Composition of primitive binary quadratic forms

($\Longleftrightarrow$ $Cl^+(\Delta) = \mathrm{Quad}_{prim}(\Delta)/SL_2(\mathbb{Z})$ is a (finite) abelian group)

Ex: (1) $(x^2-\Delta y^2)(x'^2-\Delta y'^2) = (xx'-\Delta yy')^2 - \Delta(xy'+yx')^2$ $\qquad (\sqrt{\Delta}\notin\mathbb{Z})$

(2) $(2x^2+2xy+3y^2)(2x'^2+2x'y'+3y'^2) = (2xx'+xy'+yx'+3yy')^2 + 5(xy'-yx')^2$ $\qquad$ (Lagrange)

Def: $h \in \mathrm{Quad}_{prim}(\Delta)$ is a composition of $f, f' \in \mathrm{Quad}_{prim}(\Delta)$ if $\exists\ A_i = a_i xx' + b_i xy' + c_i yx' + d_i yy'$ such that $f(x,y)f'(x',y') = h(A_1(x,y,x',y'), A_2(x,y,x',y'))$ $(a_i, b_i, c_i, d_i \in \mathbb{Z})$

Legendre: considered only $GL_2(\mathbb{Z})$-equivalence of quadratic forms and discovered examples in which f and f' had several compositions belonging to different $GL_2(\mathbb{Z})$-equivalence classes.

Ex: $(\Delta = -56)$ $\quad \mathrm{Quad}_{prim}(-56)/SL_2(\mathbb{Z}) = \{[f_1], [f_2], \underbrace{[f_3], [f_4]}_{GL_2(\mathbb{Z})\text{-equivalent}}\}$

$f_1 = x^2+14y^2$, $f_2 = 2x^2+7y^2$, $f_{3,4} = 3x^2 \pm 2xy + 5y^2$

$$(3x^2+2xy+5y^2)(3x'^2+2x'y'+5y'^2) = 2(xx'-2xy'-2yx'-3yy')^2 + 7(xx'+xy'+yx'-yy')^2$$

($\Rightarrow$ f_2 is a composition of f_3 with f_3)

$$(3x^2+2xy+5y^2)(3x'^2-2x'y'+5y'^2) = (3xx'-xy'+yx'-5yy')^2 + 14(xy'+yx')^2$$

($\Rightarrow$ f_1 is a composition of f_3 with f_4)

Gauss: introduced $SL_2(\mathbb{Z})$-equivalence and defined h to be a direct composition of f, f' if $f(1,0) = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$ and $f'(1,0) = \begin{vmatrix} a_1 & a_2 \\ c_1 & c_2 \end{vmatrix}$ (letting $x=1, y=0$ resp. $x'=1, y'=0$ gives

$$f(1,0)f'(x',y') = h(a_1x'+b_1y', a_2x'+b_2y') \Rightarrow f(1,0)^2\Delta(f') = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}^2 \Delta(h)$$
$$f(x,y)f'(1,0) = h(a_1x+c_1y, a_2x+c_2y) \Rightarrow f'(1,0)^2\Delta(f) = \begin{vmatrix} a_1 & a_2 \\ c_1 & c_2 \end{vmatrix}^2 \Delta(h)\ )$$

He showed that direct composition preserves $SL_2(\mathbb{Z})$-equivalence (and is associative), hence defines the structure of an abelian group on $Cl^+(\Delta)$.

the neutral element = the class of the principal form $\begin{cases} [1, 0, -\Delta/4] \\ [1, 1, (1-\Delta)/4] \end{cases}$

the inverse of the class of $f = [a,b,c]$ is — " — $f\left|\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right. = [c, b, a]$.

This explained why Legendre's composition was multivalued:

$\mathrm{Quad}_{prim}(\Delta)/GL_2(\mathbb{Z}) = Cl^+(\Delta)/$(identify each element with its inverse).

If $(A, +)$ is any abelian group, then $A/$(— " —) inherits from A a group law $\Longleftrightarrow$ $\forall\, a, b \in A$ $a+b = (-a)+b \Longleftrightarrow \forall a \in A$ $2a = 0$.

Above, $Cl^+(-56) \simeq \mathbb{Z}/4\mathbb{Z}$, generated by the class of f_3 (= the inverse of — " — f_4).

Dirichlet: found a more direct description of the group law on $Cl^+(\Delta)$.

Def. Two forms $f_j = [a_j, b_j, c_j] \in \mathrm{Quad}_{\mathrm{prim}}(\Delta)$ $(j=1,2)$ are concordant if (1) $a_1a_2 \neq 0$; (2) $b_1 = b_2 = b$; (3) $f_3 = [a_1a_2, b, \frac{\Delta - b^2}{4a_1a_2}]$ has coefficients in $\mathbb{Z}$

Note: if $\gcd(a_1,a_2)=1 \Rightarrow [(1),(2) \Rightarrow (3)]$.

Prop.–Def. $\forall\, C_1, C_2 \in Cl^+(\Delta)$ $\forall\, M \in \mathbb{Z}_{>0}$: (1) $\exists\, f_j = [a_j, b_j, c_j] \in C_j$ $(j=1,2)$ such that f_1, f_2 are concordant and $\gcd(a_1,a_2) = 1 = \gcd(a_j, M)$ (—"—).

(2) $\underbrace{\text{the } SL_2(\mathbb{Z})\text{-equivalence class of } f_3}$ depends only on C_1 and C_2.

(3) the Dirichlet composition of C_1, C_2 defines an abelian group law on $Cl^+(\Delta)$.

Pf: See [J.W.S. Cassels, Rational Quadratic forms, ch. 14, Thm. 2.1, Lemma 2.3–4]

We are going to give another description of the abelian group law on $Cl^+(\Delta)$, by translating everything in terms of arithmetic of the quadratic ring $\mathcal{O}_\Delta = \mathbb{Z}[\frac{\Delta + \sqrt{\Delta}}{2}]$.

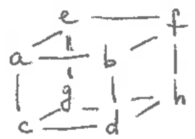
Exercise: What happens if $\sqrt{\Delta} \in \mathbb{Z} \setminus \{0\}$? If $\Delta = 0$?

Other definitions of composition:

Kneser (1982) – using modules over Clifford algebras

Bhargava (2001) – using "Rubik's cube" $2\times2\times2$:

a cube C



$(a, \ldots, h \in \mathbb{Z})$

can be sliced in three different ways into pairs of matrices in $M_2(\mathbb{Z})$:

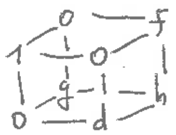
$$M_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad M_2 = \begin{pmatrix} a & c \\ e & g \end{pmatrix} \quad M_3 = \begin{pmatrix} a & e \\ b & f \end{pmatrix}$$

$$N_1 = \begin{pmatrix} e & f \\ g & h \end{pmatrix} \quad N_2 = \begin{pmatrix} b & d \\ f & h \end{pmatrix} \quad N_3 = \begin{pmatrix} c & g \\ d & h \end{pmatrix}$$

$\Rightarrow$ 3 quadratic forms $Q_i^C(x,y) = -\det(M_i x - N_i y)$ with the same $\Delta(Q_i^C) = \Delta$

If all three forms Q_i^C are primitive, then $\underbrace{[Q_1^C]}_{\text{the class of } Q_1^C} \cdot [Q_2^C] \cdot [Q_3^C] =$ the neutral element of $Cl^+(\Delta)$

Ex:



$Q_1 = [-d, h, fg]$

$Q_2 = [-g, h, df]$

$\xRightarrow{\text{Dirichlet}}$ the composition of $[Q_1]$ and $[Q_2]$ is the class of $[dg, h, -f]$

$Q_3 = [-f, h, dg]$, $Q_3 | \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = [dg, h, -f]$

In fact, Bhargava defined an abelian group law on the set of $SL_2(\mathbb{Z}) \times SL_2(\mathbb{Z}) \times SL_2(\mathbb{Z})$-equivalence classes of "primitive" cubes of discriminant Δ and showed that this abelian group is isomorphic to $\{(C_1, C_2, C_3) \in Cl^+(\Delta)^3 \mid C_1C_2C_3 = 1\}$ via the map (cube) $\longmapsto ([Q_1], [Q_2], [Q_3])$.

Classes of forms vs classes of ideals

Ex: $(\Delta = 40,\ \mathcal{O}_\Delta = \mathbb{Z}[\sqrt{10}])$

$x^2 - 10y^2 = \underbrace{(x+\sqrt{10}\,y)}_{\alpha}\underbrace{(x-\sqrt{10}\,y)}_{\alpha'} = \dfrac{N(\alpha)}{1}$ $\qquad |\mathcal{O}_\Delta/I| = (\mathcal{O}_\Delta : I)$

$\alpha \in \mathbb{Z}\cdot 1 + \mathbb{Z}\cdot\sqrt{10} = I = \mathcal{O}_\Delta$, basis $1, \sqrt{10}$

$2x^2 - 5y^2 = \dfrac{(2x+\sqrt{10}\,y)(2x-\sqrt{10}\,y)}{2} = \dfrac{N(\alpha)}{2}$

$\alpha \in \mathbb{Z}\cdot 2 + \mathbb{Z}\cdot\sqrt{10} = J$, — " — $2, \sqrt{10}$

$|\mathcal{O}_\Delta/J| = (\mathcal{O}_\Delta : J)$

I, J are non-zero ideals of $\mathcal{O}_\Delta$.

Recall: an ideal of a (commutative) ring A is an additive subgroup $I \subset (A, +)$ such that $\forall \beta \in A \quad \beta I \subset I$.

Why ideals? the origin a most amazing idea of Kummer, according to which a failure of unique factorisation such that $3\cdot 7 = (4+i\sqrt{5})(4-i\sqrt{5})$ could be remedied by introducing "ideal numbers" P_j, Q_j such that $3 = P_1P_2$, $7 = Q_1Q_2$, $4+i\sqrt{5} = P_1Q_1$, $4-i\sqrt{5} = P_2Q_2$, for which there would be unique factorisation [Kummer worked with much more complicated objects, namely, $\mathbb{Z}[\zeta_p]$ for p = prime].

The mysterious objects P_j, Q_j are "divisors" — one can say when a divisor divides an honest number. The next step was carried out by Dedekind, who identified a divisor D with the set of all numbers divisible by D, which then became an ideal in the sense of the above definition.

The ring $\mathbb{Z}[i\sqrt{5}] = \mathcal{O}_{-20}$ (and $\mathcal{O}_\Delta$ if Δ = fundamental discriminant) is a Dedekind ring: every $(\neq 0)$ ideal can be written uniquely as a product of $(\neq 0)$ prime ideals.

Quadratic forms $\longleftrightarrow$ Ideals $\qquad (\sqrt{\Delta} \notin \mathbb{Z})$

$(I, \underbrace{(\alpha_1, \alpha_2)}_{\text{positive basis of } I})$

$\mathrm{Quad}(\Delta) \ni f(x,y) = \dfrac{N(\alpha_1 x + \alpha_2 y)}{N(I)} = \dfrac{(\alpha_1 x + \alpha_2 y)(\alpha_1' x + \alpha_2' y)}{(\mathcal{O}_\Delta : I)}$ $\qquad I$ (fractional) $\mathcal{O}_\Delta$-ideal

This will induce bijections

$\mathrm{Quad}(\Delta)/SL_2(\mathbb{Z}) \xleftrightarrow{\sim}$ suitable equivalence classes of $\mathcal{O}_\Delta$-ideals

$\cup \qquad\qquad \cup$

$\mathcal{C}\ell^+(\Delta) = \mathrm{Quad}_{prim}(\Delta)/SL_2(\mathbb{Z}) \xleftrightarrow{\sim}$ — " — of $\underbrace{\text{proper } \mathcal{O}_\Delta\text{-ideals} = \text{invertible } \mathcal{O}_\Delta\text{-ideals}}_{\text{abelian group under the product of ideals}}$

Def (: Generalised index): $X, Y \subset U$ abelian groups such that $(U:X), (U:Y) < \infty$

$\Longrightarrow \quad (X:Y) = \dfrac{(U:Y)}{(U:X)} \in \mathbb{Q}_{>0}$ does not depend on U, is equal to $|X/Y|$ if $X \supset Y$ and $(X:Z) = (X:Y)(Y:Z)$

Ex: (1) $(2\mathbb{Z} : 3\mathbb{Z}) \quad \left(= (\mathbb{Z} : \frac{3}{2}\mathbb{Z})\right) = \dfrac{(\mathbb{Z}:3\mathbb{Z})}{(\mathbb{Z}:2\mathbb{Z})} = \dfrac{3}{2}$

(2) If $X, Y \simeq \mathbb{Z}^n$ are both contained in a $\mathbb{Q}$-vector space of dim $= n$, then $X = \bigoplus_{i=1}^{n} \mathbb{Z}x_i$, $Y = \bigoplus_{j=1}^{n} \mathbb{Z}y_j$, $y_j = \sum_{i=1}^{n} a_{ij}x_i$, $A = (a_{ij}) \in GL_n(\mathbb{Q})$, $(X:Y) = |\det(A)|$

Def. $(\mathcal{O}_\Delta \subset K = \mathbb{Q}(\sqrt{\Delta}), \sqrt{\Delta} \notin \mathbb{Z})$ An additive subgroup $I \subset (K, +)$ is a fractional $\mathcal{O}_\Delta$-ideal if $\exists\, \beta \in K^*$ $\beta I \subset \mathcal{O}_\Delta$ is a non-zero ideal of $\mathcal{O}_\Delta$.

[this makes sense for an arbitrary integral domain A and its fraction field K.]

Ex: $\forall \alpha \in K^* \quad (\alpha) = \alpha\mathcal{O}_\Delta \quad$ (= the principal fract. ideal generated by α)

Note: $(\alpha) = (\beta) \iff (\alpha\beta^{-1}) = (1) = \mathcal{O}_\Delta \iff \alpha\beta^{-1} \in \mathcal{O}_\Delta^*$.

Prop.-Def. (1) Any fractional $\mathcal{O}_\Delta$-ideal I is of the form $I = \mathbb{Z}\alpha_1 \oplus \mathbb{Z}\alpha_2$, where $\alpha_1, \alpha_2 \in K$ are linearly independent over $\mathbb{Q}$.

(2) The norm $N(I) = (\mathcal{O}_\Delta : I) \in \mathbb{Q}_{>0}$ is defined.

(3) $\forall \beta \in K^* \quad N(\beta I) = |N(\beta)|\, N(I) \quad (\Longrightarrow N((\alpha)) = |N(\alpha)| = |\alpha\alpha'|)$

Note: $N(I)$ depends not only on I, but also on $\mathcal{O}_\Delta$!!

$\forall n \geq 1 \quad \underbrace{\mathcal{O}_\Delta \supset \mathcal{O}_{\Delta n^2}}_{\text{index } n}$ (ex: $\mathbb{Z}[i] = \mathbb{Z}\cdot 1 + \mathbb{Z}\cdot i = \mathcal{O}_{-4} \supset \mathbb{Z}[ni] = \mathbb{Z}\cdot 1 + \mathbb{Z}\cdot ni = \mathcal{O}_{-4n^2}$), I is also an $\mathcal{O}_{\Delta n^2}$-ideal (fractional)

and $(\mathcal{O}_{\Delta n^2} : I) = (\mathcal{O}_\Delta : I)/(\mathcal{O}_\Delta : \mathcal{O}_{\Delta n^2}) = (\mathcal{O}_\Delta : I)/n$

Pf: (1), (2) $\exists \beta \in K^*$ $\underbrace{(\beta I, +)}_{\text{contains } 0 \neq \alpha,\ \alpha\sqrt{\Delta}} \subset (\mathcal{O}_\Delta, +) \simeq \mathbb{Z}^2 \Longrightarrow$ result.

3) Fix bases $I = \mathbb{Z}\alpha_1 \oplus \mathbb{Z}\alpha_2$, $K = \mathbb{Q}\gamma_1 \oplus \mathbb{Q}\gamma_2$. There are matrices $M, N, P \in GL_2(\mathbb{Q})$ such that $\beta(\alpha_1, \alpha_2) = (\alpha_1, \alpha_2)M$, $\beta(\gamma_1, \gamma_2) = (\gamma_1, \gamma_2)N$, $(\alpha_1, \alpha_2) = (\gamma_1, \gamma_2)P \Longrightarrow M = P^{-1}NP$, $(I : \beta I) = |\det(M)| = |\det(N)|$. If $\beta = u + v\sqrt{\Delta}$ $(u, v \in \mathbb{Q})$, take $\gamma_1 = 1$, $\gamma_2 = \sqrt{\Delta}$; then $\beta(1\ \sqrt{\Delta}) = (1\ \sqrt{\Delta}) \underbrace{\begin{pmatrix} u & \Delta v \\ v & u \end{pmatrix}}_{N}$, $\det(N) = u^2 - \Delta v^2 = N(\beta)$. So: $N(\beta I) = \underbrace{(\mathcal{O}_\Delta : I)}_{N(I)} \underbrace{(I : \beta I)}_{|N(\beta)|}$

Lemma. $\boxed{\mathbb{Q}(\sqrt{\Delta})^{*2} \cap \mathbb{Q}^* = \mathbb{Q}^{*2} \cup \Delta\mathbb{Q}^{*2}}$

Pf: $\underbrace{(u + v\sqrt{\Delta})^2}_{\alpha^2} = (u^2 + \Delta v^2) + 2uv\sqrt{\Delta} \in \mathbb{Q}^* \iff \begin{cases} u = 0 \neq v & \alpha^2 = \Delta v^2 \\ \text{OR} & \\ v = 0 \neq u & \alpha^2 = u^2 \end{cases}$

Def. Let $\alpha_1, \alpha_2 \in K$ be linearly independent over $\mathbb{Q}$. The multiplier ring of $M = \mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_2 = \mathbb{Z}\alpha_1 \oplus \mathbb{Z}\alpha_2 \subset K$ is $E(M) = \{\beta \in K \mid \beta M \subset M\}$ (subring of K).

Note: $\forall \alpha \in K^*$ $E(\alpha M) = E(M)$; $\quad E(\mathcal{O}_\Delta) = \mathcal{O}_\Delta$.

$(\Longrightarrow E(\mathbb{Z}\alpha_1 \oplus \mathbb{Z}\alpha_2) = E(\mathbb{Z}\cdot 1 \oplus \mathbb{Z}\cdot \frac{\alpha_2}{\alpha_1}))$

Key calculation: **Prop.** If $\alpha \in K \setminus \mathbb{Q}$, $M = \mathbb{Z}\cdot 1 \oplus \mathbb{Z}\cdot\alpha$, then $E(M) = \mathbb{Z}\cdot 1 \oplus \mathbb{Z}\cdot a\alpha$, where $a\alpha^2 + b\alpha + c = 0$, $a, b, c \in \mathbb{Z}$, $\gcd(a,b,c) = 1$ $\quad (\Longrightarrow (E(M):M) = (a\mathbb{Z}:\mathbb{Z}) = 1/|a|)$

Pf. If $\beta = u + v\alpha \in K$ $(u, v \in \mathbb{Q})$, then:

$$\beta \in E(M) \iff \begin{cases} \beta\cdot 1 \in M \\ \underbrace{\beta\alpha}_{u\alpha + v\alpha^2 = (u - bv/a)\alpha - cv/a} \in M \end{cases} \iff \begin{cases} u, v \in \mathbb{Z} \\ u - bv/a,\ cv/a \in \mathbb{Z} \end{cases} \iff u \in \mathbb{Z},\ \underbrace{av, bv, cv \in a\mathbb{Z}}_{\iff v \in a\mathbb{Z}}$$

Orientation: (1) notation: $\sqrt{\Delta} = \begin{cases} \sqrt{\Delta} > 0 & \Delta > 0 \\ i\sqrt{|\Delta|},\ \sqrt{|\Delta|} > 0 & \Delta < 0 \end{cases}$ $\quad \Big|\quad \mathrm{Irr}(u + v\sqrt{\Delta}) = v$ $\quad (u, v \in \mathbb{Q})$

(2) $1, \sqrt{\Delta}$ is a positive basis of K (over $\mathbb{Q}$); for $\alpha_j = u_j + v_j\sqrt{\Delta}$ $\quad (j = 1, 2)$, $\alpha_2/\alpha_1 \notin \mathbb{Q}$

α_1, α_2 is a —"— of $\mathbb{Z}\alpha_1 \oplus \mathbb{Z}\alpha_2 = \mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_2 \subset K$ if $(\alpha_1\ \alpha_2) = (1\ \sqrt{\Delta})\underbrace{\begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix}}_{\det > 0}$

$(\iff \mathrm{Irr}(\alpha_1\alpha_2') < 0)$

From ideals to forms

Def. $I = \mathbb{Z}\alpha_1 \oplus \mathbb{Z}\alpha_2$ fractional $\mathcal{O}_\Delta$-ideal, α_1, α_2 positive basis of I

$$f_{I,(\alpha_1,\alpha_2)}(x,y) = \frac{N(\alpha_1 x + \alpha_2 y)}{N(I)} = \frac{(\alpha_1 x + \alpha_2 y)(\alpha_1' x + \alpha_2' y)}{(\mathcal{O}_\Delta : I)}$$

Note: (1) If we replace Δ by Δn^2 $(n \in \mathbb{Z}_{\geq 1}) \Longrightarrow f$ is multiplied by n.

(2) $\beta \in K^*$, $N(\beta) > 0 \Longrightarrow \beta\alpha_1, \beta\alpha_2$ positive basis of I, $f_{\beta I,(\beta\alpha_1,\beta\alpha_2)} = \underbrace{\frac{N(\beta)}{|N(\beta)|}}_{1} f_{I,(\alpha_1,\alpha_2)}$

(3) $N(\beta) < 0 \Longrightarrow \beta\alpha_1, -\beta\alpha_2$ —"—, $f_{\beta I,(\beta\alpha_1,-\beta\alpha_2)}(x,y) = \underbrace{\frac{N(\beta)}{|N(\beta)|}}_{(-1)} \underbrace{f_{I,(\alpha_1,\alpha_2)}(x,-y)}_{f_{I,(\alpha_1,\alpha_2)}|\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}$

Notation: $I(\mathcal{O}_\Delta) = \{\text{fractional ideals of } \mathcal{O}_\Delta\}$

$\cup$

$P(\mathcal{O}_\Delta) = \{(\beta) = \beta\mathcal{O}_\Delta \mid \beta \in K^*\}$

$\cup$

$P^+(\mathcal{O}_\Delta) = \{$ —"— $,\ N(\beta) > 0\}$

Quotients and double quotients

<u>Given</u>: • X = set, • G, H = groups ($e_G \in G$, $e_H \in H$ the neutral elements)

• <u>left action</u> of G on X: $g \in G, x \in X \rightsquigarrow gx \in X$, $(gg')x = g(g'x)$, $e_G x = x$

• <u>right action</u> of H on X: $x \in X, h \in H \rightsquigarrow xh \in X$, $x(hh') = (xh)h'$, $xe_H = x$

• <u>compatibility of actions</u>: $g(xh) = (gx)h$ $(= gxh)$

<u>Our case</u>: $X = \{(I, (\alpha_1, \alpha_2)) \mid I \in I(\mathcal{O}_\Delta), (\alpha_1, \alpha_2) = \text{positive basis of } I\}$

$G = K^*_{N>0} = \{\beta \in K^* \mid N(\beta) > 0\}$, $H = SL_2(\mathbb{Z})$, $\beta(I, (\alpha_1, \alpha_2))h = (\beta I, (\beta\alpha_1, \beta\alpha_2)h)$

In this case both G and H act freely on X: $g \neq e_G, h \neq e_H \Rightarrow \forall x \; gx \neq x \neq xh$

<u>Quotients</u>: $G \backslash X = X/$the equivalence relation $x \sim gx$ $(\forall x \in X \; \forall g \in G)$

X/H = ——— " ——— $x \sim xh$ (— " — $\forall h \in H$)

$G\backslash X/H$ = ——— " ——— $x \sim gxh$ $(\forall x \; \forall g \; \forall h)$

$$\begin{array}{ccccc} G\backslash X & \longleftarrow & X & \longrightarrow & X/H \\ \downarrow & & \downarrow & & \downarrow \\ (G\backslash X)/H & = & G\backslash X/H & = & G\backslash(X/H) \end{array}$$

<u>Thm</u> $(\sqrt{\Delta} \notin \mathbb{Z})$ (1) The map $\underline{X \longrightarrow I(\mathcal{O}_\Delta)}$, $(I, (\alpha_1, \alpha_2)) \longmapsto I$ is surjective and its fibres are precisely the H-orbits in $X \Rightarrow$ it identifies $\underline{X/H \xrightarrow{\sim} I(\mathcal{O}_\Delta)}$

(2) The map $\underline{X \longrightarrow \mathrm{Quad}(\Delta)}$, $(I, (\alpha_1, \alpha_2)) \longmapsto f_{I,(\alpha_1,\alpha_2)}(x,y)$ is surjective and its fibres are the $K^*_{N>0}$-orbits in $X \Rightarrow$ it identifies $\underline{G\backslash X \xrightarrow{\sim} \mathrm{Quad}(\Delta)}$.

(3) $f_{I,(\alpha_1,\alpha_2)}(x,y) \in \underline{\mathrm{Quad}_{prim}(\Delta)} \Longleftrightarrow \underbrace{\text{the inclusion } \mathcal{O}_\Delta \subset E(I) \text{ is an equality}}$

<u>Def</u>: I is a <u>proper</u> fractional $\mathcal{O}_\Delta$-ideal

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<u>Cor.</u>

$$\begin{array}{ccc} \mathrm{Quad}(\Delta) = K^*_{N>0}\backslash X & \longleftarrow \; X \; \longrightarrow & X/SL_2(\mathbb{Z}) = I(\mathcal{O}_\Delta) \\ \downarrow & & \downarrow \\ \mathrm{Quad}(\Delta)/SL_2(\mathbb{Z}) & = & K^*_{N>0}\backslash I(\mathcal{O}_\Delta) = P^+(\mathcal{O}_K)\backslash I(\mathcal{O}_K) \\ \cup & & \cup \qquad\qquad \cup \\ \mathrm{Quad}_{prim}(\Delta)/SL_2(\mathbb{Z}) & = & K^*_{N>0}\backslash I_{proper}(\mathcal{O}_\Delta) = P^+(\mathcal{O}_K)\backslash I_{proper}(\mathcal{O}_K) \end{array}$$

<u>Pf</u>. (1) If (β_1, β_2) = another positive basis of $I \Rightarrow \exists h \in SL_2(\mathbb{Z})$ $(\beta_1, \beta_2) = (\alpha_1, \alpha_2)h$ (and vice versa)

(2), (3) We can assume $(I, (\alpha_1, \alpha_2))$ by $(\alpha_1^{-1} I, (1, \pm \alpha_2 \alpha_1^{-1}))$ and assume that $I = \mathbb{Z}\cdot 1 \oplus \mathbb{Z}\cdot\alpha$, $a\alpha^2 + b\alpha + c = 0$, $a, b, c \in \mathbb{Z}$,

positive basis $\qquad \gcd(a,b,c) = 1$

$$f = f_{I,(1,\alpha)} = \frac{(x+\alpha y)(x+\alpha' y)}{N(I)} = \frac{ax^2 - bxy + cy^2}{a(\mathcal{O}_\Delta : I)}$$ What is $\Delta(f) = ?$

We (now) prove that $\Delta(f) = \Delta$ ($\Rightarrow$ surjectivity of $X \longrightarrow \mathrm{Quad}(\Delta)$).

$\Delta(\alpha) := b^2 - 4ac = (a(\alpha-\alpha'))^2 \in (K^{*2} \cap \mathbb{Q}^*) \setminus \mathbb{Q}^{*2} = \Delta \mathbb{Q}^{*2} \Rightarrow \exists n \in \mathbb{Q}_{>0}$ $\quad \underline{\Delta = \Delta(\alpha) n^2}$

$I = \mathcal{O}_\Delta$-ideal (fractional) $\Rightarrow \mathcal{O}_\Delta \subset E(I) = \mathbb{Z}\cdot 1 \oplus \mathbb{Z}\cdot a\alpha = \mathcal{O}_{\Delta(\alpha)} \Rightarrow \underline{n \in \mathbb{Z},\ n>0}$.

$(\mathcal{O}_\Delta : I)^{-1} = (I : \mathcal{O}_{\Delta(\alpha)})(\mathcal{O}_{\Delta(\alpha)} : \mathcal{O}_\Delta) = |a| n \Rightarrow f = \mathrm{sgn}(a) n (ax^2 - bxy + cy^2)$

$\Rightarrow \begin{cases} \underline{\Delta(f) = \Delta(\alpha) n^2 = \Delta}, \quad f \in \mathrm{Quad}(\Delta) \\ f \text{ is primitive} \iff n = 1 \iff \mathcal{O}_\Delta = E(I). \end{cases}$

The H-actions on $\mathrm{Quad}(\Delta)$ and $G\backslash X$ are the same: $f_{I,(\alpha_1,\alpha_2)}|h = f_{I,(\alpha_1,\alpha_2)h}$

Finally, if $\underbrace{f_{I,(\alpha_1,\alpha_2)}}_{\text{positive}} = \underbrace{f_{J,(\beta_1,\beta_2)}}_{\text{positive}} \Rightarrow \beta_1/\beta_2 = \alpha_1/\alpha_2$ or $\underbrace{\alpha_1'/\alpha_2'}_{\text{impossible if positive}}$

$\Rightarrow \gamma = \alpha_1/\beta_1 = \alpha_2/\beta_2 \in K^*$, $N(\gamma) = \dfrac{\alpha_1\alpha_2'}{\beta_1\beta_2'} \in \mathbb{Q}^*$, $\left.\begin{matrix}\mathrm{Im}(\alpha_1\alpha_2') < 0 \\ \mathrm{Im}(\beta_1\beta_2') < 0\end{matrix}\right\} \Rightarrow N(\gamma) > 0$,

$\Rightarrow (J,(\beta_1,\beta_2)) \in K^*_{N>0}(I,(\alpha_1,\alpha_2))$.

Invertible ideals and the (abelian) group law on $Cl^+(\Delta)$

<u>Def.</u> $I, J \subset (K,+)$ additive subgroups

$IJ := \left\{ \sum_{k=1}^{n} x_k y_k \mid x_k \in I, y_k \in J, n \geq 0 \right\} \subset (K,+)$ additive subgroup
$(= JI)$

$(I_1 I_2)J = I_1(I_2 J)$, $\quad \underbrace{\left(\sum_{i=1}^{m} \mathbb{Z}a_i\right)}_{I} \underbrace{\left(\sum_{j=1}^{n} \mathbb{Z}b_j\right)}_{J} = \sum_{i,j} \mathbb{Z}a_i b_j$

I, J $\mathcal{O}_\Delta$-ideals $\Rightarrow$ so is IJ
(fractional)

<u>Ex:</u> $K = \mathbb{Q}(i)$, $\Delta = -16$, $\mathcal{O}_\Delta = \mathbb{Z}[2i] = \mathbb{Z}\cdot 1 \oplus \mathbb{Z}\cdot 2i$

$I = 2\mathbb{Z}[i] = \mathbb{Z}\cdot 2 \oplus \mathbb{Z}\cdot 2i \supsetneq J = \mathbb{Z}\cdot 2 \oplus \mathbb{Z}\cdot 4i = 2\mathcal{O}_\Delta$, $E(I) = \mathbb{Z}[i]$, $E(J) = \mathbb{Z}[2i]$

$I\cdot I\ (= I^2) = \mathbb{Z}\cdot 4 \oplus \mathbb{Z}\cdot 4i = I\cdot J = \mathbb{Z}\cdot 4 + \mathbb{Z}\cdot 8i + \mathbb{Z}\cdot 4i + \mathbb{Z}\cdot(-8) = \mathbb{Z}\cdot 4 \oplus \mathbb{Z}\cdot 4i$

<u>Def.</u> A fractional $\mathcal{O}_\Delta$-ideal I is <u>invertible</u> if $\exists$ fractional $\mathcal{O}_\Delta$-ideal J $\quad IJ = \underbrace{(1)}_{\mathcal{O}_\Delta}$

<u>Prop</u> (True for <u>any integral domain</u>) Let I = fractional $\mathcal{O}_\Delta$-ideal.

(1) $I^{-1} := \{\beta \in K \mid \beta I \subset \mathcal{O}_\Delta\}$ is a fractional $\mathcal{O}_\Delta$-ideal and $II^{-1} \subset (1)$.

(2) I is invertible $\iff II^{-1} = (1) \Rightarrow J$ in Def. above is $J = I^{-1}$
$\Rightarrow E(I) = \mathcal{O}_\Delta$ (I is a <u>proper</u> $\mathcal{O}_\Delta$-ideal)

<u>Pf.</u> (1) $II^{-1} \subset (1)$ is automatic; if $0 \neq \alpha \in I \Rightarrow (\alpha) \subset I \Rightarrow (\alpha^{-1}) = (\alpha)^{-1} \supset I^{-1} \Rightarrow \alpha I^{-1} \subset \mathcal{O}_\Delta$ ($\mathcal{O}_\Delta$-ideal).

2) If $IJ = (1) \Rightarrow J \subset I^{-1} \Rightarrow (1) = IJ \subset II^{-1} \subset (1) \Rightarrow II^{-1} = (1)$,
$I^{-1} = I^{-1}(1) = I^{-1}IJ = (1)J = J$. If $\beta I \subset I \Rightarrow \beta IJ \subset IJ \Rightarrow \beta\cdot 1 \in \mathcal{O}_\Delta$.

<u>Cor.</u> Invertible ideals (of any domain) form a group under multiplication.

Prop. (Specific to $K=\mathbb{Q}(\sqrt{\Delta})$). Let $I \in I(\mathcal{O}_\Delta)$.

(1) $I' = \{\alpha' \mid \alpha \in I\} \in I(\mathcal{O}_\Delta)$; (2) $II' = (E(I):I)\,E(I)$

(3) I is invertible $\Longleftrightarrow$ $E(I) = \mathcal{O}_\Delta$ ($\Leftrightarrow$ I is a proper fractional $\mathcal{O}_\Delta$-ideal)

(4) $I, J \in I(\mathcal{O}_\Delta)$ invertible $\Longrightarrow$ $N(IJ) = N(I)N(J)$.

Pf: (1) Clear; (2) Can replace I by βI; then $I = \mathbb{Z}\cdot 1 \oplus \mathbb{Z}\cdot\alpha$,

$I' = \mathbb{Z}\cdot 1 \oplus \mathbb{Z}\cdot\alpha'$, $a\alpha^2 + b\alpha + c = 0$, $\gcd(a,b,c)=1$, $E(I) = \mathbb{Z}\cdot 1 \oplus \mathbb{Z}\cdot a\alpha \supset \mathcal{O}_\Delta$

$$\Rightarrow aII' = \mathbb{Z}\cdot a + \mathbb{Z}\cdot a\alpha + \mathbb{Z}\cdot \underbrace{a\alpha'}_{-b-a\alpha} + \mathbb{Z}\cdot \underbrace{a\alpha\alpha'}_{c} = \mathbb{Z}\cdot a\alpha + \underbrace{\mathbb{Z}a + \mathbb{Z}b + \mathbb{Z}c}_{\mathbb{Z}\gcd(a,b,c) = \mathbb{Z}} = E(I)$$

$(E(I):I) = (I:E(I))^{-1} = |a|^{-1}$.

(3) ($\Rightarrow$) always holds; ($\Leftarrow$) If $E(I) = \mathcal{O}_\Delta \overset{(2)}{\Longrightarrow} J = (\mathcal{O}_\Delta : I)^{-1} I'$ satisfies $IJ = \mathcal{O}_\Delta$.

(4) Apply (2) and (3) to I, J and IJ: $II' = N(I)\mathcal{O}_\Delta$, $JJ' = N(J)\mathcal{O}_\Delta$,

$(IJ)(IJ)' = N(IJ)\mathcal{O}_\Delta \Longrightarrow N(I)N(J)/N(IJ) \in \mathcal{O}_\Delta^* \cap \mathbb{Q}^*_{>0} = \{1\}$.

Cor. $\underbrace{I_{inv}(\mathcal{O}_\Delta)}_{\text{abelian group under multiplication}} = \{\text{invertible } I \in I(\mathcal{O}_\Delta)\} = I_{proper}(\mathcal{O}_\Delta)$

abelian group under multiplication, $N: I_{inv}(\mathcal{O}_\Delta) \longrightarrow \mathbb{Q}^*_{>0}$ group morphism

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Def: $\mathrm{Pic}^+(\mathcal{O}_\Delta) = I_{inv}(\mathcal{O}_K)/P^+(\mathcal{O}_\Delta)$ the narrow Picard group of $\mathcal{O}_\Delta$

$\mathrm{Pic}(\mathcal{O}_\Delta) = I_{inv}(\mathcal{O}_K)/P(\mathcal{O}_\Delta)$ the —— " ——

Thm above: $Cl^+(\Delta) = \mathrm{Quad}_{prim}(\Delta)/SL_2(\mathbb{Z}) \xrightarrow{\sim} \mathrm{Pic}^+(\mathcal{O}_\Delta)$ finite abelian group

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If $\Delta < 0$: $N(K^*) \subset \mathbb{Q}^*_{>0}$, $P^+(\mathcal{O}_\Delta) = P(\mathcal{O}_\Delta)$, $\mathrm{Pic}^+(\mathcal{O}_\Delta) = \mathrm{Pic}(\mathcal{O}_\Delta)$. **Def:** $Cl(\Delta) = Cl^+(\Delta)$

If $\Delta > 0$ ($\sqrt{\Delta} \notin \mathbb{Z}$): $\mathcal{O}_\Delta^* = \{\pm 1\} \times \varepsilon^{\mathbb{Z}}$, $\varepsilon =$ the fundamental unit of $\mathcal{O}_\Delta$

$$\begin{array}{c} \cup \\ \mathcal{O}^*_{\Delta, N=1} \end{array} = \{\pm 1\} \times \begin{cases} \varepsilon^{\mathbb{Z}} & \text{if } N(\varepsilon) = 1 \\ \varepsilon^{2\mathbb{Z}} & \text{if } N(\varepsilon) = -1 \end{cases}$$

$pr: \mathrm{Pic}^+(\mathcal{O}_\Delta) \longrightarrow \mathrm{Pic}(\mathcal{O}_\Delta)$, $IP^+(\mathcal{O}_\Delta) \longmapsto IP(\mathcal{O}_\Delta)$ is surjective

$\mathrm{Ker}(pr) = P(\mathcal{O}_\Delta)/P^+(\mathcal{O}_\Delta)$, $K^*/\mathcal{O}_\Delta^* \xrightarrow{\sim} P(\mathcal{O}_\Delta)$, $K^*_{N>0}/\mathcal{O}^*_{\Delta,N=1} \xrightarrow{\sim} P^+(\mathcal{O}_\Delta)$

$\beta\mathcal{O}_\Delta^* \longmapsto (\beta)$ $\qquad$ $\beta\mathcal{O}^*_{\Delta,N=1} \longmapsto (\beta)$

$$\Rightarrow \mathrm{Ker}(pr) \simeq K^*/\mathcal{O}_\Delta^* K^*_{N>0} \xrightarrow[\sim]{\operatorname{sgn}\circ N} \mathbb{Z}^*/N(\mathcal{O}_\Delta^*) = \begin{cases} \{1\} & \text{if } N(\varepsilon) = -1 \\ \{\pm 1\} & \text{if } N(\varepsilon) = 1 \end{cases}$$

Back to quadratic forms:

$Cl(\Delta) = Cl^+(\Delta)$ if $\Delta < 0$

$$\begin{array}{ccc} Cl^+(\Delta) & \xrightarrow{\sim} & \mathrm{Pic}^+(\mathcal{O}_\Delta) \\ \downarrow pr & & \downarrow pr \\ Cl(\Delta) & \xrightarrow{\sim} & \mathrm{Pic}(\mathcal{O}_\Delta) \end{array}$$

finite abelian groups

If $\Delta > 0$: $Cl(\Delta) = Cl^+(\Delta)/$ the equivalence relation $\left(f \sim -f \mid \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right)$

$$|Cl^+(\Delta)| = |Cl(\Delta)| \cdot \begin{cases} 1 & N(\varepsilon) = -1 \\ 2 & N(\varepsilon) = 1 \end{cases}$$

Ex: $\Delta = 4\cdot 34$, $Cl^+(\Delta) \simeq \mathbb{Z}/4\mathbb{Z}$, $\varepsilon = 35 + 6\sqrt{34}$, $N(\varepsilon) = 1$, $Cl(\Delta) \simeq \mathbb{Z}/2\mathbb{Z}$

Solutions of $f(x,y)=m$ $(x,y\in\mathbb{Z})$ and ideals

Assume: $f\in \mathrm{Quad}_{prim}(\Delta)$, $\sqrt{\Delta}\notin\mathbb{Z}$, $m\in\mathbb{Z}\setminus\{0\}$.
After multiplying both f and m by -1 we can assume that $m>0$.
$\exists$ invertible fractional (or even honest) $\mathcal{O}_\Delta$-ideal I and a positive basis α_1,α_2 of $I=\mathbb{Z}\alpha_1+\mathbb{Z}\alpha_2$ such that

$$f(x,y)=\frac{N(\alpha_1 x+\alpha_2 y)}{N(I)}.$$

let $[I]\in \mathrm{Pic}^+(\mathcal{O}_\Delta)$ be the class of I. ($\Leftrightarrow$ the class $[f]\in Cl^+(\Delta)$)

So: $f(x,y)=m$, $x,y\in\mathbb{Z}\iff \alpha=\alpha_1x+\alpha_2\in I$, $N(\alpha)=N(I)m$ (>0).
As $(\alpha)\subset I$, $J=(\alpha)I^{-1}\subset\mathcal{O}_\Delta$ is an invertible $\mathcal{O}_\Delta$-ideal,
$N(J)=N((\alpha))\,N(I^{-1})=|N(\alpha)|/N(I)=m$, $[J]=[I]^{-1}$.
this gives a bijection

$$\{\alpha\in I \mid N(\alpha)=N(I)m\}/\mathcal{O}^*_{\Delta,N=1}\ \overset{\sim}{\longleftrightarrow}\ \left\{\begin{array}{l} J\subset\mathcal{O}_\Delta \text{ invertible } \mathcal{O}_\Delta\text{-ideal} \\ N(J)=m,\ [J]=[I]^{-1}\end{array}\right\}$$

Fundamental discriminants

Recall: $\Delta\equiv 0,1 \pmod 4$ is a fundamental discriminant
$\iff \Delta\neq\Delta' n^2$, $\Delta'\equiv 0,1\pmod 4$, $n\in\mathbb{Z}_{>1}\iff \Delta=\begin{cases} d & d\equiv 1\pmod 4\\ 4d & d\equiv 2,3\pmod 4\end{cases}$, d square-free

Prop. Assume that $\Delta\neq 1$ is a fundamental discriminant ($\Rightarrow\sqrt{\Delta}\notin\mathbb{Z}$).
(1) If $\Delta'\in\mathbb{Z}\setminus\{0\}$ and $\sqrt{\Delta'}\in\mathbb{Q}(\sqrt{\Delta})\Rightarrow\exists\, n\in\mathbb{Z}_{\geq 1}$ $\Delta'=\Delta n^2$
(2) Every fractional $\mathcal{O}_\Delta$-ideal is invertible ("$\mathcal{O}_\Delta$ is a Dedekind ring").
For such rings we call $\mathrm{Pic}(\mathcal{O}_\Delta)=Cl(\mathcal{O}_\Delta)$ (resp. $\mathrm{Pic}^+(\mathcal{O}_\Delta)=Cl^+(\mathcal{O}_\Delta)$) the (ideal) class group (resp. the narrow (ideal) class group).

(3) $\forall n\geq 1$
$$\frac{|Cl(\Delta n^2)|}{|Cl(\Delta)|}=\frac{n}{(\mathcal{O}^*_\Delta:\mathcal{O}^*_{\Delta n^2})}\prod_{\substack{p\mid n\\ p\nmid\Delta}}\left(1-\left(\frac{\Delta}{p}\right)\frac{1}{p}\right)\qquad (p \text{ prime})$$

Pf: (1) Clear; (2) $I\in I(\mathcal{O}_\Delta)\Rightarrow E(I)=\mathcal{O}_{\Delta'}$, $\Delta'=\Delta/m^2$ for some $m\geq 1$
Δ fundamental $\Rightarrow m=1\Rightarrow I$ invertible. (3) Later.

Cor. If $\Delta<-4$ is fundamental, then $\mathcal{O}^*_\Delta=\mathcal{O}^*_{\Delta n^2}=\{\pm 1\}\Rightarrow$

$$\frac{|Cl(\Delta n^2)|}{|Cl(\Delta)|}=n\prod_{\substack{p\mid n\\ p\nmid\Delta}}\left(1-\left(\frac{\Delta}{p}\right)\frac{1}{p}\right),\qquad (p \text{ prime})$$

An alternative approach

Some authors allow negative definite forms for $\Delta<0$. This does not change $Cl(\Delta)$, but $Cl^+(\Delta)$ becomes $Cl(\Delta)\times\{\pm1\}$ $(\Delta<0)$.

How does it work: $\Delta\equiv 0,1 \pmod 4$, $\sqrt{\Delta}\notin\mathbb{Z}$

Def: $I_{or}(\mathcal{O}_\Delta)=\{\text{oriented fractional } \mathcal{O}_\Delta\text{-ideals}\}=\{(I,\varepsilon)\mid I\in I(\mathcal{O}_\Delta),\ \varepsilon\in\{\pm1\}\}=I(\mathcal{O}_\Delta)\times\{\pm1\}$

$P_{or}(\mathcal{O}_\Delta)=\{\text{principal} - \text{"} - \}=\{((\beta),\operatorname{sgn}(N(\beta)))\mid \beta\in K^*\}\subset I_{or}(\mathcal{O}_\Delta)$

$I_{or,inv}(\mathcal{O}_\Delta)=I_{inv}(\mathcal{O}_\Delta)\times\{\pm1\}\quad(\supset P_{or}(\mathcal{O}_\Delta))$

$\operatorname{Pic}_{or}(\mathcal{O}_\Delta)=I_{or,inv}(\mathcal{O}_\Delta)/P_{or}(\mathcal{O}_\Delta)$ (the oriented Picard group of $\mathcal{O}_\Delta$)

If $\Delta<0$: $\forall\beta\in K^*$ $\operatorname{sgn}(N(\beta))=+1$, $\operatorname{Pic}_{or}(\mathcal{O}_\Delta)=\operatorname{Pic}(\mathcal{O}_\Delta)\times\{\pm1\}$

If $\Delta>0$:

$$\begin{array}{ccccc}
\overbrace{P^+(\mathcal{O}_\Delta)}^{K^*_{N>0}/\mathcal{O}^*_{\Delta,N=1}} & \subset & I_{inv}(\mathcal{O}_\Delta)=I_{inv}(\mathcal{O}_\Delta)\times\{1\} & & I_{inv}(\mathcal{O}_\Delta)/P^+(\mathcal{O}_\Delta)=\operatorname{Pic}^+(\mathcal{O}_\Delta)\\
\cap & & \cap & \Longrightarrow & \downarrow \qquad\qquad \downarrow\\
\underbrace{P_{or}(\mathcal{O}_\Delta)}_{K^*/\mathcal{O}^*_{\Delta,N=1}} & \subset & I_{or,inv}(\mathcal{O}_\Delta) & & I_{or,inv}(\mathcal{O}_\Delta)/P_{or}(\mathcal{O}_\Delta)=\operatorname{Pic}_{or}(\mathcal{O}_\Delta)
\end{array}$$

$\exists\beta\in K^*\ N(\beta)<0 \Longrightarrow \operatorname{Pic}^+(\mathcal{O}_\Delta)\xrightarrow{\sim}\operatorname{Pic}_{or}(\mathcal{O}_\Delta)$ isomorphism

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Modifying earlier definitions: "mod" = modified

$\operatorname{Quad}(\Delta)^{mod}=\{f=ax^2+bxy+cy^2\mid a,b,c\in\mathbb{Z},\ b^2-4ac=\Delta\}$

$X^{mod}=\{(I,\varepsilon,(\alpha_1,\alpha_2))\mid I=\mathbb{Z}\alpha_1\oplus\mathbb{Z}\alpha_2\in I(\mathcal{O}_\Delta),\ \varepsilon\in\{\pm1\},\ \underbrace{\varepsilon=\text{the orientation of }(\alpha_1,\alpha_2)}_{\operatorname{sgn}(\operatorname{Irr}(\alpha_1\alpha_2'))=-\varepsilon}\}$

$\beta\in G^{mod}=K^*\supset K^*_{N>0}=G$, $h\in H^{mod}=GL_2(\mathbb{Z})\supset SL_2(\mathbb{Z})=H$,

actions: $\beta(I,\varepsilon,(\alpha_1,\alpha_2))h=(\beta I,\varepsilon\operatorname{sgn}(\beta),(\beta\alpha_1,\beta\alpha_2)h)$

quadratic form: $f_{I,\varepsilon,(\alpha_1,\alpha_2)}(x,y)=\varepsilon\dfrac{N(\alpha_1x+\alpha_2y)}{N(I)}\in\operatorname{Quad}(\Delta)^{mod}$

$f_{\beta(I,\varepsilon,(\alpha_1,\alpha_2))h}=\det(h)\left(f_{I,\varepsilon,(\alpha_1,\alpha_2)}\mid h\right)$ (since $((\alpha_1,\alpha_2)h)\binom{x}{y}=(\alpha_1,\alpha_2)(h\binom{x}{y})$)

$$\begin{array}{ccccccc}
f_{I,\varepsilon,(\alpha_1,\alpha_2)} & \longleftarrow\!\shortmid & (I,\varepsilon,(\alpha_1,\alpha_2)) & \longmapsto & (I,\varepsilon) & \longmapsto & I\\
\pitchfork & & \pitchfork & & \pitchfork & & \pitchfork\\
\operatorname{Quad}(\Delta)^{mod}\simeq G^{mod}\backslash X & \longleftarrow & X^{mod} & \longrightarrow & X^{mod}/H\simeq I_{or}(\mathcal{O}_\Delta) & \longrightarrow & X^{mod}/H^{mod}\simeq I(\mathcal{O}_\Delta)\\
\downarrow & & \downarrow & & \downarrow & & \downarrow\\
\operatorname{Quad}(\Delta)^{mod}/SL_2(\mathbb{Z}) & \simeq & G^{mod}\backslash X^{mod}/H & \simeq & K^*\backslash I_{or}(\mathcal{O}_\Delta) & \longrightarrow & \underbrace{G^{mod}\backslash X^{mod}/H^{mod}}_{K^*\backslash I(\mathcal{O}_\Delta)}\\
\cup & & & & \cup & & \cup\\
\underbrace{\operatorname{Quad}(\Delta)^{mod}_{prim}/SL_2(\mathbb{Z})}_{Cl^+(\Delta)^{mod}} & & \overset{\sim}{\longleftrightarrow} & & \underbrace{K^*\backslash I_{or,inv}(\mathcal{O}_\Delta)}_{\operatorname{Pic}_{or}(\mathcal{O}_\Delta)} & & \underbrace{K^*\backslash I_{inv}(\mathcal{O}_\Delta)}_{\operatorname{Pic}(\mathcal{O}_\Delta)}\\
\rightsquigarrow Cl(\Delta) & & & \overset{\sim}{\longleftrightarrow} & & \searrow & \operatorname{Pic}(\mathcal{O}_\Delta)
\end{array}$$

quotient by the action of $H^{mod}/H=GL_2(\mathbb{Z})/SL_2(\mathbb{Z})\xrightarrow[\sim]{\det}\{\pm1\}\ni\lambda$ given by $f\longmapsto\lambda f\mid\begin{pmatrix}1&0\\0&\lambda\end{pmatrix}$.

$Cl^+(\Delta)^{mod}=Cl^+(\Delta)\times\begin{cases}\{1\} & \Delta>0\\ \{\pm1\} & \Delta<0.\end{cases}$