LILLE
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Beilinson's Conjectures

Dream: values of L-functions $\overset{??}{\longleftrightarrow}$ arithmetic invariants

Example 1 (Dedekind) $[K:\mathbb{Q}]<\infty$

$$\zeta_K(s)=\prod_{\mathfrak{p}}\left(1-(N\mathfrak{p})^{-s}\right)^{-1}$$ abs. convergent for $Re(s)>1$, has hol. cont. to $\mathbb{C}-\{1\}$

For $s\to 1$, $$\zeta_K(s)\sim(s-1)^{-1}\cdot\frac{2^{r_1}(2\pi)^{r_2}h_K R_K}{w_K\,|D_K|^{1/2}}$$

$K\otimes_{\mathbb{Q}}\mathbb{R}\xrightarrow{\sim}\mathbb{R}^{r_1}\times\mathbb{C}^{r_2}$, $h_K=|Cl_K|$, $w_K=|(\mathcal{O}_K^*)_{tors}|$

Functional equation (Hecke): $\Gamma_{\mathbb{R}}(s)=\pi^{-s/2}\Gamma(\frac{s}{2})$

$\Gamma_{\mathbb{C}}(s)=\Gamma_{\mathbb{R}}(s)\Gamma_{\mathbb{R}}(s+1)=2(2\pi)^{-s}\Gamma(s)$

$$\hat{\zeta}_K(s):=|D_K|^{s/2}\Gamma_{\mathbb{R}}(s)^{r_1}\Gamma_{\mathbb{C}}(s)^{r_2}=\hat{\zeta}_K(1-s)$$

For $s\to 0$, $$\boxed{\zeta_K(s)\sim s^{r_1+r_2-1}\cdot\left(\frac{-h_K R_K}{w_K}\right)},$$

Regulator map:

$$\begin{tikzcd}
\varepsilon \arrow[d, phantom, "\cap"] \arrow[rr, mapsto] & & (\log|\sigma(\varepsilon)|)_\sigma \arrow[d, phantom, "\cap"] & \\
\mathcal{O}_K^* \arrow[rr] \arrow[dr, "reg"] & & (\mathbb{R}^{\mathrm{Hom}(K,\mathbb{C})})^+ \simeq \mathbb{R}^{r_1+r_2} \arrow[dr, "\Sigma"] & \\
& \mathrm{Ker}(\Sigma)\simeq\mathbb{R}^{r_1+r_2-1} \arrow[ur, hook] & & \mathbb{R}
\end{tikzcd}$$

"+" = (+1)-eigenspace for complex conjugation

$$\begin{aligned}
r:=r_1+r_2-1 &= rk_{\mathbb{Z}}(\mathcal{O}_K^*)\\
&= \dim_{\mathbb{R}}(\mathrm{Ker}(\Sigma))\\
&= \mathrm{ord}_{s=0}\,\zeta_K(s)
\end{aligned}$$

$$\begin{aligned}
R_K &= vol\Big(\mathrm{Ker}(\Sigma)/\underbrace{reg(\mathcal{O}_K^*)}_{\text{lattice in } \mathrm{Ker}(\Sigma)}\Big)\\
&= \det\big((r\times r)\text{-matrix with entries } \log|\sigma_i(\varepsilon_j)|\big)
\end{aligned}$$

Beilinson's conjectures :

"motivic L-function" $\qquad L(M,s) \underset{\mathbb{Q}^\times}{\sim} s^{r_M} \cdot \det(\text{reg})\ , \ s \to 0$

reg: "arithmetic cohomology" $\longrightarrow$ "analytic cohomology"

$\simeq \mathbb{Q}^{r_M}$ $\qquad\qquad$ $\simeq \mathbb{R}^{r_M}$

motivic cohomology
alg. K-theory
higher Chow groups

(provided $s=0$ < centre of symmetry of the functional equation for $L(M,s)$)

Example 2 (Birch, Swinnerton-Dyer)

$E/\mathbb{Q}$ elliptic curve, conductor $N = N_E$

[illegible] [illegible] $\rightsquigarrow \exists$ nice model $\mathcal{E}/\mathbb{Z}[\frac{1}{N}]$

$p \nmid N$ prime $\qquad |\mathcal{E}(\mathbb{F}_p)| = (1-\alpha_p)(1-\beta_p) = 1 - a_p + p$

$$\beta_p = \overline{\alpha_p} \ , \quad \alpha_p \beta_p = p$$

$$L(E,s) = \prod_{p \nmid N} [(1-\alpha_p p^{-s})(1-\beta_p p^{-s})]^{-1} \prod_{p \mid N} (1 - a_p p^{-s})^{-1} = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

$\uparrow$ $a_p = 0, \pm 1$

abs. convergent for $\mathrm{Re}(s) > \frac{3}{2}$

Wiles + ... : $\quad f_E(\tau) = \sum_{n=1}^{\infty} a_n q^n \in \underline{\underline{S_2(\Gamma_0(N))}} \quad (q = e^{2\pi i \tau})$

$\Longrightarrow$ $\quad L(E,s)$ has hol. cont. to $\mathbb{C}$,

$$\hat{L}(E,s) := N^{s/2} \Gamma_{\mathbb{C}}(s) L(E,s) = \pm \hat{L}(E, 2-s)$$

Conjecture of BSD:

As $s \to 1$, $$L(E,s) \sim (s-1)^r \frac{\Omega_E^+ \, |Ш(E/\mathbb{Q})| \, R_E}{|E(\mathbb{Q})_{tors}|^2} \underbrace{\prod_{p|N\infty} c_p}_{\in \mathbb{N}}$$

$r = rk_{\mathbb{Z}} \, E(\mathbb{Q})$

Ω_E^+ – period, $\Omega_E^+ = \int_{E(\mathbb{R})^0} \omega$

$R_E = \det(\,(r \times r)\text{-matrix of } \hat{h}(P_i, P_j)\,)$

P_i – generators of $E(\mathbb{Q})/tors$

$\hat{h} : E(\mathbb{Q}) \times E(\mathbb{Q}) \longrightarrow \mathbb{R}$ height pairing

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Analogy:

	$\zeta_K(s)$	$L(E,s)$
pair of points	$s = 0, 1$	$s = 1$
order of vanishing	$r_1 + r_2 - 1$, -1	r
period	$(2\pi)^{r_2}$ (at $s=1$)	Ω_E^+
arithmetic cohomology	$\mathcal{O}_K^*$	$E(\mathbb{Q})$
	Cl_K	$Ш(E/\mathbb{Q})$
linear	reg (= $\log|\cdot|$)	height $\hat{h}$
	R_K	R_E

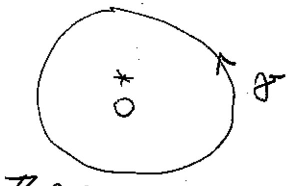
bilinear ($s=1$ central pt)

Next: "motivic" interpretation of each term in terms of

$H_1(\mathbb{G}_{m/K})$ $H_1(E)$

$\boxed{\mathbb{G}_m = \mathbb{G}_{m/K}}$

$\mathbb{G}_m(\mathbb{C}) = \mathbb{C}^*$



$H_1(\mathbb{C}^*, \mathbb{Z}) = \mathbb{Z}\cdot\gamma$

$\omega = \frac{dz}{z}$

$\int_\gamma \omega = 2\pi i$

<u>logarithm</u> :
$$\begin{tikzcd} \mathbb{C}^* \arrow[r, "\log", "\sim"'] & \mathbb{C}/2\pi i\mathbb{Z} \\ a \arrow[r, mapsto] & \int_1^a \omega \text{ (mod periods)} \end{tikzcd}$$

$$|\log| : \mathbb{C}^* \xrightarrow{\log} \mathbb{C}/2\pi i\mathbb{Z} \longrightarrow \mathbb{C}/2\pi i\mathbb{R} \underset{\mathrm{Re}}{\overset{\sim}{\Leftarrow}} \mathbb{R}$$

$$(\mathbb{C}^*)_n = \mu_n(\mathbb{C}) \cong \underbrace{H_1(\mathbb{C}^*, \mathbb{Z}/n\mathbb{Z})}_{\text{free of rk}=1 \text{ over } \mathbb{Z}/n\mathbb{Z}}$$

$$\begin{tikzcd} (\mathbb{C}^*)_n \arrow[d, "\log"', "\wr"] & H_1(\mathbb{C}^*, \mathbb{Z}/n\mathbb{Z}) \arrow[dl, "\wr"] \\ \frac{2\pi i}{n}\mathbb{Z}/2\pi i\mathbb{Z} & \end{tikzcd}$$

(K-any field)

<u>(1) K=C</u>

topology

differentials

periods

$\left(= \int_{\text{closed path}} \right)$

$\int_{\text{any path}}$

torsion points

$\exists$ Galois action on them !!

$\boxed{\begin{array}{l} E : y^2 = x^3 + Ax + B \quad (A,B \in K) \\ \quad + \text{pt } O \text{ at infinity} \end{array}}$

$E(\mathbb{C})$

$H_1(E(\mathbb{C}), \mathbb{Z}) = \mathbb{Z}\gamma_1 \oplus \mathbb{Z}\gamma_2$

$$\underbrace{\underbrace{\omega = \frac{dx}{y}}_{1^{st}\text{ kind}} , \quad \eta = \frac{x\,dx}{y}}_{(1^{st}+)2^{nd}\text{ kind}}$$

$$\int_{\gamma_j} \left\{ \begin{matrix} \omega \\ \eta \end{matrix} \right\} = \left\{ \begin{matrix} \omega_j \\ \eta_j \end{matrix} \right\} \qquad \begin{pmatrix} \omega_1 & \eta_1 \\ \omega_2 & \eta_2 \end{pmatrix}$$

Legendre : $\begin{vmatrix} \omega_1 & \eta_1 \\ \omega_2 & \eta_2 \end{vmatrix} = 2\pi i$

<u>Abel-Jacobi map</u> (elliptic logarithm):

$$\begin{tikzcd} E(\mathbb{C}) \arrow[r, "A\text{-}J", "\sim"'] & \mathbb{C}/\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 = \mathbb{C}/L \\ P \arrow[r, mapsto] & \int_O^P \omega \text{ (mod periods)} \end{tikzcd}$$

$$\begin{tikzcd} E(\mathbb{C})_n \arrow[r, "\cong"] \arrow[d, "A\text{-}J", "\wr"'] & H_1(E(\mathbb{C}), \mathbb{Z}/n\mathbb{Z}) \arrow[dl, "\wr"] \\ \frac{1}{n}L/L & \end{tikzcd}$$

(free of rk=2 over $\mathbb{Z}/n\mathbb{Z}$)

$\boxed{\mathbb{G}_m/K}$

$$K \text{ --- } K(\mu_n)$$

$$\begin{tikzcd} \mathrm{Gal}(K(\mu_n)/K) \arrow[r, hook] & \mathrm{Aut}(\mu_n) \simeq (\mathbb{Z}/n\mathbb{Z})^* \\ G_K \arrow[u, two heads] \arrow[ur, "\chi_n"'] & \end{tikzcd}$$

$$\det(1-\mathrm{Fr}(v)^{geom}t \mid \mu_n) \equiv 1-(Nv)^{-1}t \pmod n$$

fix $a \in K^*$

$$0 \longrightarrow \mu_n \longrightarrow (K^{sep})^* \xrightarrow{n} (K^{sep})^* \longrightarrow 0$$

$$K^* \otimes \mathbb{Z}/n\mathbb{Z} \overset{\delta}{\xrightarrow{\sim}} H^1(G_K, \mu_n)$$
$$\overset{\Psi}{a \otimes 1}$$

$$K \text{ --- } K(\mu_n) \text{ --- } K(\mu_n, \sqrt[n]{a})$$

$$\mu_n = \langle \zeta_n \rangle, \quad \sigma : \left\{ \begin{array}{l} \zeta_n \longmapsto \zeta_n^{\chi_n(\sigma)} \\ \sqrt[n]{a} \longmapsto \sqrt[n]{a}\,\zeta_n^{a(\sigma)} \end{array} \right\}$$

$$\rho_a : \sigma \longmapsto \begin{pmatrix} \chi_n(\sigma) & a(\sigma) \\ 0 & 1 \end{pmatrix}$$

$$a(\sigma\tau) = \chi_n(\sigma)a(\tau) + a(\sigma)$$

$$\begin{array}{ccc} a \otimes 1 & \in & K^* \otimes \mathbb{Z}/n\mathbb{Z} \\ \downarrow & & \downarrow\wr \\ [a(\sigma)] & \in & H^1(G_K, \mu_n) \\ \downarrow & & \downarrow\wr \\ [\rho_a] & \in & \mathrm{Ext}^1_{\mathbb{Z}/n\mathbb{Z}[G_K]}(\mathbb{Z}/n\mathbb{Z}, \mu_n) \end{array}$$

$$0 \rightarrow \mu_n \longrightarrow \rho_a \longrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow 0$$

(2) K arbitrary

$(n, \mathrm{char}(K)) = 1$

$G_K = \mathrm{Gal}(K^{sep}/K)$

torsion points + Galois repr.

L-factors $\overset{[K:\mathbb{Q}]<\infty}{(v \nmid n)}$

Kummer theory

Extensions of Galois repr.

Action of $\sigma \in G_K$

$\boxed{E/K}$

$$K \text{ --- } K(E_n)$$

$$\rho_{E,n} : G_K \twoheadrightarrow \mathrm{Gal}(K(E_n)/K) \hookrightarrow \mathrm{Aut}(E_n) \simeq GL_2(\mathbb{Z}/n\mathbb{Z})$$

$$\det(1-\mathrm{Fr}(p)^{geom}t \mid E_n) \equiv \left(1-\frac{\alpha t}{p^r}\right)\left(1-\frac{\beta t}{p^r}\right) \pmod n \quad \begin{array}{l} K=\mathbb{Q} \\ p \nmid N_E \cdot n \end{array}$$

fix $P \in E(K)$

$$0 \longrightarrow E_n \longrightarrow E(K^{sep}) \xrightarrow{n} E(K^{sep}) \longrightarrow 0$$

$$E(K) \otimes \mathbb{Z}/n\mathbb{Z} \overset{\delta}{\hookrightarrow} H^1(G_K, E_n)$$
$$\overset{\Psi}{P \otimes 1}$$

$$K \text{ --- } K(E_n) \text{ --- } K(Q), \quad \begin{array}{l} nQ = P \\ Q \in E(K^{sep}) \end{array}$$

σ : $\rho_{E,n}$ on E_n

$$Q \longmapsto Q + a(\sigma), \quad a(\sigma) \in E_n$$

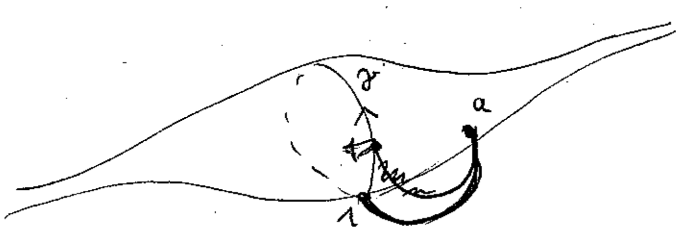
$$\rho_P : \sigma \longmapsto \begin{pmatrix} \boxed{\rho_{E,n}(\sigma)} & \boxed{a(\sigma)} \\ 0 \quad 0 & 1 \end{pmatrix}$$

$$\begin{array}{ccc} P \otimes 1 & \in & E(K) \otimes \mathbb{Z}/n\mathbb{Z} \\ \downarrow & & \downarrow \\ [a(\sigma)] & \in & H^1(G_K, E_n) \\ \downarrow & & \downarrow\wr \\ [\rho_P] & \in & \mathrm{Ext}^1_{\mathbb{Z}/n\mathbb{Z}[G_K]}(\mathbb{Z}/n\mathbb{Z}, E_n) \end{array}$$

$$0 \longrightarrow E_n \longrightarrow \rho_P \longrightarrow \mathbb{Z}/n\mathbb{Z} \rightarrow 0$$

$\boxed{\mathbb{G}_{m/K}}$

$a \in K^* \setminus \{1\}$



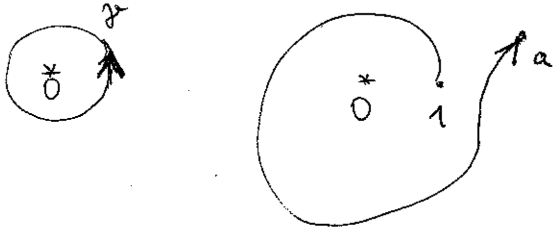
"Topological" interpretation

$A = \mathbb{Z}, \mathbb{Z}/n\mathbb{Z}, \mathbb{R}, \mathbb{C}$

$K \hookrightarrow \overline{K} \hookrightarrow \mathbb{C}$

$$0 \to H_1(\mathbb{C}^*, A) \to H_1(\mathbb{C}^*, \{1,a\}; A) \to H_0(\{1,a\}, A) \to H_0(\mathbb{C}^*, A) \to 0$$

$$\longrightarrow A \cdot ((a) - (1)) \longrightarrow 0$$



$\downarrow$

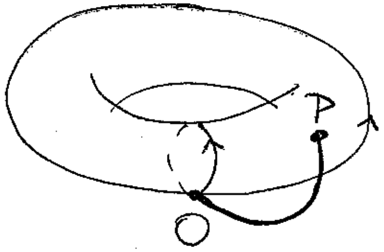
(1) $\underline{K = \mathbb{C}, A = \mathbb{Z}}$: $\left(1 \longrightarrow a \right) \xmapsto{\int \omega} \log(a) \in \mathbb{C}/2\pi i \mathbb{Z}$

(2) $\underline{A = \mathbb{Z}/n\mathbb{Z}}$: G_K acts on everything & we get

$$0 \to \mu_n \to \rho_a \to \mathbb{Z}/n\mathbb{Z} \to 0$$

$\boxed{E_{/K}}$

$P \in E(K) \setminus \{O\}$



$$0 \to H_1(E(\mathbb{C}), A) \to H_1(E(\mathbb{C}), \{O,P\}; A) \to$$

$$\to A \cdot ((P) - (O)) \to 0$$

$\downarrow$

(1) $\underline{A = \mathbb{Z}}$ $\underline{K = \mathbb{C}}$ $\left(O \longrightarrow P \right) \xmapsto{\int \omega} AJ(P) \in \mathbb{C}/L$

(2) $\underline{A = \mathbb{Z}/n\mathbb{Z}}$: we get

$$0 \to E_n \to \rho_P \to \mathbb{Z}/n\mathbb{Z} \to 0$$

Motives :

K - field , $\overline{X}_{\bar{K}} = X \otimes_K K^{sep}$

Pure motives (Grothendieck): there should be a universal "cohomology theory"

$$\begin{tikzcd}
(\text{Smooth Proj.}/K) \arrow[r] & (\text{pure) motives}_{/K}
\end{tikzcd}$$

$$\begin{tikzcd}
X \arrow[r, mapsto] & h^i(X)=M \arrow[r] \arrow[dr] \arrow[ddr] & H^i_{et}(\overline{X}, -) \arrow[r] & L(M,s) \\
 & & H^i(X(\mathbb{C}), -) & K \hookrightarrow \mathbb{C} \\
 & & H^i_{dR}(X/K) & \text{integration} \arrow[d] \\
 & & & \text{periods} \arrow[dl] \\
 & & L_\infty(M,s) &
\end{tikzcd}$$

motive pure of wt $= i$ realizations

(+ "direct factors" of $h^i(X)$)

$\Lambda(M,s) = L_\infty(M,s)\, L(M,s)$

Expected : $\Lambda(M,s) \longleftrightarrow \Lambda(M^*(1), -s)$

Ex : (1) $\mathbb{Z}(1)$ $(= h^1(\mathbb{G}_m)^* = h^2(\mathbb{P}^1)^*)$ (wt $= -2$)

$$\begin{tikzcd}
\mathbb{Z}(1) \arrow[r] \arrow[dr] \arrow[ddr] & H_1(\mathbb{G}_m, \mathbb{Z}/n\mathbb{Z}) = \mu_n \arrow[loop right, "G_K"] \\
 & \mathbb{Z}\cdot\gamma \\
 & (\mathbb{Z}\cdot\omega)^*
\end{tikzcd}$$

$\left.\begin{matrix} \mathbb{Z}\cdot\gamma \\ (\mathbb{Z}\cdot\omega)^* \end{matrix}\right\} \longrightarrow 2\pi i$

$(n, \mathrm{char}(K)) = 1$

(2) $h_1(E)$ $(= h^1(E)^*)$ (wt $= -1$)

$$\begin{tikzcd}
h_1(E) \arrow[r] \arrow[dr] \arrow[ddr] & H_1(\overline{E}, \mathbb{Z}/n\mathbb{Z}) = E_n \arrow[loop right, "G_K"] \\
 & \mathbb{Z}\gamma_1 \oplus \mathbb{Z}\gamma_2 \\
 & (\mathbb{Z}\omega \oplus \mathbb{Z}\eta)^*
\end{tikzcd}$$

$\left.\begin{matrix} \mathbb{Z}\gamma_1 \oplus \mathbb{Z}\gamma_2 \\ (\mathbb{Z}\omega \oplus \mathbb{Z}\eta)^* \end{matrix}\right\} \longrightarrow \begin{pmatrix} \omega_1 & \eta_1 \\ \omega_2 & \eta_2 \end{pmatrix}$

Mixed motives (Deligne, Beilinson): there should be universal

$$\begin{tikzcd}
(\text{Schemes of f.t.} + \text{sep.}/K) \arrow[r] & (\text{mixed motives}_{/K}) = \mathcal{MM}_K \\
X \arrow[r, mapsto] & h^i(X)
\end{tikzcd}$$

$M \in \mathcal{MM}_K$ weight filtration $W_i M$, $W_i M / W_{i-1} M$ pure of wt $= i$

$\rightsquigarrow$ (non-trivial) extensions, e.g. $0 \to W_i/W_{i-1} \to W_{i+1}/W_{i-1} \to W_{i+1}/W_i \to 0$

Variant : universal triangulated functor

$X \longmapsto R\Gamma(X) \in \mathcal{D}(K)$ Δ-category with t-structure

$\mathcal{MM}_K$ = heart of $\mathcal{D}(K)$

$\mathcal{D}^b(\mathcal{MM}_K) \longrightarrow \mathcal{D}(K)$

Motivic cohomology : $\mathbb{1} = h^0(\operatorname{Spec}(K))$ trivial motive

$$H^i_{\mathcal{M}}(K, M) := \operatorname{Ext}^i_{\mathcal{MM}_K}(\mathbb{1}, M) \quad \left(= \operatorname{Hom}_{D(K)}(\mathbb{1}, M[i])\right)$$

Beilinson's conjecture (vague form) M pure of $wt(M) < -2$

$$L(M^*(1), s) \underset{\mathbb{Q}^*}{\sim} s^{r_M} \det(\mathrm{reg})$$

"period of a mixed motive"

$$\mathrm{reg} : \underbrace{H^1_{\mathcal{M},f}(K, M)}_{\mathbb{Q}^{r_M}} \longrightarrow \underbrace{\text{Hodge realization of } H^1_{\mathcal{M}}}_{\mathbb{R}^{r_M}}$$

$\mathrm{reg}\otimes 1 : H^1_{\mathcal{M},f}(\ )\otimes_{\mathbb{Q}}\mathbb{R} \xrightarrow{\sim} \mathbb{R}_{Hs}$]

Ex :

$$H^1_{\mathcal{M}}(K, \mathbb{Z}(1)) \overset{?}{=} K^*$$
$$H^1_{\mathcal{M}}(K, h_1(E)) \overset{?}{=} E(K)$$

⊛

L - functions of pure motives

$[K:\mathbb{Q}] < \infty$ $\qquad \overline{X} = X \otimes_K \overline{K}$

M pure motive over K, with coeff. in $\mathbb{Q}$

$M = $ "direct factor" of $h^i(X)(n) = h^i(X) \otimes \mathbb{Z}(1)^{\otimes n}$

$X_{/K}$ smooth proj.

Ex : $X = E$ ell. curve $\Longrightarrow$ $h^1(E)(1) \simeq h_1(E)$

Realizations of $M = h^i(X)(n)$ (pure of $wt = i - 2n$)

(1) Étale ℓ-adic : $H^i_{et}(\overline{X}, \mathbb{Q}_\ell) := \left(\varprojlim_r H^i_{et}(\overline{X}, \mathbb{Z}/\ell^r\mathbb{Z})\right) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$

$$M_\ell = H^i_{et}(\overline{X}, \mathbb{Q}_\ell) \otimes_{\mathbb{Z}_\ell} \underbrace{\mathbb{Z}_\ell(1)^{\otimes n}}_{\mathbb{Z}_\ell(n)}, \quad \mathbb{Z}_\ell(1) = \varprojlim_r \mu_{\ell^r} = T_\ell(\mathbb{G}_m)$$

$\mathbb{Q}_\ell[G_K]$ - module

Ex : $X_{/K}$ curve, geom. irred. $\Longrightarrow$ $H^i_{et}(\overline{X}, \mathbb{Z}_\ell)(1) = \begin{cases} \mathbb{Z}_\ell(1) & i = 0 \\ T_\ell \operatorname{Jac}(X) & i = 1 \\ \mathbb{Z}_\ell & i = 2 \\ 0 & i > 2 \end{cases}$

⊛ We need $\underline{\mathcal{O}_K^* \subset K^*}$: $j : \operatorname{Spec}(K) \hookrightarrow \operatorname{Spec}(\mathcal{O}_K)$

$j_* : \mathcal{MM}_K \longrightarrow \mathcal{MM}_{\mathcal{O}_K}$

$H^1_{\mathcal{M},f}(K, M) := \operatorname{Ext}^1_{\mathcal{MM}_{\mathcal{O}_K}}(\mathbb{1}, j_* M)$; $H^1_{\mathcal{M},f}(K, \mathbb{Z}(1)) \overset{?}{=} \mathcal{O}_K^*$

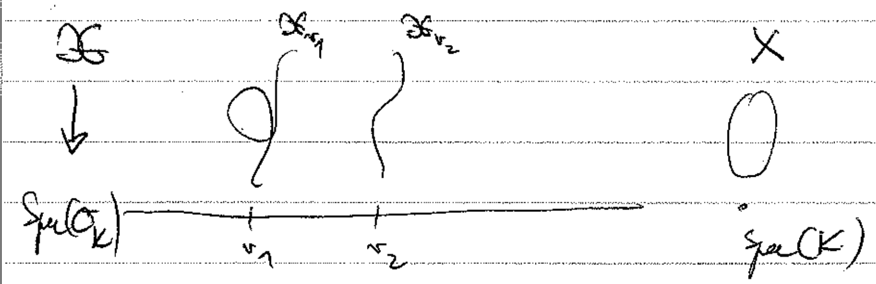
Expected : if K/ℚ is fin. gen., $\operatorname{Ext}^i_{\mathcal{MM}_K}(M, N) = 0$ for $i > 1 + \operatorname{tr.deg.}(K/\mathbb{Q})$

$\exists$ proper model $\mathfrak{X} \longrightarrow \mathrm{Spec}(\mathcal{O}_K)$ of X

which is smooth over

$\mathrm{Spec}(\mathcal{O}_{K,S}) = \mathrm{Spec}(\mathcal{O}_K) \setminus S$

(S finite set of non-arch. primes of K)



$v \rightsquigarrow \mathfrak{X}_v = \mathfrak{X} \otimes_{\mathcal{O}_K} k(v)$ special fibre of $\mathfrak{X}$ at v

$\overline{\mathfrak{X}_v} := \mathfrak{X} \otimes_{k(v)} \overline{k(v)}$

Fix $\overline{K} \hookrightarrow \overline{K_v}$; $G_K \hookleftarrow \underbrace{G_{K_v}}\supset I_v$

~~Facts~~:

$G_{K_v}/I_v = \langle Fr(v) \rangle$ geom. Frobenius

Def: for $v \nmid \ell$, $P_{v,\ell} := \det(1 - t\,Fr(v) \mid M_\ell^{I_v}) \in \mathbb{Q}_\ell[t]$

$$= \prod_j (1 - \alpha_{v,\ell,j}\, t)$$

Facts: (1) $(\forall v \nmid \ell)$ $\exists$ canonical G_{K_v}/I_v-map

$$H^i_{et}(\overline{\mathfrak{X}_v}, \mathbb{Q}_\ell)(n) \longrightarrow H^i_{et}(\overline{X}, \mathbb{Q}_\ell(n))^{I_v} = M_\ell^{I_v}$$

(2) If $v \notin S$, $v \nmid \ell \Rightarrow$

(a) map in (1) is an isom.

(b) $M_\ell^{I_v} = M_\ell$ $\Big\}$ (SGA 4)

(c) $P_v := P_{v,\ell} \in \mathbb{Q}[t]$ & is indep. of ℓ

(d) $(\forall \sigma : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C})$ $|\sigma(\alpha_{v,\ell,j})| = (Nv)^{W/2}$ $\Big\}$ Deligne

($w = i - 2n$

as $Fr(v) \mid_{\mathbb{Z}_\ell(1)} = (Nv)^{-2/2}$)

(3) If $v \in S$, $v \nmid \ell \Rightarrow$ monodromy-weight conjecture predicts

- (2c) holds
- (2d) is replaced by $|\sigma(\alpha_{v,\ell,j})| = (Nv)^{w_j/2}$,

$w_j \in \{w, w-1, \dots, w-i\}$.

Def: $$L(M,s) := \prod_v \underbrace{P_v(M, (Nv)^{-s})^{-1}}_{L_v(M,s)} = \sum_{n\geq 1} \frac{a_n}{n^s}, \quad a_n \in \mathbb{Q}$$

assuming (2c) for $v \in S$.

Note: (i) $L_S(M,s) = \prod_{v \notin S} L_v(M,s)$ is well-defined & abs. conv. for $\mathrm{Re}(s) > \frac{w}{2} + 1$.

(ii) If (3) holds, then the same is true for $L(M,s)$ & $L_v(M,s)$ can have poles only for $\mathrm{Re}(s) = \begin{cases} \frac{w}{2} & (v \notin S) \\ \frac{w}{2}, \frac{w-1}{2}, \dots, \frac{w-i'}{2} & (v \in S). \end{cases}$

Relation to ζ-functions

$\mathcal{Y} \downarrow$ of f.t. $\mathrm{Spec}(\mathbb{Z})$ $\quad y \in |\mathcal{Y}| = \{$closed pts of $\mathcal{Y}\}$

$k(y)$ finite field, $N(y) := |k(y)|$

$$\zeta(\mathcal{Y}, s) := \prod_{y \in |\mathcal{Y}|} (1 - N(y)^{-s})^{-1}$$

Ex: $\mathcal{Y} = \mathrm{Spec}(A)$, $A = \mathbb{Z}[T_1, \dots, T_r]/I$

$$\zeta(\mathcal{Y}, s) = \prod_{\substack{m \subset A \\ \text{max. ideal}}} (1 - (\# A/m)^{-s})^{-1}$$

$\Rightarrow \quad \zeta(\mathrm{Spec}(\mathcal{O}_K), s) = \zeta_K(s)$

Fact: $\mathcal{X}_S := \mathcal{X} \otimes_{\mathcal{O}_K} \mathcal{O}_{K,S} \Rightarrow \zeta(\mathcal{X}_S, s) = \prod_{i=0}^{2d} L_S(h^i(X), s)^{(-1)^{i-1}}$, $d = \dim(X)$
(Grothendieck)

$\boxed{\text{WHAT IS A MOTIVE? IT HAS } \zeta(-)\text{ !!}}$

Ex: (0) $M = h^0(\mathrm{Spec}(K))$, $\quad L(M,s) = \zeta_K(s)$

(1) $E_{/K}$ ell. curve, $\quad L(h^1(E), s) = L(E,s)$

(2) $X = \mathbb{P}^N_K$ $\quad H^i_{et}(\mathbb{P}^N_{\overline{K}}, \mathbb{Z}_\ell) = \begin{cases} \underline{\mathbb{Z}_\ell(-j)}, & i = 2j, \ 0 \leq j \leq N \\ \text{gen. by class } (\mathbb{P}^{N-j}_{\overline{K}})(-j) & \\ 0 & \text{otherwise} \end{cases}$

$$L(h^i(\mathbb{P}^N_K), s) = \begin{cases} \zeta_K(s-j), & i = 2j, \ 0 \leq j \leq N \\ 1 & \text{otherwise} \end{cases}$$

$$\zeta(\mathbb{P}^N_{\mathcal{O}_K}, s) = \prod_{j=0}^{N} \zeta_K(s-j)$$

Restriction of scalars: K/F (finite)

$$\begin{tikzcd} X \arrow[r, "\pi"] \arrow[dr, "\pi_F"'] & \mathrm{Spec}(K) \arrow[d] \\ & \mathrm{Spec}(F) \end{tikzcd}$$

π smooth proj.

motive over K : $M = h^i(X)(n)$ (via π)

— " — F : $R_{K/F}(M) =$ — " — (via π_F)

$$R_{K/F}(M)_\ell = \mathrm{Ind}_{G_K}^{G_F}(M_\ell)$$

Properties:

- $L_v(M(m), s) = L_v(M, s+m)$
- $L_v(M_1 \otimes M_2, s) = L_v(M_1, s)\, L_v(M_2, s)$
- $L_{v_F}(R_{K/F}(M), s) = \prod_{v | v_F} L_v(M, s)$

$L_v(M,s)$ for $v | \infty$

(2) **de Rham realization:** $X_{/K}$ smooth proj., $M = h^i(X)(n)$

$$M_{dR} = \mathbb{H}^i_{Zar}(X, \Omega^\bullet_{X/K}) \qquad K\text{-vector space}$$

Hodge filtration: $F^k M_{dR} = \mathbb{H}^i_{Zar}(X, \Omega^{\geq k+n}_{X/K})$

$F^k M_{dR} \to M_{dR}$ is *injective*, $gr^k_F(M_{dR}) = H^{i-k-n}_{Zar}(X, \Omega^{k+n}_{X/K})$

Ex: $M = h^1(X)$: $\underbrace{M_{dR} = F^0 M_{dR}}_{\{\text{diff. of } 2^{nd} \text{ kind}\} / \{df \mid f \in K(X)^*\}} \supset \underbrace{F^1 M_{dR}}_{H^0_{Zar}(X, \Omega^1_{X/K})} \supset F^2 M_{dR} = 0$

$\mathrm{res}_D(\cdot) = 0 \quad \forall$ divisor $D \subset X$

(3) **Betti realization:** $\sigma : K \hookrightarrow \mathbb{C}$, $M_{\sigma,B} = H^i((X \otimes_{K,\sigma} \mathbb{C})^{an}, \underbrace{\mathbb{Q}(n)}_{(2\pi\sqrt{-1})^n \mathbb{Q}})$ $\mathbb{Q}$-v. sp.

Comparison isomorphisms:

(A) ℓ prime, $\begin{tikzcd} K \arrow[r, hook, "\sigma"] \arrow[d, hook] & \mathbb{C} \\ \bar{K} \arrow[ur, "\bar\sigma"'] & \end{tikzcd}$, $I_{\ell,\bar\sigma} : M_{\sigma,B} \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \xrightarrow{\sim} M_\ell$

Ex: $M = h^1(E)(1)$ ("$= h_1(E)$"), $(E \otimes_{K,\sigma} \mathbb{C})^{an} = \mathbb{C}/L$

$M_{\sigma,B} = L \otimes \mathbb{Q}$, $M_\ell = L \otimes \mathbb{Q}_\ell$.

(B) Integration, Hodge theory: $\sigma : K \hookrightarrow \mathbb{C}$

$$I_{\infty,\sigma} : M_{\sigma,B} \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\sim} M_{dR} \otimes_{\sigma,K} \mathbb{C} \xrightarrow{\sim} \bigoplus_{p+q=w} H^{p,q}$$

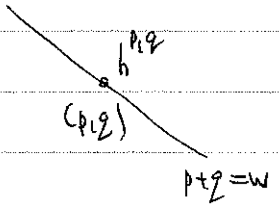
$$\left(\lambda\mu \int_\gamma \omega \longleftarrow\!\shortmid\ \gamma^* \otimes \mu\right) \longleftarrow\!\shortmid\ \omega \otimes \lambda$$

$$F^k M_{dR} \otimes \mathbb{C} \xrightarrow{\sim} \bigoplus_{p \geq k} H^{p,q}$$

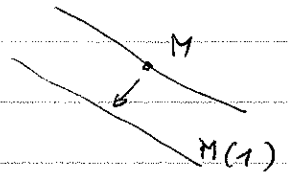
$$\overline{H^{p,q}} = H^{q,p}$$

("$-$" w.r.t. $M_{\sigma,B} \otimes_{\mathbb{Q}} \mathbb{R}$)

Hodge numbers: $h^{p,q} = \dim_{\mathbb{C}} H^{p,q}$ $(= h^{q,p})$



Tate motive: $\mathbb{Q}(1)$, $w=-2$, $h^{-1,-1} = 1$, $\mathbb{Q}(1)_B = 2\pi i \mathbb{Q}$



For $K \overset{\sigma}{\hookrightarrow} \mathbb{R} \hookrightarrow \mathbb{C}$: cplx conj. on $(X \otimes_{\sigma,K} \mathbb{C})(\mathbb{C})$ & $\mathbb{Q}(n)$

induces involution F_σ on $M_{\sigma,B}$ s.t.

(a) $(F_\sigma \otimes 1)(H^{p,q}) = H^{q,p}$

(b) $F_\sigma \otimes c \xleftrightarrow{I_{\infty,\sigma}} 1 \otimes c$ on $M_{dR} \otimes \mathbb{C}$

(c) If $w \in 2\mathbb{Z}$, put $h^{w/2,\pm} := \dim_{\mathbb{C}} (H^{w/2,w/2})^{F_\sigma \otimes 1 = \pm(-1)^{w/2}}$

(d) $F_\sigma = -1$ on $\mathbb{Q}(1)_B$

Local L-factors at $v | \infty$: v induced by $\sigma : K \hookrightarrow \mathbb{C}$

Goal: $L_v(M,s) =$ product of Γ-factors depending only on $h^{p,q}$ (or $h^{\frac{w}{2},\pm}$) for $M_{\sigma,B} \otimes$, s.t. $L_v(M,s)$ have same 3 properties as for $v \nmid \infty$

Apply $R_{K/\mathbb{Q}}$ — we can assume $K = \mathbb{Q}$, $v = \infty$.

Apply Tate twists & decompose $M_B \otimes \mathbb{C}$ into $\{(p,q),(q,p)\}$ or $\{(p,p)\}$:

(a) $(0,k)$ — $(k,0)$, $k \geq 1$: $L(f,s)$, $f \in S_{k+1}(\Gamma(N))$; $L_v = \Gamma_{\mathbb{C}}(s)$

(b) $\cdot(0,0)$, $F_\infty = +1$: $\zeta_{\mathbb{Q}}(s)$; $L_v = \Gamma_{\mathbb{R}}(s)$

(c) $\cdot(0,0)$, $F_\infty = -1$: $\dfrac{\zeta_{\mathbb{Q}(i)}(s)}{\zeta_{\mathbb{Q}}(s)} = L(s, (\frac{-1}{\cdot}))$; $L_v = \dfrac{\Gamma_{\mathbb{C}}(s)}{\Gamma_{\mathbb{R}}(s)} = \Gamma_{\mathbb{R}}(s+1)$

This gives :

v complex : $L_v(M,s) = \prod_{p+q=w} \Gamma_{\mathbb{C}}(~~s~~ \min(p,q))^{h^{p,q}}$

v real : $L_v(M,s) = \prod_{p<q} \Gamma_{\mathbb{C}}(s-p)^{h^{p,q}} \times \begin{cases} \Gamma_{\mathbb{R}}(s-\frac{w}{2})^{h^{\frac{w}{2},+}} \Gamma_{\mathbb{R}}(s-\frac{w}{2}+1)^{h^{\frac{w}{2},-}} & \text{if } w\in 2\mathbb{Z} \\ 1 & \text{if } 2\nmid w \end{cases}$

Def : $L_\infty(M,s) = \prod_{v|\infty} L_v(M,s)$, $\Lambda(M,s) = L_\infty(M,s) L(M,s)$

Expected functional equation : $\boxed{\Lambda(M,s) \overset{?}{=} a \cdot b^s \Lambda(M^*(1),-s)}$ $M \subset h^i(X)(n)$

What is $L_v(M^*(1),s) = ?$ $M = h^i(X)(n)$

fix $X \overset{i}{\hookrightarrow} \mathbb{P}^N$ $d = \dim(X)$

~~polarization~~ $L := c_1(i^* \mathcal{O}_{\mathbb{P}^N}(1)) \in H^2_{et}(\overline{X}, \mathbb{Q}_\ell)(1)$

hyperplane section class

isomorphisms of G_K-modules

$$\begin{tikzcd} H^i_{et}(\overline{X},\mathbb{Q}_\ell) \arrow[r, "\cup L^{d-i}", "\sim"'] & H^{2d-i}_{et}(\overline{X},\mathbb{Q}_\ell)(d-i) \arrow[d, "\wr", "\text{Poincaré dual.}"'] \\ & H^i_{et}(\overline{X},\mathbb{Q}_\ell)^*(-i) \end{tikzcd}$$

(Deligne)
("hard Lefschetz")

(& similarly for other realizations).

So : $\forall$ ~~$? \in \{\ell, dR$~~ , $? = \begin{cases} \ell \\ dR \\ \sigma, B \end{cases}$, $M = h^i(X)(n)$

$M^*(1)_? = h^i(X)^*_?(1-n) \xrightarrow{\sim} h^i(X)_?(i+1-n) = M_?(i+1-2n) = M_?(w+1)$

$\Rightarrow (\forall v)$ $L_v(M^*(1),-s) = L_v(M, w+1-s)$

$\boxed{\Lambda(M,s) \overset{?}{=} a \cdot b^s \Lambda(M, w+1-s)}$ central pt $= \frac{w+1}{2}$

~~near central pt $= \frac{w}{2}+1$~~

~~So : $w=-1 \Leftrightarrow s=0$ is central~~

~~$w=-2 \Leftrightarrow s=0$ is near central~~

~~$w<-2 \Leftrightarrow s=0$ is $>$ near central~~

~~(i.e. in the region of abs conv)~~

$(E \hookrightarrow \mathrm{End}(M))$

Motives with coefficients in E $([E:\mathbb{Q}] < \infty)$

Note: $L(s,\chi) \longleftrightarrow L(1-s,\chi^{-1})$, not $L(1-s,\chi)$

these are "pieces" of $h^i(X)(n) \otimes E$

Ex: F/K finite Galois , $G = G(F/K)$

N motive over K (e.g. $h^i(X)(n)$)

$\rightsquigarrow N_{/F}$ — " — ($h^i(X \otimes_K F)(n)$)

$M := \mathrm{Res}_{F/K}(N_{/F}) \xrightarrow{\sim} N \otimes_{\mathbb{Q}} \mathbb{Q}[G]$

given $\chi : G \longrightarrow E^*$, $e_\chi := \frac{1}{|G|} \sum_{g \in G} \chi^{-1}(g) g \in E[G]$

$M \otimes [\chi] := e_\chi (M \otimes_{\mathbb{Q}} E)$

Ex: $\{$Artin motives $/K$ with coeff. in $E\} \longleftrightarrow \mathrm{Rep}_E(G_K)$

(above: $N = h^0(\mathrm{Spec}(F))$)
χ non-abelian

$$M \longleftrightarrow V$$

$$M_\ell \longleftrightarrow V \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$$

$$M_{\sigma, B} \longleftrightarrow \sigma^* V \in \mathrm{Rep}_E(G_{K_\sigma})$$

$$M_{dR} \longleftrightarrow (V \otimes_{\mathbb{Q}} \overline{K})^{G_K}$$

Ex: $K = \mathbb{Q}$, $M \overset{=[\chi]}{\longleftrightarrow} V = E[\chi]$ (E with $G_{\mathbb{Q}}$-action via $\chi : G_{\mathbb{Q}} \longrightarrow E^*$)

$f = \mathrm{cond}(\chi)$, $\underbrace{G(\chi)}_{\text{period}} = \sum_{a \in (\mathbb{Z}/f\mathbb{Z})^*} \chi(a) \otimes e^{2\pi i a/f} \in (E(\chi) \otimes \overline{\mathbb{Q}})^{G_{\mathbb{Q}}} = M_{dR}$

under comparison map I_∞

L-function: M_ℓ free over $E \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$, $F^k M_{dR}$ free over $E \otimes_{\mathbb{Q}} K$

$v \nmid \infty$: $\det_{E \otimes_{\mathbb{Q}} \mathbb{Q}_\ell}(1 - Fr(v) t \mid M_\ell^{I_v}) \in E[t]$

$\rightsquigarrow L(M,s) = \sum \frac{a_n}{n^s}$, $a_n \in \overline{E}$

$(\forall \tau : E \hookrightarrow \mathbb{C})$ $L(\tau, M, s) = \sum \frac{\tau(a_n)}{n^s}$

$v \mid \infty$: $L_v(M,s)$ indep. of τ

$\Rightarrow \Lambda(M,s)$ with values in $\mathbb{C}^{\mathrm{Hom}(E,\mathbb{C})} \xleftarrow{\sim} E \otimes \mathbb{C}$

$= (\Lambda(\tau, M, s))_{\tau : E \hookrightarrow \mathbb{C}}$ $(\tau(e) \otimes \lambda)_\tau \longleftarrow e \otimes \lambda$.

Note : $M^\vee(1) \xrightarrow{\sim} M(w+1)$ need NOT preserve E-structures !

Ex : $M \in h^1(A)$, A abelian variety, $E \subset \mathrm{End}(A)_{\mathbb{Q}}$

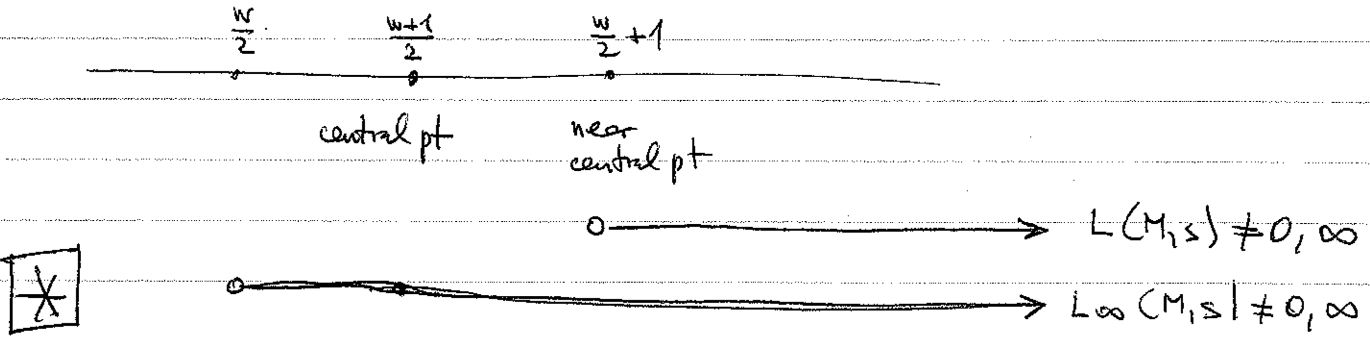
Rosatti involution$|_E$ appears

Orders of vanishing

Γ-factors: $n \in \mathbb{Z}$

$\mathrm{ord}_{s=0}\,\Gamma_{\mathbb{C}}(s-n) = \begin{cases} -1 & n=0,1,2,\dots \\ 0 & \text{otherwise} \end{cases}$ | $\mathrm{ord}_{s=0}\,\Gamma_{\mathbb{R}}(s-n) = \begin{cases} -1 & n=0,2,4,\dots \\ 0 & \text{otherwise} \end{cases}$

M - pure motive of wt = w over K:



Tate's conjecture: (1) The only possible pole of $L(M,s)$ is at the near central point $\frac{w}{2}+1$, provided $w \in 2\mathbb{Z}$

(Ex: $\zeta_K(s)$, $w=0$, $\frac{w}{2}+1=1$)

(2) ~~P there is a po~~ ~~The~~ Assume that $s=0$ is the near central pt ($\Longleftrightarrow$ $w=-2$). Then

$$-\mathrm{ord}_{s=0} L(M,s) = \dim_{\mathbb{Q}} \underbrace{\mathrm{Hom}_{\mathcal{MM}_K}(\mathbb{1}, M^*(1))}_{H^0_{\mathcal{M}}(K, M^*(1))}$$

From now on: - $K = \mathbb{Q}$

- M is pure of $wt = w \leq -1$ ($\Longleftrightarrow$ $s=0 \geq \frac{w+1}{2}$)

(so $M^*(1)$ is pure of $wt = -2-w \geq -1$) & has coeff. in $\mathbb{Q}$

- assume funct. eq.

Calculation: $\mathrm{ord}_{s=0} L_\infty(M,s) = 0$

$$-\mathrm{ord}_{s=0} L_\infty(M^*(1),s) = \dim_{\mathbb{Q}}(M_{dR}/F^0) - \dim_{\mathbb{Q}}(M_B^+)$$

($M_B^\pm = M_B^{F_\infty = \pm 1}$)

$\boxed{*}$ $w=-1 \Longleftrightarrow s=0$ is central

$w=-2 \Longleftrightarrow s=0$ is near central

$w<-2 \Longleftrightarrow s=0 >$ —"—, i.e. in the region of abs. convergence.

Deligne's Conjecture

Ex : $E_{/\mathbb{Q}}$ ell. curve $\qquad M = h_1(E) = h^1(E)(1)$

$$M_B^{\pm} = H_1(E(\mathbb{C}), \mathbb{Q})^{F_\infty \overset{?}{=} \pm 1} = \mathbb{Q}\cdot\gamma_{\pm}$$

$\omega = \frac{dx}{y}, \quad \eta = \frac{x\,dx}{y} \qquad \int_{\gamma_\pm} \begin{Bmatrix} \omega \\ \eta \end{Bmatrix} = \begin{Bmatrix} \omega_\pm \\ \eta_\pm \end{Bmatrix} \qquad \omega_+, \eta_+ \in \mathbb{R}$

$$\omega_-, \eta_- \in \underbrace{\mathbb{R}(1)}_{2\pi i \mathbb{R}}$$

$$\begin{vmatrix} \omega_+ & \omega_- \\ \eta_+ & \eta_- \end{vmatrix} = 2\pi i$$

$h^1(E)_B$: $\mathbb{Q}$-basis $\gamma_\pm^*$

$M_B^{\pm} = h^1(E)(1)_B^{\pm}$: $\mathbb{Q}$-basis $\quad 2\pi i\, \gamma_\mp^*$

$M_{dR} = h^1(E)_{dR}$: $\mathbb{Q}$-basis $\quad \omega, \eta$

$F^0 M_{dR} = F^1 H^1_{dR}(E/\mathbb{Q})$: $\mathbb{Q}$-basis ω

Comparison isomorphism : $\quad I_\infty : M_B \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\sim} M_{dR} \otimes_{\mathbb{Q}} \mathbb{C}$

$$\gamma_+^* \otimes \omega_+ + \gamma_-^* \otimes \omega_- \longleftrightarrow \omega$$

$$\gamma_+^* \otimes \eta_+ + \gamma_-^* \otimes \eta_- \longleftrightarrow \eta$$

$$2\pi i\, \gamma^* \longleftrightarrow -\eta_+ \omega + \omega_+ \eta$$

$$2\pi i\, \gamma_+^* \longleftrightarrow \eta_- \omega - \omega_- \eta$$

I_∞ has matrix $\begin{pmatrix} \omega_+ & -\eta_+ \\ -\omega_- & \eta_- \end{pmatrix}$, BSD has only ω_+ !

We must cross out 2nd row & column!

Deligne's period map : $\quad \begin{array}{l} M_B^+ \otimes \mathbb{R} \\ M_B^- \otimes \mathbb{R}(-1) \end{array} \longleftrightarrow M_{dR} \otimes \mathbb{R}$

$$\alpha_M = I_\infty^+ : \underbrace{M_B^+ \otimes_{\mathbb{Q}} \mathbb{R}} \longrightarrow \underbrace{(M_{dR}/F^0)} \otimes_{\mathbb{Q}} \mathbb{R}$$

Above : basis $2\pi i\, \gamma_-^*$ $\qquad$ basis η mod $\langle \omega \rangle$, matrix (ω_+)

Ex : $\quad N = \mathrm{Sym}^2(M) = (\mathrm{Sym}^2 h^1(E))(2)$

$$L(h^1(E), s) = L(E, s) = \prod_{p \nmid N} \left[(1-\alpha_p p^{-s})(1-\beta_p p^{-s})\right]^{-1} \prod_{p \mid N} (1 - a_p p^{-s})^{-1}$$

$$\underset{\substack{\shortparallel \\ L(N, s-2)}}{L(\mathrm{Sym}^2 h^1(E), s)} = \prod_{p \nmid N} \left[(1-\alpha_p^2 p^{-s})(1-\alpha_p\beta_p p^{-s})(1-\beta_p^2 p^{-s})\right]^{-1} \prod_{p \mid N} (\cdots)^{-1}$$

$$\mathrm{Sym}^2 \begin{pmatrix} \omega_+ & -\eta_+ \\ -\omega_- & \eta_- \end{pmatrix} = \begin{pmatrix} \omega_+^2 & -\omega_+\eta_+ & \cancel{\eta_+^2} \\ \cancel{-\omega_+\omega_-} & \cancel{\eta_+\omega_- + \eta_-\omega_+} & \cancel{-\eta_+\eta_-} \\ \omega_-^2 & -\omega_-\eta_- & \cancel{\eta_-^2} \end{pmatrix}$$

$$\alpha_N \longleftrightarrow \begin{pmatrix} \omega_+^2 & -\omega_+\eta_+ \\ \omega_-^2 & -\omega_-\eta_- \end{pmatrix}$$

<u>Facts</u> : (1) $\mathrm{wt}(M) < 0 \Rightarrow \mathrm{Ker}(\alpha_M) \subseteq (F^0 \cap \overline{F}^0)(M_B \otimes \mathbb{C}) = 0$

$\Rightarrow$ <u>α_M is injective</u>

(2) $\mathrm{ord}_{s=0} L_\infty(M, s) = \dim_{\mathbb{R}} \mathrm{Ker}(\alpha_M) = 0$

$- \mathrm{ord}_{s=0} L_\infty(M^*(1), s) = \dim_{\mathbb{R}} \mathrm{Coker}(\alpha_M)$.

<u>Def</u> : (i) $s=0$ is a <u>critical value</u> for M

(Deligne) $\iff \mathrm{ord}_{s=0} L_\infty(M, s) = \mathrm{ord}_{s=0} L_\infty(M^*(1), s) = 0$

(ii) If true, put $c^+(M) := \det(\alpha_M) \in \mathbb{R}^*/\mathbb{Q}^*$

$\det(-)$ w.r.t. $\mathbb{Q}$-structures $\det_{\mathbb{Q}}(M_B^+)$, $\det_{\mathbb{Q}}(M_{dR}/F^0)$.

<u>Ex</u> : For $M = h^1(E)(1)$, $s=0$ is critical, $c^+(M) = \omega^+$

For $N = \mathrm{Sym}^2(M)$, —"— , $c^+(N) = \omega_+\omega_-(-2\pi i)$

<u>Deligne's Conjecture</u> : If M is critical at $s=0$, then

$$L(M, 0) \in c^+(M)\mathbb{Q}.$$

(Variant for motives with coeff. in E: $c^+(M) \in (\mathbb{R} \otimes_{\mathbb{Q}} E)^* / E^*$

$L(M, 0) \in c^+(M)E \subset \mathbb{C} \otimes_{\mathbb{Q}} E$.)

<u>Ex</u> : $M = R_{K/\mathbb{Q}}(\mathbb{Q}(n))$, $L(M, s) = \zeta_K(s+n)$, $n \geq 1$

$M_B = \mathbb{Q}^{\mathrm{Hom}(K, \mathbb{C})} \cdot (2\pi i)^n$, $M_{dR} = K = F^{-n} M_{dR} \supset F^{-n+1} M_{dR} = 0$

$L_\infty(M, s) = \Gamma_{\mathbb{R}}(s+n)^{r_1} \Gamma_{\mathbb{C}}(s+n)^{r_2}$ so $F^0 M_{dR} = 0$

$$\dim_{\mathbb{Q}} M_B^+ = \begin{cases} r_1 + r_2 & 2 \mid n \\ r_2 & 2 \nmid n \end{cases}, \quad \dim_{\mathbb{R}} \mathrm{Coker}(\alpha_M) = \begin{cases} r_2 & 2 \mid n \\ r_1 + r_2 & 2 \nmid n \end{cases}$$

Coker(α_M) – cohomological interpretation

$A \subset \mathbb{R}$ subring , $X_{\mathbb{C}} = X \otimes_{\mathbb{Q}} \mathbb{C}$

$$H^i(X_{\mathbb{C}}^{an}, \underbrace{A(n)}_{(2\pi\sqrt{-1})^n A}) \longrightarrow H^i_{dR}(X/\mathbb{Q})/F^n \otimes_{\mathbb{Q}} \mathbb{C}$$

$$\| \quad H^i_{dR}(X_{\mathbb{C}}/\mathbb{C})/F^n$$

$$\| \text{ GAGA} \quad \mathbb{H}^i(X_{\mathbb{C}}^{an}, \Omega^{<n}_{an})$$

new complex: $$\Big[\underbrace{A(n)}_{deg=0} \longrightarrow \mathcal{O}_{an} \xrightarrow{d} \Omega^1_{an} \longrightarrow \dots \xrightarrow{d} \underbrace{\Omega^{n-1}_{an}}_{deg=n}\Big] = A(n)_{\mathcal{D}}$$

"$\mathcal{D}$" – Deligne

~~exact~~ Def: Deligne cohomology over $\mathbb{C}$

$$H^i_{\mathcal{D}}(X_{\mathbb{C}}, A(n)) = \mathbb{H}^i(X^{an}_{\mathbb{C}}, A(n)_{\mathcal{D}})$$

exact sequences: $0 \longrightarrow \Omega^{<n}_{an}[-1] \longrightarrow A(n)_{\mathcal{D}} \longrightarrow A(n) \longrightarrow 0$

$$\dots \longrightarrow H^i(X^{an}_{\mathbb{C}}, A(n)) \longrightarrow H^i_{dR}(X/\mathbb{Q})/F^n \otimes_{\mathbb{Q}} \mathbb{C} \longrightarrow H^{i+1}_{\mathcal{D}}(X_{\mathbb{C}}, A(n)) \longrightarrow$$
$$\longrightarrow H^{i+1}(\qquad) \longrightarrow \dots$$

action of F_∞: obvious action on $(X^{an}_{\mathbb{C}}, A(n))$

& $(X^{an}_{\mathbb{C}}, \Omega^{\bullet}_{an})$

Deligne cohomology over $\mathbb{R}$:

Def: $$H^i_{\mathcal{D}}(X_{\mathbb{R}}, A(n)) := H^i_{\mathcal{D}}(X_{\mathbb{C}}, A(n))^{F_\infty = 1}$$

$$\dots \longrightarrow H^i(X^{an}_{\mathbb{C}}, A(n))^{F_\infty = 1} \xrightarrow[\alpha_{h^i(X)(n)}]{} H^i_{dR}(X/\mathbb{Q})/F^n \otimes_{\mathbb{Q}} \mathbb{R} \longrightarrow H^{i+1}_{\mathcal{D}}(X_{\mathbb{R}}, A(n)) \longrightarrow$$
$$\dots \longrightarrow H^{i+1}(\qquad) \xrightarrow[\alpha_{h^{i+1}(X)(n)}]{}$$

Special case: $A = \mathbb{R}$:

$X/\mathbb{Q}$ smooth proj.

So: ~~If $w+1 = i+1-2n < 0$, then~~

$$0 \longrightarrow \mathrm{Coker}(\alpha_{h^i(X)(n)}) \longrightarrow H^{i+1}_{\mathcal{D}}(X_{/\mathbb{R}}, \mathbb{R}(n)) \longrightarrow \mathrm{Ker}(\alpha_{h^{i+1}(X)(n)}) \longrightarrow 0$$

Cor: If $w = i - 2n < -1$, then $\mathrm{Ker}(\alpha_{h^{i+1}(X)(n)}) = 0$, hence

$(M = h^i(X)(n))$ $\quad \mathrm{Coker}(\alpha_M) \xrightarrow{\sim} H^{i+1}_{\mathcal{D}}(X_{/\mathbb{R}}, \mathbb{R}(n))$

<u>Ex</u> : <u>$w=-1$</u> , i.e. $i = 2n-1$; $A = \mathbb{Z}$

<u>special case</u> : $i = n = 1 = \dim(X)$ $\qquad Y = X_{\mathbb{C}}^{an}$ cpt Riemann surface

$h^1(X)(1) \simeq h_1(X)$

$$\begin{tikzcd}
0 \arrow[r] & \underbrace{\mathrm{Coker}\big(H_1(Y,\mathbb{Z}) \longrightarrow H^0(Y,\Omega^1_{an})^*\big)}_{\gamma \longmapsto (\omega \mapsto \int_\gamma \omega)} \arrow[r] & H^2_{\mathcal{D}}(X_{\mathbb{C}},\mathbb{Z}(1)) \arrow[r] & \underbrace{H_0(Y,\mathbb{Z})}_{\mathbb{Z}} \arrow[r] & 0 \\
0 \arrow[r] & \mathrm{Jac}(Y) = Cl^0(Y) \arrow[r] & Cl(Y) \arrow[r, "\deg"] \arrow[u, "\wr"] & \mathbb{Z} \arrow[r] \arrow[u, "id"] & 0
\end{tikzcd}$$

$$\begin{tikzcd}
0 \arrow[r] & \text{Griffiths' Jacobian} \arrow[r] & H^{2n}_{\mathcal{D}}(X_{\mathbb{C}},\mathbb{Z}(n)) \arrow[r] & \underbrace{H^{2n}(X_{\mathbb{C}}^{an},\mathbb{Z}(n)) \cap F^n}_{\text{Hodge classes}} \arrow[r] & 0 \\
0 \arrow[r] & \underbrace{CH^n(X_{\mathbb{C}})_0}_{\mathrm{Ker}(cl_B)} \arrow[r] \arrow[u, "\substack{\text{Griffiths'} \\ \text{A-J map}}"'] & CH^n(X_{\mathbb{C}}) \arrow[u] \arrow[ur, "cl_B"'] & = \dfrac{\{\sum n_i Y_i \mid Y_i \subset X_{\mathbb{C}} \text{ int., codim} = n\}}{\langle \mathrm{div}(f) \mid f \in k(Z)^*, Z \subset X_{\mathbb{C}}\ {}_{\text{codim}=n-1} \rangle} &
\end{tikzcd}$$

<u>Central point</u> (conj. of Bloch – Beilinson)

$X_{/\mathbb{Q}}$ sm. proj. $\quad d = \dim(X)$ $\qquad w = -1$: $i = 2n-1$

$L(h^{2n-1}(X), s)$ $\qquad s \longrightarrow n$

$$CH^n(X)_0 := \mathrm{Ker}\big(CH^n(X) \longrightarrow CH^n(X_{\mathbb{C}}) \xrightarrow{cl_B} H^{2n}(X_{\mathbb{C}}^{an},\mathbb{Z}(n))\big)$$

height pairing : $CH^n(X)_0 \times CH^{d+1-n}(X)_0 \xrightarrow{\langle , \rangle} \mathbb{R}$

<u>Conj.</u> : (i) $\mathrm{ord}_{s=n} L(h^{2n-1}(X), s) = \mathrm{rk}_{\mathbb{Z}} CH^n(X)_0$

(ii) $\langle , \rangle$ is non-deg. mod torsion

(iii) leading coefficient $\underset{\mathbb{Q}^*}{\equiv} c^+(h^{2n-1}(X)(n)) \cdot \det\langle , \rangle$

(Jannsen, LNM 1400)

~~absolut~~

geometric cohomology	absolute (arithmetic) cohomology
étale coh. of $\overline{X} = X \otimes_K K^{sep}$	étale coh. of X
Betti de Rham coh. $/\mathbb{C}$	absolute Hodge coh. (= Deligne – Beilinson)
crystalline coh.	syntomic coh.

Ex : (étale) $X_{et} \xrightarrow{\pi} \underbrace{\mathrm{Spec}(K)_{et}}$ π sep., of f.t.

sheaf $\mathcal{F}$ | sheaves : discrete G_K-modules

Leray : $E_2^{i,j} = H^i_{et}(\mathrm{Spec}(K)_{et}, \underbrace{R^j\pi_*\mathcal{F}}) \Rightarrow H^{i+j}_{et}(X, \mathcal{F})$

G_K-module $H^j_{et}(\overline{X}, \mathcal{F})$

So $E_2^{i,j} = H^i(G_K, H^j_{et}(\overline{X}, \mathcal{F}))$

Special case : $\mathcal{F} = \mathbb{Z}/\ell^r\mathbb{Z}(n)$, $\ell \neq \mathrm{char}(K)$

$${}^{et}E_2^{i,j} = H^i(G_K, H^j_{et}(\overline{X}, \mathbb{Q}_\ell(n))) \Rightarrow H^{i+j}_{et}(X, \mathbb{Q}_\ell(n))$$

(cont. coh.)

(π proper & smooth $\Rightarrow$ ${}^{et}E_2 = {}^{et}E_\infty$ (Deligne + de Jong + ...))

Expected motivic version : $X_{mot} \xrightarrow{\pi} \underbrace{\mathrm{Spec}(K)_{mot}}_{\text{sheaves}}$ $\mathcal{MM}_K$

$${}^{mot}E_2^{i,j} = H^i_{\mathcal{M}}(K, \underbrace{R^j\pi_*(\mathbb{Q}(n))}_{h^j(X)(n)}) \Rightarrow H^{i+j}_{\mathcal{M}}(X, \mathbb{Q}(n))$$

Special case : X/K smooth proj., $[K:\mathbb{Q}] < \infty$

$$0 \to H^1_{\mathcal{M}}(K, h^i(X)(n)) \to H^{i+1}_{\mathcal{M}}(X, \mathbb{Q}(n)) \to \underbrace{H^0_{\mathcal{M}}(K, h^{i+1}(X)(n))}_{=0 \text{ if } w = i-2n \neq -1} \to 0$$

The "Hodge realization" of this sequence (if $K=\mathbb{Q}$) should be

$$0 \longrightarrow \mathrm{Coker}(\alpha_{h^i(X)(n)}) \longrightarrow H^{i+1}_{\mathcal{D}}(X_{\mathbb{R}}, \mathbb{R}(n)) \longrightarrow \mathrm{Ker}(\alpha_{h^{i+1}(X)(n)}) \to 0,$$

provided $w = i-2n \leq -1$

Hodge version : $A \subset \mathbb{R}$ noeth. subring, $A \otimes \mathbb{Q}$ field

Mixed Hodge structures with coeff. in A $\quad$ ($H^i(X, A(n))$, $X/\mathbb{C}$ sep. f.t.)

(over $\mathbb{C}$) : $A\text{-MHS}_{\mathbb{C}}$: • H A-module of f.t. ("Betti real.")

- weight filtr. $W_i H_{\mathbb{Q}} \subset W_{i+1} H_{\mathbb{Q}} \subset \cdots$
- Hodge filtr. $F^p H_{\mathbb{C}} \supset F^{p+1} H_{\mathbb{C}} \supset \cdots$

s.t. $gr_i^W(H_{\mathbb{Q}})$ is pure of wt $= i$, i.e.

$$gr_i^W(H_{\mathbb{C}}) \simeq \bigoplus_{p+q=i} (F^p \cap \overline{F}^q)(gr_i^W H_{\mathbb{C}}).$$

Ex : $X/\mathbb{C}$ smooth proj. $\Rightarrow$ $H^i(X, A(n))$ pure of wt $= i-2n$

(over $\mathbb{R}$) : $A\text{-MHS}_{\mathbb{R}}$: • also involution $F_\infty : H \longrightarrow H$ s.t.

$$F_\infty(W_i H_{\mathbb{Q}}) = W_i H_{\mathbb{Q}}$$
$$(F_\infty \otimes c)(F^p H_{\mathbb{C}}) = F^p H_{\mathbb{C}}$$

Fact$_{\mathbb{C}}$ (~~Beilinson~~ Carlson) If $H = W_0 H$, then

$$\mathrm{Ext}^i_{A\text{-MHS}_{\mathbb{C}}}(A(0), H) = H^i\left([\underbrace{H}_{0} \longrightarrow \underbrace{H_{\mathbb{C}}/F^0 H_{\mathbb{C}}}_{1}] \right)$$

(see Jannsen, LNM 1400, Ch. 9)

(cf. $[D_{\mathrm{cris}} \xrightarrow{(\varphi-1,\,\mathrm{can})} D_{\mathrm{cris}} \oplus D_{dR}/F^0]$)

Ex : X smooth proj. curve$/\mathbb{C}$, $H = H_1(X, \mathbb{Z}) = H^1(X, \mathbb{Z})^*$

$$(H_{\mathbb{C}}/F^0 H_{\mathbb{C}})^* = F^0(H^*(1)_{\mathbb{C}}) = F^1 H^*_{\mathbb{C}} = H^0(X, \Omega^1_{X/\mathbb{C}})$$

$$\mathrm{Ext}^1_{\mathbb{Z}\text{-MHS}_{\mathbb{C}}}(\mathbb{Z}(0), H_1(X, \mathbb{Z})) = \mathrm{Coker}(H_1(X, \mathbb{Z}) \to H^0(X, \Omega^1)^*) = \mathrm{Jac}(X)$$

(as in lect. 1, X=E) $\qquad \gamma \longmapsto (\omega \mapsto \int_\gamma \omega)$

Cor: $X \to \mathrm{Spec}(\mathbb{C})$ smooth projective, then the long exact sequence for Deligne cohomology becomes

$$0 \to \mathrm{Ext}^1_{A\text{-MHS}_{\mathbb{C}}}(A(0), H^i(X, A(n))) \to H^{i+1}_{\mathcal{D}}(X_{\mathbb{C}}, A(n)) \to$$

$$\to \mathrm{Hom}_{A\text{-MHS}_{\mathbb{C}}}(A(0), H^{i+1}(X, A(n))) \to 0$$

, provided $w = i-2n \leq -1$

Version over $\mathbb{R}$: put $H^{\pm} = H^{F_\infty = \pm 1}$, $H_{dR} = H_{\mathbb{C}}^{F_\infty \otimes c = 1}$

Fact$_{\mathbb{R}}$: If $\frac{1}{2} \in A$ and $H = W_0 H$, then

$$\mathrm{Ext}^i_{A\text{-MHS}_{\mathbb{R}}}(A(0), H) = H^i\left(\left[\underbrace{H^+}_{0} \longrightarrow \underbrace{H_{dR}/F^0}_{1}\right]\right)$$

Special case: $A = \mathbb{R}$, $X_{/\mathbb{R}}$ smooth proj., $M = h^i(X)(n)$

$H = M_B \otimes_{\mathbb{Q}} \mathbb{R}$

$$\| $$

$$\left[M_B^+ \otimes \mathbb{R} \xrightarrow{\alpha_M} (M_{dR}/F^0) \otimes \mathbb{R}\right]$$

So $\begin{Bmatrix} \mathrm{Ker} \\ \mathrm{Coker} \end{Bmatrix}(\alpha_M) = \begin{Bmatrix} \mathrm{Hom} \\ \mathrm{Ext}^1 \end{Bmatrix}_{\mathbb{R}\text{-MHS}_{\mathbb{R}}}(\mathbb{R}(0), M_B \otimes \mathbb{R})$, provided $w \leq 0$

Assume $w < -1$: regulator maps

$X_{/\mathbb{Q}}$ smooth proj.
$\mathcal{X}_{/\mathbb{Z}}$ "nice" model

$\circledast$

$$\begin{tikzcd}
H^1_{\mathcal{M},f}(\mathbb{Q}, M) \arrow[r, hook, bend left] & H^1_{\mathcal{M}}(\mathbb{Q}, \underbrace{h^i(X)(n)}_{M}) \arrow[r, "\sim"] \arrow[d, "\mathrm{reg}"] & H^{i+1}_{\mathcal{M}}(X, \mathbb{Q}(n)) \arrow[d, "\mathrm{reg}"] & \text{Image of } H^{i+1}_{\mathcal{M}}(\mathcal{X}, \mathbb{Q}(n)) \arrow[l, squiggly] \\
 & \underbrace{\mathrm{Coker}(\alpha_M)}_{\mathrm{Ext}^1_{\mathbb{R}\text{-MHS}_{\mathbb{R}}}(\mathbb{R}(0), M_B \otimes \mathbb{R})} \arrow[r, "\sim"] & H^{i+1}_{\mathcal{D}}(X_{\mathbb{R}}, \mathbb{R}(n)) &
\end{tikzcd}$$

<u>Assume</u>: $w < -1$ & $L(M,s)$ has no pole at $s=0$ (if $w=-2$).

Funct. eqn $\Rightarrow r_{M^*(1)} := \mathrm{ord}_{s=0} L(M^*(1), s) = \dim_{\mathbb{R}}(\text{bottom row in } (*))$

<u>Beilinson's Conjecture</u>: $\mathrm{reg} \otimes 1$ induces an isomorphism

$$(f\text{-subspace of top row in } (*)) \otimes_{\mathbb{Q}} \mathbb{R} \xrightarrow{\sim} \text{bottom row in } (*),$$

and

$$L(M,0)\,\mathbb{Q}^* = \det(\mathrm{reg}) \in \mathbb{R}^*/\mathbb{Q}^*$$

<u>Here</u>: $\det(\mathrm{reg}) \in \mathbb{R}^*/\mathbb{Q}^*$ is w.r.t. $\det_{\mathbb{Q}} H^1_{\mathcal{M},f}(\mathbb{Q}, M)$ & the $\mathbb{Q}$-str. $\det_{\mathbb{Q}}(M_{dR}/F^0) \otimes \det_{\mathbb{Q}}(M_B^+)^{-1}$ on $\det_{\mathbb{R}}(\mathrm{Coker}(\alpha_M))$.

<u>Rmks</u>: (0) $\exists$ version with coeff. in E.

(1) If $s=0$ is critical for M, then $\det(\mathrm{reg}) = c^+(M)$.

(2) Functional equation + computation of $\det(I_\infty)$ gives an equivalent formulation

$$\left.\frac{L(M^*(1), s)}{s^{r_{M^*(1)}}}\right|_{s=0} \mathbb{Q}^* = \det{}'(\mathrm{reg}) \in \mathbb{R}^*/\mathbb{Q}^*,$$

where $\det'(\mathrm{reg})$ is w.r.t. $\mathbb{Q}$-structure on $\det_{\mathbb{R}}(\mathrm{Coker}(\alpha_M))$ coming from duality $\mathrm{Coker}(\alpha_M)^* \simeq \mathrm{Ker}(\alpha_{M^*(1)})$, i.e.

$$\det_{\mathbb{Q}}(M^*(1)_{dR}/F^0) \otimes \det_{\mathbb{Q}}(M^*(1)_B^+)^{-1}.$$

(3) If there is a pole at $s=0$ ($\Rightarrow w=-2$), then one adds an extra term related to Tate's conjecture.

<u>Other incarnations of motivic cohomology</u>

<u>Topology</u>: X nice top. space

Chern character

$$\mathrm{ch}: K_{top}(X)_{\mathbb{Q}} \xrightarrow{\sim} \bigoplus_{n \geq 0} \underbrace{H^{2n}(X, \mathbb{Q})}_{gr^n_\gamma(\mathrm{LHS})}$$

$$K^{-1}_{top}(X)_{\mathbb{Q}} \xrightarrow{\sim} \bigoplus_{n \geq 1} \overbrace{H^{2n-1}(X, \mathbb{Q})}$$

Beilinson : defined for $X_{/K}$ smooth (more generally, regular)

$$H^i_{\mathcal{M}}(X, \mathbb{Q}(n)) = gr^n_\gamma\left(K_{2n-i}(X)_{\mathbb{Q}}\right)$$

regulator = Chern character

Special case : $i = 2n$ $\quad H^{2n}_{\mathcal{M}}(X, \mathbb{Q}(n)) = CH^n(X)_{\mathbb{Q}}$ (Grothendieck)

p-adic version : p-adic regulators have values in syntomic (= "absolute p-adic Hodge") coh.

ξ-elements

$\exists$ objects of $\det_{\mathbb{Q}} H^1_{\mathcal{M},f}(\mathbb{Q}, M)$ ($\otimes$ sth.) computing $L(M,0)$ exactly.

They appear in compatible families (Euler systems).

Ex : (1) Cyclotomic "units" :

$$\frac{d}{ds} \sum_{\substack{n\in\mathbb{Z} \\ n\equiv a \,(\mathrm{mod}\, N)}} |n|^{-s} \Bigg|_{s=0} = - \underset{\substack{\uparrow \\ (?-\frac{1}{2}?)}}{} \log\left|1-e^{2\pi i a/N}\right|$$

$$h^+_K \doteq \left[\text{all units} : \text{cyclotomic units}\right]$$

(2) Heegner pts : $K = \mathbb{Q}(\sqrt{-D})$

$E_{/\mathbb{Q}}$

$$\frac{L'(E/K,1)}{\Omega} \doteq \hat{h}(P_K, P_K)$$

$$Ш(E/K) \doteq \left(E(K) : \mathbb{Z}P_K\right)^2$$

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(3) Polylogarithms : probably no time left

"explicit" description of: - motivic cohomology
- regulator maps