See: A. BAKER, A Concise Introduction to the Theory of Numbers

Continued fractions

Ex: $\frac{7}{4} = 1 + \cfrac{1}{\frac{4}{3}} = \underbrace{1 + \cfrac{1}{1+\cfrac{1}{3}}}_{[1,1,3]}$; $\alpha = \frac{1+\sqrt5}{2} = 1 + \frac{1}{\alpha} = 1 + \cfrac{1}{1+\cfrac{1}{\alpha}} = 1 + \cfrac{1}{1+\cfrac{1}{1+\dots}} = [1,1,1,1,\dots]$

Algebraic formulas: Möbius transformations: $A = \begin{pmatrix} a & b \\ c & d\end{pmatrix} \in GL_2(\mathbb{C})$, $x \in \mathbb{C}$ $\quad A(x) = \frac{ax+b}{cx+d}$

$(AA')(x) = A(A'(x))$

$$\frac{d}{dx} A(x) = \frac{ad-bc}{(cx+d)^2} = \frac{\det(A)}{(cx+d)^2}$$

Notation: $[a_0, a_1, \dots, a_n, x] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots + \underbrace{a_n + \cfrac{1}{x}}_{\frac{a_n x + 1}{x} = \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix}(x)}}}$

$\Rightarrow [a_0, \dots, a_n, x] = [a_0, \dots, a_{n-1}, \begin{pmatrix} a_n & 1 \\ 1 & 0\end{pmatrix}(x)] = [a_0, \dots, a_{n-2}, \begin{pmatrix} a_{n-1} & 1 \\ 1 & 0\end{pmatrix}\begin{pmatrix} a_n & 1 \\ 1 & 0\end{pmatrix}(x)] = \dots = \left(\begin{pmatrix} a_0 & 1 \\ 1 & 0\end{pmatrix} \cdots \begin{pmatrix} a_n & 1 \\ 1 & 0\end{pmatrix}\right)(x)$

Conclusion: $[a_0, \dots, a_n, x] = \begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1}\end{pmatrix}(x)$, $\begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1}\end{pmatrix} = \begin{pmatrix} a_0 & 1 \\ 1 & 0\end{pmatrix} \cdots \begin{pmatrix} a_n & 1 \\ 1 & 0\end{pmatrix}$

$\begin{pmatrix} p_n & * \\ q_n & *\end{pmatrix} = \begin{pmatrix} p_{n-1} & * \\ q_{n-1} & *\end{pmatrix}\begin{pmatrix} a_n & 1 \\ 1 & 0\end{pmatrix} = \begin{pmatrix} * & p_{n-1} \\ * & q_{n-1}\end{pmatrix}$

$\det\begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1}\end{pmatrix} = (-1)^{n-1}$, $\begin{pmatrix} p_n \\ q_n \end{pmatrix} = \begin{pmatrix} p_{n-1} & p_{n-2} \\ q_{n-1} & q_{n-2}\end{pmatrix}\begin{pmatrix} a_n \\ 1\end{pmatrix}$, $\begin{pmatrix} p_n & p_{n-2} \\ q_n & q_{n-2}\end{pmatrix} = \begin{pmatrix} p_{n-1} & p_{n-2} \\ q_{n-1} & q_{n-2}\end{pmatrix}\begin{pmatrix} a_n & 0 \\ 1 & 1\end{pmatrix}$, $\det\begin{pmatrix} p_n & p_{n-2} \\ q_n & q_{n-2}\end{pmatrix} = (-1)^n a_n$

n	-2	-1	0	1	2
a_n			a_0	a_1	a_2
p_n	0	1	a_0	a_0a_1+1	$a_0a_1a_2+a_2+a_0$
q_n	1	0	1	a_1	a_1a_2+1

$\frac{d}{dx}[a_0, \dots, a_n, x] = \frac{(-1)^{n-1}}{(q_n x + q_{n-1})^2}$, $\underbrace{[a_0, \dots, a_n, \infty]}_{[a_0, \dots, a_n]} = \frac{p_n}{q_n}$

$[a_0, \dots, a_n, x] - \frac{p_n}{q_n} = \frac{(-1)^n}{q_n(q_n x + q_{n-1})}$

Continued fractions of real numbers

Input $\alpha = \alpha_0 \in \mathbb{R}$. let $a_0 = \lfloor \alpha_0 \rfloor$. If $a_0 \neq \alpha_0 \Rightarrow \alpha_0 = a_0 + \frac{1}{\alpha_1}$ $(\alpha_1 > 1)$, $\alpha_0 = [a_0, \alpha_1]$.
let $a_1 = \lfloor \alpha_1 \rfloor$. If $a_1 \neq \alpha_1 \Rightarrow \alpha_1 = a_1 + \frac{1}{\alpha_2}$ $(\alpha_2 > 1)$, $\alpha_0 = [a_0, a_1, \alpha_2]$, etc.

If $\alpha \in \mathbb{Q}$, this process must stop (Euclid's algorithm) $\Rightarrow \alpha = [a_0, \dots, a_n] = \frac{p_n}{q_n}$ $\quad a_0 \in \mathbb{Z}$, $1 \le a_1, \dots, a_n \in \mathbb{Z}$

If $\alpha \notin \mathbb{Q}$, this never stops, all $a_n \neq \alpha_n$; so $\forall n \ge 0$ $\quad \alpha_0 = [a_0, \dots, a_n, \alpha_{n+1}] = \frac{p_n \alpha_{n+1} + p_{n-1}}{q_n \alpha_{n+1} + q_{n-1}}$, $\alpha_{n+1} > 1$.

Def. the rational numbers $\frac{p_n}{q_n} = [a_0, \dots, a_n]$ are called the convergents to α $\quad (a_i \in \mathbb{Z}, \ a_j \ge 1 \ \forall j \neq 0)$

Properties: (1) $\forall n \ge 1$ $a_n \ge 1 \Rightarrow \lim_{n\to\infty} q_n = +\infty$, $q_1 < q_2 < \dots$

(2) $\frac{p_0}{q_0} < \frac{p_2}{q_2} < \dots < \alpha < \dots < \frac{p_3}{q_3} < \frac{p_1}{q_1}$; (3) $\left| \frac{p_n}{q_n} - \frac{p_{n+1}}{q_{n+1}} \right| = \frac{1}{q_n q_{n+1}} < \frac{1}{q_n^2} \Rightarrow$ (4) $\left| \alpha - \frac{p_n}{q_n} \right| < \frac{1}{q_n^2}$

$\lim_{n\to\infty} \frac{p_n}{q_n} = \alpha$

(5) $\forall n \ge 0$ $\exists m \in \{n, n+1\}$ $\left| \alpha - \frac{p_m}{q_m} \right| < \frac{1}{2q_m^2}$

Pf: $\alpha - \frac{p_n}{q_n}$, $\alpha - \frac{p_{n+1}}{q_{n+1}}$ have opposite signs $\Rightarrow \left|\alpha - \frac{p_n}{q_n}\right| + \left|\alpha - \frac{p_{n+1}}{q_{n+1}}\right| = \frac{1}{q_n q_{n+1}} < \frac{1}{2}\left(\frac{1}{q_n^2} + \frac{1}{q_{n+1}^2}\right)$

Exercise: $\forall n \ge 0$ $\exists m \in \{n, n+1, n+2\}$ $\left| \alpha - \frac{p_m}{q_m} \right| < \frac{1}{\sqrt5 q_m^2}$ (for $\alpha = \frac{1+\sqrt5}{2}$ this cannot be improved)

(6) $\forall n \ge 0$ $\quad |q_n \alpha - p_n| = \frac{1}{q_n \alpha_{n+1} + q_{n-1}} \Rightarrow \left| \alpha - \frac{p_n}{q_n} \right| < \frac{1}{a_{n+1} q_n^2}$

(7) $\forall n > 1$ $\quad q_n \alpha_{n+1} + q_{n-1} > q_n + q_{n-1} = (a_n + 1) q_{n-1} + q_{n-2} > q_{n-1}\alpha_n + q_{n-2} \Rightarrow |q_n \alpha - p_n| < |q_{n-1}\alpha - p_{n-1}|$

(8) If $1 \le q < q_{n+1}$, $p \in \mathbb{Z} \Rightarrow |q\alpha - p| \ge |q_n \alpha - p_n|$. Pf: $\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} p_n & p_{n+1} \\ q_n & q_{n+1}\end{pmatrix}\begin{pmatrix} u \\ v\end{pmatrix}$ $\quad u, v \in \mathbb{Z}$, $u \neq 0$.

If $v \neq 0 \Rightarrow u, v$ have opposite signs $\Rightarrow$ so do $q_n\alpha - p_n$, $q_{n+1}\alpha - p_{n+1} \Rightarrow$

$|q\alpha - p| = |u(q_n\alpha - p_n) + v(q_{n+1}\alpha - p_{n+1})| \ge |q_n\alpha - p_n| \cdot |u| \ge |q_n \alpha - p_n|$ (equality $\Leftrightarrow v = 0$, $u = 1$, $p = p_n$, $q = q_n$)

(9) If $p,q \in \mathbb{Z}, q>0$, $\left|\alpha - \frac{p}{q}\right| < \frac{1}{2q^2} \Longrightarrow \exists n \geq 0 \quad \frac{p}{q} = \frac{p_n}{q_n}$.

Pf: If $q_n \leq q < q_{n+1}$, then $\left|\frac{p}{q} - \frac{p_n}{q_n}\right| \leq \left|\alpha - \frac{p}{q}\right| + \left|\alpha - \frac{p_n}{q_n}\right| \overset{(8)}{\leq} \underbrace{|q\alpha - p|}_{<\frac{1}{2q}} \left(\underbrace{\frac{1}{q}}_{\leq \frac{1}{q_n}} + \frac{1}{q_n}\right) < \frac{1}{qq_n} \Longrightarrow \frac{p}{q} = \frac{p_n}{q_n}$

(Ultimately) periodic continued fractions

Ex: $[1,1,1,1,\dots] = [\overline{1}] = \frac{1+\sqrt{5}}{2}$

Prop. If $\alpha \in \mathbb{R}$, $\alpha = [a_0, \dots, a_{k-1}, \underbrace{\overline{a_k, \dots, a_{k+m-1}}}_{\text{repeats itself}}]$, then $\exists A, B, C \in \mathbb{Z}$, $A \neq 0$ $\quad A\alpha^2 + B\alpha + C = 0$.

Pf: $\alpha = [a_0, \dots, a_{k-1}, \alpha_k] = \begin{pmatrix} p_{k-1} & p_{k-2} \\ q_{k-1} & q_{k-2} \end{pmatrix}(\alpha_k)$, so we can replace α by α_k and assume that $\alpha = [\overline{a_0, \dots, a_{k-1}}]$ is purely periodic. Then $\alpha = [a_0, \dots, a_{k-1}, \alpha] = \begin{pmatrix} r & s \\ t & u \end{pmatrix}(\alpha) = \frac{r\alpha + s}{t\alpha + u}$
$\Longrightarrow$ quadratic equation $\quad t\alpha^2 + (u-r)\alpha - s = 0$, $\quad t,u,r,s \in \mathbb{Z}$.

Ex: $\alpha = [1,2,2,2,\dots] = [1,\overline{2}]$, $\quad \alpha + 1 = [\overline{2}] = 2 + \frac{1}{\alpha+1} \Longrightarrow (\alpha+1)^2 = 2\alpha + 3$, $\alpha^2 = 2 \Rightarrow \alpha = \sqrt{2}$

Goal: prove a converse result and characterise purely periodic continued fractions.

Assume: $\alpha = \alpha_0 \notin \mathbb{Q}$ is a root of $A_0X^2 + B_0X + C_0 = A(X-\alpha_0)(X-\alpha_0') = 0$ $\quad A_0, B_0, C_0 \in \mathbb{Z}$, $A_0 \neq 0$

Write $f_0(x,y) = A_0x^2 + B_0xy + C_0y^2$. We want to find $\Delta = B_0^2 - 4A_0C_0 > 0$, equations for all α_n in $\alpha_0 = [a_0, \dots, a_n, \alpha_{n+1}]$. As $\alpha_n = a_n + \frac{1}{\alpha_{n+1}}$, $\alpha_n' = a_n + \frac{1}{\alpha_{n+1}'}$,

$f_n(x_n,y_n) = A_n(x_n - \alpha_n y_n)(x_n - \alpha_n' y_n) = \frac{A_n}{\alpha_{n+1}\alpha_{n+1}'}\left(-\alpha_{n+1}\underbrace{(x_n - a_n y_n)}_{y_{n+1}} + \underbrace{y_n}_{x_{n+1}}\right)\left(-\alpha_{n+1}'(x_n - a_n y_n) + y_n\right)$

Let $\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & -a_n \end{pmatrix}\begin{pmatrix} x_n \\ y_n \end{pmatrix} \quad \left(\Longleftrightarrow \begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix}\right)$, then

$\underbrace{A_n x_n^2 + B_n x_n y_n + C_n y_n^2}_{A_n(a_nx_{n+1}+y_{n+1})^2 + B_n(a_nx_{n+1}+y_{n+1})x_{n+1} + C_nx_{n+1}^2} = f_n(x_n,y_n) = f_{n+1}(x_{n+1},y_{n+1}) = A_{n+1}x_{n+1}^2 + B_{n+1}x_{n+1}y_{n+1} + \underbrace{C_{n+1}}_{A_n}y_{n+1}^2$

$\Delta(f_{n+1}) = B_{n+1}^2 - 4A_{n+1}C_{n+1} = \Delta(f_n)\left|\begin{matrix} 0 & 1 \\ 1 & -a_n \end{matrix}\right|^2 = \Delta(f_n) = \dots = \Delta$

$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \underbrace{\begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix}}_{\begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix}}\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix}$, $\quad f_0(x_0,y_0) = \underbrace{f_0(p_n,q_n)}_{A_{n+1}}x_{n+1}^2 + \dots + \underbrace{f_0(p_{n-1},q_{n-1})}_{C_{n+1} = A_n}y_{n+1}^2$

Claim: $\exists M \quad \forall n \geq 0 \quad |A_n|, |B_n|, |C_n| \leq M$.

Pf: $f_0(\alpha,1) = 0 \Longrightarrow \frac{A_{n+1}}{q_n^2} = f_0\left(\frac{p_n}{q_n},1\right) - f_0(\alpha,1) = \underbrace{A_0\left(\left(\frac{p_n}{q_n}\right)^2 - \alpha^2\right) + B_0\left(\frac{p_n}{q_n} - \alpha\right)}_{\left(\frac{p_n}{q_n} - \alpha\right)\left(A_0\left(\frac{p_n}{q_n} + \alpha\right) + B_0\right)}$

$\left|\frac{p_n}{q_n} - \alpha\right| < \frac{1}{q_n^2} \Longrightarrow \left|\frac{p_n}{q_n} + \alpha\right| < 2|\alpha| + 1$

$\Longrightarrow \underline{|A_{n+1}| < |B_0| + (2|\alpha|+1)|A_0|}$; $\Longrightarrow |C_n| = |A_{n-1}| < (\text{const.})$.

$\Longrightarrow B_n^2 = \Delta + 4A_nC_n \leq \text{const.}$

Cor. If $\alpha_0 \notin \mathbb{Q}$ is a root of $A_0X^2 + B_0X + C_0 = 0$ $\quad (A_0, B_0, C_0 \in \mathbb{Z})$, then $\exists\, m < n$ such that $(A_m, B_m, C_m) = (A_n, B_n, C_n)$ and $\alpha_m = \alpha_n$

$\Longrightarrow \alpha = [a_0, \dots, a_{m-1}, \overline{a_m, \dots, a_{n-1}}]$ is ultimately periodic.

<u>Def</u>. If $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ is a root of $AX^2+BX+C=0$ $(A,B,C \in \mathbb{Z})$, let α' be the other root : $AX^2+BX+C = A(X-\alpha)(X-\alpha')$. We say that <u>$\alpha$ is reduced</u> if <u>$\alpha > 1 > -\alpha' > 0$</u>. <u>Ex</u>: if $d \in \mathbb{Z}$, $d>1$, $\sqrt{d} \notin \mathbb{Z}$ $(\Rightarrow \sqrt{d} \notin \mathbb{Q}) \Rightarrow$ $\boxed{\alpha = \sqrt{d} + \lfloor \sqrt{d} \rfloor \text{ is reduced}}$
$(\alpha' = \sqrt{d} - \lfloor \sqrt{d} \rfloor)$

<u>Prop</u>. (1) If $\alpha = \alpha_0 = [a_0, a_1, \dots]$ is reduced, then $\forall n \geq 1$ α_n is reduced.
<u>Pf</u>: $\forall n \geq 1$ $a_n \geq 1$. $\alpha_n > 1$. If α_n is reduced $\Rightarrow -1 < \alpha_n' < 0 \Rightarrow \alpha_n' - a_n < -1 \Rightarrow \alpha_{n+1}' = \dfrac{1}{\alpha_n' - a_n} > -1$.

(2) $\underbrace{\text{the continued fraction of } \alpha_0 \text{ is purely periodic}}_{\alpha_0 = [a_0, \dots, a_{k-1}, \alpha_0] \text{ for some } k} \Longleftrightarrow \alpha_0$ is reduced.

<u>Pf</u> ($\Rightarrow$) If $\alpha_0 = \alpha_{n+1}$ for some $n \geq 0 \Rightarrow \alpha_0$ is a root of $g(X) = q_n X^2 + (q_{n-1} - p_n)X - p_{n-1} = (X-\alpha_0)(X-\alpha_0')$
$\begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix}(\alpha_{n+1})$ But $g(-1) = q_n + p_n - q_{n-1} - p_{n-1} > 0 > -p_{n-1} = g(0) \Rightarrow -1 < \alpha_0' < 0$.

($\Leftarrow$) Assume α_0 reduced. We have $\alpha_m = \alpha_n$ for some $1 \leq m < n \Rightarrow a_m = a_n$.
Each α_k reduced by (1), so $-1 < \alpha_k' = a_k + \dfrac{1}{\alpha_{k+1}'} < 0 \Rightarrow a_k = \left\lfloor -\dfrac{1}{\alpha_{k+1}'} \right\rfloor$
$\Rightarrow a_{m-1} = a_{n-1} \Rightarrow \alpha_{m-1} = \alpha_{n-1}$. Induction $\Longrightarrow \alpha_0 = \alpha_{n-m}$.

<u>Ex</u>: (1) $d \in \mathbb{Z}$, $d > 1$, $\sqrt{d} \notin \mathbb{Z}$: $\sqrt{d} + \lfloor \sqrt{d} \rfloor$ reduced $\Rightarrow \sqrt{d} = [a_0, \overline{a_1, \dots, a_m}]$, $a_0 = \lfloor \sqrt{d} \rfloor = \frac{1}{2} a_m$

<u>$d=7$</u> $\alpha_0 = \sqrt{7}$, $a_0 = \lfloor \alpha_0 \rfloor = 2$, $\alpha_1 = \dfrac{1}{\sqrt{7}-2} = \dfrac{\sqrt{7}+2}{3}$, $a_1 = \lfloor \alpha_1 \rfloor = 1$, $\alpha_2 = \dfrac{3}{\sqrt{7}-1} = \dfrac{\sqrt{7}+1}{2}$, $a_2 = \lfloor \alpha_2 \rfloor = 1$,
$\alpha_3 = \dfrac{2}{\sqrt{7}-1} = \dfrac{\sqrt{7}+1}{3}$, $a_3 = \lfloor \alpha_3 \rfloor = 1$, $\alpha_4 = \dfrac{3}{\sqrt{7}-2} = \sqrt{7}+2$, $a_4 = \lfloor \alpha_4 \rfloor = 4$, $\alpha_5 = \dfrac{1}{\sqrt{7}-2} = \alpha_1$

$\sqrt{7} = [2, \overline{1,1,1,4}]$

n	-2	-1	0	1	2	3	4	5	6	7
a_n			2	1	1	1	4	1	1	1
p_n	0	1	2	3	5	8	37	45	82	137
q_n	1	0	1	1	2	3	14	17	31	48
$p_n^2 - 7q_n^2$			-3	2	-3	1	-3	2	-3	1

$(8+3\sqrt{7})^2 = 137 + 48\sqrt{7}$

$\underbrace{(8+3\sqrt{7})(8-3\sqrt{7})}_{8^2 - 7\cdot 3^2} = 1$

(2) $d \in \mathbb{Z}$, $d > 1$, $\sqrt{d} \notin \mathbb{Z}$, $d \equiv 1 \pmod 4 \Rightarrow \dfrac{1+\sqrt{d}}{2} + \left\lfloor \dfrac{\sqrt{d}-1}{2} \right\rfloor$ reduced $\Rightarrow \dfrac{1+\sqrt{d}}{2} = [a_0, \overline{a_1, \dots, a_m}]$, $a_m = 2a_0 - 1$

<u>$d=13$</u> $\alpha_0 = \dfrac{\sqrt{13}+1}{2}$, $a_0 = \lfloor \alpha_0 \rfloor = 2$, $\alpha_1 = \dfrac{2}{\sqrt{13}-3} = \dfrac{\sqrt{13}+3}{2}$, $a_1 = \lfloor \alpha_1 \rfloor = 3$, $\alpha_2 = \dfrac{2}{\sqrt{13}-3} = \dfrac{\sqrt{13}+3}{2} = \alpha_1$

$\dfrac{\sqrt{13}+1}{2} = [2, \overline{3}]$

n	-2	-1	0	1	2
a_n			2	3	3
p_n	0	1	2	7	23
q_n	1	0	1	3	10
$p_n^2 - p_n q_n - 3q_n^2$			-1	1	-1

Lemma Let $\alpha=[a_0,a_1,\dots]\in\mathbb{R}\setminus\mathbb{Q}$. If $\exists\, h=\begin{pmatrix}P&R\\Q&S\end{pmatrix}\in GL_2(\mathbb{Z})$, $Q>S>0$, and $\exists\,\beta>1$ such that $\alpha=h(\beta)=\frac{P\beta+R}{Q\beta+S}$, then $\exists\, k\ge1$ $h=\begin{pmatrix}p_k&p_{k-1}\\q_k&q_{k-1}\end{pmatrix}$ and $\beta=\alpha_{k+1}=[a_{k+1},a_{k+2},\dots]$.

Pf. $PS-QR=\det(h)\in\mathbb{Z}^\times=\{\pm1\}\Rightarrow(P,Q)=1$; as $Q\ge2\Rightarrow P/Q\notin\mathbb{Z}\Rightarrow$ we can write $\frac{P}{Q}=[b_0,\dots,b_k]$ with $k\ge1$, $b_k\ge2$. After possibly replacing $[b_0,\dots,b_k]$ by $[b_0,\dots,b_{k-1},1]$ we can assume that $\det(h)=(-1)^{k-1}$. Write $\frac{P_j}{Q_j}=[b_0,\dots,b_j]$ $(0\le j\le k)$ in the usual way. We have $\frac{P_k}{Q_k}=\frac{P}{Q}$, $(P,Q)=1$, $Q>0\Rightarrow \underline{P_k=P,\ Q_k=Q}$.

As $\begin{vmatrix}P&P_{k-1}\\Q&Q_{k-1}\end{vmatrix}=\begin{vmatrix}P&R\\Q&S\end{vmatrix}$ $(=(-1)^{k-1})$ $\Rightarrow P(S-Q_{k-1})=Q(R-P_{k-1})\Rightarrow Q\mid(S-Q_{k-1})$ $(P,Q)=1$

But $Q>S>0$, $Q=Q_k>Q_{k-1}>0\Rightarrow|S-Q_{k-1}|<Q\Rightarrow$ $\underline{S=Q_{k-1}\Rightarrow R=P_{k-1}}$

So $h=\begin{pmatrix}P_k&P_{k-1}\\Q_k&Q_{k-1}\end{pmatrix}\Rightarrow\alpha=[b_0,\dots,b_k,\beta]$.

As $\beta>1\Rightarrow\beta=[b_{k+1},b_{k+2},\dots]$, $b_{k+1}\ge1$ $\Big\}\Rightarrow\alpha=[b_0,b_1,\dots]\Rightarrow\forall n\ge0\ a_n=b_n$ and $\beta=\alpha_{k+1}$.

Def. $\alpha=[a_0,a_1,\dots]$, $\alpha'=[a'_0,a'_1,\dots]\in\mathbb{R}\setminus\mathbb{Q}$ are equivalent if $\exists\,k,l\ge0\ \forall n\ge1\ a_{k+n}=a'_{l+n}$ $(\iff\exists\,k,l\ge0\ \ \alpha_{k+1}=\alpha'_{l+1})$.

Prop. $\alpha,\alpha'\in\mathbb{R}\setminus\mathbb{Q}$ are equivalent $\iff\exists\,g=\begin{pmatrix}A&B\\C&D\end{pmatrix}\in GL_2(\mathbb{Z})$ $\begin{pmatrix}A,B,C,D\in\mathbb{Z}\\ \det(g)=\pm1\end{pmatrix}$ $\alpha=g(\alpha')=\frac{A\alpha'+B}{C\alpha'+D}$

Pf ($\Rightarrow$) $\alpha=h_k(\alpha_{k+1})$, $h_k=\begin{pmatrix}p_k&p_{k-1}\\q_k&q_{k-1}\end{pmatrix}$, $\alpha'=h'_l(\alpha'_{l+1})$, $h'_l=\begin{pmatrix}p'_l&p'_{l-1}\\q'_l&q'_{l-1}\end{pmatrix}$.

If $\alpha_{k+1}=\alpha'_{l+1}\Rightarrow\alpha=h_kh_l'^{-1}(\alpha')$. But $h_k,h'_l\in GL_2(\mathbb{Z})\Longrightarrow h_kh_l'^{-1}\in GL_2(\mathbb{Z})$.

($\Leftarrow$) If $\alpha=g(\alpha')$, $g=\begin{pmatrix}A&B\\C&D\end{pmatrix}\in GL_2(\mathbb{Z})$, we can assume that $C\alpha'+D$ (after possibly replacing g by $-g$). We have $\forall l\ge0$ $\alpha=(gh'_l)(\alpha'_{l+1})$. Write

$$gh'_l=\begin{pmatrix}Ap'_l+Bq'_l & Ap'_{l-1}+Bq'_{l-1}\\ Cp'_l+Dq'_l & Cp'_{l-1}+Dq'_{l-1}\end{pmatrix}=\begin{pmatrix}A_l&B_l\\C_l&D_l\end{pmatrix}.$$

If $l\gg0$ is big enough,

$$D_{l+1}=C_l=(C\alpha'+D)q'_l+C(p'_l-q'_l\alpha)\ge\underbrace{(C\alpha'+D)}_{>0}+C(p'_l-q'_l\alpha)>0$$

$(|p'_l-q'_l\alpha|<1/q'_l)$

$$\Rightarrow p'_l(C_l-D_l)>p'_{l-1}C_l-p'_lD_l=D(p'_{l-1}q'_l-q'_{l-1}p'_l)=(-1)^lD$$

So, if $l\gg0$ and $(-1)^l=\operatorname{sgn}(D)$, then $\underline{C_l>D_l>0}$.

Lemma above then applies to $h=gh'_l$ and $\beta=\alpha'_{l+1}\Rightarrow\exists\,k\ge1\ \alpha_{k+1}=\alpha'_{l+1}$.

Solutions of $f(x,y) = Ax^2 + Bxy + Cy^2 = \pm 1$

$A, B, C \in \mathbb{Z}$
$\Delta = B^2 - 4AC$

If there is a solution $u \in \mathbb{Z}^2$ of $f(u) = \pm 1 \implies u$ primitive $\implies \exists v \in \mathbb{Z}^2$ such that u, v is a positive basis of $\mathbb{Z}^2$. The change of variables $f'(x', y') = f(x'u + y'v)$ transforms f (properly) to $f' = \pm x^2 + \cdots$. After $x \mapsto x + ny$ we obtain, finally, a properly equivalent form $\pm(x^2 + 0 \cdot xy + \cdots)$ or $\pm(x^2 + xy + \cdots)$.

Summary: if f represents ± 1, then $\pm f$ is properly equivalent to the principal form

$$x^2 - (\Delta/4)y^2 \quad \text{if } \Delta \equiv 0 \pmod 4$$
$$x^2 + xy + \frac{1-\Delta}{4}y^2 \quad \text{if } \Delta \equiv 1 \pmod 4$$

The case $\Delta < 0$: $x^2 + \frac{|\Delta|}{4}y^2 = 1$ has solutions ($x, y \in \mathbb{Z}$) $(x,y) = (\pm 1, 0)$ (and $(0, \pm 1)$ if $\Delta = -4$)

$\underbrace{x^2 + xy + \frac{1+|\Delta|}{4}y^2}_{\frac{(2x+y)^2 + |\Delta| y^2}{4}} = 1$ — " — $(x,y) = (\pm 1, 0)$ (and $(0, \pm 1)$, $\pm(1, -1)$ if $\Delta = -3$)

The case $\Delta > 0$, $\sqrt{\Delta} \notin \mathbb{Z}$:
$$f(x,y) = \left\{ \begin{matrix} x^2 - \frac{\Delta}{4}y^2 \\ x^2 + xy + \frac{1-\Delta}{4}y^2 \end{matrix} \right\} = (x - \alpha y)(x - \alpha' y)$$
$\alpha = \sqrt{\Delta}/2$, $\alpha' = -\sqrt{\Delta}/2$
$\alpha = \frac{1+\sqrt{\Delta}}{2}$, $\alpha' = \frac{1-\sqrt{\Delta}}{2}$

Thm (1) $[\alpha] = [a_0, \overline{a_1, \ldots, a_m}]$ (2) the solutions of $f(x,y) = \pm 1$, $x, y \in \mathbb{Z}$, $x, y > 0$ are given precisely by the convergents $x_\ell = p_{\ell m - 1}$, $y_\ell = q_{\ell m - 1}$, $x_\ell - y_\ell \alpha = (x_1 - y_1 \alpha)^\ell$ $(\ell = 1, 2, \ldots)$
$f(x_\ell, y_\ell) = (-1)^{m\ell}$. All solutions with $x, y \in \mathbb{Z}$: $x - y\alpha = \pm (x_1 - y_1\alpha)^k$, $k \in \mathbb{Z}$

Pf. (1) Already proved. (2) If $x^2 - \frac{\Delta}{4}y^2 = \pm 1$ ($x, y \geq 1$) $\implies \left|\frac{x}{y} - \frac{\sqrt{\Delta}}{2}\right| \cdot \underbrace{\left|\frac{x}{y} + \frac{\sqrt{\Delta}}{2}\right|}_{> 1+1 = 2} = \frac{1}{y^2} \implies \left|\frac{x}{y} - \frac{\sqrt{\Delta}}{2}\right| < \frac{1}{2y^2} \implies \exists n\ \frac{x}{y} = \frac{p_n}{q_n}$.

If $\left(x - \frac{1+\sqrt{\Delta}}{2}y\right)\left(x - \frac{1-\sqrt{\Delta}}{2}y\right) = \pm 1$, $y > 0$, $x \geq 0$ $\implies \left|\frac{x}{y} + \frac{\sqrt{\Delta} - 1}{2}\right| \geq 2$ (if $\Delta \neq 5$) $\implies \left|\frac{x}{y} - \frac{1+\sqrt{\Delta}}{2}\right| < \frac{1}{2y^2} \implies \exists n\ \frac{x}{y} = \frac{p_n}{q_n}$

As $\alpha = [a_0, \ldots, a_n, \alpha_{n+1}] = \begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix} (\alpha_{n+1}) \implies \alpha_{n+1} = \begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix}^{-1} (\alpha) = \begin{pmatrix} q_{n-1} & -p_{n-1} \\ -q_n & p_n \end{pmatrix} (\alpha) = \frac{q_{n-1}\alpha - p_{n-1}}{p_n - q_n \alpha} \cdot \frac{p_n - q_n \alpha'}{p_n - q_n \alpha'} =$

$\boxed{\alpha' = u - \alpha,\ u = \{0, 1\}}$

$= \frac{(q_{n-1}p_n - p_{n-1}q_n)(\alpha + c_n)}{f(p_n, q_n)} = \frac{\alpha + c_n}{\underbrace{(-1)^{n-1} f(p_n, q_n)}_{> 0}}$ for some $c_n \in \mathbb{Z}$. Therefore, we have the following equivalences:

(α_{n+1} reduced)

$|f(p_n, q_n)| = 1 \iff f(p_n, q_n) = (-1)^{n-1} \iff$
$\iff \alpha_{n+1} - \alpha \in \mathbb{Z} \iff \alpha_{n+2} = \alpha_1 \iff \exists \ell \geq 1\ n+1 = \ell m$

It remains to prove the relation between $x_\ell - y_\ell \alpha$ and $x_1 - y_1 \alpha$, but this follows from
$$\frac{x_{\ell+1} - y_{\ell+1}\alpha}{x_\ell - y_\ell \alpha} = \prod_{n=\ell m}^{\ell m + m - 1} \frac{p_n - q_n \alpha}{p_{n-1} - q_{n-1}\alpha} = \prod_{n=\ell m}^{\ell m + m - 1}\left(-\frac{1}{\alpha_{n+1}}\right) = \prod_{n=0}^{m-1}\left(-\frac{1}{\alpha_{n+1}}\right) = \prod_{n=0}^{m-1} \frac{p_n - q_n \alpha}{p_{n-1} - q_{n-1}\alpha} = x_1 - y_1 \alpha.$$

The case $\Delta = 5$: $\alpha = \frac{1+\sqrt{5}}{2} = [\overline{1}]$, the solutions of $x^2 - xy - y^2 = \pm 1$ ($x, y \in \mathbb{Z}$) are given by $x - \alpha y = \pm \alpha^\ell$ ($\ell \in \mathbb{Z}$)

Ex: $\Delta = 7$: the solutions of $x^2 - 7y^2 = \pm 1$ are given by
$$x + y\sqrt{7} = \pm (8 + 3\sqrt{7})^n \qquad n \in \mathbb{Z}$$
$$(\implies \quad x^2 - 7y^2 = +1)$$

Cor of Thm. The fundamental unit $\varepsilon \in \mathcal{O}_\Delta^{+}$ satisfies $N(\varepsilon) = (-1)^m$, m = length of the period of the continued fraction of $\left\{ \begin{matrix} \sqrt{\Delta}/2 \\ \frac{1+\sqrt{\Delta}}{2} \end{matrix} \right\}$.

Note: $\varepsilon \in \mathcal{O}_\Delta^*$ can be determined from the continued fraction of any α such that $a\alpha^2 + b\alpha + c = 0$, $b^2 - 4ac = \Delta$, $\gcd(a,b,c) = 1$:

write $\alpha = [a_0, \dots, a_{k-1}, \beta]$, $\beta = [\overline{b_0, \dots, b_{\ell-1}}] = [b_0, \dots, b_{\ell-1}, \beta] = M(\beta)$

$M = \begin{pmatrix} b_0 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} b_{\ell-1} & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \in GL_2(\mathbb{Z})$, $\det(M) = (-1)^\ell$, $A\beta^2 + B\beta + C = 0$, $B^2 - 4AC = \Delta$, $\gcd(A, B, C) = 1$

Then $M\begin{pmatrix} \beta \\ 1 \end{pmatrix} = (r\beta + s)\begin{pmatrix} \beta \\ 1 \end{pmatrix}$, $\lambda = r\beta + s$, $\lambda' = r\beta' + s$ are the eigenvalues of M

$\lambda(\mathbb{Z}\beta + \mathbb{Z}) \subset \mathbb{Z}\beta + \mathbb{Z} \overset{\text{exercise}}{\Longrightarrow} \lambda \in \mathcal{O}_\Delta$ (uses)

$N(\lambda) = \lambda\lambda' = \det(M) = (-1)^\ell \Longrightarrow \lambda \in \mathcal{O}_\Delta^*$.

Ex: $\alpha = \frac{\sqrt{34} + 4}{9} = [\overline{1, 10, 1, 4}] = \beta = [1, 10, 1, 4, \beta] = M(\beta)$

$M = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 10 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 4 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 59 & 12 \\ 54 & 11 \end{pmatrix}$, $\lambda^2 - 70\lambda + 1 = 0$

$35^2 - 34 \cdot 6^2 = 1$ $\qquad$ $\lambda, \lambda' = 35 \pm 6\sqrt{34}$

Distribution of continued fractions

Consider the space $X = [0,1] \setminus \mathbb{Q}$ with the Lebesgue measure μ ($\Rightarrow \mu(X) = 1$).
Every $x \in X$ has a continued fraction expansion
$x = [0, a_1, a_2, \dots] = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \dots}}$. Eeach a_i is a function $a_i : X \longrightarrow \mathbb{N}_+$.

We want to estimate the measure (= probability) of various subsets of X defined in terms of the a_i's.

Basic estimates (F. Bernstein, 1911): let $k \in \mathbb{N}_+$

(1) $\mu(a_1 \geq k) = \mu(\text{open interval with endpoints } \frac{1}{k} \text{ and } 0) = \frac{1}{k}$; $\mu(a_1 < k) = 1 - \frac{1}{k}$

(2) $(n \geq 1)$ $\mu(\underbrace{a_1, \dots, a_n}_{\text{fixed}} ; a_{n+1} \geq k) = \mu(\text{— " —} \underbrace{[0, a_1, \dots, a_n]}_{\frac{p_n}{q_n}} \text{ and } \underbrace{[0, a_1, \dots, a_n, k]}_{\frac{kp_n + p_{n-1}}{kq_n + q_{n-1}}}) =$

$$= \left| \frac{p_n q_{n-1} - q_n p_{n-1}}{q_n (kq_n + q_{n-1})} \right| = \frac{1}{q_n (kq_n + q_{n-1})}$$

(3) $\mu(a_1 \text{ fixed}) = \frac{1}{a_1} - \frac{1}{a_1+1} = \frac{1}{a_1(a_1+1)} = \frac{1}{q_1(q_1+1)} < \frac{1}{q_1^2} = \frac{1}{a_1^2}$

(4) $\mu(a_1, \dots, a_n \text{ fixed}) = \mu(a_1, \dots, a_n ; a_{n+1} \geq 1) = \frac{1}{q_n(q_n + q_{n-1})} < \frac{1}{q_n^2} \leq \frac{1}{(a_1 \cdots a_n)^2}$

(5) $\frac{1}{k} < \frac{\mu(a_1, \dots, a_n \text{ fixed}; a_{n+1} \geq k)}{\mu(a_1, \dots, a_n \text{ fixed})} = \frac{q_n + q_{n-1}}{kq_n + q_{n-1}} < \frac{2}{k+1}$

(6) $1 - \frac{2}{k+1} < \frac{\mu(a_1, \dots, a_n \text{ fixed}; a_{n+1} < k)}{\mu(a_1, \dots, a_n \text{ fixed})} < 1 - \frac{1}{k}$

(7) The bounds in (6) do not depend on $a_1, \dots, a_n$. It follows that, for $1 \leq n_1 < \dots < n_r$ $(n_r > 1)$ and $k_1, \dots, k_r \in \mathbb{N}_+$

$$\prod_{i=1}^{r} \left(1 - \frac{2}{k_i + 1}\right) < \mu(a_{n_1} < k_1, \dots, a_{n_r} < k_r) < \prod_{i=1}^{r} \left(1 - \frac{1}{k_i}\right)$$

Thm (F. Bernstein, 1911). Fix a map $g : \mathbb{N}_+ \longrightarrow (1, \infty)$. The set $B_\infty = \{x \in X \mid \exists \infty \text{ many } n \geq 1 \text{ such that } a_n \geq g(n)\}$ has measure

$$\mu(B_\infty) = \begin{cases} 0 & \text{if } \sum_1^\infty \frac{1}{g(n)} < \infty \\ 1 & \text{if } \sum_1^\infty \frac{1}{g(n)} = \infty \end{cases}$$

Pf. Let $A_n = \{x \in X \mid a_n < g(n)\}$, $B_n = X \setminus A_n = \{x \in X \mid a_n \geq g(n)\}$. We know that $\mu(B_n) \overset{(5)}{<} \frac{2}{g(n)+1}$. If $\sum \frac{1}{g(n)} < \infty$, then $\sum \mu(B_n) < \infty \xRightarrow{\text{Borel-Cantelli}} \mu(B_\infty) = 0$.

As $B_\infty = \bigcap_{m \geq 1} \bigcup_{n \geq m} B_n$, $X \setminus B_\infty = \bigcup_{m \geq 1} \bigcap_{n \geq m} A_n$. If $\sum \frac{1}{g(n)} = \infty$, then

$\forall m \geq 1$ $\quad \mu\left(\bigcap_{n \geq m} A_n\right) \overset{(7)}{\leq} \prod_{n \geq m} \left(1 - \frac{1}{g(n)}\right) = 0 \Rightarrow \mu(X \setminus B_\infty) \leq \sum_{m \geq 1} \mu\left(\bigcap_{n \geq m} A_n\right) = 0$

$\Rightarrow \mu(B_\infty) = \mu(X) = 1$.

Prop. (Khintchine, 1924) If $A > \exp(\sqrt{2\ln 2})$, then $\mu(\{x \in X \mid \limsup_{n\to\infty} \sqrt[n]{a_1 \cdots a_n} \le e^A\}) = 1$ (the constant is not optimal).

Pf: For $n, k \in \mathbb{N}_+$, let $E_n(k) = \{x \in X \mid a_1 \cdots a_n \ge k\}$. We have (by (4))

$$\mu(E_n(k)) < \underbrace{\sum_{\substack{a_1, \ldots, a_n \in \mathbb{N}_+ \\ a_1 \cdots a_n \ge k}} \frac{1}{(a_1 \cdots a_n)^2}}_{} = S_{n,k}$$

Lemma: $S_{n,k} \le \frac{2^n}{k} \sum_{i=0}^{n-1} \frac{(\ln k)^i}{i!}$

Pf: $\prod_{i=1}^n \frac{1}{a_i^2} \le 2^n \prod_{i=1}^n \frac{1}{a_i(a_i+1)} = 2^n \prod_{i=1}^n \int_{a_i}^{a_i+1} \frac{dx_i}{x_i^2}$; let $X_n(k) = \{x_1, \ldots, x_n \ge 1, x_1 \cdots x_n \ge k\}$

$$\Longrightarrow S_{n,k} \le 2^n \int_{X_n(k)} \frac{dx_1 \cdots dx_n}{(x_1 \cdots x_n)^2} = \frac{2^n}{k} \sum_{i=0}^{n-1} \frac{(\ln k)^i}{i!} \Longrightarrow \text{Lemma.}$$

For $A > 1$, $\mu(E_n(e^{An})) < 2^n e^{-An} \sum_{i=0}^{n-1} \underbrace{\frac{(An)^i}{i!}}_{< \frac{(An)^n}{n!}} < 2^n e^{-An}\, n \frac{(An)^n}{n!}$

Stirling: $n! > (\text{const}) \sqrt{n}\, n^n e^{-n} \Longrightarrow \mu(E_n(e^{An})) < (\text{const}) \sqrt{n}\, (2Ae^{1-A})^n$

If $A > e^{\sqrt{2\ln 2}}$, then $A - 1 - \ln A > \frac{1}{2}(\ln A)^2 > \ln 2 \Longrightarrow \mu(E_n(e^{An})) < \underbrace{(\text{const})\sqrt{n}\, e^{-\alpha n}}_{\alpha > 0}$

$$\Longrightarrow \sum_{n \ge 1} \mu(E_n(e^{An})) < \infty.$$

Borel–Cantelli Lemma $\Longrightarrow \mu(\{x \in X \mid \exists \infty \text{ many } n \ \sqrt[n]{a_1 \cdots a_n} \ge e^A\}) = 0$

$$\mu(\{x \in X \mid \exists n_0\ \forall n > n_0\ \sqrt[n]{a_1 \cdots a_n} < e^A\}) = 1$$

Cor. If $A > e^{\sqrt{2\ln 2}}$, then $\mu(\{x \in X \mid \exists n_0\ \forall n > n_0\ \sqrt[n]{q_n} < 2e^A\}) = 1$.

Pf: $q_n = a_n q_{n-1} + q_{n-2} \le (a_n + 1) q_{n-1} \le 2 a_n q_{n-1} \Longrightarrow q_n \le 2^n a_1 \cdots a_n$.

Thm (Khintchine, 1924). Let $f : [1, \infty) \longrightarrow (0, \infty)$ be a function such that $x f(x)$ is non-increasing and $\sum_{m=1}^\infty f(m) = \infty$. Then

$$\mu(\{x \in X \mid \exists \infty \text{ many } q \in \mathbb{N}_+ \text{ such that } ((qx)) < f(q)\}) = 1.$$

Pf. Let $C = 2e^A$, $A > e^{\sqrt{2\ln 2}}$ and consider $g(n) = \frac{1}{C^n f(C^n)}$.

$$\infty = \sum_{m \ge 1} f(m) = \sum_{n \ge 0} \sum_{C^n \le m < C^{n+1}} f(m) \le \sum_{n \ge 0} f(C^n)\, C^{n+1} \Rightarrow \sum_{n \ge 1} \frac{1}{g(n)} = \infty \xrightarrow{\text{Bernstein}}$$

$\mu(Z_1) = 1$, $Z_1 = \{x \mid \exists \infty\ n \quad a_{n+1} \ge g(n)\}$. By Cor., $Z_2 = \{x \mid \exists n_0\ \forall n > n_0\ \sqrt[n]{q_n} < C\}$ has $\mu(Z_2) = 1 \Longrightarrow \mu(Z_1 \cap Z_2) = 1$. If $x \in Z_1 \cap Z_2$ and $n > n_0$, then

$$|q_n x - p_n| \le \frac{1}{q_{n+1}} < \frac{1}{a_{n+1} q_n} \le \frac{1}{q_n g(n)} = \frac{C^n f(C^n)}{q_n} \le \frac{q_n f(q_n)}{q_n} = f(q_n).$$

The limit distribution function

For $x = [0, a_1, a_2, \ldots] \in X$ and $n \in \mathbb{N}_+$, let $x_n = [0, a_{n+1}, a_{n+2}, \ldots] \in X$; then $x_{n+1} = \left\{\frac{1}{x_n}\right\} = \frac{1}{x_n} - \left[\frac{1}{x_n}\right]$. Consider the function $m_n(t) = \mu(x_n < t)$ $(0 \le t \le 1)$

$(m_n(t) = \mu(\{x \in X \mid x_n < t\}))$.

As $x_{n+1} < t \iff x_n \in \coprod_{k\ge1} \left(\frac{1}{k}, \frac{1}{k+t}\right)$, $m_{n+1}(t) = \sum_{k\ge1} \left(m_n\left(\frac{1}{k}\right) - m_n\left(\frac{1}{k+t}\right)\right)$

If $\lim_{n\to\infty} m_n(t) = m(t)$ exists, then $\underset{m(1)=1}{m(t)} = \sum_{k\ge1} \left(m\left(\frac{1}{k}\right) - m\left(\frac{1}{k+t}\right)\right)$

<u>Fact</u>: $m(t)$ exists and $m(t) = \ln(1+t)/\ln 2$

This was asserted – without proof – by Gauss, proved by Kuzmin with $|m - m_n| = O(e^{-c\sqrt{n}})$ and P. Lévy with $|m - m_n| = O(e^{-cn})$.

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the set $E_n^{(k)} = \{x \in X \mid \frac{1}{k+1} < x_n \le \frac{1}{k}\} = \{x \in X \mid a_{n+1} = k\}$ has measure

$$\mu(E_n^{(k)}) = m_n\left(\tfrac{1}{k}\right) - m_n\left(\tfrac{1}{k+1}\right) \xrightarrow{n\to\infty} \Big(\underbrace{\ln\left(1+\tfrac{1}{k}\right) - \ln\left(1+\tfrac{1}{k+1}\right)}_{\ln\left(1+\frac{1}{k(k+2)}\right)}\Big)/\ln 2$$

$$\mu(E_{n_1}^{(k_1)} \cap E_{n_2}^{(k_2)}) - \mu(E_{n_1}^{(k_1)})\,\mu(E_{n_2}^{(k_2)}) = O\left(\frac{1}{k_1^2 k_2^2} e^{-c(n_2-n_1)}\right)$$

if $n_2 - n_1$ is big ("almost independence")

<u>Thm</u> (Khintchine) If $f : \mathbb{N}_+ \longrightarrow (0,\infty)$ is an increasing (but not too rapidly) function, then

$$\mu\left(\left\{x \in X \,\middle|\, \lim_{n\to\infty} \frac{f(a_1) + \cdots + f(a_n)}{n} = \frac{1}{\ln 2} \sum_{k\ge1} f(k) \ln\left(1 + \frac{1}{k(k+2)}\right)\right\}\right) = 1$$

<u>Ex</u>: <u>$f(x) = \ln(x)$</u>: $\mu(\{x \in X \mid \lim_{n\to\infty} \sqrt[n]{a_1 \cdots a_n} = C\}) = 1$,

$$\ln C = \frac{1}{\ln 2} \sum_{k\ge1} \ln k \, \ln\left(1 + \frac{1}{k(k+2)}\right)$$
