Continued fractions and irrationality of π, e^k $(k \in \mathbb{Z})$

1730's: Euler: (1) $\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}} = [0, \frac{1}{x}, \frac{3}{x}, \frac{5}{x}, \frac{7}{x}, \ldots]$

(2) $\frac{e+1}{e-1} \overset{x=\frac{1}{2}}{=} [2,4,6,8,\ldots] \Rightarrow e = [2,1,2,1,1,4,1,1,6,1,1,8,1,1,\ldots]$

(3) $\frac{e^2+1}{e^2-1} \overset{x=1}{=} [1,3,5,7,\ldots] \Rightarrow e^2 = [7,2,1,1,3,18,5,1,1,6,30,8,1,1,9,42,\ldots]$

(4) k $\qquad e^{1/k} = [1, k-1, 1, 1, 3k-1, 1, 1, 5k-1, 1, 1, 7k-1, 1, 1, \ldots]$

Euler did not state explicitly that these identities imply $e, e^2 \notin \mathbb{Q}$.

More generally, he expressed $[\frac{c}{x}, \frac{c+1}{x}, \frac{c+2}{x}, \ldots]$ as a ratio of two explicit power series (see below).

1760's Lambert: (1) can be written as

$$\tanh(x) = \cfrac{x}{1 + \cfrac{x^2}{3 + \cfrac{x^2}{5 + \cdots}}} \quad \Longrightarrow \quad \tan(x) = \cfrac{x}{1 - \cfrac{x^2}{3 - \cfrac{x^2}{5 - \cfrac{x^2}{\cdots}}}}$$

$\Longrightarrow \forall x \in \mathbb{Q} \setminus \{0\} \quad \tanh(x), \tan(x) \notin \mathbb{Q}$

$\Longrightarrow \forall x \in \mathbb{Q} \setminus \{0\} \quad e^x \notin \mathbb{Q}, \quad \pi \notin \mathbb{Q}$ (since $\tan(\frac{\pi}{4}) = 1 \in \mathbb{Q}$).

Some gaps in Lambert's reasoning were filled in by Legendre , who also showed that $\pi^2 \notin \mathbb{Q}$ by this method.

Remarks: (a) The "usual" proof of $e \notin \mathbb{Q}$ was given by Fourier (1815):

if $e = \frac{m}{n}$, then $n!e \in \mathbb{Z}$, but $n!e = \underbrace{\sum_{k \le n} \frac{n!}{k!}}_{S_1 \in \mathbb{Z}} + \underbrace{\sum_{k > n} \frac{n!}{k!}}_{S_2}$,

$$|S_2| = \sum_{\ell \ge 1} \frac{n!\,\ell!}{(n+\ell)!} \frac{1}{\ell!} \le \frac{1}{n+1} \sum_{\ell \ge 1} \frac{1}{\ell!} = (e-1)/(n+1) < 1$$ – contradiction.

(b) Variant (Siegel): $\forall n \ge 1$ odd

$$\mathbb{Z} \ni a_n = \sum_{k \le n} (-1)^k \frac{n!}{k!} < n!\,e^{-1} = \sum_{k \ge 0} (-1)^k \frac{n!}{k!} < a_{n+1} = a_n + \frac{1}{n+1} < a_n + 1 \Rightarrow \frac{n!}{e} \notin \mathbb{Z}.$$

$\Rightarrow e^{-1} \notin \mathbb{Q}$.

(c) Clever tricks (Liouville, 1840's) involving the series for $e^{\pm 1}, e^{\pm 2}$ imply that $\forall a, b, c \in \mathbb{Z}^3 \setminus \{(0,0,0)\}$ $\quad ae + b + ce^{-1} \neq 0 \neq ae^2 + b + ce^{-2}$.

Hypergeometric series

$${}_kF_\ell\left(\begin{matrix}a_1,\dots,a_k\\ b_1,\dots,b_\ell\end{matrix};z\right)=\sum_{n\ge 0}\frac{(a_1)_n\cdots(a_k)_n}{(b_1)_n\cdots(b_\ell)_n}\frac{z^n}{n!}\quad,\qquad \begin{matrix}(a)_0=1\\ (a)_n=a(a+1)\cdots(a+n-1)\end{matrix}$$

$(a_i, b_j, z\in\mathbb{C};\quad b_j\notin\{0,-1,-2,\dots\})$.

Special case: $\displaystyle f(c;z)=\sum_{n\ge0}\frac{z^n}{n!(c)_n}={}_0F_1\left(\begin{matrix}\emptyset\\ c\end{matrix};z\right)$ $\qquad c\in\mathbb{C}$ fixed, $c\neq 0,-1,-2,\dots$

Properties of $f(c;z)$: (0) the coefficients tend to 0 very rapidly $\Rightarrow$ $f(c;z)$ is holomorphic for $z\in\mathbb{C}$ and all its derivatives can be computed by differentiating the series term by term.

(1) $f(\frac12;z^2)=\cosh(2z)$, $\quad f(\frac32;z^2)=\frac{\sinh(2z)}{2z}$

(2) $$\frac{1}{(c)_n}=\frac{c+n}{(c)_{n+1}}=\frac{1}{(c+1)_n}+\frac{n}{(c)_{n+1}}\Longrightarrow f(c;z)=f(c+1;z)+\underbrace{\sum_{n\ge0}\frac{z^{n+1}}{(c)_{n+2}\,n!}}_{\frac{z}{c(c+1)}f(c+2;z)}$$

(3) $$\frac{f(c;z^2)}{f(c+1;z^2)}\cdot\frac{c}{z}=\frac{c}{z}+\cfrac{1}{\frac{f(c+1;z^2)}{f(c+2;z^2)}\cdot\frac{c+1}{z}}=$$
$$=\left[\frac{c}{z},\frac{c+1}{z},\dots,\frac{c+k-1}{z},\frac{f(c+k;z^2)}{f(c+k+1;z^2)}\,\frac{c+k}{z}\right]\qquad(\forall k\ge1)$$

(4) $$f'(c;z)=\sum_{n\ge1}\frac{z^{n-1}}{(c)_n(n-1)!}=\sum_{n\ge0}\frac{z^n}{(c)_{n+1}n!}=\frac1c f(c+1;z)$$

(5) $$zf''(c;z)=\frac{z}{c(c+1)}f(c+2;z)=f(c;z)-cf'(c;z)\Longrightarrow\left(z\left(\frac{d}{dz}\right)^2+c\frac{d}{dz}-1\right)f(c;z)=0$$

(6) $$0=4\left(z^2\left(\frac{d}{dz^2}\right)^2+c\frac{d}{dz^2}-1\right)f(c;z^2)=\left(\left(\frac{d}{dz}\right)^2+\frac{2c-1}{z}\frac{d}{dz}-4\right)f(c;z^2)$$

(7) Bessel functions: $\displaystyle J_\lambda(z)=\frac{1}{\Gamma(\lambda+1)}\left(\frac z2\right)^\lambda K_\lambda(z),\ K_\lambda(z)=\sum_{n\ge0}\frac{(-z^2/2)^n}{n!(\lambda+1)_n}=f(\lambda+1;-\frac{z^2}{2})$

$(\lambda\neq0,-1,-2,\dots)$
$$\left(\left(\frac{d}{dz}\right)^2+\frac{2\lambda+1}{z}\frac{d}{dz}+1\right)K_\lambda(z)=0,\qquad f(c;z^2)=K_{c-1}(i\sqrt2\,z)$$

(8) The convergents of (3): $\displaystyle\frac zc\left[\frac cz,\frac{c+1}{z},\dots,\frac{c+n-1}{z}\right]=\frac{P_n(c;z)}{Q_n(c;z)}$

$$P_n(c;z)=\sum_{k=0}^{[n/2]}\frac{(n-k)!}{(n-2k)!(c+n-k)_k}\frac{z^{2k}}{k!(c)_k},\qquad Q_n(c;z)=P_{n-1}(c+1;z)$$

$$\lim_{n\to\infty}P_n(c;z)=\underbrace{\sum_{k\ge0}\frac{z^{2k}}{k!(c)_k}}_{f(c;z)},\qquad\lim_{n\to\infty}Q_n(c;z)=f(c+1;z)$$

For fixed c, the convergence is uniform on any compact subset of $\mathbb{C}\Rightarrow\boxed{\left[\frac cz,\frac{c+1}{z},\frac{c+2}{z},\dots\right]=\frac cz\,\frac{f(c;z^2)}{f(c+1;z^2)}}$ converges for $z\in\mathbb{C}\setminus(\{\text{zeroes of } f(c+1;z^2)\}\cup\{0\})$ (uniformly on compact sets)

Generalised continued fractions

$$\frac{b_1|}{|a_1} + \frac{b_2|}{|a_2} + \dots + \frac{b_n|}{|a_n} + \dots = \cfrac{b_1}{a_1 + \cfrac{b_2}{a_2 + \cfrac{b_3}{a_3 + \dots}}} \qquad (*)$$

Note: $\frac{b_1|}{|a_1} + \frac{b_2|}{|a_2} + \frac{b_3|}{|a_3} + \dots = \frac{\lambda_1 b_1|}{|\lambda_1 a_1} + \frac{\lambda_1\lambda_2 b_2|}{|\lambda_2 a_2} + \frac{\lambda_2\lambda_3 b_3|}{|\lambda_3 a_3} + \frac{\lambda_3\lambda_4 b_4|}{|\lambda_4 a_4} + \dots$

We know: $\frac{f(c;z^2)}{f(c+1;z^2)} = \frac{z}{c}\left[\frac{c}{z}, \frac{c+1}{z}, \frac{c+2}{z}, \dots\right] = \frac{1|}{|1} + \frac{z/c|}{|(c+1)/z} + \frac{1|}{|(c+2)/z} + \frac{1|}{|(c+3)/z} + \dots =$

$= \frac{1|}{|1} + \frac{z^2/c|}{|c+1} + \frac{z^2|}{|c+2} + \frac{z^2|}{|c+3} + \dots \implies$

$$\boxed{f(c;t)/f(c+1;t) = \frac{1|}{|1} + \frac{t/c|}{|c+1} + \frac{t|}{|c+2} + \frac{t|}{|c+3} + \dots} \quad \left(\begin{array}{l}\text{if } c \neq 0,-1,-2,\dots;\ t \in \mathbb{C} \\ f(c+1;t) \neq 0\end{array}\right)$$

Cor: if $c = \frac{a}{b} \in \mathbb{Q}$, $t = \frac{p}{q} \in \mathbb{Q}$, then $\quad (a,b,p,q \in \mathbb{Z};\ b,q \geq 1)$

$$f(c;t)/f(c+1;t) = \frac{1|}{|1} + \frac{pb^2|}{|a(a+b)q} + \frac{ab^2p|}{|a+2b} + \frac{b^2p|}{|(a+3b)q} + \frac{b^2p|}{|a+4b} + \frac{b^2p|}{|(a+5b)q} + \dots \quad (**)$$

Convergents of $(*)$: $\frac{b_1|}{|a_1} + \dots + \frac{b_n|}{|a_n} = \cfrac{b_1}{a_1 + \cfrac{b_2}{\ddots + \cfrac{b_n}{a_n}}} = \frac{q_n}{p_n}$

n	0	1	2
p_n	1	a_1	$a_1a_2+b_2$
q_n	0	b_1	b_1a_2

$$\begin{pmatrix} a_1 & 1 \\ b_1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_n & 1 \\ b_n & 0 \end{pmatrix} = \begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix}$$

$$\begin{pmatrix} p_{n+1} \\ q_{n+1} \end{pmatrix} = \begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix}\begin{pmatrix} a_{n+1} \\ b_{n+1} \end{pmatrix}, \quad \begin{vmatrix} p_{n+1} & p_{n-1} \\ q_{n+1} & q_{n-1} \end{vmatrix} = a_{n+1}\begin{vmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{vmatrix} = a_{n+1}(-b_1)\cdots(-b_n)$$

Proposition 1. Assume that $\forall n \geq 1$ $a_n, b_n \in \mathbb{R}$, $a_n \geq b_n \geq 1$. Then

(1) $\forall n \geq 1$ $p_n \geq n$, $\quad 0 < \frac{q_2}{p_2} < \frac{q_4}{p_4} < \dots < \dots < \frac{q_3}{p_3} < \frac{q_1}{p_1} \leq 1$, $\quad p_n \geq a_1 \cdots a_n \geq b_1 \cdots b_n$,

$\left|\frac{q_{n-1}}{p_{n-1}} - \frac{q_n}{p_n}\right| \leq \frac{1}{p_{n-1}}$, $\quad \alpha = \lim_{n\to\infty} \frac{q_n}{p_n}$ exists, $\quad 0 < \alpha < 1$.

(2) If, in addition, $\forall n \geq 1$ $a_n, b_n \in \mathbb{Z}$, then $\alpha \notin \mathbb{Q}$.

Pf (1) Follows from the formulas above.

(2) $\forall m \geq 1$ $\quad \frac{b_m|}{|a_m} + \frac{b_{m+1}|}{|a_{m+1}} + \dots = \alpha_m$ exists, $0 < \alpha_m < 1$, $\alpha_m = \frac{b_m}{a_m + \alpha_{m+1}}$,

$\alpha_{m+1} = b_m/\alpha_m - a_m$. If ($\alpha \in \mathbb{Q}$, then) $\alpha_m = \frac{u}{v} \in \mathbb{Q}$ $\quad (u,v \in \mathbb{Z},\ u,v > 0)$; then

$0 < u < v$ and $u\alpha_{m+1} = b_m v - a_m u \in \mathbb{Z} \implies$ (the denominator of α_{m+1}) $<$ (the denominator of α_m). Infinite descent $\implies$ contradiction.

Cor. If $c \in \mathbb{Q}$, $-c \notin \mathbb{N}$, $t \in \mathbb{Q}$, $t > 0 \Rightarrow \alpha = \dfrac{f(c;t)}{f(c+1;t)} \notin \mathbb{Q} \cup \{\infty\}$.

Pf: If $t = \frac{p}{q} > 0$, then $a + nb \geq b^2 p$ for $n \geq n_0$. Apply
($c = a/b$, $b, q > 0$) Prop. 1 to $\dfrac{b_{n_0}|}{|a_{n_0}} + \dfrac{b_{n_0+1}|}{|a_{n_0+1}} + \cdots$ from (**) $\Rightarrow$
$\alpha_{n_0} \notin \mathbb{Q} \cup \{\infty\} \Rightarrow \alpha \notin \mathbb{Q} \cup \{\infty\}$.

Cor ($c = \frac{1}{2}$) $\forall t \in \mathbb{Q}_{>0}$ $\quad 0 < \dfrac{f(\frac{1}{2};t)}{f(\frac{3}{2};t)} = \dfrac{2\sqrt{t}}{\tanh(2\sqrt{t})} \notin \mathbb{Q}$.

($\Rightarrow \forall u \in \mathbb{Q} \setminus \{0\}$ $\tanh(u) \notin \mathbb{Q} \Rightarrow e^{2u} \notin \mathbb{Q}$).

Proposition 2. Assume that $\forall n \geq 1$ $a_n, b_n \in \mathbb{R}$, $a_n \geq 1 + |b_n| \geq 2$.

(1) $\forall m \geq 1$ $\forall n \geq 1$ $\dfrac{b_m|}{|a_m} + \cdots \dfrac{b_{m+n}|}{|a_{m+n}}$ has the same sign as $\dfrac{b_m}{a_m}$ and absolute value $|\cdot| \leq 1$.

(2) If $\alpha = \lim\limits_{n\to\infty} \dfrac{b_1|}{|a_1} + \cdots + \dfrac{b_n|}{|a_n}$ exists ($\Longleftrightarrow \forall m \geq 1$ $\alpha_m = \lim\limits_{n\to\infty} \dfrac{b_m|}{|a_m} + \cdots + \dfrac{b_{m+n}|}{|a_{m+n}}$ exists
$\Longleftrightarrow \exists m \geq 1$ ——— " ———)

then: (2a) $\forall m$ $\quad |\alpha_m| \leq 1$

(2b) If $\forall n \geq 1$ $a_n, b_n \in \mathbb{Z}$ and $\alpha \in \mathbb{Q}$, then
$\exists m$ $\forall n \geq m$ $\quad \alpha_n = -1$, $b_n = 1 - a_n$.

Ex: $\dfrac{-1|}{|2} + \dfrac{-1|}{|2} + \dfrac{-1|}{|2} + \cdots = -1$

Pf: (1) Easy induction. (2a) Follows from (1).

(2b) If $\alpha \in \mathbb{Q}$, then $\forall m$ $\alpha_m \in \mathbb{Q}$. As in the proof of Proposition 1, (2a) implies that (the denominator of α_n) $\geq$ (the one of α_{n+1})
$\Rightarrow$ equality for all $n \geq m \Rightarrow |\alpha_n| = 1$ $\forall n \geq m \Rightarrow$
$0 < a_n + \alpha_{n+1} = b_n \alpha_n = |b_n| < a_n \Rightarrow \alpha_{n+1} = -1$.

Cor. If $c \in \mathbb{Q}$, $-c \notin \mathbb{N}$, $t \in \mathbb{Q}_{>0}$, then $\alpha = f(c;-t)/f(c+1;-t) \notin \mathbb{Q} \cup \{\infty\}$.

Pf: If $c = a/b$, $t = p/q$, $a, b, p, q \in \mathbb{Z}$, $b, q > 0$, then Prop. 2 applies to

$$\alpha_{n_0} = \frac{b_{n_0}|}{|a_{n_0}} + \frac{b_{n_0+1}|}{|a_{n_0+1}} + \cdots, \text{ where } \alpha = \frac{b_1|}{|a_1} + \frac{b_2|}{|a_2} + \cdots = \frac{1|}{|1} + \frac{-pb^2|}{|a(a+b)q} + \frac{-ab^2p|}{|a+2b} + \frac{-b^2p|}{|(a+3b)q} + \frac{-b^2p|}{|a+4b} + \cdots$$

(for some $n_0 \geq 1$) $\Rightarrow \alpha_{n_0} \notin \mathbb{Q} \cup \{\infty\} \Rightarrow \alpha \notin \mathbb{Q} \cup \{\infty\}$.

Cor ($c = \frac{1}{2}$) $\forall t \in \mathbb{Q}_{>0}$ $\quad f(\frac{1}{2};-t)/f(\frac{3}{2};-t) = \dfrac{2\sqrt{t}}{\tan(2\sqrt{t})} \notin \mathbb{Q} \cup \{\infty\}$

Cor. $\pi^2 \notin \mathbb{Q}$. Pf: If $\pi^2 = t \in \mathbb{Q}_{>0}$ then $\underbrace{\qquad}_{\infty}$, contradiction.