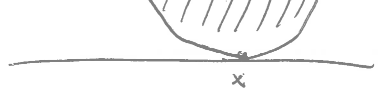
$U_r^* := U_r \cup \{\infty\}$

Fundamental system of neighbourhoods in $\mathcal{H}^*_\Gamma$:

(a) for $\tau \in \mathcal{H}$ — neighbourhoods in $\mathcal{H}$

(b) for $x \in \text{cusps}(\Gamma)$ — $\sigma^{-1}(U_r^*)$, $r > 0$, $\sigma \in G$ any such that $\sigma(x) = \infty$.





Note: (1) $\mathcal{H}^*_\Gamma$ is Hausdorff

(2) Γ acts on $\mathcal{H}^*_\Gamma$ by homeomorphisms

So we have $\Gamma \backslash \mathcal{H} \underbrace{\subseteq}_{\text{induced topology}} \Gamma \backslash \mathcal{H}^*_\Gamma$ with quotient topologies

---

Goal: $\Gamma \backslash \mathcal{H}^*_\Gamma$ is Hausdorff.

Lemma (Shimizu) Assume $\infty \in \mathrm{cusps}(\Gamma)$. Let $\Gamma_\infty \cdot \{\pm 1\} = \pm \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}^{\mathbb{Z}}$, $h > 0$. If $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, $|ch| < 1$, then $c = 0$ and $\gamma \in \Gamma_\infty$.

Pf. Define $\gamma_n \in \Gamma$ by $\gamma_0 = \gamma$, $\gamma_{n+1} = \gamma_n \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \gamma_n^{-1}$, $\gamma_n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}$.

Then $a_{n+1} - 1 = -a_n (c_n h)$, $b_{n+1} = a_n^2 h$

$c_{n+1} = -c_n^2 h$, $d_{n+1} - 1 = a_n (c_n h)$

$\implies c_n = -c(ch)^{2^n - 1}$, $|a_n| \leq |a| + n$

$$\left. \begin{array}{l} |a_n| \leq |a| + n \\ |a_{n+1} - 1| = |d_{n+1} - 1| \leq (|a| + n)\, |ch|^{2n} \end{array} \right\} \implies \gamma_n \xrightarrow{n \to +\infty} \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}$$

$\{\pm 1\} \cdot \Gamma$ discrete $\implies \exists n \quad \gamma_n = \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \implies c = 0$.

Cor. $(\forall \tau \in \mathcal{H})\ (\forall \gamma \in \Gamma \smallsetminus \Gamma_\infty)\quad \mathrm{Im}(\tau)\, \mathrm{Im}(\gamma(\tau)) \leq h^2$.

Pf. $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $|c| \geq \frac{1}{h} \implies \underbrace{\mathrm{Im}(\tau)\, \mathrm{Im}(\gamma(\tau))}_{} = \left(\frac{\mathrm{Im}(\tau)}{|c\tau + d|} \right)^2 \leq \frac{1}{c^2} \leq h^2$.

Lemma. Let $x \in \mathrm{cusps}(\Gamma)$. Then:

(1) $\exists\ U \ni x$ open in $\mathcal{H}^*_\Gamma$ such that $\{\gamma \in \Gamma \mid \gamma(U) \cap U \neq \emptyset\} = \Gamma_x$.

(2) $\forall$ compact $K \subset \mathcal{H}$ $\exists\ V \ni x$ open $\forall \gamma \in \Gamma$ $V \cap \gamma(K) = \emptyset$.

Pf. (1) We can assume $x = \infty$; take $U = U^*_h = \{\tau \in \mathcal{H} \mid \mathrm{Im}(\tau) > h\} \cup \{\infty\}$

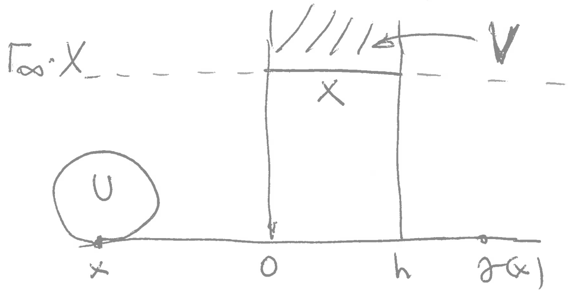
(2) ——— " ———; $\exists\ A, B \quad A < \mathrm{Im}(\tau) < B \quad \forall \tau \in K$

Take $V = U_{\max(B,\, h^2/A)}$; for $\tau \in K$

$$\left. \begin{array}{l} \gamma \in \Gamma_\infty \implies \mathrm{Im}(\gamma(\tau)) = \mathrm{Im}(\tau) < B \\ \gamma \in \Gamma \smallsetminus \Gamma_\infty \implies \mathrm{Im}(\gamma(\tau)) \leq \frac{h^2}{\mathrm{Im}(\tau)} \leq \frac{h^2}{A} \end{array} \right\} \implies \gamma(\tau) \notin V.$$

Prop. $\Gamma\backslash\mathcal{H}^*_\Gamma$ with quotient topology is Hausdorff.

Pf. We know that $\Gamma\backslash\mathcal{H}$ is Hausdorff.
By (2) of lemma above we can separate $\tau\in\mathcal{H}$ from a cusp.
Let $x, y \in \mathrm{cusps}(\Gamma)$, $x\neq\gamma(y)$ $\forall\gamma\in\Gamma$. We can assume $y=\infty$.
Fix $u>0$ and put $X=\{\tau\in\mathcal{H}\mid \mathrm{Im}(\tau)=u,\ 0\le\mathrm{Re}(\tau)\le h\}$
$\left(\Gamma_\infty\cdot\{\pm1\}=\pm\begin{pmatrix}1&h\\0&1\end{pmatrix}^{\mathbb{Z}}\right)$ $V=\{\tau\in\mathcal{H}\mid \mathrm{Im}(\tau)\ge u$, —"—, —"— $\}$



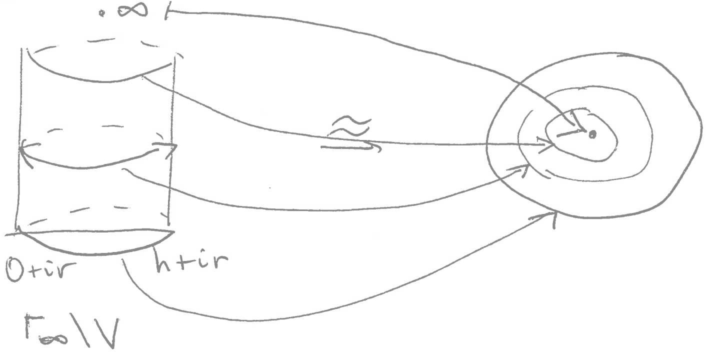
We know: $\exists\, U\ni x$ open in $\mathcal{H}^*_\Gamma$ such that $\Gamma\cdot U\cap X=\emptyset$. Assume $\exists\gamma\in\Gamma$ $\gamma(U)\cap V\neq\emptyset$.
As $\gamma(x)\neq\infty$, $\gamma(x)\in\mathbb{R}\Rightarrow\gamma(U)\cap\Gamma_\infty\cdot X\neq\emptyset$
$\Rightarrow\Gamma_\infty\cdot\gamma(U)\cap X\neq\emptyset$ – contradiction.

Prop. $\Gamma\backslash\mathcal{H}^*_\Gamma$ is locally compact.

Pf. Let $x\in\mathrm{cusps}(\Gamma)$; we can assume $x=\infty$. By (1) of Lemma above, $\exists\, r>0$ such that $V=\{\tau\in\mathcal{H}\mid\mathrm{Im}(\tau)\ge r\}\cup\{\infty\}$ satisfies
$\{\gamma\in\Gamma\mid\gamma(V)\cap V\neq\emptyset\}=\Gamma_\infty$. As a result, under $\pi:\mathcal{H}^*_\Gamma\to\Gamma\backslash\mathcal{H}^*_\Gamma$,

$$\Gamma_\infty\backslash V=\pi(V)=\pi\big(\underbrace{\{\infty\}\cup\{\tau\in\mathcal{H}\mid\mathrm{Im}(\tau)\ge r,\ 0\le\mathrm{Re}(\tau)\le h\}}_{\text{compact}}\big)$$

In fact, $\tau\longmapsto e^{2\pi i\tau/h}$, $\infty\longmapsto 0$ is a homeomorphism $\Gamma_\infty\backslash V\xrightarrow{\approx}$ closed disc of radius $e^{-2\pi r/h}$.



Def. Γ is a "Fuchsian group of the first kind" if $\Gamma\backslash\mathcal{H}^*_\Gamma$ is compact.

Facts: assume that $\Gamma \subset SL_2(\mathbb{R})$ is a discrete subgroup such that $\underline{\overline{\Gamma}\backslash\mathcal{H}^*_\Gamma \text{ is compact}}$ (Γ is a Fuchsian subgroup of the 1^{st} kind). then:

(1) $\mathrm{vol}(\overline{\Gamma}\backslash\mathcal{H}) < \infty$

(2) $\exists$ fundamental domain $E \subset \mathcal{H}$ of $\overline{\Gamma}$ which is the interior of a convex hyperbolic polygon with finitely many sides

(3) Γ is finitely generated

(4) $\overline{\Gamma}$ has a presentation of the following form:

generators: $A_1, \dots, A_g, B_1, \dots, B_g, \underbrace{C_1, \dots, C_r}_{\text{elliptic}}, \underbrace{P_1, \dots, P_t}_{\text{parabolic}}$ $(g, r, t \geq 0)$

relations: $C_i^{m_i} = 1$ $(m_i \geq 1)$

$$[A_1, B_1] \cdots [A_g, B_g]\, C_1 \cdots C_r\, P_1 \cdots P_t = 1$$

Ex: $\overline{\Gamma} = \overline{SL_2(\mathbb{Z})}$, $C_1 = \overline{S}$, $C_2 = \overline{ST}$, $P_1 = \overline{T}^{-1}$

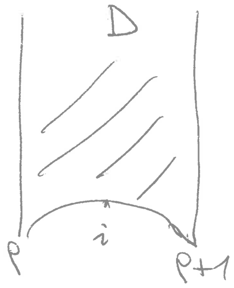
$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ $\quad m_1 = 2$, $m_2 = 3$

Prop. (1) $\underbrace{SL_2(\mathbb{Z})}_{\Gamma} \backslash \mathcal{H}^*_\Gamma$ is compact.

(2) Let $\Gamma' \subset \Gamma$ be a subgroup of finite index ($\Longrightarrow$ {cusps of Γ'} = {cusps of Γ}). Then:
[$\Gamma \backslash \mathcal{H}^*_\Gamma$ is compact $\Longleftrightarrow$ $\Gamma' \backslash \mathcal{H}^*_{\Gamma'}$ is compact].

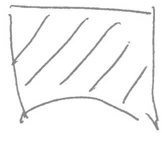
Pf: (1)

$\Gamma = SL_2(\mathbb{Z})$



$$\mathcal{H}^*_\Gamma = \mathcal{H} \cup \Gamma\cdot\infty$$

$$\Gamma \backslash \mathcal{H}^*_\Gamma = \pi(\underbrace{\overline{D} \setminus U_r}) \cup \pi(\underbrace{U_r^*}_{\text{disc (closed)} \Rightarrow \text{compact}}) \qquad \forall r > 0$$



(2) $\mathcal{H}^*_\Gamma = \mathcal{H}^*_{\Gamma'}$

$\boxed{\Leftarrow}$ $\Gamma \backslash \mathcal{H}^*_\Gamma \longleftarrow \Gamma' \backslash \mathcal{H}^*_\Gamma$ is surjective, continuous

$\boxed{\Rightarrow}$ $\forall x \in \underbrace{\Gamma \backslash \mathcal{H}^*_\Gamma}$ $\exists$ $U \subset \mathcal{H}^*_\Gamma$ open such that $\overline{U}$ is compact, $\pi_\Gamma(U) \ni x$.

Compactness of $\Longrightarrow$ $\exists$ $U_1, \dots, U_m \subset \mathcal{H}^*_\Gamma$ open with $\overline{U_j}$ compact and $\Gamma \backslash \mathcal{H}^*_\Gamma = \bigcup_{j=1}^m \pi_\Gamma(U_j) = \bigcup_{j=1}^m \pi_\Gamma(\overline{U_j})$. Write $\Gamma = \bigcup_{k=1}^n \Gamma' \gamma_k$

$$\Longrightarrow \Gamma' \backslash \mathcal{H}^*_\Gamma = \bigcup_{j=1}^m \bigcup_{k=1}^n \pi_{\Gamma'}(\underbrace{\gamma_k \overline{U_j}}_{\text{compact}}).$$

Cor. For every $\Gamma \subset SL_2(\mathbb{R})$ commensurable with $SL_2(\mathbb{Z})$ (i.e., such that $\Gamma \cap SL_2(\mathbb{Z})$ has finite index in both $SL_2(\mathbb{Z})$ and Γ) $\Gamma \backslash \mathcal{H}^*_\Gamma$ is compact.

Rmk. Siegel showed that

$$\Gamma \backslash \mathcal{H}^*_\Gamma \text{ is compact} \Longleftrightarrow \overbrace{\mathrm{vol}(\Gamma \backslash \mathcal{H})}^{\text{hyperbolic area of a fundamental domain}} < \infty$$

Prop. If $\Gamma \backslash \mathcal{H}^*_\Gamma$ is compact, then $\Gamma \backslash$ {cusps of Γ} and $\Gamma \backslash$ {elliptic points of Γ} are finite sets.

Pf. $\forall x \in \mathcal{H}^*_\Gamma$ $\exists$ $U \ni x$ open in $\mathcal{H}^*_\Gamma$ $\{\gamma \in \Gamma \mid \gamma(U) \cap U \neq \emptyset\} = \Gamma_x$.
Thus {elliptic points} and {cusps} are discrete Γ-invariant (and closed) subsets of $\mathcal{H}^*_\Gamma$ $\Longrightarrow$ their images in the compact set $\Gamma \backslash \mathcal{H}^*_\Gamma$ (with quotient topology) are discrete and compact $\Longrightarrow$ finite.

$\Gamma\backslash\mathcal{H}^*_\Gamma$ as a Riemann surface

$\Gamma \subset G = \mathrm{SL}_2(\mathbb{R})$ discrete subgroup, $\overline{\Gamma} = \Gamma/(\Gamma\cap\{\pm I\}) \subset \mathrm{SL}_2(\mathbb{R})/\{\pm I$

$\mathcal{H}^* = \mathcal{H} \cup \mathrm{cusps}(\Gamma)$, $\underline{\pi : \mathcal{H}^* \longrightarrow \Gamma\backslash\mathcal{H}^*}$ locally compact, Hausdorff

<u>Goal</u>: define local holomorphic coordinates on $\Gamma\backslash\mathcal{H}^*$.

Fix $x \in \mathcal{H}^*$. $\exists\, U \ni x$ open such that $\Gamma_x = \{\gamma\in\Gamma \mid \gamma(U)\cap U \neq \emptyset\}$

$\Gamma_x \cdot U = U$

Then $\Gamma_x\backslash U \hookrightarrow \Gamma\backslash\mathcal{H}^*$ is an open neighbourhood of $\pi(x)$.

<u>Holomorphic structure on $\Gamma_x\backslash U$</u>: characterised by

$\forall$ Riemann surface Y, $\forall$ continuous map $\Gamma_x\backslash U \xrightarrow{f} Y$

$\Big[$ f is holomorphic $\iff$ $U \xrightarrow{\pi} \Gamma_x\backslash U \xrightarrow{f} Y$ is holomorphic $\Big]$

(and Γ_x-invariant)

<u>Case 1</u>: <u>$x = \tau_0 \in \mathcal{H}$</u>: ~~Cayley~~ map $\alpha = \alpha_{\tau_0}$: $\tau \longmapsto z = \dfrac{\tau-\tau_0}{\tau-\overline{\tau_0}}$

$$\begin{tikzcd} \tau \arrow[r, mapsto] \arrow[d, phantom, "\cap" rotate=90] & z \arrow[d, phantom, "\cap" rotate=90] \\ \mathcal{H} \arrow[r, "\sim"] & D \end{tikzcd}$$

unit disc (open)

$\alpha\Gamma_x\alpha^{-1} = \{z \longmapsto \zeta^k z \mid k \in \mathbb{Z}/e_x\mathbb{Z}\}$

$e_x = |\overline{\Gamma}_x|$, $\zeta = \exp(2\pi i/e_x)$

For suitable U as above,

$$\boxed{\begin{tikzcd} U \arrow[r, "\alpha", "\sim"'] \arrow[d, "\pi"] & D_r \arrow[d] & \ni & z \arrow[d, mapsto] \\ \Gamma_x\backslash U \arrow[r, "\sim"] & \langle\zeta\rangle\backslash D_r & \ni & z^{e_x} \end{tikzcd}}$$

$D_r \subset D$ disc of radius $r<1$

Local coordinate at $\pi(x) = \pi(\tau_0)$:

$$\pi(\tau) \longmapsto (\alpha_{\tau_0}(\tau))^{e_x} = \left(\frac{\tau-\tau_0}{\tau-\overline{\tau_0}}\right)^{e_{\tau_0}}$$

<u>Case 2</u>: <u>$x \in \mathrm{cusps}(\Gamma)$</u>: fix $\sigma \in G$ $\sigma(x) = \infty$

$$(\sigma\Gamma_x\sigma^{-1})\cdot\{\pm 1\} = \pm\begin{pmatrix}1 & h\\ 0 & 1\end{pmatrix}^{\mathbb{Z}}$$

For suitable $U = \sigma^{-1}(U_r^*)$,

$$\begin{tikzcd} U \arrow[d, "\pi"] & \ni & \tau \arrow[drr, mapsto] & & x \arrow[drr, mapsto] & & \\ \Gamma_x\backslash U \arrow[r, "\sim"] & D_R & \ni & e^{2\pi i\sigma(\tau)/h} & & & 0 \end{tikzcd}$$

$R = e^{-2\pi r/h}$

D_R open disc of radius R (centre at 0)

<u>Local coordinate at $\pi(x)$</u> :

$\pi(\tau) \longmapsto e^{2\pi i\sigma(\tau)/h}$

$\pi(x) \longmapsto 0$

Ex : $\Gamma = SL_2(\mathbb{Z})$: local coordinates :

at $x = \infty$: $e^{2\pi i \tau}$ | at $\tau_0 \notin \Gamma\cdot i \cup \Gamma\cdot\rho$: $\tau - \tau_0$

at $\tau_0 = i$: $\left(\frac{\tau - i}{\tau + i}\right)^2$ | at $\tau_0 = \rho$: $\left(\frac{\tau-\rho}{\tau-\bar{\rho}}\right)^3$

In general : $\mathcal{H} \xrightarrow{\pi_\Gamma} \Gamma\backslash\mathcal{H}$, $\tau_0 \longmapsto \pi(\tau_0)$ the ramification index of π_Γ at $\tau_0 \in \mathcal{H}$ is $e_{\tau_0} = |\bar{\Gamma}_{\tau_0}|$

Proposition : (1) $A_0 = \mathcal{M}(SL_2(\mathbb{Z})\backslash\mathcal{H}^*)$

(2) $j \in \mathcal{M}(SL_2(\mathbb{Z})\backslash\mathcal{H}^*)$ has divisor $\mathrm{div}(j) = (\pi(\rho)) - (\pi(\infty))$, hence induces an isomorphism of Riemann surfaces

$$SL_2(\mathbb{Z})\backslash\mathcal{H}^* \xrightarrow{\sim} \mathbb{P}^1(\mathbb{C})$$

(3) $\mathcal{M}(SL_2(\mathbb{Z})\backslash\mathcal{H}^*) = \mathbb{C}(j)$

(4) $\forall f \in A_k \quad \sum_{\tau \in (SL_2(\mathbb{Z})\backslash\mathcal{H}) \cup \{\infty\}} \frac{\mathrm{ord}_\tau(f)}{e_\tau} = \frac{k}{12} \qquad (e_\infty = 1)$

Pf. (1) By definition.

(2) $j(\tau) = \frac{1}{q} + \cdots$ has simple pole at $\pi(\infty)$, no other pole, and a zero (of order 1, necessarily) at $\pi(\rho)$.

(3) $\mathcal{M}(\mathbb{P}^1(\mathbb{C})) = \mathbb{C}(z)$ (z = standard coordinate on $\mathbb{C}$)

(4) $g = f^{12}/\Delta^k \in A_0$ satisfies (when viewed as element of $\mathcal{M}(SL_2(\mathbb{Z})\backslash\mathcal{H}^*)$)

$$0 = \sum_{x \in SL_2(\mathbb{Z})\backslash\mathcal{H}^*} \mathrm{ord}_x(g) \quad , \quad \mathrm{ord}_{\pi(\infty)}(g) = 12\, \mathrm{ord}_\infty(f) - k$$

$$\mathrm{ord}_{\pi(\tau)}(g) = \frac{1}{e_\tau}\mathrm{ord}_\tau(g \circ \pi) = \frac{12}{e_\tau}\mathrm{ord}_\tau(f)$$

Rmk : if $\alpha : X \longrightarrow Y$ is a non-constant holomorphic map between Riemann surfaces and if $0 \neq f \in \mathcal{M}(Y)$, then $\forall\, x \in X$ $\underline{\mathrm{ord}_x(f \circ \alpha) = e_x\, \mathrm{ord}_{\alpha(x)}(f)}$, where e_x is the ramification index of α at x.

Maps $X_{\Gamma'} \longrightarrow X_\Gamma$

Data: $\Gamma \subset G = SL_2(\mathbb{R})$ discrete subgroup such that $X_\Gamma := \Gamma \backslash \mathcal{H}^*_\Gamma$ is compact.

$\Gamma' \subset \Gamma$ subgroup of finite index

then: $X_{\Gamma'} = \Gamma' \backslash \mathcal{H}^*_{\Gamma'} = \Gamma' \backslash \mathcal{H}^*_\Gamma$ is also compact and there are holomorphic projection maps, with f proper

$$\begin{tikzcd}
\mathcal{H}^*_\Gamma \arrow[r, "\pi_{\Gamma'}"] \arrow[dr, "\pi_\Gamma"'] & X_{\Gamma'} \arrow[d, "f"] \\
& X_\Gamma
\end{tikzcd}$$

(4) there is a bijection

$$\overline{\Gamma'} \backslash \overline{\Gamma} / \overline{\Gamma}_x \xrightarrow{\sim} f^{-1}(\pi_\Gamma(x))$$
$$\gamma \longmapsto \pi_{\Gamma'}(\gamma(x))$$

lemma. (1) $\deg(f) = (\overline{\Gamma} : \overline{\Gamma'})$

(2) If $y = \pi_{\Gamma'}(z)$ $(z \in \mathcal{H}^*_\Gamma)$, then the ramification index of f at y is equal to $e_y = (\overline{\Gamma}_z : \overline{\Gamma'_z})$.

(3) If $\Gamma' \triangleleft \Gamma$, then $\overline{\Gamma} / \overline{\Gamma'}$ acts transitively on $f^{-1}(x)$, for each $x \in X_\Gamma$, hence e_y depends only on $f(y)$.

Pf. (1) If $\overline{\Gamma} = \coprod_j \overline{\Gamma'} \gamma_j$ (disjoint union), then

$$f^{-1}(\pi_\Gamma(z)) = \{ \underbrace{\pi_{\Gamma'}(\gamma_j(z))}_{\text{distinct for generic } z} \}$$

(2) (a) If $z \in \mathcal{H}$:

$$e_y = |\overline{\Gamma}_z| / \underbrace{|\overline{\Gamma'_z}|}_{\text{ramification index of } \pi_{\Gamma'} \text{ at } z}$$

(b) If $z \in \mathrm{cusps}(\Gamma)$: can assume $z = \infty$

$$\overline{\Gamma}_z = \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}^{\mathbb{Z}}, \quad \overline{\Gamma'_z} = \begin{pmatrix} 1 & nh \\ 0 & 1 \end{pmatrix}^{\mathbb{Z}} \quad \Longrightarrow e_y = n = (\overline{\Gamma}_z : \overline{\Gamma'}_z)$$

local coord.: $e^{2\pi i \tau/h}$ at $\pi_\Gamma(z)$ $\qquad e^{2\pi i \tau/nh}$ at $\pi_{\Gamma'}(z) = y$

(3) If $\overline{\Gamma} = \coprod_j \overline{\Gamma'} \gamma_j$, then $f^{-1}(\pi_\Gamma(z)) = \{\pi_{\Gamma'}(\gamma_j(z))\}$

and

$$\begin{tikzcd}
X_{\Gamma'} \arrow[rr, "\gamma_k \gamma_j^{-1}"] \arrow[dr] & & X_{\Gamma'} \arrow[dl] \\
& X_\Gamma &
\end{tikzcd}$$

is well-defined, sending $\pi_{\Gamma'}(\gamma_j(z))$ to $\pi_{\Gamma'}(\gamma_k(z))$.

(4) the map is well-defined and surjective. If $\gamma_1, \gamma_2 \in \overline{\Gamma}$ and $\gamma_1(x) = \gamma' \gamma_2(x)$ $(\gamma' \in \overline{\Gamma'})$, then $\underbrace{\gamma_1^{-1} \gamma' \gamma_2}_{\delta} \in \overline{\Gamma}_x$ and $\gamma'^{-1} \gamma_1 \delta = \gamma_2$

$\Longrightarrow$ injectivity.

Theorem. Let $\Gamma \subset \Gamma(1) = SL_2(\mathbb{Z})$ be a subgroup of finite index.

Let $\mu := (\overline{SL_2(\mathbb{Z})} : \overline{\Gamma})$, $\nu_\infty := |\Gamma \backslash \underbrace{\text{cusps}(\Gamma)}_{\mathbb{P}^1(\mathbb{Q})}|$,

$\nu_2 := |\Gamma \backslash \{\text{elliptic points of } \Gamma \text{ above } i\}|$

$\nu_3 := |\Gamma \backslash \{\text{elliptic points of } \Gamma \text{ above } \rho\}|$.

Then the compact Riemann surface $X_\Gamma = \Gamma \backslash \mathcal{H}^*_\Gamma$ has genus

$$\boxed{g(X_\Gamma) = 1 + \frac{\mu}{12} - \frac{\nu_2}{4} - \frac{\nu_3}{3} - \frac{\nu_\infty}{2}}$$

Pf:

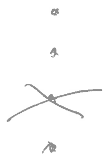
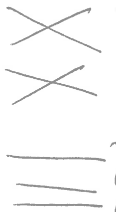
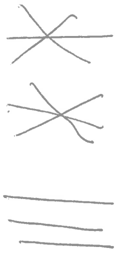
$$\begin{tikzcd} \mathcal{H}^*_\Gamma = \mathcal{H} \cup \mathbb{P}^1(\mathbb{Q}) \arrow[r, "\pi_\Gamma"] \arrow[dr, "\pi = \pi_{\Gamma(1)}"'] & X_\Gamma \arrow[d, "f"] & \\ & X_{\Gamma(1)} \arrow[r, "j", "\sim"'] & \mathbb{P}^1(\mathbb{C}) \end{tikzcd}$$

Riemann–Hurwitz formula for f:

$$2g(X_\Gamma) - 2 = (2\underbrace{g(X_{\Gamma(1)})}_{0} - 2)\underbrace{\deg(f)}_{\mu} + \sum_{y \in X_\Gamma} (e_y - 1)$$

Ramification of f:

$$\begin{tikzcd} X_\Gamma \arrow[d, "f"] & \ni y \arrow[d, mapsto] \\ X_{\Gamma(1)} & \ni x \end{tikzcd}$$

	at cusps	at i	at ρ
	 $\}\nu_\infty$	 e=2 $\}\frac{\mu-\nu_2}{2}$ e=1 $\}\nu_2$	 e=3 $\}\frac{\mu-\nu_3}{3}$ e=1 $\}\nu_3$
x :	$\pi(\infty)$	$\pi(i)$	$\pi(\rho)$
$\sum_{f(y)=x}(e_y-1)$:	$\mu-\nu_\infty$	$\frac{\mu-\nu_2}{2}$	$\frac{2}{3}(\mu-\nu_3)$

(since $\sum_{f(y)=x} e_y = \mu$)

So $2g(X_\Gamma) - 2 = -2\mu + (\mu - \nu_\infty) + (\frac{\mu-\nu_2}{2}) + \frac{2}{3}(\mu - \nu_3)$

$= \frac{\mu}{6} - \nu_\infty - \frac{\nu_2}{2} - \frac{2}{3}\nu_3$

Principal congruence subgroups $\Gamma(N)$

$N \geq 1$, $\quad \Gamma(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \;\middle|\; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod N \right\}$

Notation: $\quad Y(N) := \Gamma(N) \backslash \mathcal{H} \subset \Gamma(N) \backslash \mathcal{H}^* = X(N)$

Note: (1) $-I \in \Gamma(N) \iff N \leq 2$

(2) $\Gamma(N) = \operatorname{Ker}(SL_2(\mathbb{Z}) \longrightarrow SL_2(\mathbb{Z}/N\mathbb{Z})) \lhd \Gamma(1) = SL_2(\mathbb{Z})$

Exercise (important) $\quad SL_2(\mathbb{Z}) \longrightarrow SL_2(\mathbb{Z}/N\mathbb{Z})$ is surjective.

Cor: $\mu = (\overline{\Gamma(1)} : \overline{\Gamma(N)}) = |SL_2(\mathbb{Z}/N\mathbb{Z})| \cdot \begin{cases} 1, & N = 1, 2 \\ 1/2, & N \geq 3 \end{cases}$

Prop. (1) $N = \prod p_i^{n_i} \Longrightarrow |SL_2(\mathbb{Z}/N\mathbb{Z})| = \prod |SL_2(\mathbb{Z}/p_i^{n_i}\mathbb{Z})|$
(p_i – distinct primes)

(2) $|SL_2(\mathbb{Z}/p^{n+1}\mathbb{Z})| = p^3 \, |SL_2(\mathbb{Z}/p^n\mathbb{Z})| \qquad \forall n \geq 1 \quad \forall$ prime p

(3) $|SL_2(\mathbb{Z}/p\mathbb{Z})| = p^3 - p \qquad \forall$ prime p

(4) $|SL_2(\mathbb{Z}/N\mathbb{Z})| = N^3 \prod_{p|N} \left(1 - \frac{1}{p^2}\right) \qquad$ (p prime)

Pf: (1) Chinese remainder thm.

(2) Hensel's Lemma: $SL_2(\mathbb{Z}/p^{n+1}\mathbb{Z}) \longrightarrow SL_2(\mathbb{Z}/p^n\mathbb{Z})$ is surjective, with $\operatorname{Ker} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + p^n M \pmod{p^{n+1}} \;\middle|\; M \in \operatorname{Lie}(SL_2)(\mathbb{Z}/p\mathbb{Z}) \right\}$

$M = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \qquad a, b, c \in \mathbb{Z}/p\mathbb{Z}$

(3) $SL_2(\mathbb{Z}/p\mathbb{Z})$ acts transitively on $\mathbb{P}^1(\mathbb{Z}/p\mathbb{Z})$, with

$SL_2(\mathbb{Z}/p\mathbb{Z})_\infty = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \;\middle|\; a, b \in \mathbb{Z}/p\mathbb{Z},\ a \neq 0 \right\}$

$\Longrightarrow |SL_2(\mathbb{Z}/p\mathbb{Z})| = (p+1)(p-1)p$.

(4) Follows from (1) – (3).

Prop. (1) If $N \geq 2 \Longrightarrow \Gamma(N)$ has no elliptic points $\Longrightarrow \nu_2 = \nu_3 = 0$.

(2) Each cusp y of $\Gamma(N)$ and the ramification index e_y of $X(N) \longrightarrow X(1)$ at y is equal to N.

(3) $\nu_\infty = \mu / N$.

Pf. (1) Every elliptic element of $\Gamma(1)$ is conjugate there to some $\gamma \in (\Gamma(1)_\rho \cup \Gamma(1)_i) \setminus \{\pm I\} = \left\{ \pm \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \pm \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}, \pm \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \right\}$ – none of them lie in a conjugate of $\Gamma(N)$ inside $\Gamma(1)$; but $\Gamma(N) \lhd \Gamma(1)$.

(2) $\Gamma(N)_\infty = \begin{pmatrix} 1 & N \\ 0 & 1 \end{pmatrix}^{\mathbb{Z}}$ $(\cdot \{\pm 1\}$ if $N \leq 2) \Longrightarrow e_\infty = N$ and ∞ is regular.
the same is true for all cusps, since $\Gamma(N) \lhd \Gamma(1)$.

(3) $\Longleftarrow$ (2).

Thm. $\forall N > 1$ the projection $X(N) \longrightarrow X(1) = \mathbb{P}^1(\mathbb{C})$ has

$$\text{degree} = \mu = \begin{cases} 6, & N=2 \\ \frac{N^3}{2}\prod_{p\mid N}\left(1-\frac{1}{p^2}\right), & N>2 \end{cases}$$

$$\nu_\infty = |\underbrace{\text{cusps of } X(N)}_{X(N)\smallsetminus Y(N)}| = \frac{\mu}{N} = \begin{cases} 3, & N=2 \\ \frac{N^2}{2}\prod_{p\mid N}\left(1-\frac{1}{p^2}\right), & N>2 \end{cases}$$

$\nu_2 = \nu_3 = 0$ (no elliptic points)

$$\text{genus} \quad g(X(N)) = \begin{cases} 0, & N=2 \\ 1+\frac{\mu}{12N}(N-6), & N>2 \end{cases}$$

Cor. $g(X(N)) = 0 \iff N \leq 5.$

Exercise. Describe explicitly the set of cusps $X(N)\smallsetminus Y(N)$

Remarks on $X(N)$ for $\boxed{N \leq 5}$:

$$SL_2(\mathbb{Z}/N\mathbb{Z})/\{\pm I\} \xrightarrow{\sim} \left\{\begin{array}{lll} & & \nu_\infty \\ S_3 \simeq D_6 & N=2 & 3 \\ A_4 & N=3 & 4 \\ S_4 & N=4 & 6 \\ A_5 & N=5 & 12 \end{array}\right.$$

$\exists$ holomorphic isomorphisms

$$\begin{tikzcd} X(N) \arrow[r, "\sim"] \arrow[d] & \mathbb{P}^1(\mathbb{C}) \arrow[r, phantom, "="] \arrow[d] & S^2 \arrow[d, "\pi_N"] \\ X(1) \arrow[r, "\sim"] & \mathbb{P}^1(\mathbb{C}) \arrow[r, phantom, "="] & S^2 \end{tikzcd}$$

π_N ramified above 3 points of $X(1)$, with ramification indices $e = (2, 3, N)$

the Ramification points of $X(N)$ with $e = N$ are the cusps. Geometrically, they are the vertices of a regular polyhedron:

tetrahedron $N=3$
octahedron $N=4$
icosahedron $N=5$

(= the centres of faces of the dual polyhedron).

Exercise. Describe the remaining ramification points in terms of this polyhedron (or its dual).

Reference. F. Klein, Lectures on the icosahedron.

Example : $\Gamma = \Gamma_0(N)$

Def: $\Gamma_0(N) := \left\{ \gamma \in SL_2(\mathbb{Z}) \mid \gamma \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} (\mathrm{mod}\ N) \right\} \ni \pm I$

$Y_0(N) := \Gamma_0(N) \backslash \mathcal{H} \subset \Gamma_0(N) \backslash \mathcal{H}^* = X_0(N)$

Ex: $N = p$, p prime : $\Gamma = \Gamma_0(p) \subset \Gamma(1) = SL_2(\mathbb{Z})$

coset space $\Gamma(1)/\Gamma = \overline{\Gamma(1)}/\overline{\Gamma} \xrightarrow[\sim]{(\mathrm{mod}\, p)} SL_2(\mathbb{F}_p) / \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\} \xrightarrow{\sim} \mathbb{P}^1(\mathbb{F}_p)$

Consider $X_0(p) \longrightarrow X(1)$: $\qquad g \longmapsto g(\infty)$

degree: $\mu = \deg = |\mathbb{P}^1(\mathbb{F}_p)| = p+1$

cusps : $\overline{\Gamma_\infty} = \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\rangle$, $\overline{\Gamma_\infty} \backslash \mathbb{P}^1(\mathbb{F}_p) = \{\infty, 0\}$ $\qquad$ 2 cusps

$\nu_\infty = 2$

elliptic points above i : $\overline{\Gamma_i} = \left\langle \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\rangle$ $\qquad \left(\frac{-1}{2}\right) := 0$

$$\nu_2 = \left|\mathbb{P}^1(\mathbb{F}_p)^{\overline{\Gamma_i}}\right| = \left|\{a \in \mathbb{F}_p \mid \underset{\underset{a^2+1}{\Updownarrow}}{a = -1/a}\}\right| = 1 + \left(\frac{-1}{p}\right)$$

elliptic points above ρ : $\overline{\Gamma_\rho} = \left\langle \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \right\rangle$ $\qquad \left(\frac{-3}{2}\right) := -1$

$$\nu_3 = \left|\mathbb{P}^1(\mathbb{F}_p)^{\overline{\Gamma_\rho}}\right| = \left|\{b \in \mathbb{F}_p \mid \underset{\underset{b^2+b+1=0}{\Updownarrow}}{b = -1/(b+1)}\}\right| = 1 + \left(\frac{-3}{p}\right)$$

Conclusion :

$p \ (\mathrm{mod}\ 12)$	2	3	1	5	7	11
ν_2	1	0	2	2	0	0
ν_3	0	1	2	0	2	0
$\frac{\nu_2}{4} + \frac{\nu_3}{3}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{7}{6}$	$\frac{1}{2}$	$\frac{2}{3}$	0
$g(\Gamma_0(p))$	0	0	$\frac{p-13}{12}$	$\frac{p-5}{12}$	$\frac{p-7}{12}$	$\frac{p+11}{12}$

$$g(X_0(p)) = \left[\frac{p+1}{12}\right] + \begin{cases} -1, & 1+p \equiv 2 \ (\mathrm{mod}\ 12) \\ 0, & 1+p \not\equiv 2 \ (\mathrm{mod}\ 12) \end{cases} = \dim S_{p+1}(SL_2(\mathbb{Z}))$$

$g(X_0(p)) = 0 \iff p = 2, 3, 5, 7, 13 \quad (\iff (p-1) \mid 12)$

$g(X_0(p)) = 1 \iff p = 11, 17, 19$

$g(X_0(p)) = 2 \iff p = 23, 29, 31, 37$

Exercise: for $\Gamma_0(N)$ $(N \geq 2)$

$$\mu = N \prod_{p \mid N} \left(1 + \frac{1}{p}\right)$$

$$\nu_2 = \begin{cases} 0, & 4 \mid N \\ \prod_{p \mid N} \left(1 + \left(\frac{-1}{p}\right)\right), & 4 \nmid N \end{cases}$$

$$\nu_3 = \begin{cases} 0, & 9 \mid N \\ \prod_{p \mid N} \left(1 + \left(\frac{-3}{p}\right)\right), & 9 \nmid N \end{cases}$$

$$\nu_\infty = \sum_{d \mid N} \varphi\left(\left(d, \frac{N}{d}\right)\right) = \prod_{p \mid N} \sum_{i=0}^{\mathrm{ord}_p(N)} \varphi\left(p^{\min(i,\, \mathrm{ord}_p(N) - i)}\right)$$

---

for $\Gamma_1(N) = \left\{ \alpha \in SL_2(\mathbb{Z}) \mid \alpha \equiv \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} (\mathrm{mod}\ N) \right\}$, $N \geq 4$:

$(\overline{\Gamma_1(N)} = \overline{\Gamma_0(N)}$ if $N = 2, 3)$

$$\mu = \frac{\varphi(N)\, N}{2} \prod_{p \mid N} \left(1 + \frac{1}{p}\right)$$

$$\nu_2 = \nu_3 = 0$$

$$\nu_\infty = \begin{cases} 3, & N = 4 \\ \frac{1}{2} \sum_{d \mid N} \varphi(d)\, \varphi\left(\frac{N}{d}\right), & N > 4 \end{cases}$$

Exercise. (1) Describe explicitly the cusps of $X_0(N)$.

(2) Does $X_1(N)$ have any irregular cusps?

Properties of $Y_\Gamma = \Gamma\backslash\mathcal{H} \subset X_\Gamma = \Gamma\backslash(\mathcal{H} \cup \{\text{cusps of } \overline{\Gamma}\})$

$\Gamma \subset SL_2(\mathbb{R})$ discrete subgroup, $\overline{\Gamma} = \operatorname{Im}(\Gamma \longrightarrow SL_2(\mathbb{R})/\{\pm I\})$

(1) If X_Γ is compact ("Γ is a Fuchsian subgroup of the 1st kind") then X_Γ is the set of complex points of a non-singular projective curve over $\mathbb{C}$, and Y_Γ is obtained from X_Γ by removing finitely may points (the cusps of X_Γ).

(2) If $\overline{\Gamma}$ contains no element of finite order except I, then it acts freely discontinuously on $\mathcal{H}$ and the projection $\mathcal{H} \xrightarrow{\pi_\Gamma} Y_\Gamma$ is a covering.

As $\pi_1(\mathcal{H}) = \{1\}$, the projection π_Γ identifies $\mathcal{H}$ with the universal covering of Y_Γ and Γ with the fundamental group $\pi_1(Y_\Gamma)$ (for any base point of Y_Γ).

---

(3) Example of (2): $\boxed{\Gamma = \Gamma(2)}$, $Y_{\Gamma(2)} = Y(2)$

$$\begin{tikzcd}
\mathcal{H} \arrow[r, "\lambda"] \arrow[d, "\pi"] & \mathbb{C} \setminus \{0,1\} \\
\Gamma(2)\backslash\mathcal{H} \arrow[ur, dashed, "\sim"', "\tilde{\lambda}"] &
\end{tikzcd}$$

defines a holomorphic isomorphism

In particular, λ is a covering map and $\mathcal{H} \simeq D = \{|w| < 1\}$ is a universal covering of $\mathbb{C} \setminus \{0,1\}$.

(4) Application: Picard's small theorem:

Any holomorphic map $f: \mathbb{C} \longrightarrow \mathbb{C} \setminus \{0,1\}$ is constant.

Pf.

$$\begin{tikzcd}
 & \mathcal{H} \simeq D \arrow[d, "\lambda"] \arrow[d, phantom, shift left=6, "\text{covering}"] \\
\underbrace{\mathbb{C}}_{\text{simply connected}} \arrow[ur, dashed, "F"] \arrow[r, "f"] & \mathbb{C} \setminus \{0,1\}
\end{tikzcd}$$

$\exists$ holomorphic map $F: \mathbb{C} \longrightarrow \mathcal{H}$ such that $\lambda \circ F = f$, since $\mathbb{C}$ is simply connected and λ is a covering.

Composed with the Cayley map $c: \mathcal{H} \xrightarrow{\sim} D$, $c \circ F: \mathbb{C} \longrightarrow D$ is constant by Liouville's theorem $\Rightarrow$ F is constant $\Rightarrow$ f is constant.

---

(5) <u>Variant</u>: $J = 12^{-3} j = \dfrac{g_2^3}{g_2^3 - 27 g_2^3} : \mathcal{H} \to \mathbb{C}$ is ramified at elliptic points of $\Gamma = SL_2(\mathbb{Z})$, i.e., at $\Gamma i \cup \Gamma \rho$. However, the restriction of J to $\mathcal{H} \setminus (\Gamma i \cup \Gamma \rho)$ is a covering of $\mathbb{C} \setminus \{J(\rho), J(i)\} = \mathbb{C} \setminus \{0,1\}$ (but not the universal one). the argument of (4) applies and gives a proof of Picard's small theorem using J rather than λ.

---

(6) <u>Abstract version</u>: if $\Gamma \subset SL_2(\mathbb{R})$ is a discrete subgroup, then any holomorphic map $\mathbb{C} \longrightarrow \Gamma \backslash (\mathcal{H} - \{\text{elliptic points of } \Gamma\})$ is constant.

---

(7) <u>Picard's big theorem</u>: any holomorphic map
$f: \mathcal{D}^* = \{0 < |w| < 1\} \longrightarrow \mathbb{C} \setminus \{0,1\}$ extends to
a holomorphic map $\mathcal{D} = \{|w| < 1\} \longrightarrow \mathbb{P}^1(\mathbb{C})$.

---

(8) <u>Exercise</u>. If Γ has no elliptic points (i.e., if we are in (2)), then any holomorphic map $\mathcal{D}^* \longrightarrow Y_\Gamma$ extends to a holomorphic map $\mathcal{D} \longrightarrow X_\Gamma$.

[<u>Hint</u>: consider

$$\begin{tikzcd} & \mathcal{H} \arrow[d, "\pi_\Gamma = \text{covering}"] & \text{universal} \\ \mathcal{D}^* \arrow[r, "f"] & Y_\Gamma & \end{tikzcd}$$

and the quotient of $\mathcal{H}$ by the image of
$\mathbb{Z} = \pi_1(\mathcal{D}^*) \xrightarrow{f_*} \pi_1(Y_\Gamma) = \Gamma$.]

---

(9) <u>Note</u>: Picard's big theorem = Exercise (8) for $\Gamma = \Gamma(2)$.