Dirichlet's theorem on primes, ζ- and L-functions

Thm (Euler) $\sum_{p\in P} \frac{1}{p} = \infty$

Pf: $\forall x \geq 1$ $(x\in\mathbb{N})$

$$\prod_{p\leq x}\left(1-\frac{1}{p}\right)^{-1} = \prod_{p\leq x}\left(1+\frac{1}{p}+\frac{1}{p^2}+\dots\right) \geq \sum_{n=1}^{x}\frac{1}{n} > \int_1^x \frac{dt}{t} = \log(x)$$

$-\log(1-t) = \sum_{k\geq 1}\frac{t^k}{k}$ $(|t|<1)$

$$\Longrightarrow \underbrace{\sum_{p\leq x} \underbrace{-\log\left(1-\frac{1}{p}\right)}_{\frac{1}{p}+\sum_{k\geq 2}\frac{1}{kp^k}}}_{\sum_{p\leq x}\frac{1}{p} + \underbrace{\sum_{k\geq 2}\sum_{p\leq x}\frac{1}{kp^k}}_{\leq \sum_{n\geq 2}\sum_{k\geq 2}\frac{1}{n^k} = \sum_{n\geq 2}\frac{1}{n(n-1)} = 1}} \geq \log\log(x)$$

$$\Longrightarrow \sum_{p\leq x}\frac{1}{p} \geq \log\log|x| - 1$$

Variant: for $\mathrm{Re}(s)>1$, $\underbrace{\prod_{p\in P}\left(1-\frac{1}{p^s}\right)^{-1}}_{\text{"Euler's product"}} = \sum_{n=1}^{\infty}\frac{1}{n^s} = \zeta(s)$ both sides absolutely convergent (and holomorphic for $\mathrm{Re}(s)>1$)

(the Riemann ~~zeta~~-function)

$\zeta(s) \sim \frac{1}{s-1}$ as $s\to 1$ $\quad\left(\lim_{s\to 1}(s-1)\zeta(s) = 1\right)$

$$\Longrightarrow \sum_{p\in P}\underbrace{-\log\left(1-\frac{1}{p^s}\right)}_{\frac{1}{p^s}+\sum_{k\geq 2}\frac{1}{kp^{ks}}} = \log\zeta(s), \quad \sum_{p\in P}\sum_{k\geq 2}\left|\frac{1}{kp^{ks}}\right| \leq 1, \text{ as above}$$

(principal branch of $\log(1-t)$)

$$\Longrightarrow \sum_{p\in P}\frac{1}{p^s} \sim \log\frac{1}{s-1} \quad \text{as } s\to 1$$

Dirichlet: considered $P_{a\,(\mathrm{mod}\, m)} = P_{\equiv a[m]} = \{p\in P \mid p\equiv a \pmod m\}$

$m\geq 1$, $\gcd(a,m) = 1$

Ex $(m=4)$: $L(s) = \prod_{p\equiv 1[4]}\left(1-\frac{1}{p^s}\right)^{-1}\prod_{p\equiv 3[4]}\left(1+\frac{1}{p^s}\right)^{-1} = 1-\frac{1}{3^s}+\frac{1}{5^s}-\frac{1}{7^s}+\dots$

(Dirichlet L-function) (absolutely convergent if $\mathrm{Re}(s)>1$)

$$\zeta(s)L(s) = \prod_{p\equiv 1[4]}\left(1-\frac{1}{p^s}\right)^{-2}\prod_{p\equiv 3[4]}\left(1-\frac{1}{p^{2s}}\right)^{-1}\left(1-\frac{1}{2^s}\right)^{-1}.$$

As $s\to 1$:

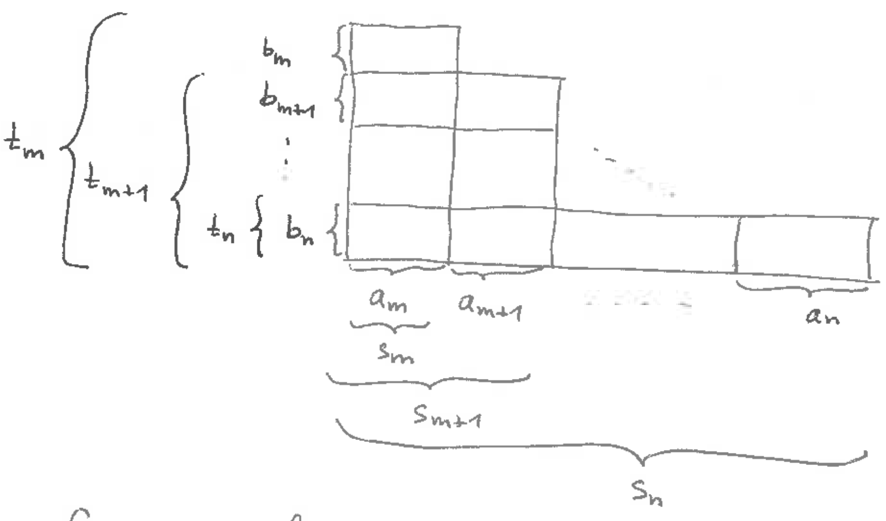
$L(s) \to 1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\dots = \frac{\pi}{4} \neq 0, \infty$, $\log\zeta(s) \sim \log\frac{1}{s-1} \sim \sum_{p\in P}\frac{1}{p^s}$

$$\underbrace{\log(\zeta(s)L(s))}_{\sim\log\frac{1}{s-1}} \sim 2\sum_{p\equiv 1[4]}\frac{1}{p^s} + \underbrace{\sum_{p\equiv 3[4]}\frac{1}{p^{2s}}}_{\text{bounded and holomorphic}} \sim 2\sum_{p\equiv 1[4]}\frac{1}{p^s}$$

$$\Longrightarrow \sum_{p\equiv 1[4]}\frac{1}{p^s} \sim \sum_{p\equiv 3[4]}\frac{1}{p^s} \sim \frac{1}{2}\log\frac{1}{s-1}.$$

Abel's summation and convergence of Dirichlet series $\sum_{n=1}^{\infty} \frac{a_n}{n^s}$

Abel's summation: $$\sum_{k=m}^{n} a_k t_k = \sum_{k=m}^{n} s_k b_k, \quad s_k = a_m + \dots + a_k, \quad t_k = b_k + \dots + b_n \qquad (*)$$



Convergence of power series:

Thm (Abel) If $a_n \in \mathbb{C}$, $\sum a_n \longrightarrow a \in \mathbb{C}$, then $\sum_{1}^{\infty} a_n x^n$ converges uniformly for $x \in [0,1]$ ($\Rightarrow$ to a continuous function $f(x)$ satisfying $\lim_{x \to 1-} f(x) = a$).

Pf. Fix $x \in [0,1]$ and apply $(*)$ with $t_k = x^k$ ($\Rightarrow b_n = x^n$, $b_k = x^k - x^{k+1}$ if $m \le k < n$):

$$\left| \sum_{k=m}^{n} a_k x^k \right| = \left| \sum_{k=m}^{n} s_k b_k \right| \le \left(\max_{m \le k \le n} \big| \underbrace{a_m + \dots + a_k}_{s_k} \big| \right) \underbrace{\sum_{k=m}^{n} |b_k|}_{\sum_{k=m}^{n} b_k = t_m = x^m} \quad \left(\begin{matrix} < \varepsilon x^m \\ \text{if } m \ge N(\varepsilon) \end{matrix} \right)$$

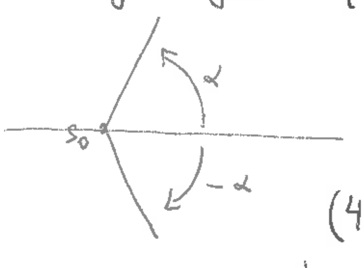
$\left(\forall \varepsilon > 0 \ \exists N(\varepsilon) \ \forall k > m \ge N(\varepsilon) \quad |a_m + \dots + a_k| < \varepsilon \right)$

Convergence of Dirichlet series $f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$, $a_n \in \mathbb{C}$ $\left(\frac{1}{n^s} = e^{-s \log(n)} \right)$.

Thm (1) If $\exists C \ \forall n \ |a_n| \le C \Longrightarrow f(s)$ converges absolutely (and uniformly on compact subsets) on $\{ s \in \mathbb{C} \mid \mathrm{Re}(s) > 1 \}$ ($\xRightarrow{\text{Weierstrass}}$ to a holomorphic function).

(2) If f converges for $s_0 = \sigma_0 + i t_0 \Longrightarrow f(s)$ converges uniformly in every region $\{ s \in \mathbb{C} \mid \mathrm{Re}(s) \ge \sigma_0, \ |\mathrm{Arg}(s - s_0)| \le \alpha \}$ $\quad 0 < \alpha < \frac{\pi}{2}$ fixed.

$\Downarrow$



(3) $f(s)$ converges to a holomorphic function in $\{ s \in \mathbb{C} \mid \mathrm{Re}(s) > \sigma_0 \}$

$\Downarrow$

(4) $\exists \ \sigma(f) \in \mathbb{R} \cup \{\pm\infty\}$ such that $f(s)$ converges to a holomorphic function if $\mathrm{Re}(s) > \sigma(f)$ and diverges if $\mathrm{Re}(s) < \sigma(f)$.

(5) If $\exists C \ \forall N \ \left| \sum_{n=1}^{N} a_n \right| \le C \Longrightarrow \sigma(f) \le 0$.

Pf. (1) If $\mathrm{Re}(s)=\sigma \geq 1+\delta$, $\delta>0$ fixed $\Rightarrow \left|\sum a_n n^{-s}\right| \leq C \sum \overbrace{|n^{-s}|}^{n^{-\sigma}} \leq C \sum n^{-(1+\delta)} \leq C \int_1^\infty \frac{dt}{t^{1+\delta}} < \infty$

(2) Replace s by $s-s_0 \rightsquigarrow s=0$, $\sum a_n$ converges. Fix $s=\sigma+it$, $\sigma>0$.
Abel's summation with $t_k = k^{-s}$ ($b_n = n^{-s}$, $b_k = k^{-s}-(k+1)^{-s}$ if $m \leq k < n$):

$$\sum_{k=m}^{n} a_k k^{-s} = \sum_{k=m}^{n} s_k b_k, \qquad s_k = a_m + \cdots + a_k$$

Lemma. If $0<\alpha<\beta$, $s=\sigma+it$, $\sigma>0$, then

$$\left|e^{-\alpha s} - e^{-\beta s}\right| = \left| s \int_\alpha^\beta e^{-ts}\,dt \right| \leq |s| \int_\alpha^\beta \underbrace{|e^{-ts}|}_{e^{-t\sigma}}\,dt = \frac{|s|}{\sigma}\left(e^{-\alpha\sigma} - e^{-\beta\sigma}\right)$$

Therefore $\displaystyle\sum_{k=m}^{n} |b_k| \leq \frac{|s|}{\sigma}\left(\sum_{k=m}^{n-1}\left(k^{-\sigma}-(k+1)^{-\sigma}\right) + n^{-\sigma}\right) = \frac{|s|}{\sigma} m^{-\sigma} \quad \left(\leq \frac{|s|}{\sigma}\right)$

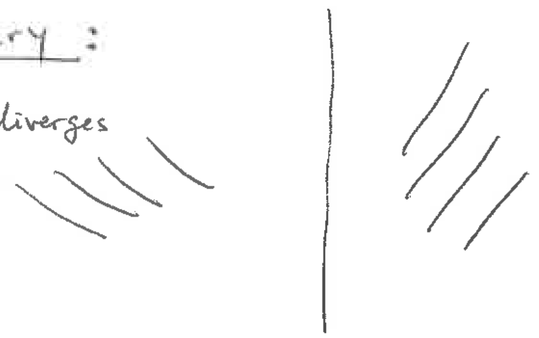
$$\left|\sum_{k=m}^{n} a_k k^{-s}\right| \leq \underbrace{\left(\max_{m\leq k\leq n} |s_k|\right)}_{<\,\varepsilon \text{ if } n>m\geq N(\varepsilon)} \frac{|s|}{\sigma} \underbrace{m^{-\sigma}}_{\leq 1} \quad \Longrightarrow \text{ result}$$

(2) $\Rightarrow$ (3) Weierstrass's thm; (3) $\Rightarrow$ (4) automatic

(5) $\forall k\ |s_k| \leq 2C \Longrightarrow \left|\sum_{k=m}^{n} a_k k^{-s}\right| \leq \frac{2C|s|}{\sigma} m^{-\sigma} \xrightarrow[m\to+\infty]{} 0$ for fixed $s=\sigma+it$, $\sigma>0$.

Summary:

$f(s)$ diverges



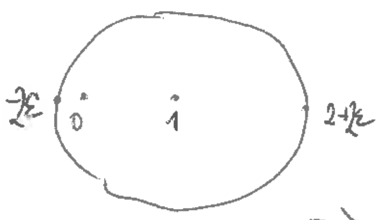
$f(s)=\sum_{n=1}^\infty \frac{a_n}{n^s}$ converges to a holomorphic function

$\mathrm{Re}=\sigma(f)$

What about the boundary line $\mathrm{Re}(s)=\sigma(f)$?

Thm (E. Landau) If $\forall n\ a_n \geq 0$ and $f(s)=\sum_{n=1}^\infty \frac{a_n}{n^s}$ has $\sigma(f)\in\mathbb{R}$, then f does not converge at $s=\sigma(f)$.

Pf. After replacing s by $s-\sigma(f)$ we can assume that $\sigma(f)=0$.
If f converges at $s=0 \Rightarrow$ it is holomorphic in a disc $|s-1| < 1+\frac{\varepsilon}{2}$, for some $\varepsilon>0 \Longrightarrow f(s)=\sum_{k=0}^\infty \frac{1}{k!}(s-1)^k f^{(k)}(1)$ if



As $f^{(k)}(s)=\sum_{n=1}^\infty \frac{a_n(-\log n)^k}{n^s}$ for $\mathrm{Re}(s)>0$ (Weierstrass)

$\Longrightarrow f(-\varepsilon) = \sum_{k=0}^\infty \frac{(1+\varepsilon)^k}{k!} \sum_{n=1}^\infty \frac{a_n}{n} \log(n)^k$ positive terms, converges

$\Longrightarrow f(-\varepsilon) = \sum_{n=1}^\infty \frac{a_n}{n} \underbrace{\sum_{k=0}^\infty \frac{(1+\varepsilon)^k}{k!}\log(n)^k}_{e^{(1+\varepsilon)\log n}} = \sum_{n=1}^\infty a_n n^\varepsilon$ converges $\Rightarrow \sigma(f) \leq -\varepsilon$ contradiction.

Prop. $\exists$ holomorphic function f on $\{\mathrm{Re}(s)>0\}$ such that
$$f(s)+\frac{1}{s-1}=\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s} \quad \text{if } \mathrm{Re}(s)>1.$$

Pf: For $\mathrm{Re}(s)>1$,
$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}=\sum_{n=1}^{\infty}s\int_n^{\infty}\frac{dt}{t^{s+1}}=s\int_1^{\infty}\Big(\sum_{n\le t}1\Big)\frac{dt}{t^{s+1}}=s\int_1^{\infty}\frac{[t]\,dt}{t^{s+1}}$$
$$\int_1^{\infty}\frac{[t]\,dt}{t^{s+1}}=\int_1^{\infty}\frac{t\,dt}{t^{s+1}}-\int_1^{\infty}\frac{(t-[t])\,dt}{t^{s+1}}=\frac{1}{s-1}-g(s), \quad g(s)=\int_1^{\infty}\frac{(t-[t])\,dt}{t^{s+1}}$$
holomorphic for $\mathrm{Re}(s)>0$

Take $f(s)=1-s\,g(s)$.

Cor: $\lim_{s\to 1}(s-1)\zeta(s)=1 \quad (\mathrm{Re}(s)>1)$

Infinite products

Recall: if $b_n\in\mathbb{C}$, $b_n\neq -1$, $\prod_{n=1}^{\infty}(1+b_n)$ is convergent if $\lim_{n\to\infty}\prod_{k=1}^{n}(1+b_k)$ exists and is non-zero. It is absolutely convergent if $\prod_{n=1}^{\infty}(1+|b_n|)$ is convergent ($\Longleftrightarrow \sum_{n=1}^{\infty}|a_n|$ is convergent)

$\Rightarrow \prod_{n=1}^{\infty}(1+b_n)$ is convergent and its value does not change if we reorder the terms of the product.

Prop. If $f(s)=\sum_1^{\infty}a_n n^{-s}$ and $g(s)=\sum_1^{\infty}b_n n^{-s}$ converge absolutely for $\mathrm{Re}(s)>\sigma_0$

$\Rightarrow$ so does $h(s)=\sum_1^{\infty}c_n n^{-s}$, where $c_n=\sum_{dd'=n}a_d b_{d'}$, and $h(s)=f(s)g(s)$.

Pf: $\sum_{n=1}^{\infty}\left|\frac{c_n}{n^s}\right|\le\sum_{d,d'=1}^{\infty}\frac{|a_d|\,|b_{d'}|}{d^{\sigma}\,d'^{\sigma}}<\infty$ if $s=\sigma+it$, $\sigma>\sigma_0$. Use Cauchy's thm on a product of absolutely convergent series.

Note: (1) If $f(s)=\sum_1^{\infty}\frac{a_n}{n^s}$ converges for $s_0=\sigma_0+it_0 \Longrightarrow \frac{a_n}{n^{\sigma_0}}\longrightarrow 0$

$\Longrightarrow \exists C\ \forall n\ |a_n|\le C n^{\sigma_0} \longrightarrow f(s)$ converges absolutely for $\mathrm{Re}(s)>\sigma_0+1$.

(2) As a formal product, $\zeta(s)\underbrace{\sum_{n=1}^{\infty}\frac{a_n}{n^s}}_{f(s)}=\underbrace{\sum_{n=1}^{\infty}\frac{b_n}{n^s}}_{g(s)}$, where
$$b_n=\sum_{dd'=n}a_d=\sum_{d\mid n}a_d$$
$$\zeta(s)^{-1}=\prod_{p\in\mathcal{P}}\left(1-\frac{1}{p^s}\right)=\sum_{n=1}^{\infty}\frac{\mu(n)}{n^s}, \quad f(s)=\zeta(s)^{-1}g(s)$$
$$\Downarrow$$
$$a_n=\sum_{dd'=n}\mu(d)\,b_{d'}=\sum_{d\mid n}\mu(d)\,b_{n/d}$$
(the Möbius formula)

Def. A function $a : \mathbb{Z}_{>0} \longrightarrow \mathbb{C}$ is multiplicative if $\gcd(m,n)=1 \Rightarrow a(mn)=a(m)a(n)$

it is strictly multiplicative if $\forall m,n \geq 1$

Prop. let $a : \mathbb{Z}_{>0} \longrightarrow \mathbb{C}$ be bounded ($\exists c \ \forall n\geq 1 \ |a(n)| \leq c$) and multiplicative.

$\Longrightarrow$ $f(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}$ is absolutely convergent for $Re(s)>1$ and is equal there to an absolutely convergent product $\prod_{p\in P} f_p(s)$, where $f_p(s) = \sum_{k=0}^{\infty} \frac{a(p^k)}{p^{ks}}$

is —— " ——

Pf. For $x\geq 1$, let $N_x = \{\prod p^{a_p} \in \mathbb{N} \mid p\in P,\ p\leq x\}$. Then $\forall x\in\mathbb{Z}_{>0}$

$\prod_{p\leq x} f_p(s) = \sum_{n\in N_x} \frac{a(n)}{n^s}$ (unique factorisation in $\mathbb{Z}_{>0}$) and

$$\left| f(s) - \prod_{p\leq x} f_p(s) \right| \leq \sum_{n>x} \frac{|a(n)|}{n^s} \xrightarrow{x\to\infty} 0 \quad \text{for fixed } s \text{ with } Re(s)>1$$

Cor1: If $a : \mathbb{Z}_{>0} \longrightarrow \mathbb{C}$ is strictly multiplicative $\overset{a(1)\neq 0}{\text{and}}$ bounded, then

$$f(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s} = \prod_{p\in P} \left(1 - \frac{a(p)}{p^s}\right)^{-1} \quad \text{(absolute convergence for } Re(s)>1\text{)}.$$

Cor 2: For $\sigma > 1$, $\log \zeta(\sigma) = \log \prod_p (1-p^{-\sigma})^{-1} = \sum_p -\log(1-p^{-\sigma}) =$

$$= \sum_p \sum_{k\geq 1} \frac{1}{kp^{k\sigma}} = \sum_{p\in P} \frac{1}{p^\sigma} + \underbrace{\sum_p \sum_{k\geq 2} \frac{1}{kp^{k\sigma}}}_{\leq 1 \text{ (see above)}} = \sum_{p\in P} \frac{1}{p^\sigma} + O(1) \text{ as } \sigma\to 1+$$

Cor3 $$\lim_{\sigma\to 1+} \left(\sum_{p\in P} \frac{1}{p^\sigma} \right) \Big/ \log \frac{1}{\sigma - 1} = 1$$

Generalisation to number fields

let $[K:\mathbb{Q}] = m < \infty$, $I_+(O_K) = \{ I \subset O_K \mid I \neq (0) \text{ ideal} \}$

A function $a : I_+(O_K) \longrightarrow \mathbb{C}$ is $\left\{\begin{array}{l}\text{strictly multiplicative} \\ \text{multiplicative}\end{array}\right\}$ if $a(IJ)=a(I)a(J)$

for $\left\{\begin{array}{l}\text{all } I,J \in I_+(O_K) \\ I,J \text{ such that } \gcd(I,J)=(1)\end{array}\right\}$. As $|\{I\in I_+(O_K) \mid N(I)=p\}| \leq m$ for all primes p, Prop. above yields, with minor modifications:

Prop. If a = strictly multiplicative $\overset{a((1))\neq 0}{\text{and}}$ bounded, then

$$\sum_I \frac{a(I)}{N(I)^s} = \prod_P \left(1 - \frac{a(P)}{N(P)^s}\right)^{-1},$$ with absolute convergence for $Re(s)>1$.

($P \in Spec(O_K) \smallsetminus \{(0)\}$)

Ex: $$\zeta_K(s) = \sum_{(0)\neq I\subset O_K} \frac{1}{N(I)^s} = \prod_P \left(1 - \frac{1}{N(P)^s}\right)^{-1}$$ Dedekind zeta-function of K

Ex: (1) $\zeta_{\mathbb{Q}(i)} = \frac{1}{4} \sum_{m,n\in\mathbb{Z}}' \frac{1}{(m^2+n^2)^s} = (1-2^{-s})^{-1} \prod_{p\equiv 1[4]} (1-p^{-s})^{-2} \prod_{p\equiv 3[4]} (1-p^{-2s})^{-1} = \zeta(s) \underbrace{L(s)}_{1-3^{-s}+5^{-s}-7^{-s}+\cdots}$

(2) $[K:\mathbb{Q}]=2$: $D=D_K$ $\quad \chi_D : (\mathbb{Z}/|D|\mathbb{Z})^* \longrightarrow \{\pm 1\}$ Kronecker's symbol

If $a\in\mathbb{Z}_{>0}$, $\gcd(a,2D)=1 \Longrightarrow \chi_D(a) = \left(\frac{D}{a}\right)$ Jacobi symbol

$$\zeta_K(s) = \prod_{p\mid D}(1-p^{-s})^{-1} \prod_{\left(\frac{D}{p}\right)=1} (1-p^{-s})^{-2} \prod_{\left(\frac{D}{p}\right)=-1} (1-p^{-2s})^{-1} = \underbrace{\prod_p (1-p^{-s})^{-1}}_{\zeta(s)} L(\chi_D, s),$$

$$L(\chi_D,s) = \prod_{p\nmid D} (1-\chi_D(p)p^{-s})^{-1} = \sum_{\substack{n\geq 1\\ (n,D)=1}} \chi_D(n) n^{-s}$$ Dirichlet L-series of χ_D

Dirichlet L-series

These are the series $\sum a(n) n^{-s}$ attached to periodic strictly multiplicative functions a.

Def. Let $m\in\mathbb{Z}_{>0}$. A Dirichlet character (mod m) is a group morphism $\chi : (\mathbb{Z}/m\mathbb{Z})^* \longrightarrow \mathbb{C}^*$. We extend χ to a (strictly multiplicative) function on $\mathbb{Z}_{>0}$ by $\chi(n) = \begin{cases} \chi(n \,(\mathrm{mod}\, m)) & \text{if } \gcd(n,m)=1 \\ 0 & \text{if not;}\end{cases}$

the Dirichlet L-function of χ $\quad L(\chi,s) = \sum_{\substack{n=1\\(n,m)=1}}^{\infty} \frac{\chi(n)}{n^s} = \prod_{p\nmid m}\left(1-\frac{\chi(p)}{p^s}\right)^{-1}$

is then absolutely convergent for $\mathrm{Re}(s)>1$.

Ex: (1) The constant morphism $(\mathbb{Z}/m\mathbb{Z})^* \longrightarrow \{1\} \subset \mathbb{C}^*$ yields the trivial character $\chi_0(n) = \begin{cases} 1 & \gcd(n,m)=1 \\ 0 & \text{if not}\end{cases}$)

$$L(\chi_0, s) = \left(\prod_{p\mid m}(1-p^{-s})\right)\zeta(s)$$

(2) $\chi_D : (\mathbb{Z}/|D|\mathbb{Z})^* \longrightarrow \{\pm 1\} \subset \mathbb{C}^*$, $D=D_K$, $[K:\mathbb{Q}]=2$ (see above)

Recall: (1) A character of a finite abelian group G is a group morphism $\chi : G \longrightarrow \mathbb{C}^*$ (i.e., $\chi(gh)=\chi(g)\chi(h)$ $\forall g,h\in G$).

(2) Such characters form an abelian group $\widehat{G}$, with $(\chi_1\chi_2)(g) = \chi_1(g)\chi_2(g)$.

(3) $\widehat{G\oplus H} = \widehat{G}\oplus\widehat{H}$ ($\chi : G\longrightarrow \mathbb{C}^*$, $\psi : H \longrightarrow \mathbb{C}^*$ define $G\oplus H \longrightarrow \mathbb{C}^*$, $(g,h)\longmapsto \chi(g)\psi(h)$)

(4) $G = \{g, \ldots, g^n = e\}$ cyclic of order n

$\Longrightarrow$ $\forall \chi\in\widehat{G}$ $\quad \chi(g)\in\mu_n(\mathbb{C})$ and $\chi(g^j) = \chi(g)^j$ $\forall j$.

Conversely, for every $\zeta\in\mu_n(\mathbb{C})$ $\quad \chi : g^j \longmapsto \zeta^j$ satisfies $\chi\in\widehat{G}$ and $\chi(g)=\zeta$. Therefore $\widehat{G}$ is again cyclic of order n (being isomorphic to $\widehat{G} \xrightarrow{\sim} \mu_n(\mathbb{C})$, $\chi \longmapsto \chi(g)$)

(5) G arbitrary $\Rightarrow G \simeq C_{n_1} \oplus \cdots \oplus C_{n_r}$ (C_k = cyclic of order k)

$\overset{(3)}{\Longrightarrow} \widehat{G} \simeq \widehat{C}_{n_1} \oplus \cdots \oplus \widehat{C}_{n_r} \overset{(4)}{\simeq} C_{n_1} \oplus \cdots \oplus C_{n_r} \simeq G$ (non-canonical isomorphism)

(6) $\forall g \in G \smallsetminus \{e\} \quad \exists \chi \in \widehat{G} \quad \chi(g) \neq 1$ (true if $G \simeq C_n$ by (4); in general use (5)).

(7) If $H \subset G$ is a subgroup, then the morphisms $\mathrm{incl}: H \hookrightarrow G$ and $\mathrm{pr}: G \longrightarrow G/H$ define morphisms

$$\begin{array}{ccc} \widehat{H} & \longleftarrow & \widehat{G} \\ \chi|_H & \longleftarrow & \chi \end{array} \quad \underline{\text{surjective}} \qquad\qquad \begin{array}{ccc} \widehat{G} & \longleftarrow & \widehat{G/H} \\ \psi\circ \mathrm{pr} & \longleftarrow & \psi \end{array} \quad \underline{\text{injective}}$$

(restriction of
$$\begin{array}{ccc} G & \xrightarrow{\chi} & \mathbb{C}^* \\ \cup & \nearrow_{\chi|_H} & \\ H & & \end{array}$$
)

$$G \xrightarrow{\mathrm{pr}} G/H \xrightarrow{\psi} \mathbb{C}^*, \qquad \underbrace{G \longrightarrow \mathbb{C}^*}_{\psi \circ \mathrm{pr}}$$

and $\widehat{H}$ = the quotient of $\widehat{G}$ by the subgroup $\widehat{G/H}$

Prop. (a) $\forall \chi \in \widehat{G} \quad \sum_{g \in G} \chi(g) = \begin{cases} |G| & \text{if } \chi = 1 \\ 0 & \chi \neq 1 \end{cases}$ (we know this)

(b) $\forall g \in G \quad \underbrace{\sum_{\chi \in \widehat{G}} \chi(g)}_{S} = \begin{cases} |G| & \text{if } g = e \\ 0 & \text{if } g \neq e \end{cases}$

Pf: $g \neq e \Rightarrow \exists \chi_1 \in \widehat{G} \quad \chi_1(g) \neq 1$
Write $\chi = \chi_1 \psi \Rightarrow$
$S = \sum_{\psi \in \widehat{G}} \chi_1(g)\psi(g) = \chi_1(g) S \Rightarrow S = 0$

Our case: $m = p_1^{k_1} \cdots p_r^{k_r}$, $G = (\mathbb{Z}/m\mathbb{Z})^* = \bigoplus_{i=1}^{r} (\mathbb{Z}/p_i^{k_i}\mathbb{Z})^*$

$\chi \in \widehat{G}$ is a product $\chi = \chi_1 \cdots \chi_r$, $\chi_i : (\mathbb{Z}/p_i^{k_i}\mathbb{Z})^* \longrightarrow \mathbb{C}^*$ $(*)$

[Ex: $\chi_D = \prod \chi_{D_\ell}$]

Def. χ is a primitive character (mod m) if it is not of the form
$$(\mathbb{Z}/m\mathbb{Z})^* \xrightarrow{\mathrm{pr}} (\mathbb{Z}/d\mathbb{Z})^* \longrightarrow \mathbb{C}^* \quad \text{for some } d \mid m, \ \underline{d < m}$$

Prop. Every $\chi \in (\mathbb{Z}/m\mathbb{Z})^*$ can be written uniquely as
$$\chi : (\mathbb{Z}/m\mathbb{Z})^* \xrightarrow{\mathrm{pr}} \underbrace{(\mathbb{Z}/d\mathbb{Z})^* \xrightarrow{\chi_{prim}} \mathbb{C}^*}$$
primitive character (mod d) attached to χ

$d \mid m$ is the conductor of χ, $d = f(\chi)$.

Pf: The case $m = p^k$ is immediate; the general case follows from $(*)$.

Ex: (1) χ_0 has conductor $f(\chi_0) = 1$, $(\chi_0)_{prim} = 1$

(2) Kronecker's symbol χ_D is primitive, $f(\chi_D) = |D|$.

Note: $L(\chi, s) = L(\chi_{prim}, s) \prod_{\substack{p \mid m \\ p \nmid f(\chi)}} (1 - \chi_{prim}(p) p^{-s})$

Proof of Dirichlet's thm on primes in arithmetic progressions

let $m \in \mathbb{Z}_{>0}$, $\chi \in \widehat{(\mathbb{Z}/m\mathbb{Z})^*}$, $\chi \neq \chi_0$. We know that $L(\chi, s)$ converges to a holomorphic function for $\mathrm{Re}(s) > 0$ and that $L(\chi, s) = \prod_{p \nmid m} (1 - \chi(p) p^{-s})^{-1}$ for $\mathrm{Re}(s) > 1$ (absolute convergence).

Key Proposition. $\boxed{\chi \neq \chi_0 \implies L(\chi, 1) \neq 0}$

Thm (Dirichlet): If $\gcd(a, m) = 1$, then

$$\sum_{p \in P_{\equiv a[m]}} \frac{1}{p^\sigma} = \frac{1}{\varphi(m)} \log \frac{1}{\sigma - 1} + O(1), \text{ as } \sigma \to 1+$$

$$(\implies |P_{\equiv a \pmod m}| = \infty)$$

Key Prop. $\implies$ Thm: if $s = \sigma + it$, $\sigma > 1$, then $\forall \chi \in \widehat{(\mathbb{Z}/m\mathbb{Z})^*}$

$$\log L(\chi, s) = \sum_{p \nmid m} \left(\frac{\chi(p)}{p^s} + \sum_{k \geq 2} \frac{1}{k} \left(\frac{\chi(p)}{p^s}\right)^k \right) = \sum_{p \nmid m} \frac{\chi(p)}{p^s} + \underbrace{O(1)}_{\text{as } \sigma \to 1+}$$

$$\frac{1}{\varphi(m)} \sum_{\chi} \overline{\chi}(a) \chi(p) = \begin{cases} 1 & p \equiv a \pmod m \\ 0 & p \not\equiv a \pmod m \end{cases}$$

$$\implies \sum_{p \equiv a \pmod m} \frac{1}{p^s} = \frac{1}{\varphi(m)} \sum_{\chi \in \widehat{(\mathbb{Z}/m\mathbb{Z})^*}} \overline{\chi}(a) \log L(\chi, s) + \widetilde{O(1)}$$

$\chi = \chi_0 \implies \log L(\chi_0, \sigma) \sim \log \zeta(\sigma) \sim \log\left(\frac{1}{\sigma - 1}\right)$ as $\sigma \to 1+$

$\chi \neq \chi_0 \implies$ Key Proposition says $\log L(\chi, \sigma) = O(1)$ — " —

$\implies$ result.

Pf of Key Prop. Consider $F(s) = \prod_{\chi \in \widehat{(\mathbb{Z}/m\mathbb{Z})^*}} L(\chi, s)$

$F(s) = \prod_{p \nmid m} F_p(s)$ $\quad$ ($\mathrm{Re}(s) > 1$ abs. conv.)

$$F_p(s) = \prod_{\chi} (1 - \chi(p \pmod m)) p^{-s})^{-1} = (1 - p^{-f_p s})^{-g_p}, \text{ where}$$

$$f_p = \min\{d \geq 1 \mid p^d \equiv 1 \pmod m\}, \quad f_p g_p = \varphi(m)$$

($\{\chi(p)\}$ take all values in $\mu_{f_p}(\mathbb{C})$, each with multiplicity g_p, but $\prod_{\zeta \in \mu_n(\mathbb{C})} (1 - \zeta X) = 1 - X^n$) $\implies F(s) = \sum_{n \geq 1} \frac{a_n}{n^s}$, $\underline{\underline{a_n \geq 0}}$.

We know: $L(s,\chi_0) - \frac{c}{s-1}$ is holomorphic for $Re(s)>0$ $(c\neq 0)$

$\chi\neq\chi_0$ $L(\chi,s)$ —— " ——

If $L(\chi,1)=0$ for some $\chi\neq\chi_0 \Longrightarrow F$ has holomorphic continuation to $Re(s)>0 \xRightarrow{\text{Landau}}$ it converges for $Re(s)>0$. However, the coefficients of $F(s) \geq$ the coefficients of $\prod_{p\nmid m}(1+p^{-f_p s}+\dots)^{g_p}$

$$\geq \prod_{p\nmid m}(1+p^{-\varphi(m)s}+\dots) = \sum_{\substack{n\geq 1\\(n,m)=1}} n^{-\varphi(m)s} = g(s),$$

but $g(s)$ diverges for $s=1/\varphi(m)$ – contradiction.

Note: if $p\nmid m \Longrightarrow p\,\mathcal{O}_{\mathbb{Q}(\zeta_m)} = P_1\cdots P_{g_p}$, $N(P_i)=p^{f_p}$

$$\Longrightarrow \prod_{\chi\in\widehat{(\mathbb{Z}/m\mathbb{Z})^*}} L(\chi,s) = \zeta_{\mathbb{Q}(\zeta_m)}(s)\prod_{p|m}(\text{something})$$

In fact, $\boxed{\zeta_{\mathbb{Q}(\zeta_m)}(s) = \prod_{\chi\in\widehat{(\mathbb{Z}/m\mathbb{Z})^*}} L(\chi_{prim},s)}$ (exercise).

Another approach to Key Proposition: (a) combine $L(\chi,s)$ with $L(\overline{\chi},s)$ $\Longrightarrow$ result if $\chi\neq\overline{\chi}$ $(\Longleftrightarrow \chi^2\neq\chi_0)$; (b) Dirichlet's class number formula gives an explicit non-zero value for $L(\chi,1)$ if $\chi^2=\chi_0\neq\chi$.

The following result gives a stronger version of (a).

Thm (Hadamard) (de la Vallée-Poussin) $L(\chi,1+it)\neq 0$ if $\begin{cases} t\in\mathbb{R}, & \chi^2\neq\chi_0 \\ t\in\mathbb{R}\setminus\{0\}, & \chi \text{ arbitrary.}\end{cases}$

Pf: $0\leq 2(1+\cos(\theta))^2 = 3+4\cos(\theta)+\cos(2\theta)$. Let $s=\sigma+it$, $\sigma>1$

$L(\chi,s)=\prod_{p\nmid m} L_p(\chi,s)$, $\log L_p(\chi,s) = \sum_{k\geq 1}\frac{1}{k}p^{-k\sigma}e^{-itk\log(p)}\underbrace{\chi(p^k)}_{e^{ik\beta_p}}$

$\Longrightarrow \log|L_p(\chi,\sigma+it)| = \sum_{k\geq 1}\frac{1}{k}p^{-k\sigma}\cos(k\alpha_p)$, $\alpha_p = t\log(p)-\beta_p$

$\log|L_p(\chi_0,\sigma)^3 L_p(\chi,\sigma+it)^4 L_p(\chi^2,\sigma+2it)| = \sum_{k\geq 1}\frac{1}{k}p^{-k\sigma}(\underbrace{3+4\cos(k\alpha_p)+\cos(2k\alpha_p)}_{\geq 0})$

$\Longrightarrow \forall\sigma>1\ \forall t\in\mathbb{R}$ $|L(\chi_0,\sigma)^3 L(\chi,\sigma+it)^4 L(\chi^2,\sigma+2it)|\geq 1$

Fix $t\in\mathbb{R}$, let $\sigma\longrightarrow 1+$: $L(\chi_0,\sigma)\sim C(\sigma-1)^{-3}$ $(C\neq 0)$

$L(\chi^2,\sigma+2it) = O(1)$ if $\chi^2\neq\chi_0$ (or if $\chi^2=\chi_0$, $t\neq 0$)

$\Longrightarrow L(\chi,1+it)\neq 0$

b) <u>What if $\chi^2=\chi_0\neq\chi$?</u> In this case $\chi_{prim}=\chi_D$, $D=D_K$, $[K:\mathbb{Q}]=2$

<u>Thm</u> (Dirichlet's class number formula for quadratic fields).. If $[K:\mathbb{Q}]=2$, $D=D_K$, then $\zeta_K(s)=\zeta(s)L(\chi_D,s)$ satisfies:

$$\lim_{s\to 1}(s-1)\zeta_K(s)=L(\chi_D,1)=\begin{cases}\frac{2h\log(\varepsilon)}{D^{1/2}}, & D>0 \quad \mathcal{O}_K^*=\{\pm1\}\times\varepsilon^{\mathbb{Z}},\ \varepsilon>1\\ \frac{2\pi h}{w|D|^{1/2}}, & D<0 \quad \mathcal{O}_K^*=\mu_w,\ w=\begin{cases}6 & D=-3\\ 4 & D=-4\\ 2 & D<-4\end{cases}\end{cases}$$

$h=h_K=|\mathcal{C}\ell_K|$

<u>Ex</u>: $D=-4$, $1-\frac{1}{3}+\frac{1}{5}-\dots=\frac{2\pi}{4|4|^{1/2}}=\frac{\pi}{4}$

This formula was generalised by Kummer to cyclotomic fields $K=\mathbb{Q}(\zeta_m)$, and by Dedekind to arbitrary number fields.

<u>Thm</u> (Dedekind) If $[K:\mathbb{Q}]<\infty$, then

$$\lim_{s\to 1}(s-1)\zeta_K(s)=\frac{2^{r_1}(2\pi)^{r_2}hR}{w|D|^{1/2}}$$

$K_{\mathbb{R}}\simeq\mathbb{R}^{r_1}\times\mathbb{C}^{r_2}$, $h=|\mathcal{C}\ell_K|$
$D=D_K$, $\mathcal{O}_K^*=\mu_w\times\mathbb{Z}^{r_1+r_2-1}$
$R=R_K$ the regulator of units

<u>Abelian number fields</u>: the abelian group $(\mathbb{Z}/m\mathbb{Z})^*$ is canonically isomorphic to the Galois group $G=\mathrm{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q})$, with $a\in(\mathbb{Z}/m\mathbb{Z})^*$ corresponding to $\sigma_a\in G$, $\sigma_a(\zeta_m)=\zeta_m^a$. An intermediate field $\mathbb{Q}\subset K\subset\mathbb{Q}(\zeta_m)$ corresponds to a subgroup $H=\mathrm{Gal}(\mathbb{Q}(\zeta_m)/K)\subset G$ and $\mathrm{Gal}(K/\mathbb{Q})\simeq G/H$.

[Conversely, if $\mathrm{Gal}(K/\mathbb{Q})$ is abelian $\Longrightarrow$ $K\subset\mathbb{Q}(\zeta_m)$ for some $m\geq 1$, by the "Kronecker–Weber" thm, whose first correct proof was given by Hilbert.]

In this case $\zeta_K(s)=\prod_{\substack{\chi\in\widehat{(\mathbb{Z}/m\mathbb{Z})^*}\\ \chi(H)=1}}L(\chi_{prim},s)$.

If $\chi:(\mathbb{Z}/f\mathbb{Z})^*\longrightarrow\mathbb{C}^*$ is <u>primitive</u>, then

$$L(\chi,1)=\text{“}\sum_{(n,f)=1}\frac{\chi(n)}{n}\text{”}=\sum_{n=1}^{\infty}\frac{1}{n}\frac{1}{\tau(\overline{\chi})}\sum_{a\in(\mathbb{Z}/f\mathbb{Z})^*}\overline{\chi}(a)\zeta_f^{an},\qquad \tau(\overline{\chi})=\sum_{b\in(\mathbb{Z}/f\mathbb{Z})^*}\overline{\chi}(b)\zeta_f^b$$

(Gauss sum)

$$=\frac{1}{\tau(\overline{\chi})}\sum_a\overline{\chi}(a)\underbrace{\left(\sum_{n=1}^{\infty}\frac{1}{n}\zeta_f^{an}\right)}_{-\log(1-\zeta_f^a)}=\frac{-\chi(-1)\tau(\chi)}{f}\sum_{a\in(\mathbb{Z}/f\mathbb{Z})^*}\overline{\chi}(a)\log(1-\zeta_f^a)$$

$$\Rightarrow L(\chi,1) = \begin{cases} \frac{\pi i \tau(\chi)}{f}\left(\frac{1}{f}\sum_{a\in(\mathbb{Z}/f\mathbb{Z})^*} \overline{\chi}(a)\, a\right) & \chi(-1)=-1 \\ -\frac{\tau(\chi)}{f}\sum_{a\in(\mathbb{Z}/f\mathbb{Z})^*} \overline{\chi}(a) \log|1-\zeta_f^a| & \chi(-1)=1 \end{cases}$$

Special case: $\chi=\chi_D$, $D=D_K$, $[K:\mathbb{Q}]=2$, $\tau(\chi_D)^2=D$

Gauss: $\tau(\chi_D) = \begin{cases} D^{1/2}>0 & D>0 \\ i|D|^{1/2}, \quad |D|^{1/2}>0 & D<0 \end{cases}$

($\Rightarrow$ yet another proof of QRL!)

Dirichlet's class number formula then yields:

(a) ~~D~~ $D<-4$: $h_{\mathbb{Q}(\sqrt{D})} = \frac{1}{2-\chi_D(2)} \sum_{\substack{0<x<|D|/2\\(x,D)=1}} \chi_D(x)$ $\quad(\chi_D(2)=0$ if $2|D)$

(b) $D>0$: $\varepsilon^h = \prod_{\substack{0<x<|D|/2\\(x,D)=1}} (1-\zeta_D^x)^{\chi_D(x)}$ $\quad O_K^* = \{\pm1\}\times\varepsilon^{\mathbb{Z}}$, $\varepsilon>1$, $h=h_{\mathbb{Q}(\sqrt{D})}$

Yet another approach to Key Proposition:

prove directly that $\underbrace{\lim_{s\to 1}(s-1)\zeta_{\mathbb{Q}(\zeta_m)}(s)}_{\prod_{\substack{\chi\in\widehat{(\mathbb{Z}/m\mathbb{Z})^*}\\ \chi\neq\chi_0}} L(\chi_{prim},1)} \neq 0$ (Kummer's class number formula)

The class number formulas are proved by a very nice geometric argument, due to Dirichlet. See [Borevič – Šafarevič], for example, for details. The main idea goes as follows. For each $C\in Cl_K$, fix an ideal I_C such that $[I_C]\in C^{-1}$. If $\alpha\in I\setminus\{0\}$, then $J=(\alpha)I_C^{-1}\subset O_K$ and $[J]=C$, and $(\alpha)I_C^{-1}=(\beta)I_C^{-1} \Leftrightarrow \alpha\beta^{-1}\in O_K^*$. As $N(J)=|N_{K/\mathbb{Q}}(\alpha)|\,N(I_C)^{-1}$, we get

$$\zeta_K(s) = \sum_{C\in Cl_K}\sum_{\substack{J\subset O_K\\ [J]=C}} N(J)^{-s} = \sum_C N(I_C)^s \sum_{\alpha\in(I_C\setminus\{0\})/O_K^*} |N_{K/\mathbb{Q}}(\alpha)|^{-s}$$

We know that $\mathcal{O}_K^* = \mu_w \times \varepsilon_1^{\mathbb{Z}} \cdots \varepsilon_{r_1+r_2-1}^{\mathbb{Z}}$ and that $\ell(\mathcal{O}_K^*) \subset H = \operatorname{Ker}(\mathbb{R}^{r_1+r_2} \xrightarrow{\Sigma} \mathbb{R}) \simeq \mathbb{R}^{r_1+r_2-1}$ is a lattice.

If $F \subset H$ is a fundamental domain of this lattice, its inverse image $V = \ell'^{-1}(F) \subset K_{\mathbb{R}}^* = (\mathbb{R}^*)^{r_1} \times (\mathbb{C}^*)^{r_2} \xrightarrow{\ell'} \mathbb{R}^{r_1+r_2}$ is a fundamental domain for $\mathcal{O}_K^*/\mu_w$ ($\mu_w = \{\text{roots of unity in } K\}$) in $K_{\mathbb{R}}^* \Longrightarrow w \sum_{\alpha \in (I_c \setminus \{0\})/\mathcal{O}_K^*} |N_{K/\mathbb{Q}}(\alpha)|^{-s} = \sum_{\alpha \in I_c \cap V} |N_{K/\mathbb{Q}}(\alpha)|^{-s}$

<u>General principle</u> (Dirichlet): $V \subset \mathbb{R}^n$ such that $\mathbb{R}_{>0}^* \cdot V \subset V$, $f : \mathbb{R}^n \setminus \{0\} \longrightarrow \mathbb{R}_{>0}$ homogeneous of $\deg(f) = n$ ($f(tx) = t^n f(x)$), $L \subset \mathbb{R}^n$ lattice $\Longrightarrow$

$$\lim_{s \to 1+} (s-1) \sum_{0 \neq \alpha \in V \cap L} f(\alpha)^{-s} = \frac{\mu(V \cap \{f \leq 1\})}{\mu(\mathbb{R}^n/L)}.$$

[One uses the fact that the right hand side is equal to $\lim_{k \to +\infty} \frac{|L \cap V \cap \{f \leq k\}|}{k^n}$, and makes a comparison to $(s-1)\sum_{k=1}^{\infty} \frac{1}{k^s} \to 1$.]

All that remains is to compute $\mu(\ell'^{-1}(F) \cap \{|N| \leq 1\})$.

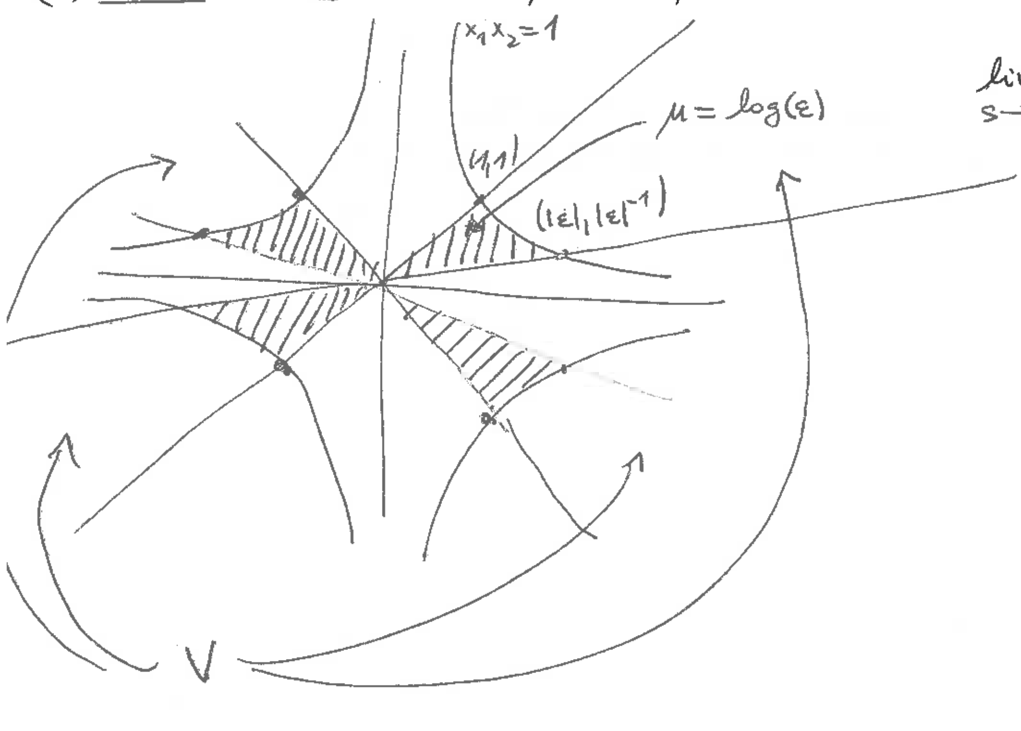
<u>Ex</u>: ($n=2$), $K = \mathbb{Q}(\sqrt{d})$, $D = D_K$

(a) <u>$D<0$</u>: $V = K_{\mathbb{R}}^* = \mathbb{C}^*$, $|N(z)| = |z|^2$, $\mu(\{z \in \mathbb{C};\ |z| \leq 1\}) = 2\mu_{class} = 2\pi$

$$\mu(\mathbb{C}/I_c) = |D|^{1/2} N(I_c)$$

$$\Longrightarrow \lim_{s \to 1+} (s-1)\zeta_K(s) = \frac{h}{w} \cdot \frac{2\pi}{|D|^{1/2}}$$

(b) <u>$D>0$</u>: $K_{\mathbb{R}}^* = \mathbb{R}^* \times \mathbb{R}^*$, $w = 2$, $|N(x_1,x_2)| = |x_1 x_2|$



$$\lim_{s \to 1+} (s-1)\zeta_K(s) = \frac{h}{w} \cdot \frac{4\log(\varepsilon)}{D^{1/2}}$$