HOMOLOGICAL ALGEBRA (M2 UPMC, 2007/08)

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0. Introduction

0.0. What is this course about?

Homological algebra studies algebraic properties of complexes and their (co)homology. Its origins lie in algebraic topology, but it has proved to be extremely important in many other areas of mathematics.

The most useful references for this course will be the following texts:

[GM] S.I. Gelfand, Yu.I. Manin, *Methods of homological algebra*, 2nd ed., Springer Monographs in Math., Springer, Berlin, 2003.

[W] C. Weibel, *An introduction to homological algebra*, Cambridge Studies in Adv. Math. **38**, Cambridge Univ. Press, Cambridge, 1994.

[S1] P. Schapira, *Categories and Homological Algebra*, lecture notes available at
`http://www.institut.math.jussieu.fr/~schapira/polycopies/`

[S2] P. Schapira, *An Introduction to Sheaves*, lecture notes available at
`http://www.institut.math.jussieu.fr/~schapira/polycopies/`

[KS] M. Kashiwara, P. Schapira, *Sheaves on manifolds*, Grund. Math. Wiss. **292**, Springer, Berlin, 1994.

[McL] S. MacLane, *Homology*, Classics in Math., Springer, Berlin, 1995.

We shall use the standard language of category theory in this course; see [S1, ch. 2,3] for the details.

0.1. Example: the space $\mathbf{C}^* = \mathbf{C} - \{0\}$ has a hole (at the origin).

This can be seen **geomerically**, by noting that the unit circle $\gamma = \{z \in \mathbf{C} \mid |z| = 1\} \subset \mathbf{C}^*$ (with its positive orientation) is a **closed** curve (i.e., its boundary $\partial\gamma = \emptyset$ is empty) which is not a boundary of a relatively compact region $U \subset \mathbf{C}^*$ (or a combination of such boundaries): $\gamma \neq \partial U$.

Analytically, we can argue that there exists a differential 1-form $\omega = a(x,y)dx + b(x,y)dy$ on $\mathbf{C}^*$ which is **closed**, i.e., for which

$$0 = d\omega = d(a\,dx + b\,dy) = da\wedge dx + db\wedge dy = \Big(\frac{\partial a}{\partial x}\,dx + \frac{\partial a}{\partial y}\,dy\Big)\wedge dx + \Big(\frac{\partial b}{\partial x}\,dx + \frac{\partial b}{\partial y}\,dy\Big)\wedge dy = \Big(\frac{\partial b}{\partial x} - \frac{\partial a}{\partial y}\Big)\,dx\wedge dy,$$

but which is not **exact**, i.e.,

$$\omega \neq df = \Big(\frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy\Big)$$

for any smooth function $f : \mathbf{C}^* \longrightarrow \mathbf{R}$ (note that $dd\,f = 0$ for any smooth function).

For example, the differential form

$$\omega = \mathrm{Im}\Big(\frac{dz}{z}\Big) = \frac{x\,dy - y\,dx}{x^2 + y^2} \qquad (z = x + iy) \tag{0.1.1}$$

has a non-trivial period integral along the unit circle γ

$$\int_\gamma \omega = \mathrm{Im}\int_\gamma \frac{dz}{z} = 2\pi \neq 0,$$

which implies both of the assertions above, as we have (by the Stokes theorem)

$$\int_{\partial U} \omega = \int_U d\omega = \int_U 0 = 0, \qquad \int_\gamma df = \int_{\partial\gamma} f = \int_\emptyset f = 0.$$

0.2. The de Rham complex

The previous example is a special case of the following construction. Let $X \subset \mathbf{R}^n$ be an non-empty open subset (resp., a smooth ($= C^\infty$) manifold of dimension n). The **de Rham complex** of X (with real coefficients) is the following sequence of real vector spaces and linear maps:

$$A^\bullet(X) = \left[A^0(X) \xrightarrow{d} A^1(X) \xrightarrow{d} \cdots \xrightarrow{d} A^n(X)\right],$$

where $A^p(X)$ is the vector space of real-valued differential p-forms on X:

$$A^0(X) = C^\infty(X, \mathbf{R}) = \{\text{smooth functions } f : X \longrightarrow \mathbf{R}\},$$
$$A^p(X) = \Big\{\sum_{|I|=p} f_I\, dx_I \mid f_I \in C^\infty(X, \mathbf{R})\Big\}, \qquad I = \{i_1 < \cdots < i_p\} \subseteq \{1, \ldots, n\}, \qquad dx_I = dx_{i_1} \wedge \cdots \wedge dx_{i_p},$$

where $x_1, \ldots, x_n$ are the standard coordinates in $\mathbf{R}^n$ (resp., local coordinates on X), and

$$df = \sum_{i=1}^n \frac{\partial f}{\partial x_i}\, dx_i, \qquad d(f_I\, dx_I) = (df_I) \wedge dx_I, \qquad dx_i \wedge dx_i = 0, \qquad dx_j \wedge dx_i = -dx_i \wedge dx_j.$$

The commutation rule

$$\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} = \frac{\partial}{\partial x_j}\frac{\partial}{\partial x_i}$$

implies that

$$\forall \omega \in A^p(X) \quad d^2\omega = dd\omega = 0.$$

In particular, the spaces of exact (resp., closed) p-forms

$$B^p(X) = \{\text{exact } p-\text{forms on } X\} := \{d\varphi \mid \varphi \in A^{p-1}(X)\}$$
$$Z^p(X) = \{\text{closed } p-\text{forms on } X\} := \{\omega \in A^p(X) \mid d\omega = 0\}$$

satisfy

$$B^p(X) \subseteq Z^p(X).$$

The degree p **de Rham cohomology** of X (with real coefficients) is defined as

$$H^p_{dR}(X, \mathbf{R}) := Z^p(X)/B^p(X).$$

This is the space of obstructions for closed p-forms to be exact: a closed p-form $\omega \in Z^p(X)$ is exact iff its cohomology class $[\omega] = \omega + B^p(X) \in H^p_{dR}(X, \mathbf{R})$ is equal to zero.

0.3. Examples of de Rham cohomology groups: (1) $H^0_{dR}(X, \mathbf{R})$ consists of the functions $f : X \longrightarrow \mathbf{R}$ that are constant on each connected component of X.

(2) $H^p(\mathbf{C}^*, \mathbf{R})$ is generated by the class of the form ω from (0.1.1) (resp., is equal to zero) for $p = 1$ (resp., for $p > 1$).

1. Complexes of R-modules

1.1. Modules

1.1.1. Let R be a ring (associative, but not necessarily with a unit). Recall that a left (resp., a right) R-module is an abelian group $(M,+)$ equipped with an operation $R \times M \longrightarrow M$, $(r,m) \mapsto rm$ (resp., $M \times R \longrightarrow M$, $(m,r) \mapsto mr$) satisfying the usual rules

$$(r+r')m = rm + r'm, \qquad r(m+m') = rm + rm', \qquad (rr')m = r(r'm)$$

resp.,

$$m(r+r') = mr + mr', \qquad (m+m')r = mr + m'r, \qquad m(rr') = (mr)r'$$

(which imply that $1m = m$ (resp., $m1 = m$) if R has a unit $1 \in R$). We write $M \in {}_R\mathrm{Mod}$ (resp., $M \in \mathrm{Mod}_R$).

1.1.2. (1) The **opposite ring** to R is the ring R^{op} equal to R as an abelian group, but with new multiplication $r \cdot_{op} r' = r'r$.
(2) If $M \in \mathrm{Mod}_R$, then $M \in {}_{R^{op}}\mathrm{Mod}$, with the operation $rm = mr$ (as $(r \cdot_{op} r')m = (r'r)m = m(r'r) = (mr')r = r(mr') = r(r'm)$).
(3) If R is commutative, then $R^{op} = R$, hence $\mathrm{Mod}_R = {}_R\mathrm{Mod}$.
(4) $\mathbf{Z}$-modules are just abelian groups.

1.1.3. Morphisms. A (homo)morphism $f : M \longrightarrow N$ (say, for $M, N \in {}_R\mathrm{Mod}$) is a map satisfying $f(m+m') = f(m) + f(m')$, $f(rm) = rf(m)$, for all $r \in R$ and $m, m' \in M$.

1.1.4. Examples of morphisms. (1) The zero map $f = 0$ ($f(m) = 0$ for all $m \in M$).
(2) The identity map $\mathrm{id}_M : M \longrightarrow M$ ($\mathrm{id}_M(m) = m$ for all $m \in M$).
(3) The inclusion $M' \hookrightarrow M$ of an R-submodule (i.e., of an abelian subgroup of $(M,+)$ satisfying $rm' \in M'$ for all $r \in R$ and $m' \in M'$).
(4) The projection $p : M \longrightarrow M/M'$ ($p(m) = m + M'$).
(5) If $f : M \longrightarrow N$ is a morphism and $M' \subset M$ and $N' \subset N$ are submodules satisfying $f(M') \subseteq N'$, then f induces a morphism $\overline{f} : M/M' \longrightarrow N/N'$, $\overline{f}(m + M') = f(m) + N'$.
(6) If $f : M \longrightarrow N$ and $g : N \longrightarrow P$ are morphisms, so is $g \circ f : M \longrightarrow P$.

1.1.5. Products and sums of modules. Let I be a set; assume that we are given, for each $i \in I$, an R-module $M_i \in {}_R\mathrm{Mod}$. The **direct product** of these modules is

$$\prod_{i \in I} M_i = \{(m_i)_{i \in I} \mid m_i \in M_i\}$$

with component-wise operations

$$(m_i) + (m_i') = (m + m_i'), \qquad r(m_i) = (rm_i).$$

The **direct sum** of the modules M_i is the following submodule of their direct product:

$$\bigoplus_{i \in I} M_i = \{(m_i)_{i \in I} \mid m_i \in M_i, \text{ only finitely many components } m_i \text{ are notzero}\} \subset \prod_{i \in I} M_i.$$

If all modules M_i are equal to the same R-module M, we write

$$M^I = \prod_{i \in I} M = \{\text{maps } I \longrightarrow M\} \supset M^{(I)} = \bigoplus_{i \in I} M = \{\text{maps } I \longrightarrow M \text{ with finite support}\}$$

(the support of a map $f : I \longrightarrow M$ is the set $\mathrm{supp}(f) = \{i \in I \mid f(i) \neq 0\}$). In the special case when $M = R \in {}_R\mathrm{Mod}$ (with operation $(r,m) = rm$ given by the product in R), then

$$R^{(I)} = \{\text{maps } I \longrightarrow R \text{ with finite support}\}$$

is the **free R-module on the set I**. In particular, $R^{(\emptyset)} = 0 = \{0\}$.

1.2. Complexes of (left) R-modules

1.2.1. Definition. A **homological** (resp., a **cohomological**) complex $(C_\bullet, \partial)$ (resp., $(C^\bullet, d)$) of (left) R-modules is a sequence of R-modules and morphisms ("differentials") such that

$$C_\bullet : \qquad \cdots \longrightarrow C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \longrightarrow \cdots, \qquad \forall n \in \mathbf{Z} \quad \partial_n \circ \partial_{n+1} = 0$$

resp.,

$$C^\bullet : \qquad \cdots \longrightarrow C^{n-1} \xrightarrow{d^{n-1}} C^n \xrightarrow{d^n} C^{n+1} \longrightarrow \cdots, \qquad \forall n \in \mathbf{Z} \quad d^n \circ d^{n-1} = 0$$

It is customary to suppress the indices and write the defining conditions as $\partial^2 = \partial \circ \partial = 0$ (resp., $d^2 = d \circ d = 0$). Any homological complex can be transformed into a cohomological one (and vice versa), using the renumbering $C^n = C_{-n}$, $d^n = \partial_{-n}$. However, complexes naturally occurring in geometry usually satisfy $\forall n < 0 \quad C_n = 0$ (resp., $\forall n < 0 \quad C^n = 0$).

1.2.2. Example. The de Rham complex $A^\bullet(X)$ of a smooth manifold X of dimension n (completed by zeros in degrees outside $\{0, 1, \ldots, n\}$)

$$\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow A^0(X) \xrightarrow{d} A^1(X) \xrightarrow{d} \cdots \xrightarrow{d} A^n(X) \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots$$

is a cohomological complex of real vector spaces.

1.2.3. Definition. Let $(C^\bullet, d)$ be a cohomological complex of R-modules. For each $n \in \mathbf{Z}$, define

$$\begin{aligned} B^n(C^\bullet) &= \{\text{the } n\text{−coboundaries of } C^\bullet\} := \{d^{n-1}y \mid y \in C^{n-1}\} \\ Z^n(C^\bullet) &= \{\text{the } n\text{−cocycles of } C^\bullet\} := \{x \in C^n \mid d^n(x) = 0\} \\ H^n(C^\bullet) &= \{\text{the } n\text{−th cohomology module of } C^\bullet\} := Z^n(C^\bullet)/B^n(C^\bullet) \end{aligned}$$

(note that $B^n(C^\bullet)$ is a sub-module of $Z^n(C^\bullet)$, as $d^n \circ d^{n-1} = 0$). For $x \in Z^n(C^\bullet)$, we denote by $[x] = x + B^n(C^\bullet) \in H^n(C^\bullet)$ the cohomology class of x; we have $[x] = [x']$ iff $x' = x + dy$ for some $y \in C^{n-1}$. We say that the complex $C^\bullet$ is **acyclic** if $\forall n \in \mathbf{Z} \quad H^n(C^\bullet) = 0$.

1.2.4. Analogously, for any homological complex $C_\bullet$ we have submodules

$$B_n(C_\bullet) = \partial_{n+1} C_{n+1} \subseteq Z_n(C_\bullet) = \{x \in C_n \mid \partial_n(x) = 0\} \subseteq C_n$$

and the homology modules

$$H_n(C_\bullet) = Z_n(C_\bullet)/B_n(C_\bullet).$$

1.2.5. Definition. A **morphism of (cohomological) complexes** $\alpha : (C^\bullet, d) \longrightarrow (C'^\bullet, d')$ is a collection of morphisms of R-modules $\alpha_n : C^n \longrightarrow C'^n$ that are compatible with the differentials in the sense that

$$\forall n \in \mathbf{Z} \quad \alpha_{n+1} \circ d^n = d'^n \circ \alpha_n, \qquad \begin{array}{ccc} C^n & \xrightarrow{\alpha_n} & C'^n \\ \big\downarrow d^n & & \big\downarrow d'^n \\ C^{n+1} & \xrightarrow{\alpha_{n+1}} & C'^{n+1} \end{array}$$

(or, suppressing the indices, $\alpha d = d' \alpha$).

In this case

$$\alpha_n(Z^n(C^\bullet)) \subseteq Z^n(C'^\bullet), \qquad \alpha_n(B^n(C^\bullet)) \subseteq B^n(C'^\bullet),$$

which implies that α_n induces a morphism of R-modules

$$H^n(\alpha) : H^n(C^\bullet) \longrightarrow H^n(C'^\bullet), \qquad [x] \mapsto [\alpha_n(x)].$$

1.2.6. Example. If, in the situation of 1.2.5, each map α_n is the inclusion into C'^n of a sub-module C^n, we say that $C^\bullet$ is a **subcomplex** of $C'^\bullet$. The modules $\cdots \longrightarrow C'^n/C^n \longrightarrow C'^{n+1}/C^{n+1} \longrightarrow \cdots$ equipped with the differentials induced by d' then form the corresponding **quotient complex** $C'^\bullet/C^\bullet$ of $C'^\bullet$.

1.2.7. Example. If $g : X \longrightarrow Y$ is a smooth map between two smooth manifolds, then the pull-back maps $g^* : A^p(Y) \longrightarrow A^p(X)$ (e.g., $g^*(f) = f \circ g$ for $f \in A^0(Y)$) define a morphism of complexes $g^* : A^\bullet(Y) \longrightarrow A^\bullet(X)$, hence morphisms of real vector spaces

$$H^p(g^*) : H^p_{dR}(Y, \mathbf{R}) \longrightarrow H^p_{dR}(X, \mathbf{R}).$$

1.2.8. Definition. A morphism of complexes $\alpha : C^\bullet \longrightarrow C'^\bullet$ is a **quasi-isomorphism** (notation: **Qis**) if $H^n(\alpha)$ is an isomorphism, for all $n \in \mathbf{Z}$.

1.3. Simplicial homology

1.3.1. Many natural complexes are of **simplicial origin**: the elements of C_n (resp., C^n) are indexed by a certain set of n-simplices [1]; the differentials ∂_n (resp., d^n) are deduced from the boundary operators on the simplices.

1.3.2. Definition. Let $v_0, \ldots, v_n \in V$ $(n \geq 0)$ be a set of $n+1$ affinely independent points in a real affine space V (i.e., the dimension of the smallest affine subspace of V containing $v_0, \ldots, v_n$ is equal to n). The **n-simplex with vertices** $v_0, \ldots, v_n$ is the convex hull of $\{v_0, \ldots, v_n\}$, namely the subset

$$\sigma = [v_0, \ldots, v_n] = \mathrm{Conv}(v_0, \ldots, v_n) = \Big\{\sum_{i=0}^{n} \lambda_i v_i \mid \lambda_i \in \mathbf{R},\ \lambda_i \geq 0,\ \sum_{i=0}^{n} \lambda_i = 1\Big\} \subset V.$$

The set of vertices $\{v_0, \ldots, v_n\}$ depends only on σ. For any non-empty subset $\{w_0, \ldots, w_m\} \subset \{v_0, \ldots, v_n\}$, the m-simplex $\mathrm{Conv}(w_0, \ldots, w_m)$ is called a **face of** σ (of codimension $n - m$).

In particular, each n-simplex has $n+1$ faces of codimension 1, each of which is an $(n-1)$-simplex.

1.3.3. Definition. A **(finite) simplicial complex in** V is a finite set K of simplices in V such that
(a) $\forall \sigma \in K$ each face of σ is an element of K;
(b) $\forall \sigma, \tau \in K$ the intersection $\sigma \cap \tau$ is a face of σ and of τ.
The set K is a disjoint union $K = K_0 \cup K_1 \cup \cdots \cup K_N$ (for suitable $N \geq 0$), where K_n consists of the n-simplices in K. We say that K **is ordered** if its set of vertices K_0 is totally ordered.

1.3.4. Examples. (1) All faces of a given n-simplex σ (including σ).
(2) All faces of a given n-simplex σ except σ itself.

1.3.5. Face maps. Let K be an ordered simplicial complex. For each $n \geq 1$ there are $n+1$ face maps (in codimension 1)

$$d_0, \ldots, d_n : K_n \longrightarrow K_{n-1}$$
$$d_i[v_0, \ldots, v_n] = [v_0, \ldots, \widehat{v_i}, \ldots, v_n] = [v_0, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n], \qquad v_0 < \cdots < v_n$$

(the notation "$\ldots \widehat{x} \ldots$" means that the term "$x$" is omitted). The face maps satisfy the following fundamental relation:

$$\forall i < j \quad d_i d_j = d_{j-1} d_i. \tag{1.3.5.1}$$

1.3.6. Example. If $v_0 < v_1 < v_2 < v_3$ and $\sigma = [v_0, v_1, v_3]$, then

$$d_2(\sigma) = [v_0, v_1], \qquad d_0 d_2(\sigma) = [v_1], \qquad d_0(\sigma) = [v_1, v_3], \qquad d_1 d_0(\sigma) = [v_1] = d_0 d_2(\sigma).$$

[1] 0-simplex = point, 1-simplex = segment, 2-simplex = triangle, 3-simplex = tetrahedron, ...

1.3.7. Definition. Let $K = K_0 \cup K_1 \cup \cdots \cup K_N$ be an ordered (finite) simplicial complex; let A be an abelian group. The **chain complex of K with coefficients in A** is the following homological complex $C_\bullet(K, A)$ of abelian groups:

$$C_n(K,A) = A^{(K_n)}, \qquad \partial_n = \sum_{i=0}^{n} (-1)^i d_i : C_n(K,A) = A^{(K_n)} \longrightarrow C_{n-1}(K,A) = A^{(K_{n-1})}$$

($C_n(K, A) = 0$ for $n > N$ and $n < 0$).

1.3.8. Example. If K consists of a 2-simplex $[v_0, v_1, v_2]$ ($v_0 < v_1 < v_2$) and all its faces, then

$$C_2 = A\cdot[v_0,v_1,v_2] \xrightarrow{\partial_2} C_1 = A\cdot[v_0,v_1] \oplus A\cdot[v_0,v_2] \oplus A\cdot[v_1,v_2] \xrightarrow{\partial_1} C_0 = A\cdot[v_0] \oplus A\cdot[v_1] \oplus A\cdot[v_2],$$

$$\partial_1([v_0,v_1]) = [v_1]-[v_0], \qquad \partial_1([v_0,v_2]) = [v_2]-[v_0], \qquad \partial_1([v_1,v_2]) = [v_2]-[v_1],$$

$$\partial_2([v_0,v_1,v_2]) = [v_1,v_2]-[v_0,v_2]+[v_0,v_1], \qquad \partial_1\circ\partial_2([v_0,v_1,v_2]) = ([v_2]-[v_1])-([v_2]-[v_0])+([v_1]-[v_0]) = 0.$$

1.3.9. Proposition-Definition. *$C_\bullet(K, A)$ is, indeed, a complex (i.e., $\partial^2 = 0$). Denote by $H_n(K, A) := H_n(C_\bullet(K, A))$ its homology ($n \in \mathbf{Z}$).*

Proof. It follows from the relation (1.3.5.1) that

$$\partial_{n-1}\circ\partial_n = \sum_{i=0}^{n-1}\sum_{j=0}^{n}(-1)^{i+j}\, d_i d_j = \sum_{n>i\geq j\geq 0}(-1)^{i+j}\, d_i d_j + \sum_{0\leq i<j\leq n}(-1)^{i+j}\, d_{j-1} d_i;$$

replacing i (resp., $j - 1$) in the second sum by j (resp., by i), we obtain

$$\partial_{n-1}\circ\partial_n = \sum_{n>i\geq j\geq 0}\left((-1)^{i+j} + (-1)^{i+j+1}\right) d_i d_j = 0.$$

1.3.10. Denote by

$$|K| = \bigcup_{\sigma\in K} \sigma \subset V$$

the subset of V (with the induced topology) consisting of the "physical points" of K. The main result of the classical simplicial homology theory states that the homology groups $H_n(K, A)$ depend only on $|K|$, in a functorial way:

(1) if K, K' are (finite, ordered) simplicial complexes, then any continuous map $f : |K| \longrightarrow |K'|$ defines a morphism of abelian groups

$$f_{n*} : H_n(K,A) \longrightarrow H_n(K',A),$$

which depends only on the homotopy class of f (see 1.5.1 below);

(2) if K, K', K'' are (finite, ordered) simplicial complexes and $f : |K| \longrightarrow |K'|$, $g : |K'| \longrightarrow |K''|$ continuous maps, then $(g \circ f)_{n*} = g_{n*} \circ f_{n*}$.

The definition of the maps f_{n*} (in fact, of a morphism of complexes $C_\bullet(K, A) \longrightarrow C_\bullet(K', A)$) is straightforward when f is a **simplicial map** (which means that f maps simplices of K to simplices of K' – possibly of a smaller dimension – and the restriction f to each simplex $\sigma \in K$ is an affine map): a simplex $[v_0 < \cdots < v_n] \in C_n(K, A)$ is mapped to 0 if $f(v_i) = f(v_j)$ for some $i \neq j$ (resp., to $\varepsilon[v'_0 < \cdots < v'_n] \in C_n(K', A)$ if the set $\{f(v_0), \ldots, f(v_n)\} = \{v'_0 < \cdots < v'_n\}$ has $n + 1$ elements; the sign $\varepsilon = \pm 1$ is equal to the sign of the permutation transforming $(f(v_0), \ldots, f(v_n))$ into $(v'_0, \ldots, v'_n)$).

In order to define f_{n*} for a general continuous map f one has to approximate f by a map that is simplicial on a sufficiently fine subdivision of K (see, for example, M.A. Armstrong, Basic Topology, Springer, 1983).

1.4. Singular homology

1.4.1. Unlike simplicial homology, singular homology is defined for arbitrary topological spaces in a manifestly functorial manner.

1.4.2. Definition. The **standard n-simplex**

$$\Delta_n = \mathrm{Conv}(e_0, \ldots, e_n) = \Big\{ \begin{pmatrix} \lambda_0 \\ \vdots \\ \lambda_n \end{pmatrix} \in \mathbf{R}^{n+1} \,\Big|\, \sum_{i=0}^{n} \lambda_i = 1,\, \lambda_i \geq 0 \Big\} \subset \mathbf{R}^{n+1}$$

is the convex hull of the endpoints of the vectors of the canonical basis $e_0, \ldots, e_n$ of $\mathbf{R}^{n+1}$. It is naturally ordered, as its vertices correspond to the elements of $[n] := \{0, \ldots, n\}$ ($e_i \longleftrightarrow i$).

1.4.3. Face maps. Any non-decreasing map $f : [m] \longrightarrow [n]$ defines an affine map

$$f : \Delta_m \longrightarrow \Delta_n, \qquad \sum_{i=0}^{m} \lambda_i e_i \mapsto \sum_{i=0}^{m} \lambda_i e_{f(i)};$$

the faces of Δ_n correspond to increasing maps f. In particular, the codimension one faces

$$d_0, \ldots, d_n : \Delta_{n-1} \longrightarrow \Delta_n$$

are given by the increasing maps $d_i : [n-1] \longrightarrow [n]$, $i \notin \mathrm{Im}(d_i)$. Recall that these maps satisfy the relations (1.3.5.1).

1.4.4. Definition (singular simplices). Let X be a topological space. A **singular n-simplex of X** ($n \geq 0$) is a continuous map $\sigma : \Delta_n \longrightarrow X$. Denote by $X(\Delta_n)$ the set of all singular n-simplices of X. For $n > 0$ there are face maps

$$X(\Delta_n) \longrightarrow X(\Delta_{n-1}), \qquad \sigma \mapsto \sigma \circ d_i \qquad (i = 0, \ldots, n).$$

1.4.5. Definition. *Let X be a topological space and A an abelian group. The* **singular chain complex of X** *is the following (homological) complex $C_\bullet = C_\bullet^{\mathrm{sing}}(X, A)$ (with $C_n = 0$ for $n < 0$):*

$$\cdots \longrightarrow C_n = A^{(X(\Delta_n))} \xrightarrow{\partial_n} C_{n-1} = A^{(X(\Delta_{n-1}))} \longrightarrow \cdots, \qquad \partial_n(\sigma) = \sum_{i=0}^{n} (-1)^i \, (\sigma \circ d_i).$$

The homology groups of this complex $H_n^{\mathrm{sing}}(X, A) := H_n(C_\bullet^{\mathrm{sing}}(X, A))$ are called the **singular homology groups of X with coefficients in** *A.*

1.4.6. In concrete terms,

$$C_n^{\mathrm{sing}}(X, A) = \Big\{ \sum_{\sigma : \Delta_n \longrightarrow X} a_\sigma \cdot \sigma \,\Big|\, \sigma \text{ continuous, } a_\sigma \in A, \text{ the sum is finite} \Big\},$$

$$\partial_n\Big(\sum_{\sigma : \Delta_n \longrightarrow X} a_\sigma \cdot \sigma \Big) = \sum_{\sigma : \Delta_n \longrightarrow X} \sum_{i=0}^{n} (-1)^i \, a_\sigma \cdot (\sigma \circ d_i).$$

The fact that $\partial^2 = 0$ follows from (1.3.5.1), as in 1.3.9.

1.4.7. Example (homology of a point). If $X = \mathrm{pt}$, then each $X(\Delta_n)$ consists of one element, $C_n^{\mathrm{sing}}(X, A) = A$ and ∂_n is given by multiplication by $\sum_{i=0}^{n} (-1)^i$; thus

$$C_\bullet^{\mathrm{sing}}(\mathrm{pt}, A) : \quad \cdots \longrightarrow A \xrightarrow{\mathrm{id}} A \xrightarrow{0} A \xrightarrow{\mathrm{id}} A \xrightarrow{0} A \ (= C_0) \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots,$$

which implies that

$$H_n^{\mathrm{sing}}(\mathrm{pt}, A) = \begin{cases} A, & n = 0, \\ 0, & n > 0. \end{cases}$$

1.4.8. Exercise. Show that $H_0^{\mathrm{sing}}(X) = A^{(\pi_0(X))}$, where $\pi_0(X)$ is the set of path components of X:

$$\pi_0(X) = X/\sim \qquad x \sim x' := \exists \sigma : [0,1] \longrightarrow X \text{ continuous, } \sigma(0) = x, \ \sigma(1) = x'.$$

1.4.9. Functoriality. Let $f : X \longrightarrow Y$ be a continuous map between two topological spaces. If $\sigma : \Delta_n \longrightarrow X$ is a singular n-simplex of X, then $f \circ \sigma : \Delta_n \longrightarrow X \longrightarrow Y$ is a singular n-simplex of Y. As $(f \circ \sigma) \circ d_i = f \circ (\sigma \circ d_i)$ for all $i = 0, \ldots, n$, the composition with f defines a morphism of complexes

$$C_\bullet^{\mathrm{sing}}(f) : C_\bullet^{\mathrm{sing}}(X, A) \longrightarrow C_\bullet^{\mathrm{sing}}(Y, A),$$

hence morphisms of abelian groups

$$H_n^{\mathrm{sing}}(f) : H_n^{\mathrm{sing}}(X, A) \longrightarrow H_n^{\mathrm{sing}}(Y, A).$$

If $g : Y \longrightarrow Z$ is continuous, then

$$H_n^{\mathrm{sing}}(g) \circ H_n^{\mathrm{sing}}(f) = H_n^{\mathrm{sing}}(g \circ f) : H_n^{\mathrm{sing}}(X, A) \longrightarrow H_n^{\mathrm{sing}}(Z, A).$$

If $Y = X$ and $f = \mathrm{id}_X$, then $H_n^{\mathrm{sing}}(\mathrm{id}_X) = \mathrm{id}$.

1.5. Homotopy invariance of singular homology

1.5.1. Definition. Let $f, g : X \longrightarrow Y$ be continuous maps between topological spaces X and Y. A **homotopy** $H : f \simeq g$ between f and g is a continuous map $H : X \times I \longrightarrow Y$ ($I = [0,1]$) such that $\forall x \in X \ H(x, 0) = f(x)$, $H(x, 1) = g(x)$. For $t \in I$, denote by $i_t : X \hookrightarrow X \times I$ the map $i_t(x) = (x, t)$. The family of continuous maps $H \circ i_t : X \longrightarrow Y$ is then a "continuous deformation" of $f = H \circ i_0$ into $g = H \circ i_1$.

1.5.2. Definition. A continuous map $f : X \longrightarrow Y$ is a **homotopy equivalence** if there exists a continuous map $g : Y \longrightarrow X$ and homotopies $gf \simeq \mathrm{id}_X$, $fg \simeq \mathrm{id}_Y$ (we say that g is a **homotopy inverse of** f).

1.5.3. Definition. A topological space X is **contractible** if there exists a homotopy equivalence $X \longrightarrow \mathrm{pt}$.

1.5.4. Examples. (1) The inclusion $i : S^n \hookrightarrow \mathbf{R}^{n+1} - \{0\}$ is a homotopy equivalence, as the retraction $r : \mathbf{R}^{n+1} - \{0\} \longrightarrow S^n$, $r(x) = x/\|x\|$ satisfies $ri = \mathrm{id}$, $ir \simeq \mathrm{id}$ (via the homotopy $H(x,t) = x/(t + (1-t)\|x\|)$).
(2) Any star-shaped subset $X \subset \mathbf{R}^n$ (i.e., such that there exists a point $x_0 \in X$ with the property that, for each point $x \in X$, the segment $[x_0, x]$ is contained in X) is contractible: if $i : \{x_0\} \hookrightarrow X$ (resp., $r : X \longrightarrow \{x_0\}$) is the inclusion (resp., the constant map), then $ri = \mathrm{id}_{\{x_0\}}$ and $ir \simeq \mathrm{id}_X$, via the homotopy $H(x, t) = tx + (1-t)x_0$.

1.5.5. Theorem. *Let $f, g : X \longrightarrow Y$ be continuous maps. If there exists a homotopy between f and g, then*

$$\forall n \geq 0 \quad H_n^{\mathrm{sing}}(f) = H_n^{\mathrm{sing}}(g) : H_n^{\mathrm{sing}}(X, A) \longrightarrow H_n^{\mathrm{sing}}(Y, A).$$

1.5.6. Corollary. *If $f : X \longrightarrow Y$ is a homotopy equivalence, then $C_\bullet^{\mathrm{sing}}(f) : C_\bullet^{\mathrm{sing}}(X, A) \longrightarrow C_\bullet^{\mathrm{sing}}(Y, A)$ is a quasi-isomorphism, i.e., the maps $H_n^{\mathrm{sing}}(f) : H_n^{\mathrm{sing}}(X, A) \longrightarrow H_n^{\mathrm{sing}}(Y, A)$ are isomorphisms (for all $n \geq 0$).*

Proof. Let $g : Y \longrightarrow X$ be a homotopy inverse of f. Then

$$H_n(f) \circ H_n(g) = H_n(f \circ g) \overset{1.5.5}{=} H_n(\mathrm{id}_Y) = \mathrm{id}_{H_n(Y)}$$
$$H_n(g) \circ H_n(f) = H_n(g \circ f) \overset{1.5.5}{=} H_n(\mathrm{id}_X) = \mathrm{id}_{H_n(X)},$$

which implies that $H_n^{\mathrm{sing}}(g) : H_n^{\mathrm{sing}}(Y, A) \longrightarrow H_n^{\mathrm{sing}}(X, A)$ is an inverse of $H_n^{\mathrm{sing}}(f)$.

1.5.7. Corollary. *If X is contractible, then*

$$H_n^{\mathrm{sing}}(X,A) = \begin{cases} A, & n = 0, \\ 0, & n > 0. \end{cases}$$

Proof of Theorem 1.5.5. The statement is a consequence of the following two Lemmas, as the composite morphisms of abelian groups

$$h_n = H \circ t_n : C_n(X,A) \xrightarrow{t_n} C_{n+1}(X \times I, A) \xrightarrow{H} C_{n+1}(Y,A)$$

(where we have denoted, for simplicity, the morphism $C_{n+1}^{\mathrm{sing}}(H)$ by H) satisfy

$$\partial h + h\partial = \partial H t + H \partial t = H(\partial t + t\partial) = h(i_1 - i_0) = g - f,$$

hence give rise to a homotopy between $C_\bullet^{\mathrm{sing}}(f)$ and $C_\bullet^{\mathrm{sing}}(g) : C_\bullet^{\mathrm{sing}}(X,A) \longrightarrow C_\bullet^{\mathrm{sing}}(Y,A)$.

1.5.8. Lemma-Definition. *Let $\alpha, \beta : (C_\bullet, \partial) \longrightarrow (C'_\bullet, \partial')$ be morphisms of (homological) complexes of R-modules. A* **homotopy** *between α and β is a collection $h = (h_n)$ of morphisms of R-modules $h_n : C_n \longrightarrow C'_{n+1}$ such that $\partial'_{n+1} \circ h_n + h_{n-1} \circ \partial_n = \beta_n - \alpha_n$, for all $n \in \mathbf{Z}$ (or, suppressing the indices, $\partial' h + h\partial = \beta - \alpha$). If such a homotopy exists, then*

$$\forall n \geq 0 \quad H_n(\alpha) = H_n(\beta) : H_n(C_\bullet) \longrightarrow H_n(C'_\bullet).$$

[Similarly, a homotopy between morphisms $\alpha, \beta : (C^\bullet, d) \longrightarrow (C'^\bullet, d')$ of cohomological complexes of R-modules is a collection $h = (h_n)$ of morphisms of R-modules $h_n : C^n \longrightarrow C'^{n-1}$ such that $d'h + hd = \beta - \alpha$. If h exists, then $H^n(\alpha) = H^n(\beta) : H^n(C^\bullet) \longrightarrow H^n(C'^\bullet)$ for all $n \in \mathbf{Z}$.]

Proof. An element of $H_n(C_\bullet)$ is of the form $[x]$, for some $x \in C_n$ such that $\partial_n(x) = 0$; then

$$H_n(\alpha)([x]) - H_n(\beta)([x]) = [\alpha_n(x)] - [\beta_n(x)] = [\partial'_{n+1}(h_n(x))] + [h_{n-1}(\partial_n(x))] = [\partial'_{n+1}(h_n(x))] + 0 = 0.$$

1.5.9. Lemma. *There exist morphisms of abelian groups $t_n : C_n(X,A) \longrightarrow C_{n+1}(X \times I, A)$ satisfying $\partial t + t\partial = i_1 - i_0$.*

Proof. It is enough to define $t_n(\sigma)$ for any singular n-simplex $\sigma : \Delta_n \longrightarrow X$ and then extend it by linearity: $t_n(\sum_\sigma a_\sigma \cdot \sigma) = \sum_\sigma a_\sigma \cdot t_n(\sigma)$. Geometrically, t_n is the "prism operator" given by

$$t_n : \sigma \mapsto \text{ a natural subdivision of the prism } \sigma \times I \text{ into singular } (n+1)\text{−simplices.}$$

In fact, it is enough to subdivide the standard prism $\Delta_n \times I$ into $n+1$ $(n+1)$-simplices; the required formula will then be a consequence of the fact that the boundary of $\Delta_n \times I$ is a sum of its vertical part $-(\partial\Delta_n) \times I$ and its horizontal part $\Delta_n \times \{1\} - \Delta_n \times \{0\}$.

The formal definition is the following: the vertices of $\Delta_n \times I$ correspond to the elements of $[n] \times [1]$. This set has a natural partial order $(i,j) \leq (i',j') \iff i \leq i', j \leq j'$. For $i \in [n]$, let $t_{n,i} : [n+1] \longrightarrow [n] \times [1]$ be the unique monotonous injective map whose image contains $\{i\} \times [1]$; as in 1.4.3, this defines an affine map $t_{n,i} : \Delta_{n+1} \longrightarrow \Delta_n \times I$ which maps the (ordered) vertices $[e_0, \ldots, e_{n+1}]$ of Δ_{n+1} into the vertices $[(e_0,0), \ldots, (e_i,0), (e_i,1), \ldots, (e_n,1)]$ of a certain $(n+1)$-simplex inside $\Delta_n \times I$. The alternating sum of these simplices

$$\sum_{i=0}^{n} (-1)^i \, [(e_0,0), \ldots, (e_i,0), (e_i,1), \ldots, (e_n,1)]$$

then gives the desired decomposition of $\Delta_n \times I$. This suggests that we should define t_n as

$$t_n(\sigma) := \sum_{i=0}^{n} (-1)^i \, (\sigma \times I) \circ t_{n,i} \in C_{n+1}(X \times I, A) \qquad (\sigma : \Delta_n \longrightarrow X),$$

where $\sigma \times I : \Delta_n \times I \longrightarrow X \times I$ is given by $(\sigma \times I)(a,t) = (\sigma(a), t)$.

We check the formula $\partial t_n + t_{n-1}\partial = i_1 - i_0$ in the case $n = 1$ and leave the general case as an exercise to the reader:

$$\begin{aligned} t_1([e_0,e_1]) &= [(e_0,0),(e_0,1),(e_1,1)] - [(e_0,0),(e_1,0),(e_1,1)], \\ \partial t_1([e_0,e_1]) &= [(e_0,1),(e_1,1)] - [(e_0,0),(e_1,0)] + [(e_0,0),(e_0,1)] - [(e_1,0),(e_1,1)] \\ t_0\partial([e_0,e_1]) &= t_0([e_1]-[e_0]) = [(e_1,0),(e_1,1)] - [(e_0,0),(e_0,1)] \\ (\partial t_1 + t_0\partial)([e_0,e_1]) &= [e_0,e_1]\times\{1\} - [e_0,e_1]\times\{0\}. \end{aligned}$$

1.5.10. Exercise. Describe a natural decomposition of $\Delta_n \times \Delta_m$ into $(m+n)$-simplices.

1.5.11. Exercise (homotopy invariance of de Rham cohomology). Let X be a smooth manifold. As the product $X \times I$ is a manifold with boundary, we denote by $A^p(X\times I)$ the space of (smooth) differential p-forms on $X \times (0,1)$ that extend to a smooth p-form on some open subset of $X \times \mathbf{R}$ containing $X \times I$. With this definition, the de Rham complex $A^\bullet(X\times I) : \cdots \longrightarrow A^p(X\times I) \xrightarrow{d} A^{p+1}(X\times I) \longrightarrow \cdots$ of $X \times I$ makes sense. If $f, g : X \longrightarrow Y$ are smooth maps to another smooth manifold Y, a **smooth homotopy** between f and g is a map $H : X \times I \longrightarrow Y$ that extends to a smooth map $U \longrightarrow Y$ on some open subset $U \subset X \times \mathbf{R}$ containing $X \times I$.

The aim of this exercise is to show that such a smooth homotopy gives rise to a homotopy between the morphisms $f^*, g^* : A^\bullet(Y) \longrightarrow A^\bullet(X)$; as a result, $H^n(f^*) = H^n(g^*) : H^n_{dR}(Y,\mathbf{R}) \longrightarrow H^n_{dR}(X,\mathbf{R})$ for all $n \in \mathbf{Z}$.

The following construction is dual to that of Lemma 1.5.9: if $\{x_i\}$ are local coordinates on X (and t is the standard coordinate on $\mathbf{R}$), define maps

$$k_n : A^n(X\times I) \longrightarrow A^{n-1}(X)$$

by the formula

$$k_n\Big(\sum_{|J|=n} \alpha_J(x,t)\, dx_J + \sum_{|J|=n-1} \beta_J(x,t)\, dt \wedge dx_J\Big) = \sum_{|J|=n-1} \Big(\int_0^1 \beta_J(x,t)\, dt\Big)\, dx_J.$$

(1) Show that

$$\forall \omega \in A^n(X\times I) \quad (dk + kd)(\omega) = i_1^*(\omega) - i_0^*(\omega) \qquad (i_t(x) = (x,t))$$

(2) Deduce from (1) that any smooth homotopy H between $f, g : X \longrightarrow Y$ gives rise to a homotopy $k \circ H^* = (k_n \circ H^* : A^n(Y) \longrightarrow A^{n-1}(X))$ between f^* and g^*.

1.6. Other constructions: relative (singular) homology, cohomology

1.6.1. Definition (relative homology). If Y is a subset of a topological space X (equipped with the induced topology), then $C^{\mathrm{sing}}_\bullet(Y,A)$ is a subcomplex of $C^{\mathrm{sing}}_\bullet(X,A)$. Denote by

$$C^{\mathrm{sing}}_\bullet(X,Y;A) = C^{\mathrm{sing}}_\bullet(X,A)/C^{\mathrm{sing}}_\bullet(Y,A)$$

the corresponding quotient complex; its homology groups $H^{\mathrm{sing}}_n(X,Y;A) = H_n(C^{\mathrm{sing}}_\bullet(X,Y;A))$ are the **relative (singular) homology groups** of the pair (X,Y) (with coefficients in the abelian group A).

1.6.2. Definition (cohomology). The **cohomological singular complex** $C^\bullet_{\mathrm{sing}}(X,A)$ of a topological space X (with coefficients in an abelian group A) is defined as

$$C^n_{\mathrm{sing}}(X,A) = A^{X(\Delta_n)} = \{\text{maps } \alpha : X(\Delta_n) \longrightarrow A\}, \qquad (d^n\alpha)(\sigma) = \sum_{i=0}^n (-1)^i\, \alpha(\sigma \circ d_i).$$

The cohomology groups $H^n_{\text{sing}}(X,A) = H^n(C^\bullet_{\text{sing}}(X,A))$ are the **(singular) cohomology groups** of X with coefficients in A. If $f : X \longrightarrow Y$ is a continuous map and $(\alpha : Y(\Delta_n) \longrightarrow A) \in C^n_{\text{sing}}(Y,A)$, then the map $\alpha \circ f : (\sigma : \Delta_n \longrightarrow X) \mapsto \alpha(f \circ \sigma)$ is an element of $C^n_{\text{sing}}(X,A)$; this defines a morphism of complexes $C^\bullet_{\text{sing}}(Y,A) \longrightarrow C^\bullet_{\text{sing}}(X,A)$.

1.6.3. Exercise. Show that $H^0_{\text{sing}}(X,A) = A^{\pi_0(X)}$.

1.6.4. Definition (relative cohomology). In the situation of 1.6.1, the cohomology groups of the subcomplex

$$C^\bullet_{\text{sing}}(X,Y;A) = \operatorname{Ker}\left(\text{restriction} : C^\bullet_{\text{sing}}(X,A) \longrightarrow C^\bullet_{\text{sing}}(Y,A)\right)$$

of $C^\bullet_{\text{sing}}(X,A)$ are the **relative (singular) cohomology groups** $H^n_{\text{sing}}(X,Y;A) = H^n(C^\bullet_{\text{sing}}(X,Y;A))$ of the pair (X,Y) (with coefficients in A).

1.6.5. Functoriality. If $f : X \longrightarrow X'$ is a continuous map between topological spaces and $Y \subset X$, $Y' \subset X'$ are subsets such that $f(Y) \subseteq Y'$, then f induces morphisms of complexes

$$C^{\text{sing}}_\bullet(X,Y;A) \longrightarrow C^{\text{sing}}_\bullet(X',Y';A), \qquad C^\bullet_{\text{sing}}(X',Y';A) \longrightarrow C^\bullet_{\text{sing}}(X,Y;A). \tag{1.6.5.1}$$

If $g : X \longrightarrow X'$ is another continuous map and $H : X \times I \longrightarrow X'$ is a homotopy between f and g satisfying $H(Y \times I) \subseteq Y'$, then the morphisms (1.6.5.1) induced by f and g are homotopic (the proof of Theorem 1.5.5 works with minor modifications).

1.6.6. These complexes are related as follows:

$$\begin{aligned}
C^{\text{sing}}_\bullet(X,Y;A) &= C^{\text{sing}}_\bullet(X,A)/C^{\text{sing}}_\bullet(Y,A)\\
C^\bullet_{\text{sing}}(Y,A) &= C^\bullet_{\text{sing}}(X,A)/C^\bullet_{\text{sing}}(X,Y;A)\\
C^{\text{sing}}_\bullet(X,A) &= C^{\text{sing}}_\bullet(X,\mathbf{Z}) \otimes_{\mathbf{Z}} A\\
C^\bullet_{\text{sing}}(X,A) &= \operatorname{Hom}^\bullet_{\mathbf{Z}}(C^{\text{sing}}_\bullet(X,\mathbf{Z}),A)
\end{aligned}$$

(strictly speaking, the last equality holds only if we change the signs of differentials; see ?? below for the sign conventions).

1.6.7. As we shall see in ?? below, the relations 1.6.6 yield the following exact sequences (the last two of which are split):

$$\cdots \longrightarrow H^{\text{sing}}_n(Y,A) \longrightarrow H^{\text{sing}}_n(X,A) \longrightarrow H^{\text{sing}}_n(X,Y;A) \longrightarrow H^{\text{sing}}_{n-1}(Y,A) \longrightarrow H^{\text{sing}}_{n-1}(X,A) \longrightarrow \cdots$$
$$\cdots \longrightarrow H^{n-1}_{\text{sing}}(X,A) \longrightarrow H^{n-1}_{\text{sing}}(Y,A) \longrightarrow H^n_{\text{sing}}(X,Y;A) \longrightarrow H^n_{\text{sing}}(X,A) \longrightarrow H^n_{\text{sing}}(Y,A) \longrightarrow \cdots$$
$$0 \longrightarrow H^{\text{sing}}_n(X,\mathbf{Z}) \otimes_{\mathbf{Z}} A \longrightarrow H^{\text{sing}}_n(X,A) \longrightarrow \operatorname{Tor}^{\mathbf{Z}}_1(H^{\text{sing}}_{n-1}(X,\mathbf{Z}),A) \longrightarrow 0$$
$$0 \longrightarrow \operatorname{Ext}^1_{\mathbf{Z}}(H^{\text{sing}}_{n+1}(X,\mathbf{Z}),A) \longrightarrow H^n_{\text{sing}}(X,A) \longrightarrow \operatorname{Hom}_{\mathbf{Z}}(H^{\text{sing}}_n(X,\mathbf{Z}),A) \longrightarrow 0$$

2. Exactness

2.1. Exactness in the category of R-modules

In this section, R is a ring, a "module" is a left R-module, a "morphism" is a morphism of left R-modules.

2.1.1. Definition. Let $f : M \longrightarrow N$ be a morphism. The **kernel** of f is $\mathrm{Ker}(f) = \{m \in M \mid f(m) = 0\}$ (a submodule of M); the **image** of f is $\mathrm{Im}(f) = \{f(m) \mid m \in M\}$ (a submodule of N); the **cokernel** of f is $\mathrm{Coker}(f) = N/\mathrm{Im}(f)$ (a quotient module of N). We say that f is a **monomorphism** if $\mathrm{Ker}(f) = 0$ ($\iff$ f is injective); an **epimorphism** if $\mathrm{Coker}(f) = 0$ ($\iff$ f is surjective); an **isomorphism** if there is a morphism $g : N \longrightarrow M$ such that $fg = \mathrm{id}_N$ and $gf = \mathrm{id}_M$ ($\iff$ f is bijective $\iff$ f is both a mono and an epimorphism).

2.1.2. Definition. A sequence of morphisms

$$M^{i_0} \longrightarrow \cdots \longrightarrow M^{i-1} \xrightarrow{f_{i-1}} M^i \xrightarrow{f_i} M^{i+1} \longrightarrow \cdots \longrightarrow M^{i_1} \qquad (2.1.2.1)$$

is **exact** at the term M^i if $\mathrm{Im}(f_{i-1}) = \mathrm{Ker}(f_i)$ (which implies that $f_i f_{i-1} = 0$ and $\mathrm{Coker}(f_{i-1}) \xrightarrow{\sim} \mathrm{Im}(f_i)$); it is **exact** if it is exact at M^i for all $i_0 < i < i_1$. In particular, an acyclic complex is the same thing as an exact sequence indexed by $i \in \mathbf{Z}$.

2.1.3. Examples. (1) $0 \longrightarrow M \xrightarrow{f} N$ is exact $\iff$ $\mathrm{Ker}(f) = 0$ $\iff$ f is a monomorphism.
(2) $M \xrightarrow{f} N \longrightarrow 0$ is exact $\iff$ $\mathrm{Im}(f) = N$ $\iff$ f is an epimorphism.
(3) $0 \longrightarrow M \xrightarrow{f} N \longrightarrow 0$ is exact $\iff$ f is both mono end epi $\iff$ f is an isomorphism.
(4) If M is a submodule of N, then $0 \longrightarrow M \xrightarrow{i} N \xrightarrow{p} N/M \longrightarrow 0$ is exact, where i and p denote the inclusion and the canonical projection, respectively.
(5) Any "short exact sequence" $0 \longrightarrow M \xrightarrow{f} N \xrightarrow{g} P \longrightarrow 0$ is naturally isomorphic to a sequence of the form considered in (4).
(6) For any morphism $f : M \longrightarrow N$, the sequence $0 \longrightarrow \mathrm{Ker}(f) \longrightarrow M \xrightarrow{f} N \longrightarrow \mathrm{Coker}(f) \longrightarrow 0$ is exact.
(7) For any short exact sequence $0 \longrightarrow M \xrightarrow{f} N \xrightarrow{g} P \longrightarrow 0$ the following conditions are equivalent (if they are satisfied, we say that the sequence is **split**):
(a) there exists a morphism $s : P \longrightarrow N$ such that $gs = \mathrm{id}_P$;
(b) there exists a morphism $r : N \longrightarrow M$ such that $rf = \mathrm{id}_M$;
(c) the 7-tuple (M, N, P, f, g, r, s) is isomorphic to $(X_1, X_1 \oplus X_2, X_2, i_1, p_2, p_1, i_2)$, where $p_i(x_1, x_2) = x_i$, $i_1(x_1) = (x_1, 0)$ and $i_2(x_2) = (0, x_2)$.
(8) Any exact sequence (2.1.2.1) can be cut into short exact sequences

$$0 \longrightarrow N^i \xrightarrow{\alpha_i} M^i \xrightarrow{\beta_i} N^{i+1} \longrightarrow 0 \qquad (i_0 < i < i_1 - 1), \qquad N^i = \mathrm{Im}(f_{i-1}) = \mathrm{Ker}(f_i), \qquad \alpha_{i+1}\beta_i = f_i,$$

and vice versa.

2.1.4. The Snake Lemma. *Let*

$$\begin{array}{ccccccc}
(0 \longrightarrow) & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow 0 \\
 & \big\downarrow{\scriptstyle\alpha} & & \big\downarrow{\scriptstyle\beta} & & \big\downarrow{\scriptstyle\gamma} & \\
0 \longrightarrow & A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & (\longrightarrow 0)
\end{array}$$

be a commutative diagram with exact rows. Then there is an exact sequence

$$(0 \longrightarrow)\ \mathrm{Ker}(\alpha) \xrightarrow{f} \mathrm{Ker}(\beta) \xrightarrow{g} \mathrm{Ker}(\gamma) \xrightarrow{\Delta} \mathrm{Coker}(\alpha) \xrightarrow{f'} \mathrm{Coker}(\beta) \xrightarrow{g'} \mathrm{Coker}(\gamma)\ (\longrightarrow 0),$$

in which the non-obvious morphism $\Delta : \mathrm{Ker}(\gamma) \longrightarrow \mathrm{Coker}(\alpha)$ *is given by "*$\Delta = f'^{-1}\beta g^{-1}$*".*

Proof. We only give a definition of Δ and leave the verification of exactness as an exercise. Let $c \in \mathrm{Ker}(\gamma)$. As g is an epimorphism, there exists $b \in B$ such that $g(b) = c$. As $g'\beta(b) = \gamma g(b) = \gamma(c) = 0$, there exists $a' \in A'$ (unique, since f' is a monomorphism) such that $f'(a') = \beta(b)$. We wish to define $\Delta(c) :=$

$a' + \alpha(A) \in \mathrm{Coker}(\alpha)$. In order to check that this definition makes sense we must analyse what happens if we take two different elements $b_1, b_2 \in B$ satisfying $g(b_i) = c$. As $b_1 - b_2 \in \mathrm{Ker}(g) = \mathrm{Im}(f)$, we have $b_1 - b_2 = f(a)$ for some $a \in A$; thus the corresponding elements $a'_i \in A'$ (where $f'(a'_i) = \beta(b_i)$) satisfy $f'(a'_1 - a'_2) = \beta(b_1 - b_2) = \beta f(a) = f'\alpha(a)$, hence $a'_1 - a'_2 = \alpha(a) \in \alpha(A)$ (since f' is a monomorphism), which implies that $\Delta(c) := a'_1 + \alpha(A) = a'_2 + \alpha(A) \in \mathrm{Coker}(\alpha)$ is, indeed, independent of the choice of b. As Δ is a composition of possibly multivalued R-linear maps, it is also R-linear.

2.1.5. Example. Let $0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$ be an exact sequence of abelian groups. For any integer $m \geq 1$, the Snake Lemma applied to the diagram

$$\begin{array}{ccccccccc} 0 \longrightarrow & X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow 0 \\ & \big\downarrow m & & \big\downarrow m & & \big\downarrow m & \\ 0 \longrightarrow & X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow 0 \end{array}$$

(in which each vertical map is given by multiplication by m) gives an exact sequence

$$0 \longrightarrow {}_mX \longrightarrow {}_mY \longrightarrow {}_mZ \longrightarrow X/mX \longrightarrow Y/mY \longrightarrow Z/mZ \longrightarrow 0, \qquad (2.1.5.1)$$

where we have put, for any abelian group A,

$${}_mA = \{a \in A \mid ma = 0\}.$$

2.1.6. Exercise. *Let $A \xrightarrow{f} B \xrightarrow{g} C$ be morphisms; then there is an exact sequence*

$$0 \longrightarrow \mathrm{Ker}(f) \longrightarrow \mathrm{Ker}(gf) \longrightarrow \mathrm{Ker}(g) \longrightarrow \mathrm{Coker}(f) \longrightarrow \mathrm{Coker}(gf) \longrightarrow \mathrm{Coker}(g) \longrightarrow 0.$$

2.1.7. Proposition. *Any exact sequence of (cohomological) complexes $0 \longrightarrow A^\bullet \xrightarrow{f} B^\bullet \xrightarrow{g} C^\bullet \longrightarrow 0$ (in the sense of 3.1.2 below) gives rise to the following cohomology exact sequence*

$$\cdots \longrightarrow H^n(A^\bullet) \xrightarrow{H^n(f)} H^n(B^\bullet) \xrightarrow{H^n(g)} H^n(C^\bullet) \xrightarrow{\delta} H^{n+1}(A^\bullet) \xrightarrow{H^{n+1}(f)} H^{n+1}(B^\bullet) \xrightarrow{H^{n+1}(g)} H^{n+1}(C^\bullet) \longrightarrow \cdots,$$

in which $\delta[c] = [f^{-1}dg^{-1}(c)]$ (more precisely, if $c \in C^n$ and $dc = 0$, then there exist $b \in B^n$ and $a \in A^{n+1}$ such that $g(b) = c$ and $f(a) = db$; then $da = 0$ and $\delta[c] = [a]$).

Proof. The Snake Lemma applied to the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A^m & \xrightarrow{f_m} & B^m & \xrightarrow{g_m} & C^m & \longrightarrow & 0 \\ & & \big\downarrow d & & \big\downarrow d & & \big\downarrow d & & \\ 0 & \longrightarrow & A^{m+1} & \xrightarrow{f_{m+1}} & B^{m+1} & \xrightarrow{g_{m+1}} & C^{m+1} & \longrightarrow & 0 \end{array}$$

for $m = n \pm 1$ implies that the rows in the commutative diagram

$$\begin{array}{cccccccc} & & A^n/dA^{n-1} & \longrightarrow & B^n/dB^{n-1} & \longrightarrow & C^n/dC^{n-1} & \longrightarrow & 0 \\ & & \big\downarrow d & & \big\downarrow d & & \big\downarrow d & & \\ 0 & \longrightarrow & Z^{n+1}(A^\bullet) & \longrightarrow & Z^{n+1}(B^\bullet) & \longrightarrow & Z^{n+1}(C^\bullet) & & \end{array}$$

are exact. Applying the Snake Lemma to the latter diagram we obtain the desired result.

2.1.8. Example. Let $C^\bullet$ be a complex of torsion-free abelian groups. For any integer $m \geq 1$, the exact sequence of complexes

$$0 \longrightarrow C^\bullet \xrightarrow{m} C^\bullet \longrightarrow C^\bullet/mC^\bullet \longrightarrow 0$$

gives rise to a long exact sequence

$$\cdots \longrightarrow H^n(C^\bullet) \xrightarrow{m} H^n(C^\bullet) \longrightarrow H^n(C^\bullet/mC^\bullet) \longrightarrow H^{n+1}(C^\bullet) \xrightarrow{m} H^{n+1}(C^\bullet) \longrightarrow \cdots,$$

hence to short exact sequences

$$0 \longrightarrow H^n(C^\bullet)/mH^n(C^\bullet) \longrightarrow H^n(C^\bullet/mC^\bullet) \longrightarrow {}_mH^{n+1}(C^\bullet) \longrightarrow 0.$$

In particular, for $C^\bullet = C_\bullet^{\mathrm{sing}}(X,\mathbf{Z})$ we obtain (using the homological notation) the following special case $A = \mathbf{Z}/m\mathbf{Z}$ of the third exact sequence from 1.6.7:

$$0 \longrightarrow H_n^{\mathrm{sing}}(X,\mathbf{Z})/mH_n^{\mathrm{sing}}(X,\mathbf{Z}) \longrightarrow H_n^{\mathrm{sing}}(X,\mathbf{Z}/m\mathbf{Z}) \longrightarrow {}_mH_{n-1}^{\mathrm{sing}}(X,\mathbf{Z}) \longrightarrow 0.$$

2.1.9. The Five Lemma. *Let*

$$\begin{array}{ccccccccc} A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & A_4 & \longrightarrow & A_5 \\ \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \downarrow{\scriptstyle f_4} & & \downarrow{\scriptstyle f_5} \\ B_1 & \longrightarrow & B_2 & \longrightarrow & B_3 & \longrightarrow & B_4 & \longrightarrow & B_5 \end{array}$$

be a commutative diagram with exact rows. If f_1, f_2, f_4 and f_5 are isomorphisms, so is f_3.

Proof. Exercise.

2.1.10. Proposition. *Let*

$$\begin{array}{ccccccccc} 0 \longrightarrow & A^\bullet & \longrightarrow & B^\bullet & \longrightarrow & C^\bullet & \longrightarrow 0 \\ & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & \\ 0 \longrightarrow & A'^\bullet & \longrightarrow & B'^\bullet & \longrightarrow & C'^\bullet & \longrightarrow 0 \end{array}$$

be a commutative diagram of morphisms of (cohomological) complexes with exact rows. If two among the vertical morphisms are Qis, so is the third.

Proof. Combine the Five Lemma and the cohomology exact sequences

$$\begin{array}{ccccccccccc} H^n(A^\bullet) & \longrightarrow & H^n(B^\bullet) & \longrightarrow & H^n(C^\bullet) & \longrightarrow & H^{n+1}(A^\bullet) & \longrightarrow & H^{n+1}(B^\bullet) & \longrightarrow & H^{n+1}(C^\bullet) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H^n(A'^\bullet) & \longrightarrow & H^n(B'^\bullet) & \longrightarrow & H^n(C'^\bullet) & \longrightarrow & H^{n+1}(A'^\bullet) & \longrightarrow & H^{n+1}(B'^\bullet) & \longrightarrow & H^{n+1}(C'^\bullet). \end{array}$$

2.2. Limits and exactness in general categories

See [S1, §1.3, 1.5.4, 1.5.11, §3.4, §4.1, §5.1, 5.2.5, 5.2.6, §5.4] for general theory.

For example, the following facts are useful.

2.2.1. [Finite] inductive (resp., projective) limits exist in a given category $\mathcal{C}$ iff coequalisers (resp., equalisers) and [finite] sums (resp., products) exist in $\mathcal{C}$. More precisely, such limits are obtained as coequalisers (resp., equalisers) of a pair of morphisms between two [finite] sums (resp., products).

2.2.2. In ${}_R\mathrm{Mod}$, arbitrary inductive and projective limits exist; they commute with the forgetful functor ${}_R\mathrm{Mod} \longrightarrow \mathrm{Ab}$. Projective limits also commute with the forgetful functor ${}_R\mathrm{Mod} \longrightarrow \mathrm{Set}$.

2.2.3. A functor $F : \mathcal{C} \longrightarrow \mathcal{C}'$ that is left (resp., right) adjoint to another functor $\mathcal{C}' \longrightarrow \mathcal{C}$ commutes with any inductive (resp., projective) limits that exist. In particular, F is right (resp., left) exact if finite inductive (resp., projective) limits exist in $\mathcal{C}$.

2.2.4. Let I be a set such that the sums (resp., products) indexed by I exist in $\mathcal{C}$. It follows from 2.2.3 applied to the diagonal functor $\mathcal{C} \longrightarrow \mathcal{C}^I$ that the sum functor $\coprod_I : \mathcal{C}^I \longrightarrow \mathcal{C}$ (resp., the product functor $\prod_I : \mathcal{C}^I \longrightarrow \mathcal{C}$) is right (resp., left) exact, provided finite inductive (resp., projective) limits exist in $\mathcal{C}$.

2.2.5. More generally, if inductive (resp., projective) limits indexed by a small category I and finite inductive (resp., projective) limits exist in $\mathcal{C}$, then the inductive limit $\varinjlim_I : \mathcal{C}^I \longrightarrow \mathcal{C}$ (resp., projective limit $\varprojlim_I : \mathcal{C}^{I^{op}} \longrightarrow \mathcal{C}$) functor is right (resp., left) exact.

2.2.6. It follows from 2.2.1 that an additive functor $F : \mathcal{A} \longrightarrow \mathcal{A}'$ between two abelian categories is right (resp., left) exact iff it commutes with cokernels (resp., kernels).

2.2.7. For any ring R and a set I, the sum and product functors $\oplus_I$, $\prod_I : ({}_R\mathrm{Mod})^I \longrightarrow {}_R\mathrm{Mod}$ are exact.

2.2.8. For any ring R and a filtered small category I, the inductive limit $\varinjlim_I : ({}_R\mathrm{Mod})^I \longrightarrow {}_R\mathrm{Mod}$ is exact.

2.2.9. For any object X of an abelian category $\mathcal{A}$, the functor $Y \mapsto \mathrm{Hom}_{\mathcal{A}}(X, Y)$ (resp., $Y \mapsto \mathrm{Hom}_{\mathcal{A}}(Y, X)$) is a left (resp., right) exact additive functor $\mathcal{A} \longrightarrow \mathrm{Ab}$ (resp., $\mathcal{A}^{op} \longrightarrow \mathrm{Ab}$). It is exact iff X is a projective (resp., an injective) object of $\mathcal{A}$. Denote by $\mathcal{P}_{\mathcal{A}}$ (resp., $\mathcal{I}_{\mathcal{A}}$) the full additive subcategory of $\mathcal{A}$ whose objects are the projective (resp., the injective) objects of $\mathcal{A}$. [Note: the morphism $\mathbf{Z} \xrightarrow{2} \mathbf{Z}$ in $\mathcal{P}_{\mathrm{Ab}}$ has trivial kernel and cokernel, but is not an isomorphism; thus $\mathcal{P}_{\mathrm{Ab}}$ is not an abelian category.]

2.2.10. An additive functor $F : \mathcal{A} \longrightarrow \mathcal{A}'$ between two abelian categories that admits an exact left adjoint preserves injectives (i.e., $F(\mathcal{I}_{\mathcal{A}}) \subseteq \mathcal{I}_{\mathcal{A}'}$).

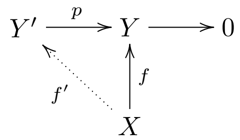
2.2.11. (Freyd-Mitchell embedding theorem). Any small abelian category $\mathcal{A}$ admits an exact fully faithful additive functor $F : \mathcal{A} \longrightarrow {}_R\mathrm{Mod}$ into a suitable category of modules.

2.2.12. It follows from 2.2.11 that everything in 2.1.2-2.1.10 (with the exception of 2.1.8) holds in an arbitrary abelian category, even though the proofs in §2.1 used explicit calculations with elements. See also [GM, Exercises to II.5] for an alternative approach.

2.3. Projective and injective modules

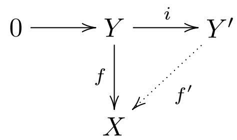
Let R be a ring. In this section we investigate basic properies of projective and injective objects in ${}_R\mathrm{Mod}$ (in particular, all R-modules will be left R-modules).

2.3.1. By definition, $X \in {}_R\mathrm{Mod}$ is **projective** iff for every epimorphism $p : Y' \longrightarrow Y$ in ${}_R\mathrm{Mod}$ and every morphism $f : X \longrightarrow Y$ there is a morphism $f' : X \longrightarrow Y'$ lifting f (i.e., $pf' = f$):



(2.3.1.1)

Similarly, X is **injective** iff for every monomorphism $i : Y \longrightarrow Y'$ in ${}_R\mathrm{Mod}$ and every morphism $f : Y \longrightarrow X$ there is a morphism $f' : Y' \longrightarrow X$ extending f (i.e., $f'i = f$):



(2.3.1.2)

2.3.2. Proposition. (1) *A direct sum of R-modules* $\bigoplus_I X_i$ *is projective* $\iff$ *each* X_i *is projective.*
(2) *Any free R-module* $R^{(I)}$ *is projective.*
(3) *For each R-module X there exists an epimorphism* $P \xrightarrow{p} X$ *in which P is a projective R-module ("the category* ${}_R\mathrm{Mod}$ *has enough projectives").*
(4) *Any short exact sequence* $0 \longrightarrow A \xrightarrow{i} B \xrightarrow{p} C \longrightarrow 0$ *in which C is a projective R-module splits.*
(5) *An R-module X is projective* $\iff$ *there is an R-module* X' *such that* $X \oplus X' = R^{(I)}$ *is free.*

Proof. (1) This follows from definition (2.3.1.1) and the universal property of $\bigoplus_I X_i$.
(2) Consider the diagram (2.3.1.1) with $X = R^{(I)}$. If $\{e_i\}$ $(i \in I)$ is the standard basis of $R^{(I)}$, choose $y_i' \in Y'$ such that $p(y_i') = f(e_i)$; the morphism $f'(\sum_{i\in I} r_i\, e_i) = \sum_{i\in I} r_i\, y_i'$ $(r_i \in R)$ then satisfies $pf' = f$.

(3) If $S \subseteq X$ is any subset that generates X as an R-module, then we can take $P = R^{(S)}$ and $p(\sum_{s \in S} r_s\, e_s) = \sum_{s \in S} r_s\, s$ $(r_s \in R)$.
(4) Consider the diagram (2.3.1.1) for the epimorphism $p : B \longrightarrow C$ and the identity map $f = \mathrm{id}_C : C \longrightarrow C$; the corresponding lift $f' : C \longrightarrow B$ then splits the given short exact sequence.
(5) The implication "$\Longleftarrow$" follows from (1) and (2). Conversely, If X is a projective R-module, construct an epimorphism $p : R^{(I)} \longrightarrow X$ from a suitable free module, as in the proof of (3). By (4), the exact sequence $0 \longrightarrow \mathrm{Ker}(p) \longrightarrow R^{(I)} \xrightarrow{p} X \longrightarrow 0$ splits, hence $R^{(I)} = X \oplus \mathrm{Ker}(p)$.

2.3.3. Exercise. Show that $\mathbf{Q}$ is not a subgroup of any free abelian group (in particular, $\mathbf{Q}$ is not projective in Ab).

2.3.4. Remarks. (1) If R is a PID (for example, $R = \mathbf{Z}$ or $K[x]$, where K is a field), then an R-module is projective iff it is free.
(2) Geometrically, projective modules of finite rank are attached to vector bundles. For example, if R is a commutative noetherian ring (with unit) and X is a finitely generated R-module, then X is projective $\Longleftrightarrow$ it is locally free on $\mathrm{Spec}(R)$ $\Longleftrightarrow$ its localisations $X_\mathfrak{m}$ at all maximal ideals $\mathfrak{m} \subset R$ are free $R_\mathfrak{m}$-modules.

2.3.5. Proposition. (1) *A direct product of R-modules $\prod_I X_i$ is injective $\Longleftrightarrow$ each X_i is injective.*
(2) *Any short exact sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ in which A is an injective R-module splits.*

Proof. Exercise (cf., 2.3.2(1),(4)).

2.3.6. Proposition (Baer's criterion). *An R-module X is injective $\Longleftrightarrow$ it has the extension property (2.3.1.2) when $Y' = R$ (i.e., $\Longleftrightarrow$ each morphism $I \longrightarrow X$, where $I \subset R$ is a left ideal, can be extended to a morphism $R \longrightarrow X$).*

Proof of "$\Longleftarrow$". (The implication "$\Longrightarrow$" is automatic). Assume that X has the extension property (2.3.1.2) with respect to all inclusions $I \subset R$ (where I is a left ideal of R). Given a monomorphism $i : Y \longrightarrow Y'$ (which we can, and will, consider to be the inclusion of a submodule) and a morphism $f : Y \longrightarrow X$, consider the set of all pairs (Z, g), where $Z \subset Y'$ is a submodule containing Y and $g : Z \longrightarrow X$ a morphism extending f (i.e., $g|_Y = f$). This set contains (Y, f) and is inductive with respect to the obvious partial order ($(g, Z) \leq (g', Z')$ iff $Z \subseteq Z'$ and $g'|_Z = g$), hence it contains a maximal element (Z, g), by Zorn's Lemma. If $Z = Y'$, then we are done. If $Z \subsetneq Y'$, fix $y' \in Y'$, $y' \notin Z$. The set $\{r \in R \mid ry' \in Z\}$ is a left ideal of R. By assumption, the morphism $I \longrightarrow X$ given by $r \mapsto g(ry')$ can be extended to a morphism $h : R \longrightarrow X$; the formula $g'(z + ry') = g(z) + h(r)$ then defines a morphism $g' : Z' = Z + Ry' \longrightarrow X$ extending g, which contradicts the maximality of (Z, g) (as $Z \subsetneq Z'$). This contradiction finishes the proof.

2.3.7. Corollary. *If R is a PID, then an R-module X is injective $\Longleftrightarrow$ it is R-divisible (i.e., $X = rX$ for all $r \in R - \{0\}$). In particular, the abelian groups $\mathbf{Q}$, $\mathbf{R}$, $\mathbf{Q}/\mathbf{Z} = \bigoplus_p \mathbf{Q}_p/\mathbf{Z}_p$, $\mathbf{Q}_p/\mathbf{Z}_p$ and $\mathbf{R}/\mathbf{Z}$ are injective in* Ab.

Proof. By 2.3.6, it is enough to verify the extension property (2.3.1.2) with respect to monomorphisms $i : R \xrightarrow{r} R$ given by multiplication by a non-zero element $r \in R$ (the case of $I = (0)$ is trivial).

2.3.8. Corollary. *If R is a field (not necessarily commutative), then each R-module is both projective and injective.*

2.3.9. Theorem. *For each R-module X there exists a monomorphism $X \longrightarrow I$ in which I is an injective R-module ("the category ${}_R$Mod has enough injectives").*

Proof. (1) Assume first that $R = \mathbf{Z}$, hence ${}_R\mathrm{Mod} = \mathrm{Ab}$. Given $X \in \mathrm{Ab}$, consider the set $D(X) := \mathrm{Hom}_\mathbf{Z}(X, \mathbf{Q}/\mathbf{Z})$ and the morphism

$$\alpha : X \longrightarrow \prod_{D(X)} \mathbf{Q}/\mathbf{Z} = (\mathbf{Q}/\mathbf{Z})^{D(X)}, \qquad \alpha(x)(f) = f(x) \qquad (f \in D(X)).$$

As $(\mathbf{Q}/\mathbf{Z})^{D(X)}$ is in jective by a combination of 2.3.5(1) and 2.3.7, it is enough to show that $\mathrm{Ker}(\alpha) = 0$. Assume that $x \in X$, $x \neq 0$; then $\mathbf{Z}x \subset X$ is a non-zero cyclic abelian group, which implies that $D(\mathbf{Z}x) \neq 0$.

Let $g : \mathbf{Z}x \longrightarrow \mathbf{Q}/\mathbf{Z}$ be a non-zero element of $D(\mathbf{Z}x)$; as $\mathbf{Q}/\mathbf{Z}$ is injective, there exists $f : X \longrightarrow \mathbf{Q}/\mathbf{Z}$ ($f \in D(X)$) extending g; then $\alpha(x)(f) = g(x) \neq 0$, hence $\alpha(x) \neq 0$. This proves that $\mathrm{Ker}(\alpha) = 0$.

(2) Let R be arbitrary. The exact forgetful functor ${}_R\mathrm{Mod} \longrightarrow \mathrm{Ab}$ is left adjoint to the functor

$$\mathrm{Ab} \longrightarrow {}_R\mathrm{Mod}, \qquad A \mapsto \mathrm{Hom}_{\mathbf{Z}}(R_R, A),$$

via the isomorphism

$$\begin{array}{rcl} \mathrm{Hom}_{\mathbf{Z}}(M, A) & \stackrel{\sim}{\longrightarrow} & \mathrm{Hom}_R(M, \mathrm{Hom}_{\mathbf{Z}}(R_R, A)) \qquad (M \in {}_R\mathrm{Mod},\ A \in \mathrm{Ab}) \\ f & \mapsto & F : (m \mapsto (r \mapsto f(rm)) \end{array}$$

(above, the notation R_R means that we consider R as a right module over itself, which makes $\mathrm{Hom}_{\mathbf{Z}}(R_R, A)$ into a left R-module in a natural way; see ??). By 2.2.10, the R-module $I := \mathrm{Hom}_{\mathbf{Z}}(R_R, \mathbf{Q}/\mathbf{Z})$ is injective.

(3) Given $X \in {}_R\mathrm{Mod}$, put $D_R(X) = \mathrm{Hom}_R(X, I)$ and define a morphism similar to that in (1):

$$\beta : X \longrightarrow \prod_{D_R(X)} I = I^{D_R(X)}, \qquad \beta(x)(F) = F(x) \qquad (F \in D_R(X)).$$

As in (1), $I^{D_R(X)}$ is an injective R-module, so it remains to show that $\mathrm{Ker}(\beta) = 0$. If $x \in X$, $x \neq 0$, then there exists a morphism $f \in \mathrm{Hom}_{\mathbf{Z}}(X, \mathbf{Q}/\mathbf{Z})$ such that $f(x) \neq 0$. The corresponding element

$$F \in \mathrm{Hom}_R(X, \mathrm{Hom}_{\mathbf{Z}}(R_R, \mathbf{Q}/\mathbf{Z})) = D_R(X)$$

then satisfies $\beta(x)(F) = F(x) : r \mapsto f(rx)$; in particular, $F(x)(1) = f(x) \neq 0$, hence $\beta(x) \neq 0$, as required.

3. Complexes in additive and abelian categories

3.1. Categories of complexes

3.1.1. The categories $C^*(\mathcal{A})$. Definition 1.2.1 of a (co)homological complex makes sense not only for R-modules, but for objects of an arbitrary **additive** category $\mathcal{A}$.

We denote by $C(\mathcal{A})$ the category whose objects are cohomological complexes $(C^\bullet, d)$ with all $C^n \in \mathcal{A}$ and whose morphisms are morphisms of complexes (as in 1.2.5). This is an additive category in which the finite direct sums are computed termwise ($(C \oplus C')^n = C^n \oplus C'^n$, $d_{C\oplus C'} = d_C \oplus d_{C'}$).

$C(\mathcal{A})$ contains as full subcategories various categories of complexes satisfying suitable boundedness conditions:

$$C^{\geq n}(\mathcal{A}) = \{C^\bullet \in C(\mathcal{A}) \mid \forall i < n \quad C^i = 0\}, \qquad C^{\leq n}(\mathcal{A}) = \{C^\bullet \in C(\mathcal{A}) \mid \forall i > n \quad C^i = 0\},$$
$$C^+(\mathcal{A}) = \bigcup_{n\in\mathbf{Z}} C^{\geq n}(\mathcal{A}), \qquad C^-(\mathcal{A}) = \bigcup_{n\in\mathbf{Z}} C^{\leq n}(\mathcal{A}), \qquad C^b(\mathcal{A}) = C^+(\mathcal{A}) \cap C^-(\mathcal{A})$$

(the last three consisting of complexes that are, respectively, bounded below, bounded above and bounded).

An additive functor $F : \mathcal{A} \longrightarrow \mathcal{A}'$ induces functors $C^*(F) : C^*(\mathcal{A}) \longrightarrow C^*(\mathcal{A}')$ (sometimes denoted, abusively, by F)

$$\left(\cdots \longrightarrow X^{n-1} \xrightarrow{d^{n-1}} X^n \xrightarrow{d^n} X^{n+1} \longrightarrow \cdots\right) \mapsto \left(\cdots \longrightarrow F(X^{n-1}) \xrightarrow{F(d^{n-1})} F(X^n) \xrightarrow{F(d^n)} F(X^{n+1}) \longrightarrow \cdots\right).$$

3.1.2. If $\mathcal{A}$ is an **abelian** category, so is each of the categories $C^*(\mathcal{A})$ ($* = \emptyset, +, -, b$), with kernels and cokernels computed termwise. This implies that a sequence of morphisms of complexes

$$0 \longrightarrow X^\bullet \xrightarrow{f} Y^\bullet \xrightarrow{g} Z^\bullet \longrightarrow 0 \tag{3.1.2.1}$$

is exact in $C(\mathcal{A})$ iff (= if and only if) the sequences

$$0 \longrightarrow X^n \xrightarrow{f_n} Y^n \xrightarrow{g_n} Z^n \longrightarrow 0 \tag{3.1.2.2}$$

are exact in $\mathcal{A}$, for all $n \in \mathbf{Z}$. We say that the sequence (3.1.2.1) in $\mathbf{C}(\mathcal{A})$ is **split exact** if all sequences (3.1.2.2) are split in $\mathcal{A}$ ($\iff$ $Y^n = X^n \oplus Z^n$). We do **not** require the splittings of (3.1.2.2) to be compatible with the differentials; in other words, $d_Y \neq d_X \oplus d_Z$, in general.

The **cohomology** of $X^\bullet \in C(\mathcal{A})$ in degree $n \in \mathbf{Z}$ is defined as in 1.2.3, namely as

$$H^n(X^\bullet) = \mathrm{Ker}(\mathrm{Coker}(X^{n-1} \xrightarrow{d^{n-1}} X^n) \xrightarrow{d^n} X^{n+1}) = \mathrm{Coker}(X^{n-1} \xrightarrow{d^{n-1}} \mathrm{Ker}(X^n \xrightarrow{d^n} X^{n+1})) \in \mathcal{A} \tag{3.1.2.3}$$

(the two expressions on the R.H.S. are canonically isomorphic to each other). The cohomology is an additive functor $H^n : C^*(\mathcal{A}) \longrightarrow \mathcal{A}$. As in 1.2.8, we say that a morphism $f : X^\bullet \longrightarrow Y^\bullet$ in $C(\mathcal{A})$ is a **quasi-isomorphism (Qis)** if $H^n(f) : H^n(X^\bullet) \longrightarrow H^n(Y^\bullet)$ is an isomorphism, for all $n \in \mathbf{Z}$.

Let $F : \mathcal{A} \longrightarrow \mathcal{A}'$ be an additive functor with values in another abelian category $\mathcal{A}'$. If F is **left exact** (resp., **right exact**), then it induces canonical morphisms in $\mathcal{A}'$

$$H^n(F(X^\bullet)) \longrightarrow F(H^n(X^\bullet)), \qquad \text{resp.,} \quad F(H^n(X^\bullet)) \longrightarrow H^n(F(X^\bullet)) \qquad\qquad (X^\bullet \in C(\mathcal{A}))$$

that give rise to a morphism of functors

$$H^n \circ C(F) \longrightarrow F \circ H^n, \qquad \text{resp.,} \quad F \circ H^n \longrightarrow H^n \circ C(F). \tag{3.1.2.4}$$

If F is **exact**, we obtain canonical isomorphisms in $\mathcal{A}'$

$$H^n(F(X^\bullet)) \xrightarrow{\sim} F(H^n(X^\bullet)) \qquad\qquad (X^\bullet \in C(\mathcal{A})).$$

3.1.3. The homotopy categories $K^*(\mathcal{A})$. Definition 1.5.8 of a homotopy between two morphisms of cohomological complexes makes sense in $C(\mathcal{A})$, for any additive category $\mathcal{A}$.

It is easy to see that, if two morphisms $f, g : X^\bullet \longrightarrow Y^\bullet$ in $C(\mathcal{A})$ are homotopic to each other ($f \simeq g$), so are $fu, gu : W^\bullet \longrightarrow Y^\bullet$ (resp., $vf, vg : X^\bullet \longrightarrow Z^\bullet$), for any morphism $u : W^\bullet \longrightarrow X^\bullet$ (resp., $v : Y^\bullet \longrightarrow Z^\bullet$) in $C(\mathcal{A})$, and also $f + w, g + w : X^\bullet \longrightarrow Y^\bullet$, for any $w : X^\bullet \longrightarrow Y^\bullet$.

This implies that it makes sense to define, for each $* = \emptyset, +, -, b$, the corresponding **homotopy category of complexes** $K^*(\mathcal{A})$ to have the same objects as $C^*(\mathcal{A})$, and morphisms given by

$$\mathrm{Hom}_{K^*(\mathcal{A})}(X^\bullet, Y^\bullet) = \mathrm{Hom}_{C^*(\mathcal{A})}(X^\bullet, Y^\bullet)/\simeq$$

(= the set of homotopy classes of morphisms of complexes from $X^\bullet$ to $Y^\bullet$). These categories are additive and $K^+(\mathcal{A}), K^-(\mathcal{A})$ and $K^b(\mathcal{A})$ are full subcategories of $K(\mathcal{A})$.

There are natural functors

$$\mathcal{A} \longrightarrow C^*(\mathcal{A}) \longrightarrow K^*(\mathcal{A})$$

(the first of which sends $X \in \mathcal{A}$ to the complex $X^\bullet$ with $X^0 = X$ and $X^n = 0$ for $n \neq 0$; the second sends $X^\bullet$ to $X^\bullet$). These functors embed $\mathcal{A}$ as a **full subcategory** of both $C^*(\mathcal{A})$ and $K^*(\mathcal{A})$, as

$$\forall X, Y \in \mathcal{A} \quad \mathrm{Hom}_{\mathcal{A}}(X, Y) = \mathrm{Hom}_{C(\mathcal{A})}(X, Y) = \mathrm{Hom}_{K(\mathcal{A})}(X, Y).$$

More precisely, we have, for reasons of degree,

$$\begin{aligned} (\forall X \in \mathcal{A})\ (\forall Y^\bullet \in C^{\geq 0}(\mathcal{A})) \quad \mathrm{Hom}_{C^+(\mathcal{A})}(X, Y^\bullet) &= \mathrm{Hom}_{K^+(\mathcal{A})}(X, Y^\bullet) \\ (\forall X \in \mathcal{A})\ (\forall Z^\bullet \in C^{\leq 0}(\mathcal{A})) \quad \mathrm{Hom}_{C^-(\mathcal{A})}(Z^\bullet, X) &= \mathrm{Hom}_{K^-(\mathcal{A})}(Z^\bullet, X). \end{aligned}$$

An additive functor $F : \mathcal{A} \longrightarrow \mathcal{A}'$ with values in another additive category $\mathcal{A}'$ preserves homotopies, hence induces additive functors $K^*(\mathcal{A}) \longrightarrow K^*(\mathcal{A}')$.

3.1.4. If $\mathcal{A}$ is an **abelian** category, then the statement of 1.5.8 can be rephrased by saying that the cohomology functors $H^n : C^*(\mathcal{A}) \longrightarrow \mathcal{A}$ factor through

$$C^*(\mathcal{A}) \longrightarrow K^*(\mathcal{A}) \xrightarrow{H^n} \mathcal{A}.$$

In particular, a **homotopy equivalence** of complexes (= a morphism of complexes $f : X^\bullet \longrightarrow Y^\bullet$ that admits a homotopy inverse, i.e., a morphism of complexes $g : Y^\bullet \longrightarrow X^\bullet$ such that there are homotopies $fg \simeq \mathrm{id}$, $gf \simeq \mathrm{id}$) is a Qis (see 1.5.6).

This is one of the reasons why it is useful to work with the homotopy category $K^*(\mathcal{A})$ rather than with the category of complexes $C^*(\mathcal{A})$. Another reason is the fact that certain constructions are not unique, but are unique up to homotopy (see 3.2 below).

3.1.5. Warning. For most abelian categories $\mathcal{A}$ the corresponding homotopy categories $K^*(\mathcal{A})$ are **not abelian**. For example, if $\mathcal{A} = \mathrm{Ab}$ is the category of abelian groups, then the morphism $\mathbf{Z} \xrightarrow{2} \mathbf{Z}$ does not have cokernel in $K(\mathrm{Ab})$.

3.1.6. Truncations. Let $\mathcal{A}$ be an abelian category. The naïve truncations of a complex

$$X^\bullet = [\cdots \longrightarrow X^{n-1} \longrightarrow X^n \longrightarrow X^{n+1} \longrightarrow \cdots] \in C(\mathcal{A})$$

are the complexes

$$\begin{aligned} \sigma_{\geq n} X^\bullet &= [\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow X^n \longrightarrow X^{n+1} \longrightarrow \cdots] \\ \sigma_{\leq n} X^\bullet &= [\cdots \longrightarrow X^{n-1} \longrightarrow X^n \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots] \end{aligned} \qquad (3.1.6.1)$$

(note that $\sigma_{\geq n} X^\bullet$ is a subcomplex of $X^\bullet$ and the quotient complex $X^\bullet / \sigma_{\geq n} X^\bullet$ is canonically isomorphic to $\sigma_{\leq n-1} X^\bullet$). More intelligent truncations are given by the formulas

$$\begin{aligned}\tau_{\geq n} X^\bullet &= [\cdots \longrightarrow 0 \longrightarrow X^n/dX^{n-1} \longrightarrow X^{n+1} \longrightarrow \cdots] \\ \tau_{\leq n} X^\bullet &= [\cdots \longrightarrow X^{n-1} \longrightarrow Z^n(X^\bullet) \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots]\end{aligned} \tag{3.1.6.2}$$

Note that $\tau_{\geq n} X^\bullet$ (resp., $\tau_{\leq n} X^\bullet$) is a quotient complex (resp., a subcomplex) of $X^\bullet$ and

$$H^m(\tau_{\geq n} X^\bullet) = \begin{cases} H^m(X^\bullet), & m \geq n \\ 0, & m < n, \end{cases} \qquad H^m(\tau_{\leq n} X^\bullet) = \begin{cases} H^m(X^\bullet), & m \leq n \\ 0, & m > n. \end{cases} \tag{3.1.6.3}$$

3.2. Derived functors (an informal introduction)

3.2.1. Assume that $F : \mathcal{A} \longrightarrow \mathcal{A}'$ is a right exact additive functor between two abelian categories. It is often the case that, for any exact sequence in $\mathcal{A}$

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0 \tag{3.2.1.1}$$

one can naturally extend the exact sequence in $\mathcal{A}'$

$$F(X) \longrightarrow F(Y) \longrightarrow F(Z) \longrightarrow 0$$

to a long exact sequence

$$\cdots \longrightarrow L_2F(Z) \longrightarrow L_1F(X) \longrightarrow L_1F(Y) \longrightarrow L_1F(Z) \longrightarrow F(X) \longrightarrow F(Y) \longrightarrow F(Z) \longrightarrow 0, \tag{3.2.1.2}$$

for suitable additive functors $L_nF : \mathcal{A} \longrightarrow \mathcal{A}'$.

3.2.2. Example. Fix an integer $m > 1$ and consider the functor

$$F = \bullet \otimes_{\mathbf{Z}} \mathbf{Z}/m\mathbf{Z} : \mathrm{Ab} \longrightarrow \mathrm{Ab}, \qquad X \mapsto X \otimes_{\mathbf{Z}} \mathbf{Z}/m\mathbf{Z} = X/mX.$$

As we shall see in ?? below, the exact sequence (2.1.5.1)

$$0 \longrightarrow {}_mX \longrightarrow {}_mY \longrightarrow {}_mZ \longrightarrow X/mX \longrightarrow Y/mY \longrightarrow Z/mZ \longrightarrow 0$$

is a special case of (3.2.1.2).

3.2.3. If the category $\mathcal{A}$ has **enough projectives** (in the sense that, for each $X \in \mathcal{A}$, there exists an epimorphism $P \longrightarrow X$ with $P \in \mathcal{P}_{\mathcal{A}}$), then the functors L_nF (the **left derived functors of** F) can be defined as follows: each object $X \in \mathcal{A}$ admits a **projective resolution**, namely a complex of projectives $P^\bullet \in C^{\leq 0}(\mathcal{P}_{\mathcal{A}})$ equipped with a Qis $P^\bullet \longrightarrow X$; then

$$L_nF(X) = H^{-n}(F(P^\bullet)). \tag{3.2.3.1}$$

Independence of $L_nF(X)$ on the choice of $P^\bullet$ and its functoriality in X follows from the fact that two projective resolutions of X are homotopically equivalent. More precisely, the rule $X \mapsto P^\bullet$ defines a "resolution functor" $\mathcal{A} \longrightarrow K^-(\mathcal{P}_{\mathcal{A}})$ and L_nF can be defined as the composition of functors

$$L_nF : \mathcal{A} \longrightarrow K^-(\mathcal{P}_{\mathcal{A}}) \xrightarrow{F} K^-(\mathcal{A}') \xrightarrow{H^{-n}} \mathcal{A}'. \tag{3.2.3.2}$$

If $X \in \mathcal{P}_{\mathcal{A}}$ is projective, then we can take $P^\bullet = X$, hence $L_nF(X) = 0$ for all $n > 0$.

As we shall see in ?? below, the left derived functors of the functor F from 3.2.2 are equal to $L_1F(X) = {}_mX$ and $L_nF = 0$ for $n > 1$.

3.2.4. In fact, the construction from 3.2.3 can be carried out not only for objects of $\mathcal{A}$, but for bounded below complexes $X^\bullet \in C^-(\mathcal{A})$ (still assuming that $\mathcal{A}$ has enough projectives): each such a complex $X^\bullet$

admits a Qis $P^\bullet \longrightarrow X^\bullet$, where $P^\bullet \in C^-(\mathcal{P}_\mathcal{A})$ is a bounded above complex of projectives, and the rule $X^\bullet \mapsto P^\bullet$ defines a functor $C^-(\mathcal{A}) \longrightarrow K^-(\mathcal{P}_\mathcal{A})$. The functors

$$\mathbf{L}_n F : C^-(\mathcal{A}) \longrightarrow K^-(\mathcal{P}_\mathcal{A}) \xrightarrow{F} K^-(\mathcal{A}') \xrightarrow{H^{-n}} \mathcal{A}' \tag{3.2.4.1}$$

are the **left hyper-derived functors of** F. For any short exact sequence of complexes in $C^-(\mathcal{A})$

$$0 \longrightarrow X^\bullet \longrightarrow Y^\bullet \longrightarrow Z^\bullet \longrightarrow 0$$

there is a long exact sequence in $\mathcal{A}'$

$$\cdots \longrightarrow \mathbf{L}_n F(X^\bullet) \longrightarrow \mathbf{L}_n F(Y^\bullet) \longrightarrow \mathbf{L}_n F(Z^\bullet) \longrightarrow \mathbf{L}_{n-1} F(X^\bullet) \longrightarrow \cdots. \tag{3.2.4.2}$$

Moreover,

$$H^n(X^\bullet) = 0 \ \ \forall n > m \implies \mathbf{L}_n F(X^\bullet) = 0 \ \ \forall n < -m. \tag{3.2.4.3}$$

The functor

$$C^-(\mathcal{A}) \longrightarrow K^-(\mathcal{P}_\mathcal{A}) \xrightarrow{F} K^-(\mathcal{A}')$$

factors through the homotopy category $K^-(\mathcal{A})$; the corresponding functor

$$\text{“}\mathbf{L}F\text{”} : K^-(\mathcal{A}) \longrightarrow K^-(\mathcal{P}_\mathcal{A}) \xrightarrow{F} K^-(\mathcal{A}') \tag{3.2.4.4}$$

is very close to the left derived functor $\mathbf{L}F$ of F in the sense of derived categories.

3.2.5. All of the above works thanks to the following key properties of the subcategory $C^-(\mathcal{P}_\mathcal{A}) \subset C^-(\mathcal{A})$:

3.2.5.1. $\forall X^\bullet \in C^-(\mathcal{A}) \ \exists P^\bullet \in C^-(\mathcal{P}_\mathcal{A}) \ \exists\, \text{Qis } P^\bullet \longrightarrow X^\bullet$.

3.2.5.2. If $f : P^\bullet \longrightarrow Q^\bullet$ is a Qis in $C^-(\mathcal{P}_\mathcal{A})$, then $F(f) : F(P^\bullet) \longrightarrow F(Q^\bullet)$ is a Qis in $C^-(\mathcal{A}')$.

3.2.5.3. If $f : X^\bullet \longrightarrow Y^\bullet$ is a morphism in $C^-(\mathcal{A})$ and $u : P^\bullet \longrightarrow X^\bullet$, $v : Q^\bullet \longrightarrow Y^\bullet$ are Qis with $P^\bullet, Q^\bullet \in C^-(\mathcal{P}_\mathcal{A})$, then there exists a morphism of complexes $f' : P^\bullet \longrightarrow Q^\bullet$ such that vf' is homotopic to fu. Moreover, f' is unique up to homotopy.

These properties, as well as (3.2.4.2-3), will be established in ?? below.

3.2.6. There is a dual theory of **right derived functors** for left exact functors $F : \mathcal{A} \longrightarrow \mathcal{A}'$, assuming that $\mathcal{A}$ has **enough injectives** (i.e., that for each $X \in \mathcal{A}$ there exists a monomorphism $X \longrightarrow I$ into an injective object $I \in \mathcal{I}_\mathcal{A}$). The properties 3.2.5.1-3 of the opposite category $\mathcal{A}^{\mathrm{op}}$ imply that

3.2.6.1. $\forall X^\bullet \in C^+(\mathcal{A}) \ \exists I^\bullet \in C^+(\mathcal{I}_\mathcal{A}) \ \exists\, \text{Qis } X^\bullet \longrightarrow I^\bullet$.

3.2.6.2. If $f : I^\bullet \longrightarrow J^\bullet$ is a Qis in $C^+(\mathcal{I}_\mathcal{A})$, then $F(f) : F(I^\bullet) \longrightarrow F(J^\bullet)$ is a Qis in $C^+(\mathcal{A}')$.

3.2.6.3. If $f : X^\bullet \longrightarrow Y^\bullet$ is a morphism in $C^+(\mathcal{A})$ and $u : X^\bullet \longrightarrow I^\bullet$, $v : Y^\bullet \longrightarrow J^\bullet$ are Qis with $I^\bullet, J^\bullet \in C^+(\mathcal{I}_\mathcal{A})$, then there exists a morphism of complexes $f' : I^\bullet \longrightarrow J^\bullet$ such that $f'u$ is homotopic to vf. Moreover, f' is unique up to homotopy.

3.2.7. Still assuming that $\mathcal{A}$ has enough injectives, the properties 3.2.6.1-3 imply that the rule $X^\bullet \mapsto I^\bullet$ defines a functor $C^+(\mathcal{A}) \longrightarrow K^+(\mathcal{I}_\mathcal{A})$. As in 3.2.3-4 we obtain the **right hyper-derived functors of** F

$$\mathbf{R}^n F : C^+(\mathcal{A}) \longrightarrow K^+(\mathcal{I}_\mathcal{A}) \xrightarrow{F} K^+(\mathcal{A}') \xrightarrow{H^n} \mathcal{A}', \qquad \mathbf{R}^n F(X^\bullet) = H^n(F(I^\bullet)) \tag{3.2.7.1}$$

giving rise to long exact sequences

$$\cdots \longrightarrow \mathbf{R}^n F(X^\bullet) \longrightarrow \mathbf{R}^n F(Y^\bullet) \longrightarrow \mathbf{R}^n F(Z^\bullet) \longrightarrow \mathbf{R}^{n+1} F(X^\bullet) \longrightarrow \cdots, \tag{3.2.7.2}$$

the **right derived functors of** F

$$R^n F : \mathcal{A} \longrightarrow C^+(\mathcal{A}) \longrightarrow K^+(\mathcal{I}_\mathcal{A}) \xrightarrow{F} K^+(\mathcal{A}') \xrightarrow{H^n} \mathcal{A}' \tag{3.2.7.3}$$

giving rise to long exact sequences

$$0 \longrightarrow F(X) \longrightarrow F(Y) \longrightarrow F(Z) \longrightarrow R^1F(X) \longrightarrow R^1F(Y) \longrightarrow R^1F(Z) \longrightarrow R^2F(X) \longrightarrow \cdots \qquad (3.2.7.4)$$

(for any exact sequence (3.2.1.1)) and the functors

$$\text{“}\mathbf{R}F\text{”} : K^+(\mathcal{A}) \longrightarrow K^+(\mathcal{I}_{\mathcal{A}}) \xrightarrow{F} K^+(\mathcal{A}'). \qquad (3.2.7.5)$$

3.2.8. In fact, everything in 3.2.4 (resp., in 3.2.7) works for an arbitrary additive functor $F : \mathcal{A} \longrightarrow \mathcal{A}'$. However, if F is not right exact (resp., left exact), then $L_0F \neq F$ (resp., $R^0F \neq F$) and F has to be replaced by L_0F (resp., by R^0F) in (3.2.1.2) (resp., in (3.2.7.4)).

3.3. The mapping cone

Let $\mathcal{A}$ be an additive category. In this section we explain one of the fundamental constructions in $C(\mathcal{A})$, which allows to deduce properties of morphisms in $C(\mathcal{A})$ from properties of objects (if $\mathcal{A}$ is abelian).

3.3.1. The shift of a complex. For $X \in C(\mathcal{A})$ and $n \in \mathbf{Z}$, we denote by $X[n] \in C(\mathcal{A})$ the following complex: $X[n]^i = X^{i+n}$ and $d^i_{X[n]} = (-1)^n d^{i+n}_X$. If $f : X \longrightarrow Y$ is a morphism in $\mathbf{C}(\mathcal{A})$, we define $f[n] : X[n] \longrightarrow Y[n]$ by $f[n]_i = f_{i+n} : X[n]^i = X^{i+n} \longrightarrow Y[n]^i = Y^{i+n}$. If $\mathcal{A}$ is abelian, then $H^i(X[n]) = H^{i+n}(X)$.

3.3.2. The mapping cone in topology. The **cone** over a topological space X is defined as the quotient $CX = X \times I/\sim$ of the cylinder $X \times I$ (where $I = [0,1]$) by the equivalence relation

$$(x,t) \sim (x',t') \iff t = t' = 1 \text{ or } (x,t) = (x',t').$$

We equip CX with the quotient topology and we view X as the bottom of CX via the inclusion map $i_0 : X \hookrightarrow CX$, $x \mapsto (x,0)$.

The **mapping cone** Cf of a continuous map $f : X \longrightarrow Y$ between topological spaces is obtained by glueing the space Y to the bottom of CX via the map f. In other words, Cf is defined as the inductive limit of the pair of arrows $CX \xleftarrow{i_0} X \xrightarrow{f} Y$ in the category Top of topological spaces. This means that Cf sits in a commutative diagram

$$\begin{array}{ccc} CX & \longrightarrow & Cf \\ \cup\!\!\uparrow & & \uparrow \\ X & \xrightarrow{f} & Y \end{array} \qquad (3.3.2.1)$$

and is universal among spaces having this property. In concrete terms, $Cf = (CX \coprod Y)/\sim'$ is the quotient of the disjoint union of CX and Y by the equivalence relation generated by $(x,0) \sim' f(x)$ $(x \in X)$.

3.3.3. Examples. (1) If $X \subset \mathbf{R}^N$ (with the induced topology), set $c = (0,\ldots,0,1) \in \mathbf{R}^N \oplus \mathbf{R} = \mathbf{R}^{N+1}$ and denote by $C_{geom}X = \{tc + (1-t)x \mid x \in X,\, \lambda \in [0,1]\} \subset \mathbf{R}^{N+1}$ the **geometric cone** over X (with the topology induced from $\mathbf{R}^{N+1}$). The map

$$C_{geom} \longrightarrow CX, \qquad tc + (1-t)x \mapsto \text{ the class of } (x,t)$$

is a continuous bijection, which is a homeomorphism if X is compact, but not in general, for example, for $X = \mathbf{R}^N$ (exercise!).
(2) The cone (and the geometric cone) over the unit sphere $S^n = \{x \in \mathbf{R}^{n+1} \mid \|x\|^2 = x_1^2 + \cdots + x_{n+1}^2 = 1\}$ is homeomorphic to the closed unit ball $D^{n+1} = \{x \in \mathbf{R}^{n+1} \mid \|x\|^2 \leq 1\}$ and also to the upper hemisphere $S^{n+1}_+ = \{x \in \mathbf{R}^{n+2} \mid \|x\|^2 = 1,\, x_{n+2} \geq 0\}$.
(3) The real projective space $\mathbf{P}^n(\mathbf{R}) = S^n/\{\pm 1\}$ can be obtained from the sphere by identifying each point $x \in S^n$ with its antipode $-x$. Any element of $\mathbf{P}^n(\mathbf{R})$ can be represented by a point on the upper hemisphere $x \in S^n_+$, and $x \neq y \in S^n_+$ have the same image in $\mathbf{P}^n(\mathbf{R})$ iff $y = -x \in \partial S^n_+ = S^{n-1} \times \{0\}$. Denote by

$f_n : S^n \longrightarrow \mathbf{P}^n(\mathbf{R})$ the canonical projection; the previous discussion implies that Cf_n is homeomorphic to $\mathbf{P}^{n+1}(\mathbf{R})$.

3.3.4. The mapping cone of a simplicial map. If $X = |K|$ for some (finite) simplicial complex K in $\mathbf{R}^N$, then $CX = C_{geom}X = |CK|$, where CK is the following simplicial complex in $\mathbf{R}^{N+1}$:

$$(CK)_n = K_n \cup \{C\sigma \mid \sigma \in K_{n-1}\}, \qquad C[v_0, \ldots, v_{n-1}] = [c, v_0, \ldots, v_{n-1}]$$

(using the notation from 3.3.3(1)). If K is ordered, we order CK by $c < v$ for any $v \in K_0$; then

$$\partial(C\sigma) = -C(\partial\sigma) + \sigma, \qquad \sigma = [v_0 < \cdots < v_{n-1}] \in K_{n-1},$$

which implies that

$$C_n(CK, A) = C_{n-1}(K, A) \oplus C_n(K, A), \qquad \partial_n(\sigma, \tau) = (-\partial_{n-1}(\sigma),\, \sigma + \partial_n(\tau)). \qquad (3.3.4.1)$$

Assume, in addition, that $Y = |L|$ for some (finite) simplicial complex L and $f : |K| \longrightarrow |L|$ is a simplicial map (see 1.3.10 above). Even though the glued object Cf will not necessarily come from from a simplicial complex, it is natural to define $C_\bullet(Cf, A)$ as the inductive limit of the pair of morphisms of complexes $C_\bullet(CK, A) \xleftarrow{i_0} C_\bullet(K, A) \xrightarrow{f} C_\bullet(L, A)$. The formula (3.3.4.1) implies that

$$C_n(Cf, A) = C_{n-1}(K, A) \oplus C_n(L, A), \qquad \partial(\sigma, \tau) = (-\partial(\sigma),\, f(\sigma) + \partial(\tau)), \qquad (3.3.4.2)$$

where $f(\sigma)$ is defined as in 1.3.10.

3.3.5. The mapping cone of a morphism of complexes. After passing to a cohomological notation, the formula (3.3.4.2) yields the following abstract algebraic definition: let $f : X \longrightarrow Y$ be a morphism in $C(\mathcal{A})$, where $\mathcal{A}$ is an additive category. The **mapping cone of** f is the following complex $\mathrm{Cone}(f) \in C(\mathcal{A})$:

$$\mathrm{Cone}(f)^n = X^{n+1} \oplus Y^n \longrightarrow \mathrm{Cone}(f)^{n+1} = X^{n+2} \oplus Y^{n+1}, \qquad d(x, y) = (-dx, f(x) + dy).$$

Alternatively, writing the elements of $\mathrm{Cone}(f)^n = X^{n+1} \oplus Y^n = X[1]^n \oplus Y^n$ as column vectors, we have

$$d_{\mathrm{Cone}(f)} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -d_X & 0 \\ f & d_Y \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} d_{X[1]} & 0 \\ f & d_Y \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Note that $\mathrm{Cone}(f)$ is, indeed, a complex, as

$$dd(x, y) = d(-dx, f(x) + dy) = (ddx, f(-dx) + df(x) + ddy) = 0.$$

If $F : \mathcal{A} \longrightarrow \mathcal{A}'$ is an additive functor into another additive category, then $F(\mathrm{Cone}(f)) = \mathrm{Cone}(F(f))$.

3.3.6. Proposition. *Let $f : X \longrightarrow Y$ be a morphism in $C(\mathcal{A})$, where $\mathcal{A}$ is an additive category. The formulas $i(y) = (0, y)$ and $p(x, y) = x$ define morphisms of complexes $i : Y \longrightarrow \mathrm{Cone}(f)$ and $p : \mathrm{Cone}(f) \longrightarrow X[1]$ satisfying $pi = 0$. If $\mathcal{A}$ is an* **abelian** *category, then i and p give rise to a split short exact sequence of complexes*

$$0 \longrightarrow Y \xrightarrow{i} \mathrm{Cone}(f) \xrightarrow{p} X[1] \longrightarrow 0.$$

For each $n \in \mathbf{Z}$, the coboundary morphism $H^{n-1}(X[1]) = H^n(X) \xrightarrow{\partial} H^n(Y)$ is equal to the morphism $H^n(f)$ induced by f.

Proof. The split exactness is clear, once we verify that both i and p are morphisms of complexes, which boils down to the formulas $di(y) = d(0, y) = (0, dy) = id(y)$ and $dp(x, y) = d_{X[1]}(x) = p(d_{X[1]}(x), f(x) + dy) = pd(x, y)$. Finally, for any $x \in X^n$ satisfying $dx = 0$, we have

$$\partial[x] = [i^{-1} d_{\mathrm{Cone}(f)} p^{-1}(x)] = [i^{-1} d(x, 0)] = [i^{-1}(0, f(x)] = [f(x)] = H^n(f)[x].$$

3.3.7. Corollary. *There is an exact sequence*

$$\cdots \longrightarrow H^{n-1}(\mathrm{Cone}(f)) \xrightarrow{H^n(p)} H^n(X) \xrightarrow{H^n(f)} H^n(Y) \xrightarrow{H^n(i)} H^n(\mathrm{Cone}(f)) \longrightarrow \cdots$$

3.3.8. Corollary. *f is a Qis ($\iff$ $H^n(f)$ is an isomorphism for all $n \in \mathbf{Z}$) $\iff$ $\mathrm{Cone}(f)$ is acyclic ($\iff$ $H^n(\mathrm{Cone}(f)) = 0$ for all $n \in \mathbf{Z}$).*

3.3.9. Functoriality of $\mathrm{Cone}(f)$. Let

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \big\downarrow g & & \big\downarrow h \\ X' & \xrightarrow{f'} & Y' \end{array}$$

be a diagram of morphisms in $\mathbf{C}(\mathcal{A})$ (where $\mathcal{A}$ is an additive category) which is commutative up to homotopy. Fix a homotopy a between $f'g$ and hf (i.e., a collection of morphisms $a_n : X^n \longrightarrow Y'^{n-1}$ in $\mathcal{A}$ such that $da + ad = hf - f'g$). The formula

$$\mathrm{Cone}(f)^n = X^{n+1} \oplus Y^n \longrightarrow \mathrm{Cone}(f')^n = X'^{n+1} \oplus Y'^n, \qquad (x, y) \mapsto (g_{n+1}(x), a_{n+1}(x) + h_n(y))$$

then defines a morphism of complexes

$$\lambda = \mathrm{Cone}(g, h, a) = \begin{pmatrix} g & 0 \\ a & h \end{pmatrix} : \mathrm{Cone}(f) \longrightarrow \mathrm{Cone}(f') \qquad (3.3.9.1)$$

which fits into the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \longrightarrow & \mathrm{Cone}(f) & \longrightarrow & X[1] & \longrightarrow & 0 \\ & & \big\downarrow h & & \big\downarrow \lambda & & \big\downarrow g[1] & & \\ 0 & \longrightarrow & Y' & \longrightarrow & \mathrm{Cone}(f') & \longrightarrow & X'[1] & \longrightarrow & 0. \end{array} \qquad (3.3.9.2)$$

In particular, if both g and h are Qis, so is λ (by 2.1.10).

If a' is another homotopy between $f'g$ and hf, the morphism $\mathrm{Cone}(g, h, a')$ is **not** necessarily homotopic to $\mathrm{Cone}(g, h, a)$ (the two morphisms are homotopic, for example, if there exists a “second order homotopy” between the homotopies a and a'). As a result, Cone **does not** define a functor from $\mathrm{Mor}(K(\mathcal{A}))$ (the category of morphisms in $K(\mathcal{A})$) to $K(\mathcal{A})$, which causes problems in the theory of derived categories.

3.3.10. Morphisms $\mathrm{Cone}(f) \longrightarrow Z$. In the topological situation of (3.3.2.1), a continuous map $Cf \longrightarrow Z$ (where Z is an arbitrary topological space) is defined by a pair of continuous maps

$$g : Y \longrightarrow Z, \qquad h : CX \longrightarrow Z, \qquad gf = hi_0.$$

The latter condition simply says that h defines a homotopy between $gf : X \longrightarrow Z$ and a constant map.

In the algebraic situation of 3.3.5, a morphism of complexes $\mathrm{Cone}(f) \longrightarrow Z$ (where $Z \in C(\mathcal{A})$) is given by a collection of maps

$$(h_{n+1}, g_n) : \mathrm{Cone}(f)^n = X^{n+1} \oplus Y^n \longrightarrow Z^n, \qquad (x, y) \mapsto h_{n+1}(x) + g_n(y)$$

that commute with the differential, i.e., such that

$$(dh, dg) = d(h, g) = (h, g)\begin{pmatrix} -d & 0 \\ f & d \end{pmatrix} = (-hd + gf, gd) \iff dg = gd, \qquad dh + hd = gf.$$

In other words, $g : Y \longrightarrow Z$ is a morphism of complexes and h is a homotopy between 0 and $gf : X \longrightarrow Z$. If we denote the corresponding morphism $\mathrm{Cone}(f) \longrightarrow Z$ by (h, g), then $g = (h, g) \circ i$, in the notation of 3.3.6.

The previous discussion can be summed up by saying that a morphism of complexes $g : Y \longrightarrow Z$ factors through $i : Y \longrightarrow \mathrm{Cone}(f)$ iff $gf : X \longrightarrow Z$ is homotopic to 0. This can be reformulated in more abstract terms as follows:

3.3.11. Proposition. *Let $f : X \longrightarrow Y$ be a morphism in $C(\mathcal{A})$, where $\mathcal{A}$ is an additive category. Then, for any $Z \in C(\mathcal{A})$, the sequences of abelian groups*

$$\mathrm{Hom}_{C(\mathcal{A})}(\mathrm{Cone}(f), Z) \xrightarrow{\circ i} \mathrm{Hom}_{C(\mathcal{A})}(Y, Z) \xrightarrow{\circ f} \mathrm{Hom}_{K(\mathcal{A})}(X, Z)$$
$$\mathrm{Hom}_{K(\mathcal{A})}(\mathrm{Cone}(f), Z) \xrightarrow{\circ i} \mathrm{Hom}_{K(\mathcal{A})}(Y, Z) \xrightarrow{\circ f} \mathrm{Hom}_{K(\mathcal{A})}(X, Z)$$

are exact.

3.3.12. The mapping cylinder in topology. In the situation of 3.3.2, the **cylinder** over a topological space X is the product $X \times I$, and the **mapping cylinder** of a continuous map $f : X \longrightarrow Y$ is the inductive limit of the pair of arrows $X \times I \xleftarrow{i_0} X \xrightarrow{f} Y$; it sits in a commutative diagram

$$\begin{array}{ccc} X \times I & \longrightarrow & \mathrm{Cyl}(f) \\ \uparrow & & \uparrow \\ X & \xrightarrow{f} & Y \end{array}$$

and is universal among spaces having this property. The inclusion $Y \hookrightarrow \mathrm{Cyl}(f)$ is a homotopy equivalence, with homotopy inverse defined by the pair $fp : X \times I \longrightarrow Y$, $\mathrm{id} : Y \longrightarrow Y$ (where $p : X \times I \longrightarrow X$ is the projection map $p(x,t) = x$). Moreover, the composite map $X \xrightarrow{i_1} X \times I \longrightarrow \mathrm{Cyl}(f)$ (where $i_1(x) = (x, 1)$) is injective.

As the cylinder $X \times I$ differs from the cone CX only at the top level $X \times \{1\}$, it is natural to translate the topological construction of $\mathrm{Cyl}(f)$ into the algebraic context of 3.3.5.

3.3.13. The mapping cylinder of a morphism of complexes. Let $f : X \longrightarrow Y$ be a morphism in $C(\mathcal{A})$, where $\mathcal{A}$ is an additive category. The **mapping cylinder of** f is the following complex $\mathrm{Cyl}(f) \in C(\mathcal{A})$:

$$\mathrm{Cyl}(f)^n = X^n \oplus X^{n+1} \oplus Y^n \longrightarrow \mathrm{Cyl}(f)^{n+1} = X^{n+1} \oplus X^{n+2} \oplus Y^{n+1}, \qquad d(x', x, y) = (dx' - x, -dx, f(x) + dy).$$

3.3.14. Proposition. *Let $f : X \longrightarrow Y$ be a morphism in $C(\mathcal{A})$, where $\mathcal{A}$ is an additive category. The formulas $\overline{f}(x) = (x, 0, 0)$, $\pi(x', x, y) = (x, y)$, $\alpha(y) = (0, 0, y)$, $\beta(x', x, y) = f(x') + y$, $h(x', x, y) = (0, x', 0)$ define a commutative diagram in $C(\mathcal{A})$*

$$\begin{array}{ccccccccc} & & 0 & \longrightarrow & Y & \xrightarrow{i} & \mathrm{Cone}(f) & \xrightarrow{p} & X[1] & \longrightarrow & 0 \\ & & & & \downarrow{\scriptstyle\alpha} & & \| & & & & \\ 0 & \longrightarrow & X & \xrightarrow{\overline{f}} & \mathrm{Cyl}(f) & \xrightarrow{\pi} & \mathrm{Cone}(f) & \longrightarrow & 0 & & \\ & & \| & & \downarrow{\scriptstyle\beta} & & & & & & \\ & & X & \xrightarrow{f} & Y & & & & & & \end{array}$$

and a homotopy $h : \mathrm{id}_{\mathrm{Cyl}(f)} \simeq \alpha\beta$; moreover, $\beta\alpha = \mathrm{id}_Y$ (hence β is a homotopy inverse of α). If $\mathcal{A}$ is an abelian category, then the first two rows of the diagram are exact and the morphisms α, β are Qis.

Proof. An easy calculation shows that all the maps are morphisms of complexes, the diagram is commutative and h is a homotopy between id and $\alpha\beta$. If $\mathcal{A}$ is abelian, the exactness of the second row follows from the definitions; as α, β are homotopy inverse to each other, they are both Qis (see the proof of 1.5.6).

3.3.15. Proposition. *Let $\mathcal{A}$ be an abelian category. If $0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$ is an exact sequence in $C(\mathcal{A})$, then there is a commutative diagram with exact rows, in which all vertical arrows are Qis:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \longrightarrow & 0 \\ & & \| & & \uparrow{\scriptstyle\beta} & & \uparrow{\scriptstyle\gamma} & & \\ 0 & \longrightarrow & X & \xrightarrow{\overline{f}} & \mathrm{Cyl}(f) & \xrightarrow{\pi} & \mathrm{Cone}(f) & \longrightarrow & 0 \end{array}$$

Proof. The formula $\gamma(x,y) = g(y)$ defines a morphism of complexes $\gamma : \mathrm{Cone}(f) \longrightarrow Z$, as $\gamma d(x,y) = \gamma(-dx, f(x)+dy) = gf(x) + g(dy) = g(dy) = d\gamma(x,y)$. The first (resp., the second) square is commutative, by 3.3.14 (resp., since $\gamma\pi(x',x,y) = \gamma(x,y) = g(y) = g(f(x')+y) = g\beta(x',x,y)$). As id_X and β are Qis (the latter by 3.3.14), the Five Lemma implies that γ is also a Qis.

3.3.16. Functoriality of $\mathrm{Cyl}(f)$**.** In the situation of 3.3.9, the formula

$$\mathrm{Cyl}(f)^n = X^n \oplus X^{n+1} \oplus Y^n \longrightarrow \mathrm{Cyl}(f')^n = X'^n \oplus X'^{n+1} \oplus Y'^n, \qquad (x,y) \mapsto (g_n(x), g_{n+1}(x), a_{n+1}(x) + h_n(y))$$

defines a morphism of complexes

$$\mathrm{Cyl}(g,h,a) = \begin{pmatrix} g & 0 & 0 \\ 0 & g & 0 \\ 0 & a & h \end{pmatrix} : \mathrm{Cyl}(f) \longrightarrow \mathrm{Cyl}(f') \tag{3.3.16.1}$$

which makes, together with the morphism (3.3.9.1), the following diagram commutative:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \xrightarrow{\overline{f}} & \mathrm{Cyl}(f) & \xrightarrow{\pi} & \mathrm{Cone}(f) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle g} & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & X' & \xrightarrow{\overline{f}'} & \mathrm{Cyl}(f') & \xrightarrow{\pi'} & \mathrm{Cone}(f') & \longrightarrow & 0. \end{array} \tag{3.3.16.2}$$

3.4. Double complexes

Let $\mathcal{A}$ be an additive category.

3.4.1. Definition. A **double complex (= bicomplex)** $K^{\bullet,\bullet}$ in $\mathcal{A}$ is a collection of objects $K^{i,j} \in \mathcal{A}$ ($i,j \in \mathbf{Z}$) and morphisms $d_I^{ij} : K^{i,j} \longrightarrow K^{i+1,j}$, $d_{II}^{ij} : K^{i,j} \longrightarrow K^{i,j+1}$ such that $d_I^2 = 0$, $d_{II}^2 = 0$, $d_I d_{II} + d_{II} d_I = 0$.

3.4.2. The total complex of a bicomplex (two versions). (1) The **small** total complex of $K^{\bullet,\bullet}$ is the complex $L^\bullet = \mathrm{Tot}^{\oplus}(K^{\bullet,\bullet})$, where $L^n = \bigoplus_{i+j=n} K^{i,j}$ (assuming the direct sum exists in $\mathcal{A}$) and the differential is equal to $d = d_I + d_{II}$.
(2) The **big** total complex of $K^{\bullet,\bullet}$ is the complex $M^\bullet = \mathrm{Tot}^{\Pi}(K^{\bullet,\bullet})$, where $M^n = \prod_{i+j=n} K^{i,j}$ (assuming the direct product exists in $\mathcal{A}$) and the differential is equal to $d = d_I + d_{II}$.

3.4.3. Example (the Dolbeault bicomplex). Let X be a complex manifold of dimension n (for example, an non-empty open subset of $\mathbf{C}^n$) with local cordinates $z_1, \ldots, z_n$. The Dolbeault bicomplex of X is formed by smooth complex-valued differential forms on X with a fixed number of holomorphic and anti-holomorphic terms:

$$A^{p,q}(X) = \Big\{ \sum_{\substack{|I|=p \\ |J|=q}} f_{IJ}\, dz_I \wedge d\overline{z}_J \mid f_{IJ} \in C^\infty(X, \mathbf{C}) \Big\}, \qquad d_I = \partial, \qquad d_{II} = \overline{\partial},$$

where, for $I = \{i_1 < \cdots < i_p\} \subseteq \{1, \ldots, n\}$,

$$dz_I = dz_{i_1} \wedge \cdots \wedge dz_{i_p}, \qquad d\overline{z}_I = d\overline{z}_{i_1} \wedge \cdots \wedge d\overline{z}_{i_p},$$

$$\partial(f\, dz_I \wedge d\overline{z}_J) = \sum_{i=1}^n \frac{\partial f}{\partial z_i}\, dz_i \wedge dz_I \wedge d\overline{z}_J, \qquad \overline{\partial}(f\, dz_I \wedge d\overline{z}_J) = \sum_{i=1}^n \frac{\partial f}{\partial \overline{z}_i}\, d\overline{z}_i \wedge dz_I \wedge d\overline{z}_J.$$

The corresponding total complex $\mathrm{Tot}^{\Pi}(A^{\bullet,\bullet}(X)) = \mathrm{Tot}^{\oplus}(A^{\bullet,\bullet}(X)) \xrightarrow{\sim} A^\bullet(X) \otimes_{\mathbf{R}} \mathbf{C}$ is canonically isomorphic to the de Rham complex of X with complex coefficients, as

$$A^n(X) \otimes_{\mathbf{R}} \mathbf{C} = \bigoplus_{p+q=n} A^{p,q}(X), \qquad d = \partial + \overline{\partial}.$$

3.4.4. Another sign convention. One can often find in the literature an alternative sign convention for double complexes, namely that of morphisms $\widetilde{d}_I^{ij} : K^{i,j} \longrightarrow K^{i+1,j}$, $\widetilde{d}_{II}^{ij} : K^{i,j} \longrightarrow K^{i,j+1}$ satisfying $\widetilde{d}_I^2 = 0$, $\widetilde{d}_{II}^2 = 0$, $\widetilde{d}_I \widetilde{d}_{II} = \widetilde{d}_{II} \widetilde{d}_I$. One can pass between 3.4.1 and 3.4.4 by putting, for example,

$$d_I = \widetilde{d}_I, \qquad d_{II}^{i,j} = (-1)^i\, \widetilde{d}_{II}^{i,j}. \tag{3.4.4.1}$$

3.4.5. The tensor product of complexes. The tensor product $X^\bullet \otimes_R Y^\bullet \in C({}_{Z(R)}\mathrm{Mod})$ of two complexes $X^\bullet \in C(\mathrm{Mod}_R)$ and $Y^\bullet \in C({}_R\mathrm{Mod})$ is the small total complex associated to the double complex

$$(X^i \otimes_R Y^j, \quad \widetilde{d}_I = d_X \otimes \mathrm{id}, \quad \widetilde{d}_{II} = \mathrm{id} \otimes d_Y)$$

using the sign rule (3.4.4.1):

$$(X^\bullet \otimes_R Y^\bullet)^n = \bigoplus_{i+j=n} X^i \otimes_R Y^j, \qquad d(x \otimes y) = dx \otimes y + (-1)^i\, x \otimes dy \qquad (x \in X^i). \tag{3.4.5.1}$$

If $f : X^\bullet \longrightarrow X'^\bullet$ is a morphism in $C(\mathrm{Mod}_R)$, then the formula $(x, x') \otimes y \mapsto (x \otimes y, x' \otimes y)$ defines an isomorphism of complexes

$$\mathrm{Cone}(X^\bullet \xrightarrow{f} X'^\bullet) \otimes_R Y^\bullet \xrightarrow{\sim} \mathrm{Cone}(X^\bullet \otimes_R Y^\bullet \xrightarrow{f \otimes \mathrm{id}} X'^\bullet \otimes_R Y^\bullet). \tag{3.4.5.2}$$

If the ring R is commutative, then $\mathrm{Mod}_R = {}_R\mathrm{Mod}$ and the formula

$$x \otimes y \mapsto (-1)^{ij}\, y \otimes x \qquad (x \in X^i,\ y \in Y^j)$$

defines an isomorphism of complexes

$$s_{12} : X^\bullet \otimes_R Y^\bullet \xrightarrow{\sim} Y^\bullet \otimes_R X^\bullet. \tag{3.4.5.3}$$

3.4.6. The $\mathrm{Hom}^\bullet$-complex. For $X^\bullet, Y^\bullet \in C(\mathcal{A})$ (where $\mathcal{A}$ is an additive category) we define

$$\mathrm{Hom}^\bullet_{\mathcal{A}}(X^\bullet, Y^\bullet) = \mathrm{Tot}^{\Pi}((\mathrm{Hom}_{\mathcal{A}}(X^{-i}, Y^j))_{i,j}) \in C(\mathrm{Ab})$$

with the following sign rules:

$$f \in \mathrm{Hom}^n_{\mathcal{A}}(X^\bullet, Y^\bullet) = \prod_{i \in \mathbf{Z}} \mathrm{Hom}_{\mathcal{A}}(X^i, Y^{i+n}), \qquad (df)(x) = d(f(x)) + (-1)^{n-1} f(dx). \tag{3.4.6.1}$$

If $g : Y^\bullet \longrightarrow Z^\bullet$ is a morphism in $C(\mathcal{A})$, then the canonical isomorphisms

$$\mathrm{Hom}_{\mathcal{A}}(X^i, \mathrm{Cone}(g)^{i+n}) = \mathrm{Hom}_{\mathcal{A}}(X^i, Y^{i+n+1} \oplus Z^{i+n}) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{A}}(X^i, Y^{i+n+1}) \oplus \mathrm{Hom}_{\mathcal{A}}(X^i, Z^{i+n})$$

define an isomorphism of complexes

$$\mathrm{Hom}^\bullet_{\mathcal{A}}(X^\bullet, \mathrm{Cone}(Y^\bullet \xrightarrow{g} Z^\bullet)) \xrightarrow{\sim} \mathrm{Cone}(\mathrm{Hom}^\bullet_{\mathcal{A}}(X^\bullet, Y^\bullet) \xrightarrow{g\circ} \mathrm{Hom}^\bullet_{\mathcal{A}}(X^\bullet, Z^\bullet)). \tag{3.4.6.2}$$

3.4.7. Proposition. *Let $X^\bullet, Y^\bullet \in \mathbf{C}(\mathcal{A})$ be complexes in an additive category $\mathcal{A}$. Then*

$$Z^0(\mathrm{Hom}^\bullet_{\mathcal{A}}(X^\bullet, Y^\bullet)) = \mathrm{Hom}_{C(\mathcal{A})}(X^\bullet, Y^\bullet)$$
$$B^0(\mathrm{Hom}^\bullet_{\mathcal{A}}(X^\bullet, Y^\bullet)) = \{f \in \mathrm{Hom}_{C(\mathcal{A})}(X^\bullet, Y^\bullet) \mid f \simeq 0\},$$

hence

$$H^0(\mathrm{Hom}^\bullet_{\mathcal{A}}(X^\bullet, Y^\bullet)) = \mathrm{Hom}_{K(\mathcal{A})}(X^\bullet, Y^\bullet).$$

Proof. An element $f \in \mathrm{Hom}^0_{\mathcal{A}}(X^\bullet, Y^\bullet) = \prod_{i \in \mathbf{Z}} \mathrm{Hom}_{\mathcal{A}}(X^i, Y^i)$ satisfies $df = 0$ iff $d(f(x)) - f(dx) = 0$, for all $x \in X^i$, which proves the first statement. The second statement follows from the fact that, for $h \in \mathrm{Hom}^{-1}_{\mathcal{A}}(X^\bullet, Y^\bullet) = \prod_{i \in \mathbf{Z}} \mathrm{Hom}_{\mathcal{A}}(X^i, Y^{i-1})$ and $x \in X^i$, we have $(dh)(x) = d(h(x)) + h(dx)$.

3.4.8. Corollary. *For $n \in \mathbf{Z}$,*

$$H^n(\mathrm{Hom}^\bullet_{\mathcal{A}}(X^\bullet, Y^\bullet)) = \mathrm{Hom}_{K(\mathcal{A})}(X^\bullet, Y^\bullet[n]).$$

Proof. There is a canonical isomorphism $\mathrm{Hom}^\bullet_{\mathcal{A}}(X^\bullet, Y^\bullet[n]) \xrightarrow{\sim} \mathrm{Hom}^\bullet_{\mathcal{A}}(X^\bullet, Y^\bullet)[n]$ (which, unlike (3.4.5.3), does not involve any signs); we apply 3.4.7 to $Y^\bullet[n]$ instead of $Y^\bullet$.

3.5. Left derived (and hyper-derived) functors

In this section we make rigorous the discussion from 3.2, by establishing the properties 3.2.5.1-3. The key points are proved in 3.5.4 and 3.5.5, the rest is fairly straightforward.

Let $\mathcal{A}$ be an abelian category. As in 2.2.9, we denote by $\mathcal{P}_{\mathcal{A}}$ the full additive subcategory of $\mathcal{A}$ consisting of the projective objects of $\mathcal{A}$. Recall that we say that $\mathcal{A}$ **has enough projectives** if for every $X \in \mathcal{A}$ there exists an epimorphism $P \longrightarrow X$ in $\mathcal{A}$ with $P \in \mathcal{P}_{\mathcal{A}}$.

3.5.1. Proposition. *If a complex $P^\bullet \in C^-(\mathcal{P}_{\mathcal{A}})$ satisfies $\forall m > n \;\; H^m(P^\bullet) = 0$, then $\tau_{\leq n} P^\bullet \in C^-(\mathcal{P}_{\mathcal{A}})$ (note that the inclusion $\tau_{\leq n} P^\bullet \hookrightarrow P^\bullet$ is a Qis, by (3.1.6.3)).*

Proof. We have $P^\bullet \in C^{\leq n+N}(\mathcal{P}_{\mathcal{A}})$ for some $N \geq 1$. As $\tau_{\leq n} P^\bullet = \tau_{\leq n} \cdots \tau_{\leq n+N-1} P^\bullet$, we can assume, using an inductive argument, that $N = 1$. By definition,

$$P^\bullet = [\cdots \longrightarrow P^{n-1} \longrightarrow P^n \xrightarrow{d^n} P^{n+1} \longrightarrow 0 \longrightarrow \cdots]$$
$$\tau_{\leq n} P^\bullet = [\cdots \longrightarrow P^{n-1} \longrightarrow Z^n \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots],$$

where $Z^n = \mathrm{Ker}(d^n)$. We must show that Z^n is projective, using the assumption that $H^{n+1}(P^\bullet) = 0$. Indeed, the sequence

$$0 \longrightarrow Z^n \longrightarrow P^n \xrightarrow{d^n} P^{n+1} \longrightarrow 0 \;\; (= H^{n+1}(P^\bullet))$$

is exact; as P^{n+1} is projective, the sequence splits, hence $P^n = Z^n \oplus P^{n+1}$. As P^n is projective, so is Z^n.

3.5.2. Corollary-Definition. *If there is a Qis $f : P^\bullet \longrightarrow X$, where $X \in \mathcal{A}$ and $P^\bullet \in C^-(\mathcal{P}_{\mathcal{A}})$, then $Q^\bullet = \tau_{\leq 0} P^\bullet \in C^{\leq 0}(\mathcal{P}_{\mathcal{A}})$ and the composite morphism $g : Q^\bullet \hookrightarrow P^\bullet \xrightarrow{f} X$ is also a Qis. In concrete terms, $g_0 : Q^0 \longrightarrow X$ is a morphism in $\mathcal{A}$ and the sequence*

$$\cdots \longrightarrow Q^{-2} \longrightarrow Q^{-1} \longrightarrow Q^0 \xrightarrow{g_0} X \longrightarrow 0$$

is exact (we say that $Q^\bullet$ is a ***projective resolution of*** *X).*

3.5.3. Proposition. *If $\mathcal{A}$ has enough projectives, then each $X \in \mathcal{A}$ has a projective resolution.*

Proof. There is an epimorphism $f_0 : P^0 \longrightarrow X$ with $P^0 \in \mathcal{P}_{\mathcal{A}}$, which gives rise to an exact sequence $0 \longrightarrow \mathrm{Ker}(f_0) \longrightarrow P^0 \xrightarrow{f_0} X$. Similarly, there is an epimorphism $f_1 : P^{-1} \longrightarrow \mathrm{Ker}(f_0)$ with $P^{-1} \in \mathcal{P}_{\mathcal{A}}$, which sits in an exact sequence $0 \longrightarrow \mathrm{Ker}(f_1) \longrightarrow P^{-1} \xrightarrow{f_1} \mathrm{Ker}(f_0) \longrightarrow 0$. Continuing this process, we obtain short exact sequences $0 \longrightarrow \mathrm{Ker}(f_n) \longrightarrow P^{-n} \xrightarrow{f_n} \mathrm{Ker}(f_{n-1}) \longrightarrow 0$ with $P^{-n} \in \mathcal{P}_{\mathcal{A}}$, which give rise (as in 2.1.3(8)) to a long exact sequence $\cdots \longrightarrow P^{-3} \longrightarrow P^{-2} \longrightarrow P^{-1} \longrightarrow P^0 \longrightarrow X \longrightarrow 0$, in which all P^{-n} are projective.

3.5.4. Proposition. *If $\mathcal{A}$ has enough projectives, then for each $X^\bullet \in C^-(\mathcal{A})$ there exists a Qis $P^\bullet \longrightarrow X^\bullet$, where $P^\bullet \in C^-(\mathcal{P}_{\mathcal{A}})$.*

Proof. After replacing $X^\bullet$ by $X^\bullet[n]$ we can assume that $X^\bullet \in C^{\leq 0}(\mathcal{A})$. We are going to construct a Qis $f : P^\bullet \longrightarrow X^\bullet$, where $P^\bullet \in C^{\leq 0}(\mathcal{P}_{\mathcal{A}})$. We use the following sufficient conditions for f to be a Qis:

$$(A_q) \qquad \mathrm{Coker}(Z^q(P^\bullet) \longrightarrow Z^q(X^\bullet)) = 0$$

(which implies that $H^q(P^\bullet) \longrightarrow H^q(X^\bullet)$ is an epimorphism) and

$$(B_q) \qquad \mathrm{Coker}(P^{q-1} \overset{j_{q-1}}{\longrightarrow} X^{q-1} \times_{Z^q(X^\bullet)} Z^q(P^\bullet)) = 0.$$

We claim that (B_q) implies that $H^q(P^\bullet) \longrightarrow H^q(X^\bullet)$ is a monomorphism. Let us first explain the morphism j_{q-1} (we consider $\mathcal{A}$ as a full subcategory of a suitable category of R-modules, so it makes sense to work with elements of X^n and P^n). The fibre product

$$\begin{array}{ccc} X^{q-1} \times_{Z^q(X^\bullet)} Z^q(P^\bullet) & \overset{\alpha}{\longrightarrow} & Z^q(P^\bullet) \\ \big\downarrow \beta & & \big\downarrow f_q \\ X^{q-1} & \overset{d}{\longrightarrow} & Z^q(X^\bullet) \end{array}$$

consists of pairs $(x, z) \in X^{q-1} \oplus Z^q(P^\bullet)$ such that $dx = f_q(z) \in Z^q(X^\bullet)$ (above, $\alpha(x, z) = z$ and $\beta(x, z) = x$). Its universal property implies that the differential $d : P^{q-1} \longrightarrow X^{q-1}$ factors through a unique morphism j_{q-1} as in (B_q) ($d = \alpha j_{q-1}$ and $f_{q-1} = \beta j_{q-1}$). If $z \in Z^q(P^\bullet)$ satisfies $f_q(z) = dx$ for some $x \in X^{q-1}$, it follows from (B_q) that $(x, z) = j_{q-1}(p)$ for some $p \in P^{q-1}$; then $z = \alpha(x, z) = \alpha j_{q-1}(p) = dp$, which proves the claim.
Moreover, (B_q) implies (A_{q-1}): if $x \in Z^{q-1}(X^\bullet)$, then $(x, 0) = j_{q-1}(p)$ for some $p \in P^{q-1}$. As $dp = \alpha(x, 0) = 0$, we have $p \in Z^{q-1}(P^\bullet)$ and $f_{q-1}(p) = \beta j_{q-1}(p) = \beta(x, 0) = x$.
As a result, it is enough to construct $f : P^\bullet \longrightarrow X^\bullet$ satisfying (A_0) (by choosing any epimorphism $f_0 : P^0 \longrightarrow X^0$ with $P^0 \in \mathcal{P}_\mathcal{A}$) and (B_q) for all $q \leq 0$ (having constructed f_q, we choose any epimorphism $j_{q-1} : P^{q-1} \longrightarrow X^{q-1} \times_{Z^q(X^\bullet)} Z^q(P^\bullet)$ with $P^{q-1} \in \mathcal{P}_\mathcal{A}$ and put $f_{q-1} = \beta j_{q-1} : P^{q-1} \longrightarrow X^{q-1}$).

3.5.5. Proposition. *Let $P^\bullet \in C^-(\mathcal{P}_\mathcal{A})$. If $Z^\bullet \in C(\mathcal{A})$ is acyclic, so is $\mathrm{Hom}^\bullet_\mathcal{A}(P^\bullet, Z^\bullet)$. In particular, $\mathrm{Hom}_{K(\mathcal{A})}(P^\bullet, Z^\bullet) = H^0(\mathrm{Hom}^\bullet_\mathcal{A}(P^\bullet, Z^\bullet)) = 0$.*

Proof. By definition, $\mathrm{Hom}^\bullet_\mathcal{A}(P^\bullet, Z^\bullet) = \mathrm{Tot}^\Pi(K^{\bullet,\bullet})$, where $K^{i,j} = \mathrm{Hom}_\mathcal{A}(P^{-i}, Z^j)$. Note that $\exists i_0\ \forall i < i_0\ \forall j\ \ K^{i,j} = 0$, since $P^\bullet \in C^-(\mathcal{A})$. Furthermore, each column $(K^{i,\bullet}, d_{II})$ of $K^{\bullet,\bullet}$ is acyclic, as it is obtained by applying an exact functor $\mathrm{Hom}_\mathcal{A}(P^{-i}, \bullet)$ to an acyclic complex $Z^\bullet$. Proposition then follows from the following Lemma.

3.5.6. Lemma. *Let $K^{\bullet,\bullet}$ be a double complex in which each column $(K^{i,\bullet}, d_{II})$ is acyclic and such that $\exists i_0\ \forall i < i_0\ \forall j\ \ K^{i,j} = 0$. Then the big total complex $\mathrm{Tot}^\Pi(K^{\bullet,\bullet})$ is acyclic (this is false, in general, for the small total complex $\mathrm{Tot}^\oplus(K^{\bullet,\bullet})$).*

Proof. Let $x = (x_{ij},\, x_{i+1,j-1},\, x_{i+2,j-2}, \ldots) \in \mathrm{Tot}^\Pi(K^{\bullet,\bullet})^{i+j}$ ($x_{ab} \in K^{a,b}$), $dx = 0$. As $d = d_I + d_{II}$, we have

$$d_{II}(x_{ij}) = 0, \qquad d_I(x_{ij}) + d_{II}(x_{i+1,j-1}) = 0, \qquad d_I(x_{i+1,j-1}) + d_{II}(x_{i+2,j-2}) = 0, \qquad \ldots$$

As $(K^{i,\bullet}, d_{II})$ is acyclic, there is $y_{i,j-1} \in K^{i,j-1}$ such that $x_{ij} = d_{II}(y_{i,j-1})$, hence

$$0 = d_{II}(x_{i+1,j-1}) + d_I d_{II}(y_{i,j-1}) = d_{II}(x_{i+1,j-1} - d_I(y_{i,j-1})).$$

As $(K^{i-1,\bullet}, d_{II})$ is acyclic, there is $y_{i+1,j-2} \in K^{i+1,j-2}$ such that $x_{i+1,j-1} - d_I(y_{i,j-1}) = d_{II}(y_{i+1,j-2})$. Continuing this process we obtain an element $y = (y_{i,j-1},\, y_{i+1,j-2},\, y_{i+2,j-3}, \ldots) \in \mathrm{Tot}^\Pi(K^{\bullet,\bullet})^{i+j-1}$ satisfying $dy = x$.

3.5.7. Proposition. *Let $X^\bullet, Y^\bullet \in C(\mathcal{A})$ and $P^\bullet, Q^\bullet \in C^-(\mathcal{P}_\mathcal{A})$.*
(1) *If $f : X^\bullet \longrightarrow Y^\bullet$ is a Qis, then the map induced by f*

$$\mathrm{Hom}_{K(\mathcal{A})}(P^\bullet, X^\bullet) \overset{f\circ}{\longrightarrow} \mathrm{Hom}_{K(\mathcal{A})}(P^\bullet, Y^\bullet)$$

is an isomorphism of abelian groups.
(2) *If $f : X^\bullet \longrightarrow Y^\bullet$ is a morphism of complexes and $u : P^\bullet \longrightarrow X^\bullet$, $v : Q^\bullet \longrightarrow Y^\bullet$ are Qis, then there is a*

morphism of complexes $f' : P^\bullet \longrightarrow Q^\bullet$, unique up to homotopy, for which the following diagram commutes up to homotopy:

$$\begin{array}{ccc} P^\bullet & \xrightarrow{u} & X^\bullet \\ \Big\downarrow{\scriptstyle f'} & & \Big\downarrow{\scriptstyle f} \\ Q^\bullet & \xrightarrow{v} & Y^\bullet \end{array}$$

(i.e., there is a homotopy $vf' \simeq fu$).
(3) If $P^\bullet$ is acyclic, then there is a homotopy $0 \simeq \mathrm{id}_{P^\bullet}$ (a "contracting homotopy" on the complex $P^\bullet$).
(4) Any Qis $\alpha : P^\bullet \longrightarrow Q^\bullet$ is a homotopy equivalence (i.e., there is a morphism of complexes $\beta : Q^\bullet \longrightarrow P^\bullet$ and homotopies $\alpha\beta \simeq \mathrm{id}_{Q^\bullet}$, $\beta\alpha \simeq \mathrm{id}_{P^\bullet}$).
(5) If $\alpha : P^\bullet \longrightarrow Q^\bullet$ is a Qis, so is $F(\alpha) : F(P^\bullet) \longrightarrow F(Q^\bullet)$, for any additive functor $F : \mathcal{A} \longrightarrow \mathcal{A}'$ with values in an abelian category $\mathcal{A}'$.

Proof. (1) As f is a Qis, its cone $\mathrm{Cone}(f)$ is acyclic, hence so is (see (3.4.6.2))

$$\mathrm{Cone}(\mathrm{Hom}^\bullet_{\mathcal{A}}(P^\bullet, X^\bullet) \xrightarrow{f\circ} \mathrm{Hom}^\bullet_{\mathcal{A}}(P^\bullet, Y^\bullet)) \xrightarrow{\sim} \mathrm{Hom}^\bullet_{\mathcal{A}}(P^\bullet, \mathrm{Cone}(X^\bullet \xrightarrow{f} Y^\bullet)),$$

by 3.5.5. In other words, the morphism of complexes

$$\mathrm{Hom}^\bullet_{\mathcal{A}}(P^\bullet, X^\bullet) \xrightarrow{f\circ} \mathrm{Hom}^\bullet_{\mathcal{A}}(P^\bullet, Y^\bullet)$$

is a Qis, hence f induces an isomorphism

$$\mathrm{Hom}_{K(\mathcal{A})}(P^\bullet, X^\bullet) = H^0(\mathrm{Hom}^\bullet_{\mathcal{A}}(P^\bullet, X^\bullet)) \xrightarrow{\sim} H^0(\mathrm{Hom}^\bullet_{\mathcal{A}}(P^\bullet, Y^\bullet)) = \mathrm{Hom}_{K(\mathcal{A})}(P^\bullet, Y^\bullet).$$

(2) By (1), the map induced by v

$$\mathrm{Hom}_{K(\mathcal{A})}(P^\bullet, Q^\bullet) \xrightarrow{v\circ} \mathrm{Hom}_{K(\mathcal{A})}(P^\bullet, Y^\bullet)$$

is bijective.
(3) As $P^\bullet \xrightarrow{0} 0$ is a Qis, we obtain from (1) an isomorphism of abelian groups

$$\mathrm{Hom}_{K(\mathcal{A})}(P^\bullet, P^\bullet) \xrightarrow{\sim} \mathrm{Hom}_{K(\mathcal{A})}(P^\bullet, 0) = 0.$$

(4) By (1), the map induced by α

$$\mathrm{Hom}_{K(\mathcal{A})}(Q^\bullet, P^\bullet) \xrightarrow{\alpha\circ} \mathrm{Hom}_{K(\mathcal{A})}(Q^\bullet, Q^\bullet)$$

is bijective, which means that there is a morphism of complexes $\beta : Q^\bullet \longrightarrow P^\bullet$, unique up to homotopy, for which $\alpha\beta \simeq \mathrm{id}_{Q^\bullet}$. As $\alpha\beta\alpha \simeq \mathrm{id}_{Q^\bullet} \circ \alpha = \alpha \circ \mathrm{id}_{P^\bullet}$, the same bijection implies that $\beta\alpha \simeq \mathrm{id}_{P^\bullet}$.
(5) By (4), α admits a homotopy inverse β; then $F(\beta)$ is a homotopy inverse of $F(\alpha)$, which implies that $F(\alpha)$ is a Qis.

3.5.8. Corollary. *If $\mathcal{A}$ has enough projectives, then there is a "resolution functor"*

$$C^-(\mathcal{A}) \longrightarrow K^-(\mathcal{P}_{\mathcal{A}}), \qquad X^\bullet \mapsto P^\bullet$$

(factoring through the functor $C^-(\mathcal{A}) \longrightarrow K^-(\mathcal{A})$), where $P^\bullet \in C^-(\mathcal{P}_{\mathcal{A}})$ is any bounded above complex of projectives equipped with a Qis $P^\bullet \longrightarrow X^\bullet$.

3.5.9. Definition. Assume that $\mathcal{A}$ has enough projectives and $F : \mathcal{A} \longrightarrow \mathcal{A}'$ is an additive functor with values in an abelian category $\mathcal{A}'$. Composing the resolution functor 3.5.8 with the functor $F = K^-(F) : K^-(\mathcal{A}) \longrightarrow K^-(\mathcal{A}')$ and with the cohomology functors $H^{-n} : K^-(\mathcal{A}') \longrightarrow \mathcal{A}'$ ($n \in \mathbf{Z}$), we obtain the following additive functors (all factoring through $C^-(\mathcal{A}) \longrightarrow K^-(\mathcal{A})$):

$$\begin{gathered} \text{“}\mathbf{L}F\text{”} : C^-(\mathcal{A}) \longrightarrow K^-(\mathcal{P}_{\mathcal{A}}) \xrightarrow{F} K^-(\mathcal{A}'), \qquad X^\bullet \mapsto P^\bullet \mapsto F(P^\bullet) \\ \mathbf{L}_n F = H^{-n} \circ \text{“}\mathbf{L}F\text{”} : C^-(\mathcal{A}) \longrightarrow K^-(\mathcal{P}_{\mathcal{A}}) \xrightarrow{F} K^-(\mathcal{A}') \xrightarrow{H^{-n}} \mathcal{A}' \\ L_n F : \mathcal{A} \longrightarrow C^-(\mathcal{A}) \xrightarrow{\mathbf{L}_n F} \mathcal{A}', \end{gathered} \tag{3.5.9.1}$$

for any Qis $P^\bullet \longrightarrow X^\bullet$ with $P^\bullet \in C^-(\mathcal{P}_\mathcal{A})$. The functors L_nF (resp., $\mathbf{L}_nF$) are the classical **left derived functors** (resp., the **left hyper-derived functors**) of F (at least when F is right exact; see 3.5.10.?? below).

3.5.10. Proposition (Properties of the left derived functors). *Under the assumptions of 3.5.9, the functors "$\mathbf{L}F$", $\mathbf{L}_n$ and L_nF have the following proprties.*
(1) *If $X \in \mathcal{P}_\mathcal{A}$, then $L_0F(X) = X$ and $\forall n > 0\ \ L_nF(X) = 0$.*
(2) *"$\mathbf{L}F$"$(X^\bullet[m]) =$ "$\mathbf{L}F$"$(X^\bullet)[m]$.*
(3) *If $f : X^\bullet \longrightarrow Y^\bullet$ is a Qis in $C^-(\mathcal{A})$, then "$\mathbf{L}F$"(f) : "$\mathbf{L}F$"$(X^\bullet) \longrightarrow$ "$\mathbf{L}F$"$(X^\bullet)$ is an isomorphism in $K^-(\mathcal{A}')$; in particular, it is a Qis (hence all maps $\mathbf{L}_nF(f) : \mathbf{L}_nF(X^\bullet) \longrightarrow \mathbf{L}_nF(Y^\bullet)$ are isomorphisms).*
(4) *If $H^m(X^\bullet) = 0$ for all $m > N$, then $\mathbf{L}_nF(X^\bullet) = 0$ for all $n < -N$. In particular, $L_nF = 0$ for all $n < 0$.*
(5) *If F is right exact, then $L_0F = F$.*
(6) *If $\mathcal{E} : 0 \longrightarrow X^\bullet \xrightarrow{f} Y^\bullet \xrightarrow{g} Z^\bullet \longrightarrow 0$ is a short exact sequence in $C^-(\mathcal{A})$, then there is a long exact sequence in $\mathcal{A}'$*

$$\cdots \longrightarrow \mathbf{L}_nF(X^\bullet) \xrightarrow{\mathbf{L}_nF(f)} \mathbf{L}_nF(Y^\bullet) \xrightarrow{\mathbf{L}_nF(g)} \mathbf{L}_nF(Z^\bullet) \longrightarrow \mathbf{L}_{n-1}F(X^\bullet) \xrightarrow{\mathbf{L}_{n-1}F(f)} \mathbf{L}_{n-1}F(Y^\bullet) \longrightarrow \cdots,$$

which is functorial in $\mathcal{E}$. In particular, for each short exact sequence $0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$ in $\mathcal{A}$ there is a long exact sequence in $\mathcal{A}'$

$$\cdots \longrightarrow L_nF(X) \longrightarrow L_nF(Y) \longrightarrow L_nF(Z) \longrightarrow L_{n-1}F(X) \longrightarrow L_{n-1}F(Y) \longrightarrow \cdots \longrightarrow L_0F(Z) \longrightarrow 0.$$

Proof. (1) We can take $P^\bullet = X$ as a projective resolution of X.
(2) If $u : P^\bullet \longrightarrow X^\bullet$ ($P^\bullet \in C^-(\mathcal{P}_\mathcal{A})$) is a Qis, so is $u[m] : P^\bullet[m] \longrightarrow X^\bullet[m]$.
(3) If $u : P^\bullet \longrightarrow X^\bullet$ is a Qis ($P^\bullet \in C^-(\mathcal{P}_\mathcal{A})$), so is $fu : P^\bullet \longrightarrow Y^\bullet$, which means that the morphism "$\mathbf{L}F$"(f) in $K^-(\mathcal{A}')$ is equal to (the homotopy class of) $\mathrm{id}_{F(P^\bullet)} : F(P^\bullet) \longrightarrow F(P^\bullet)$.
(4) Let $u : P^\bullet \longrightarrow X^\bullet$ be a Qis, where $P^\bullet \in C^-(\mathcal{P}_\mathcal{A})$; then $\tau_{\le N} P^\bullet \hookrightarrow P^\bullet \xrightarrow{u} X^\bullet$ is also a Qis and $\tau_{\le N} P^\bullet \in C^{\le N}(\mathcal{P}_\mathcal{A})$, by 3.5.1. It follows that "$\mathbf{L}F$"$(X^\bullet) = F(\tau_{\le N} P^\bullet) \in C^{\le N}(\mathcal{A}')$, hence $H^{-n}($"$\mathbf{L}F$"$(X^\bullet)) = 0$ for all $n < -N$.
(5) Let $X \in \mathcal{A}$; fix a projective resolution $P^\bullet \longrightarrow X$ of X. As F is right exact, it transforms the exact sequence $P^{-1} \xrightarrow{d^{-1}} P^0 \longrightarrow X \longrightarrow 0$ into an exact sequence $F(P^{-1}) \xrightarrow{F(d^{-1})} F(P^0) \longrightarrow F(X) \longrightarrow 0$, which implies that $(L_0F)(X) = H^0(F(P^\bullet)) = \mathrm{Coker}(F(d^{-1})) = F(X)$.
(6) Thanks to (3) and 3.3.15 we can assume that the exact sequence $\mathcal{E}$ is of the following special form:

$$0 \longrightarrow X^\bullet \xrightarrow{\overline{f}} \mathrm{Cyl}(f) \xrightarrow{\pi} \mathrm{Cone}(f) \longrightarrow 0.$$

Fix $P^\bullet, Q^\bullet \in C^-(\mathcal{P}_\mathcal{A})$ and Qis $u : P^\bullet \longrightarrow X^\bullet$, $v : Q^\bullet \longrightarrow Y^\bullet$. By 3.5.7(2) there exists a morphism $f' : P^\bullet \longrightarrow Q^\bullet$ in $C(\mathcal{A})$ and a homotopy $h : fu \simeq vf'$. The construction from 3.3.16 then gives rise to the following commutative diagram whose rows are split short exact sequences of complexes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P^\bullet & \xrightarrow{\overline{f}'} & \mathrm{Cyl}(f') & \longrightarrow & \mathrm{Cone}(f') & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle u} & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & X^\bullet & \xrightarrow{\overline{f}} & \mathrm{Cyl}(f) & \longrightarrow & \mathrm{Cone}(f) & \longrightarrow & 0, \end{array}$$

and in which all vertical arrows are Qis (u by assumption, the right vertical arrow by the remark after (3.3.9.2), and the middle one by 2.1.10). The required long exact sequence is then the cohomology exact sequence of

$$0 \longrightarrow F(P^\bullet) \longrightarrow F(\mathrm{Cyl}(f')) \longrightarrow F(\mathrm{Cone}(f')) \longrightarrow 0$$

(which is exact, as the first row of the diagram is split).

3.5.11. Definition. Assume that $\mathcal{A}$ has enough projectives and $F : \mathcal{A} \longrightarrow \mathcal{A}'$ is a right exact additive functor with values in an abelian category $\mathcal{A}'$. We say that $X \in \mathcal{A}$ is F**-acyclic** if $\forall n > 0\ \ L_nF(X) = 0$. We denote by $\mathcal{A}_F$ the full additive subcategory of $\mathcal{A}$ consisting of the F-acyclic objects. By 3.5.10(1), $\mathcal{P}_\mathcal{A} \subseteq \mathcal{A}_F$.

3.5.12. Proposition. *Under the assumptions of 3.5.11, if* $Y \in \mathcal{A}$ *and if* $X^\bullet \longrightarrow Y$ *is a Qis with* $X^\bullet \in C^{\leq 0}(\mathcal{A}_F)$ *(we say that "*$X^\bullet$ *is an* F*-acyclic resolution of* Y*"), then* $\forall n \geq 0\ \ L_nF(Y) \xrightarrow{\sim} H^{-n}(F(X^\bullet))$.

Proof. Under construction.

3.5.13. Proposition. *Under the assumptions of 3.5.11, if* $Q^\bullet \in C^-(\mathcal{A}_F)$*, then there is a Qis "*$\mathbf{L}F$*"*$(Q^\bullet) \longrightarrow F(Q^\bullet)$*, hence* $\mathbf{L}_nF(Q^\bullet) = H^{-n}(F(Q^\bullet))$ *for all* $n \in \mathbf{Z}$.

Proof. Fix a Qis $f : P^\bullet \longrightarrow Q^\bullet$ with $P^\bullet \in C^-(\mathcal{P}_{\mathcal{A}})$. Fix $m \in \mathbf{Z}$ such that $\mathrm{Cone}(f)[m] \in C^{\leq 0}(\mathcal{A}_F)$; then $\mathrm{Cone}(f)[m]$ is an F-acyclic resolution of 0, hence $F(\mathrm{Cone}(f)[m]) = \mathrm{Cone}(F(f))[m]$ is also acyclic, by 3.5.12. In particular, the map $F(f) : F(P^\bullet) \longrightarrow F(Q^\bullet)$ is a Qis; but $F(P^\bullet)$ is canonically isomorphic to "$\mathbf{L}F$"$(Q^\bullet)$ in $K^-(\mathcal{A}')$.

3.5.14. In general, the map "$\mathbf{L}F$"$(Q^\bullet) \longrightarrow F(Q^\bullet)$ is **not an isomorphism in the homotopy category** $K^-(\mathcal{A}')$. It becomes an isomorphism only in the **derived category**

$$D^-(\mathcal{A}') = K^-(\mathcal{A}')[Qis^{-1}],$$

which is obtained from the homotopy category by inverting all Qis.