EULER SYSTEMS / KOLYVAGIN SYSTEMS

What follows are my notes for the four lectures I was asked to give at the summer school on Euler Systems and Applications, held at CIB, EPFL Lausanne, from August 21 to August 25, 2017. The notes consist of several parts: (1) Motivation and basic background.

(2) The simplest case of an Euler system / Kolyvagin system in action: annihilation results, upper bounds for the order, structure theorem for $(Cl(\mathbb{Z}[\mu_p])\otimes\mathbb{Z}_p)^{(\chi)}$ ($\chi: \underbrace{Gal(\mathbb{Q}(\mu_p)/\mathbb{Q})}_{(\mathbb{Z}/p\mathbb{Z})^\times} \to \mathbb{Z}_p^\times$, $\chi(-1)=1$, $\chi\neq 1$) in terms of cyclotomic units.

(see: Rubin, appendix to Lang's book on Cyclotomic fields;
Mazur + Rubin, Kolyvagin systems;
Kolyvagin, Euler Systems, Thm 7)

(3) Statement of Kolyvagin's structure theorems for Ш$[p^\infty]$ and Sel_{p^∞} of (certain) modular elliptic curves in terms of Heegner points.

(4) An axiomatic treatment of cyclotomic Euler systems; proof of the simplest finiteness results.

(see: Kato, Kodai Math. J. 22 (1999), 313–372
Perrin-Riou, Ann. Inst. Fourier 48 (1998), 1231–1307
Rubin, Euler Systems)

(5) Remarks on Euler systems arising as p-adic étale realisations of motivic ξ-elements

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In the event, I managed to cover in my lectures only the first two topics. I hope that these notes will be useful to newcomers to this field before they embark on a more serious study of original sources.

Jan Nekovář
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Euler systems

Idealised picture (Kolyvagin, Rubin, Kato, Mazur – Rubin, ...)

Euler system (ES) $\rightsquigarrow$ (weak) Kolyvagin system (KS)

$\rightsquigarrow$ (if the KS is $\neq 0$)

$\left\{\begin{array}{l} \text{upper bounds for the } \left\{\begin{array}{l}\text{exponent}\\ \text{order}\end{array}\right\} \\ \text{sometimes full structure} \end{array}\right\}$ of some ideal class groups, Tate–Šafarevič and Selmer groups

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Motivational examples: algebraic reformulation of certain analytic class number formulas / limit formulas in terms of special elements ("zeta elements").

Ex 1 (Dirichlet's class number formula)

$K = \mathbb{Q}(\sqrt{D})$ real quadratic, $D = D_K > 0$ discriminant

$Cl_K = Cl(\mathcal{O}_K)$, $h = h_K = |Cl_K|$, $\mathcal{O}_K^\times = \{\pm 1\} \times \varepsilon^{\mathbb{Z}}$, $\varepsilon > 1$

Kronecker's symbol $\chi_D : (\mathbb{Z}/D\mathbb{Z})^\times \longrightarrow \{\pm 1\}$, $\chi_D(p) = \left(\frac{D}{p}\right)$ $\forall p \nmid 2D$ prime

$$\left.\begin{array}{l} L(1,\chi_D) \overset{\text{Dirichlet}}{=\!=} \frac{1}{\sqrt{D}} \ln(\varepsilon^{2h}) \\ L(1,\chi_D) = -\frac{\tau(\chi_D)}{D} \sum\limits_{a \in (\mathbb{Z}/D\mathbb{Z})^\times} \chi_D(a)^{-1} \ln|1-\zeta_D^a| = \frac{1}{\sqrt{D}} \ln(\varepsilon_{\mathrm{cycl}}) \end{array}\right\} \Longrightarrow \boxed{\varepsilon_{\mathrm{cycl}} = \varepsilon^{2h}}$$

$$\boxed{\zeta_n = e^{2\pi i/n}} \qquad \tau(\chi_D) := \sum_{a \in (\mathbb{Z}/D\mathbb{Z})^\times} \chi_D(a)\zeta_D^a, \quad \underbrace{\varepsilon_{\mathrm{cycl}} := \prod_{a \in (\mathbb{Z}/a\mathbb{Z})^\times} (1-\zeta_D^a)^{-\chi_D(a)}}_{\text{the cyclotomic unit of } K}$$

So $\boxed{2h_K = [\mathcal{O}_K^\times/\mathrm{tors} : \varepsilon_{\mathrm{cycl}}^{\mathbb{Z}}]}$

(Ex: $D = 5$, $h = 1$, $\varepsilon_{\mathrm{cycl}} = \left(\frac{1+\sqrt{5}}{2}\right)^2 = \varepsilon^2$)

Ex 2 (Kummer's class number formula) $p \neq 2$ prime

$K = \mathbb{Q}(\zeta_p) \supset K^+ = \mathbb{Q}(\zeta_p + \zeta_p^{-1})$

$$\mathcal{O}_{K^+}^\times \supset \mathcal{O}_{K^+,\mathrm{cycl}}^\times = \underset{\text{of } K^+}{\{\text{cyclotomic units}\}} := \mathcal{O}_{K^+}^\times \cap \langle \pm\zeta_p,\ 1-\zeta_p^a \ (a \in (\mathbb{Z}/p\mathbb{Z})^\times) \rangle$$

Class number formula for K and K^+ $\Longrightarrow$ $\boxed{h_{K^+} = [\mathcal{O}_{K^+}^\times : \mathcal{O}_{K^+,\mathrm{cycl}}^\times]}$

Ex 3 (The Gross–Zagier formula) $E_{/\mathbb{Q}}$ elliptic curve, $N = \mathrm{cond}(E)$

$\exists$ modular parameterisation $\alpha : X_0(N) \longrightarrow E$, $i\infty \longmapsto O$

$L(E/\mathbb{Q}, s) = \sum_{n\geq 1} a_n n^{-s} = L(f,s)$, $f(\tau) = \sum_{n\geq 1} a_n e^{2\pi i n\tau} \in S_2(\Gamma_0(N))$ normalised newform

$\alpha^*(\omega_E) = c\,\omega_f$, $\omega_E \in H^0(E, \Omega^1_{E/\mathbb{Q}})$ the Néron differential, $c \in \mathbb{Z}_{>0}$, $\omega_f = (2\pi i)\, f(\tau)\, d\tau$

$K = \mathbb{Q}(\sqrt{D})$ imaginary quadratic field, discriminant $D = D_K < 0$ such that every prime $l \mid N$ splits in $K/\mathbb{Q}$.

Fix ideal factorisation $N\mathcal{O}_K = \mathcal{N}\overline{\mathcal{N}}$, $\mathcal{O}_K/\mathcal{N} \simeq \mathbb{Z}/N$.

The Heegner point of conductor 1

$$x_1 := [\mathbb{C}/\mathcal{O}_K \longrightarrow \mathbb{C}/\mathcal{N}^{-1}] \in X_0(N)(\underbrace{K[1]}_{\text{the Hilbert class field of } K})$$

The basic Heegner point over K

$$y_K := \mathrm{Tr}_{K[1]/K}(\alpha(x_1)) \in E(K).$$

$$L'(E/K,1) \overset{\text{Gross–Zagier}}{=\!=} \|\omega_E\|^2\, \hat{h}(y_K)/c^2 u_K^2 |D_K|^{1/2}$$

$$L'(E/K,1) \underset{\text{BSD Conjecture}}{\overset{?}{=\!=}} \|\omega_E\|^2\, \mathrm{Tam}^2_{E/\mathbb{Q}}\, \hat{h}(y_K)\, |\text{Ш}(E/K)| / |D_K|^{1/2} [E(K) : \mathbb{Z}y_K]^2$$

$u_K = [\mathcal{O}_K^\times : \mathbb{Z}^\times]$, $\hat{h}$ = the Néron–Tate height, $\|\omega_E\|^2 = \int_{E(\mathbb{C})} \omega_E \wedge \overline{\omega_E}$

Kolyvagin : if $y_K \notin E(K)_{\mathrm{tors}}$, then $[E(K) : \mathbb{Z}y_K]$, $|\text{Ш}(E/K)| < \infty$.

Conclusion : if $y_K \notin E(K)_{\mathrm{tors}}$, then:

BSD Conjecture for $E/K \iff [E(K) : \mathbb{Z}y_K] = |\text{Ш}(E/K)|^{1/2} \cdot \mathrm{Tam}_{E/\mathbb{Q}} \cdot c \cdot u_K$

Vague picture :

$$(\text{motivic } \xi\text{-element}) \xmapsto[\text{(or the height)}]{\text{Hodge realisation } (= \text{regulator})} \text{the leading term of the relevant complex } L\text{-function}$$

cyclotomic unit $\xmapsto{\log|\cdot|}$

Heegner point $\xmapsto{\hat{h}}$

$$|\mathcal{C}\ell| \text{ or } |\text{Ш}|^{1/2} \doteq [\underbrace{\text{motivic cohomology group}}_{\mathcal{O}_K^\times,\ E(K)} : \text{subgroup generated by } \xi\text{-elements}]$$

Galois cohomology

Étale realisations of motivic ξ-elements ~~are contained~~ give rise to elements in certain Selmer groups $\subset$ Galois cohomology.

Ex: K field, $\mathrm{char}(K) \nmid n$, $\left\{\begin{array}{ll} \mathbb{G}_m & \text{multiplicative group} \\ E & \text{elliptic curve} \end{array}\right\}$ over K

Kummer maps = boundary maps attached to

$0 \to \mu_n \to (K^{sep})^\times \xrightarrow{n} (K^{sep})^\times \to 0$, $\qquad \delta : K^\times \otimes \mathbb{Z}/n \xrightarrow{\sim} H^1(K, \mu_n)$

$0 \to E[n] \to E(K) \xrightarrow{n} E(K) \to 0$, $\qquad \underline{\delta : E(K) \otimes \mathbb{Z}/n \hookrightarrow H^1(K, E[n])}$

Notation: $G_K := \mathrm{Gal}(K^{sep}/K)$, $\quad H^i(K, M) := \underbrace{H^i_{cont}(G_K, M)}_{\text{use continuous cochains}}$

M topological G_K-module

Prop. If G = profinite group, $M = \varprojlim_\alpha M_\alpha$ (with profinite topology), M_α finite discrete G-modules, with surjective transition maps $M_\alpha \leftarrow M_\beta$. Then:

(1) There is a spectral sequence

$$E_2^{p,q} = \underbrace{(R^p \varprojlim_\alpha)(H^q(G, M_\alpha))}_{= 0 \text{ if } (p > 0 \text{ and and all } \underbrace{H^q(G, M_\alpha) \text{ are finite}}_{\text{true if } q = 0})} \Longrightarrow H^{p+q}_{cont}(G, M)$$

(in particular, $H^1_{cont}(G, M) = \varprojlim_\alpha H^1(G, M_\alpha)$).

(2) If, in addition, the abelian group M has no torsion, put on $M \otimes \mathbb{Q} = \bigcup_{n \geq 1} M \otimes \frac{1}{n}\mathbb{Z} = \varinjlim_{n | n'} M \otimes \frac{1}{n}\mathbb{Z}$ the inductive limit topology. Then $\forall i$ $H^i_{cont}(G, M) \otimes \mathbb{Q} \xrightarrow{\sim} H^i_{cont}(G, M \otimes \mathbb{Q})$.

Pf. (1) $C^\bullet_{cont}(G, M) = \varprojlim_\alpha C^\bullet(G, M_\alpha)$. In the hypercohomology spectral sequences

$$\left.\begin{array}{l} {}^{I}E_1^{p,q} = H^q(i \mapsto (R^p \varprojlim_\alpha)(C^i(G, M_\alpha))) \\ {}^{II}E_2^{p,q} = (R^p \varprojlim_\alpha)(\underbrace{H^q(i \mapsto C^i(G, M_\alpha))}_{H^q(G, M_\alpha)}) \end{array}\right\} \Rightarrow (\mathbb{R}^{p+q} \varprojlim_\alpha)(C^\bullet(G, M_\alpha))$$

${}^{I}E_1^{p,q} = 0$ for $p > 0$, since all maps $C^\bullet(G, M_\alpha) \longleftarrow C^\bullet(G, M_\beta)$ are surjective.

(2) G is compact and every compact subset of $M \otimes \mathbb{Q}$ is contained in some $M \otimes \frac{1}{n}\mathbb{Z}$.

Ex: if $\mathrm{char}(K) \neq p$, $H^1(K, \underbrace{\varprojlim_n \mu_{p^n}}_{\mathbb{Z}_p(1)}) \overset{\delta}{\underset{\sim}{\longleftarrow}} K^\times \otimes \mathbb{Z}/p^n =: K^\times \hat{\otimes} \mathbb{Z}_p$

$$H^1(K, \underbrace{\mathbb{Z}_p(1) \otimes \mathbb{Q}}_{\mathbb{Q}_p(1)}) \cong K^\times \hat{\otimes} \mathbb{Q}_p := (K^\times \hat{\otimes} \mathbb{Z}_p) \otimes \mathbb{Q}.$$

$\boxed{\otimes := \otimes_{\mathbb{Z}}}$

Selmer groups

Assume: $\boxed{[K:\mathbb{Q}] < \infty}$

We are not interested in $\left\{ \begin{array}{l} H^1(K, \mu_n) \simeq K^\times \otimes \mathbb{Z}/n \\ H^1(K, E[n]) \end{array} \right\}$ but in their subgroups $\left\{ \begin{array}{l} \mathcal{O}_K^\times \otimes \mathbb{Z}/n \\ E(K) \otimes \mathbb{Z}/n \end{array} \right\}$ (E/K elliptic curves).

These can be approximated from above by intermediate "Selmer groups", defined by appropriate local conditions.

Def. A Selmer structure for a topological G_K-module M which is unramified outside a finite set S of places of K, is a collection $\mathcal{F} = (\mathcal{F}_v)$, where $\mathcal{F}_v \subset H^1(K_v, M)$ is a subgroup (for every place v of K), equal to the unramified cohomology $H^1_{ur}(K_v, M) := \mathrm{Ker}(H^1(K_v, M) \xrightarrow{res} H^1(K_v^{ur}, M))$ for $v \notin$ certain finite set of places containing $\{v | \infty\}$.

The corresponding Selmer group is defined by a cartesian diagram

$$\begin{tikzcd}
H^1_{\mathcal{F}}(K, M) \arrow[r] & \prod_v \mathcal{F}_v \\
H^1(K, M) \arrow[u, phantom, "\cap"] \arrow[ur, phantom, "\square"] \arrow[r, "(\mathrm{loc}_v)"] & \prod_v H^1(K_v, M) \arrow[u, phantom, "\cap"] \\
x \arrow[u, phantom, "\in"] \arrow[r, mapsto] & (x_v) \arrow[u, phantom, "\in"]
\end{tikzcd}$$

(i.e., $H^1_{\mathcal{F}}(K, M) = \mathrm{Ker}\left(H^1(K, M) \longrightarrow \prod_v H^1(K_v, M)/\mathcal{F}_v\right)$)

Rmks. (1) If M is unramified outside a finite set of places $S \supset \{v | \infty\}$ (i.e., $M = M^{I_v}$ $\forall v \notin S$, $I_v = \mathrm{Gal}(\overline{K_v}/K_v^{ur}) \subset G_{K_v}$ the inertia subgroup), then M is a $G_{K,S} = \mathrm{Gal}(K_S/K)$-module ($K_S :=$ the maximal subfield of $\overline{K}$ unramified over K outside S).

(2) If, in addition, $\forall v \notin S$ $\mathcal{F}_v = H^1_{ur}(K_v, M)$, the inf-res sequence $0 \longrightarrow H^1(G_{K,S}, M) \longrightarrow H^1(K, M) \longrightarrow \mathrm{Hom}(\underbrace{\mathrm{Gal}(\overline{K}/K_S)}_{\text{generated by } I_v \ (v \notin S)}, M)$ implies that

$$H^1_{\mathcal{F}}(K, M) = \mathrm{Ker}\left(H^1(G_{K,S}, M) \longrightarrow \bigoplus_{v \in S} H^1(K_v, M)/\mathcal{F}_v\right).$$

this group is of finite/cofinite type over $\mathbb{Z}_p$ if M is.

<u>Dual Selmer structure</u>: if $M = \varprojlim_{\alpha} M_\alpha$, M_α finite discrete G_K-module, the transition maps $M_\alpha \leftarrow M_\beta$ being surjective (M with profinite topology), let the <u>Cartier dual</u> of M be the discrete G_K-module

$$M^D := \mathrm{Hom}_{cont}(M, \mu) = \varinjlim_{\alpha} \mathrm{Hom}(M_\alpha, \mu) = \bigcup_{\alpha} M_\alpha^D, \text{ where}$$

$\mu := \overline{K}^\times_{tors} = \bigcup_{n \geq 1} \mu_n(\overline{K})$ has discrete topology.

<u>local duality (Tate)</u>: perfect pairings

$$H^1(K_v, M_\alpha) \times H^1(K_v, M_\alpha^D) \xrightarrow{\cup} H^2(K_v, \mu) \overset{inv_v}{\xrightarrow{\sim}} \begin{cases} \mathbb{Q}/\mathbb{Z}, & v \nmid \infty \\ \frac{1}{|G_{K_v}|}\mathbb{Z}/\mathbb{Z} \subset \mathbb{Q}/\mathbb{Z}, & v \mid \infty \end{cases}$$

$$\underbrace{\varprojlim_{\alpha} H^1(K_v, M_\alpha)}_{H^1(K_v, M)} \times \underbrace{\varinjlim_{\alpha} H^1(K_v, M_\alpha^D)}_{H^1(K_v, M^D)} \xrightarrow{\cup} \mathbb{Q}/\mathbb{Z}$$

<u>Def</u>. For a Selmer structure $\mathcal{F} = (\mathcal{F}_v)$ for $M = \varprojlim_{\alpha} M_\alpha$, $|M_\alpha| = p^{n_\alpha}$, define its <u>dual</u> to be $\mathcal{F}^D := (\mathcal{F}_v^D := (\text{the orthogonal complement } \boxed{\mathcal{F}_v^\perp} \text{ of } \mathcal{F}_v \text{ under } \cup) \subset H^1(K_v, M^D))$

this is a Selmer structure for M^D, since $H^1_{ur}(K_v, M)^\perp = H^1_{ur}(K_v, M^D)$ if $v \nmid p\infty$ (Tate).

<u>Ex:</u> (1) <u>$M = \mu_n$</u>, <u>$M^D = \mathbb{Z}/n$</u>. Consider

$$\mathcal{F}_v := \mathcal{O}_{K_v}^\times \otimes \mathbb{Z}/n \subset K_v^\times \otimes \mathbb{Z}/n \overset{\delta}{\xrightarrow{\sim}} H^1(K_v, \mu_n) \qquad (\mathcal{O}_{K_v}^\times := K_v^\times \text{ if } v \mid \infty)$$

$$\Rightarrow \mathcal{F}_v^D = \underbrace{\mathrm{Hom}_{cont}(G_{K_v}/I_v, \mathbb{Z}/n)}_{H^1_{ur}(K_v, \mathbb{Z}/n)} \subset \mathrm{Hom}_{cont}(G_{K_v}^{ab}, \mathbb{Z}/n) = H^1(K_v, \mathbb{Z}/n) \qquad (v \nmid \infty)$$

$(\mathcal{F}_v^D = 0 \text{ if } v \mid \infty)$ $\qquad (\mathcal{F}_v = H^1_{ur}(K_v, \mu_n) \text{ if } v \nmid n)$

<u>Selmer groups</u>: $0 \to \mathcal{O}_K^\times \otimes \mathbb{Z}/n \to H^1_{\mathcal{F}}(K, \mu_n) \to Cl(\mathcal{O}_K)[n] \to 0$

$$H^1(K, \mathbb{Z}/n) = \mathrm{Hom}_{cont}(G_K, \mathbb{Z}/n) \supset \mathrm{Hom}(Gal(\underbrace{K_\phi}/K), \mathbb{Z}/n) = H^1_{\mathcal{F}^D}(K, \mathbb{Z}/n)$$

(K_ϕ = the Hilbert class field of K) maximal unramified $\underbrace{\text{extension}}_{\text{(abelian)}}$ of K

<u>Class field theory</u>: $Cl(\mathcal{O}_K) \xrightarrow{\sim} Gal(K_\phi/K)$

$$H^1_{\mathcal{F}^D}(K, \mathbb{Z}/n) \cong \mathrm{Hom}(Cl(\mathcal{O}_K), \mathbb{Z}/n)$$

(2) <u>$M = E[n]$, E/K elliptic curve</u>: $\mathcal{F}_v := E(K_v) \otimes \mathbb{Z}/n \overset{\delta}{\hookrightarrow} H^1(K_v, E[n])$

$H^1_{\mathcal{F}}(K, E[n]) = Sel(E/K, n)$ Selmer group for the n-descent on E/K

$M^D \simeq M$ (Weil pairing), $\mathcal{F}^D \simeq \mathcal{F}$ (Tate).

<u>Rmk</u>: All three Selmer structures in Ex. (1), (2) are the <u>Bloch–Kato</u> Selmer structures $\mathcal{F}_v = H^1_f(K_v, -)$.

Taking limits: $n = p^k$, p prime

$\boxed{\varprojlim_k}$ Ex(1). $\mathcal{O}_K^\times \otimes \mathbb{Z}_p \xrightarrow{\sim} \underbrace{H^1_{\mathcal{F}}(K, \mathbb{Z}_p(1))}_{H^1_f(K,\, \mathbb{Z}_p(1))}$, $0 = \underbrace{H^1_{\mathcal{F}^D}(K, \mathbb{Z}_p)}_{H^1_f(K,\, \mathbb{Z}_p)}$

Ex(2). $0 \longrightarrow E(K) \otimes \mathbb{Z}_p \longrightarrow \underbrace{H^1_{\mathcal{F}}(K, T_p(E))}_{H^1_f(K,\, T_p(E))} \longrightarrow T_p \text{Ш}(E/K) \longrightarrow 0$

$T_p(X) := \varprojlim_k X[p^k]$

$\boxed{\varinjlim_k}$ (1) $0 \longrightarrow \underbrace{\mathcal{O}_K^\times \otimes \mathbb{Q}_p/\mathbb{Z}_p}_{\substack{(\mathbb{Q}_p/\mathbb{Z}_p)^{r_1+r_2-1} \\ \text{divisible}}} \longrightarrow \underbrace{H^1_{\mathcal{F}}(K, \overbrace{\mu_{p^\infty}}^{(\mathbb{Q}_p/\mathbb{Z}_p)(1)})}_{H^1_f(K,\, (\mathbb{Q}_p/\mathbb{Z}_p)(1))} \longrightarrow \underbrace{Cl(\mathcal{O}_K)[p^\infty]}_{\text{finite}} \longrightarrow 0$

$\underbrace{H^1_{\mathcal{F}^D}(K, \mathbb{Q}_p/\mathbb{Z}_p)}_{H^1_f(K,\, \mathbb{Q}_p/\mathbb{Z}_p)} \cong \mathrm{Hom}(Cl(\mathcal{O}_K), \mathbb{Q}_p/\mathbb{Z}_p)$

dual to each other

(2) $0 \longrightarrow \underbrace{E(K) \otimes \mathbb{Q}_p/\mathbb{Z}_p}_{\substack{(\mathbb{Q}_p/\mathbb{Z}_p)^{\mathrm{rk}_{\mathbb{Z}} E(K)} \\ \text{divisible}}} \longrightarrow \underbrace{H^1_{\mathcal{F}}(K, E[p^\infty])}_{H^1_f(K,\, E[p^\infty])} \longrightarrow \text{Ш}(E/K)[p^\infty] \longrightarrow 0$

$H^1_f(K, E[p^\infty])/div \xrightarrow{\sim} \text{Ш}(E/K)[p^\infty]/div$

finite, self-dual (Cassels(-Tate))

$\left\{\begin{matrix}\text{Relaxed} \\ \text{Strict}\end{matrix}\right\}$ Selmer structures: given a Selmer structure $\mathcal{F}$ for M and a finite set S of places of K, define

$$(\mathcal{F}^S)_v := \underbrace{\left\{\begin{matrix} \mathcal{F}_v, & v \notin S \\ H^1(K_v, M), & v \in S \end{matrix}\right\}}_{\text{relaxed at } S}, \quad (\mathcal{F}_S)_v := \underbrace{\left\{\begin{matrix} \mathcal{F}_v, & v \notin S \\ 0, & v \in S \end{matrix}\right.}_{\text{strict at } S}$$

$$(\mathcal{F}^S)^D = (\mathcal{F}^D)_S$$

Ex: $M = \mu_n$, $\mathcal{F}$ as above

$$0 \longrightarrow \mathcal{O}_{K,S}^\times \otimes \mathbb{Z}/n \longrightarrow H^1_{\mathcal{F}^S}(K, \mu_n) \longrightarrow Cl(\mathcal{O}_{K,S})[n] \longrightarrow 0$$

$\mathcal{O}_{K,S} := \mathcal{O}_K[1/S] = \{x \in K \mid \forall v \notin S \ (v \nmid \infty) \quad x \in \mathcal{O}_{K_v}\}$

General machinery

$[K:\mathbb{Q}]<\infty$, $p\neq 2$ (for simplicity; many things work for $p=2$, too)

$T = \mathbb{Z}_p[G_K]$ - module (continuous), T free of finite rank over $\mathbb{Z}_p$

Notation : $K \subsetneq_f F$: F is a field of finite degree over K

$\boxed{\text{Euler system (ES) of rank}=1 \text{ for } T}$ $\xleftarrow{\text{sometimes}}$ $\boxed{\text{(ES) of rank } r\geq 1 \text{ for } T}$

(Rubin, Perrin-Riou)

suitably coherent system

$c_F \in H^1_{\mathcal{F}_F}(F,T)$ for certain $K \subsetneq_f F \subset K^{ab}$

$c_F^{(r)} \in \bigwedge^r_{\mathbb{Z}_p[Gal(F/K)]} H^1_{\mathcal{F}_F}(F,T)$

Kolyvagin's derivative

$\boxed{\text{weak Kolyvagin system (KS) of rank}=1 \text{ for } T}$

sometimes

(Burns – Sano)

$\kappa_F^{\#} \in H^1_{\mathcal{F}_K^{Ram(F/K)}}(K, T/I_{F,T})$ $I_{F,T}\subset \mathbb{Z}_p$ ideal

slight modification

Mazur – Rubin

$\boxed{\text{Kolyvagin system of rank}=1}$

$\boxed{\text{(KS) of rank}=r}$

$\kappa_F^{!} \in \bigwedge^r H^1_{\mathcal{F}_K(\ldots)}(K, T/I_{F,T})$

$\cup|$ (equality if $r=1$)

special ("stub") (KS) of $rk=r$

$\|$

$\boxed{\text{Stark systems of } rk=r}$

$\kappa_F^{!} \in H^1_{\underbrace{\mathcal{F}_K(\ldots)}}(K, T/I_{F,T})$

transversal $\mathcal{F}_v$ at some v (depending on F)

sometimes

sometimes

$\boxed{\begin{array}{l}\text{upper bounds}\\ \text{for the size}\\ \text{of } H^1_{\mathcal{F}^D}(K,T^D)\\ \text{in terms of}\\ \kappa^!_K=\kappa_K=c_K\end{array}}$ $\Longleftarrow$ $\boxed{\begin{array}{l}\text{structure of}\\ H^1_{\mathcal{F}^D}(K,T^D)\\ \text{in terms of}\\ \{\kappa^!_F\}\end{array}}$

first results of this kind : Kolyvagin

abstract theory : Mazur – Rubin ($K=\mathbb{Q}$), ...

There are analogues in Iwasawa theory, for $H^1_{\mathcal{F}^D}(K_\infty, T^D)$ ($\Gamma = Gal(K_\infty/K) \simeq \mathbb{Z}_p^d$). Twisting by p-adic continuous characters of Γ enters the picture.

Upper bounds for $Cl_K = Cl(\mathcal{O}_K)$ $([K:\mathbb{Q}]<\infty)$

(1) Relations in Cl_K: $\alpha \in K^\times$, $(\alpha) = \prod_P P^{a_P} \Rightarrow \mathrm{div}(\alpha) = \sum a_P P \in \mathrm{Div}(\mathcal{O}_K)$

$$\Rightarrow \quad \boxed{\sum a_P [P] = 0 \in Cl_K}$$

(2) Relations in $Cl_K \otimes \mathbb{Z}/M$: $\beta \in K^\times \otimes \mathbb{Z}/M$, $\mathrm{div}(\beta) = \sum b_P P \in \mathrm{Div}(\mathcal{O}_K) \otimes \mathbb{Z}/M$

$$\Rightarrow \boxed{\sum b_P [P] = 0 \in Cl_K \otimes \mathbb{Z}/M}$$

(3) Producing β: for suitable $K \xhookrightarrow[f]{} F$, $G := \mathrm{Gal}(F/K)$

$$\begin{tikzcd}
K^\times \otimes \mathbb{Z}/M \arrow[r] \arrow[d, phantom, sloped, "\ni"] & (F^\times \otimes \mathbb{Z}/M)^G \arrow[l, bend right, dashed] \\
\beta & \\
& \varepsilon \in (\mathcal{O}_F^\times \otimes \mathbb{Z}/M)^G \arrow[uu] \arrow[ul, maps to] \arrow[uul, dashed]
\end{tikzcd}$$

is not far from an isomorphism, $\exists$ "inverse"

Thaine: produced upper) bounds on Cl_K by this method for $K \xhookrightarrow[f]{} F \subset \mathbb{Q}^{ab} \cap \mathbb{R}$, $\varepsilon =$ suitable cyclotomic unit

This method was further developed by Rubin and Kolyvagin (who had independently found a method for producing relations in $Ш(E/\mathbb{Q}(\sqrt{D}))[M]$ for modular elliptic curves $E/\mathbb{Q}$ using Heegner points over certain $F \hookrightarrow \mathbb{Q}(\sqrt{D})^{ab}$ $(D<0)$.

We are going to explain this method in detail in the simplest nontrivial situation, namely, $\left(\begin{matrix} p \neq 2 \\ \text{prime}\end{matrix}\right)$ for $Cl_{\mathbb{Q}(\mu_p)^+} \otimes \mathbb{Z}/p^k$ $(k \gg 0)$. This works more generally, e.g. for $Cl_{\mathbb{Q}(\mu_p)^+} \otimes \mathbb{Z}/M$ with $(M, p-1) = 1$.

The Euler system argument for $Cl_{\mathbb{Q}(\mu_p)^+}\otimes\mathbb{Z}_p$ $(p\neq 2 \text{ prime})$

<u>Notation</u> : $K=\mathbb{Q}(\mu_p)=\mathbb{Q}(\zeta_p)\supset K^+=\mathbb{Q}(\zeta_p+\zeta_p^{-1})$, $\zeta_p=e^{2\pi i/p}$, $\Delta:=\mathrm{Gal}(K/\mathbb{Q})$

<u>Δ-eigenspaces</u> : X a $\mathbb{Z}_p[\Delta]$-module, $\chi:\Delta\to\mathbb{Z}_p^\times$

$$X^{(\chi)}:=\{x\in X \mid \forall g\in\Delta\ ,\ g(x)=\chi(g)x\}=e_\chi X,\quad e_\chi=\frac{1}{|\Delta|}\sum_{g\in\Delta}\chi(g)^{-1}g$$

$\{\chi:\Delta\to\mathbb{Z}_p^\times\}=\{\omega^j \mid j\in\mathbb{Z}/(p-1)\mathbb{Z}\}$, where

ω = the <u>Teichmüller character</u> :

$$\omega:\Delta\xrightarrow[\sim]{\chi_{cycl}}\mathbb{F}_p^\times\xleftarrow{\sim}(\mathbb{Z}_p^\times)_{tors}\hookrightarrow\mathbb{Z}_p^\times\ ,\quad \omega(c)=-1$$ ($c\in\Delta$ the complex conjugation)

<u>Cyclotomic (p-)units in K and K^+:</u>

$$\begin{array}{ccl}
\mathcal{O}_K[1/p]^\times & \supset & \mathcal{O}_K[1/p]^\times_{cycl}:=\text{subgroup generated by } \pm\zeta_p,\ 1-\zeta_p^a\ (a\in(\mathbb{Z}/p\mathbb{Z})^\times)\\
\cup & & \cup\\
\mathcal{O}_K^\times & \supset & \mathcal{O}_{K,cycl}^\times:=\mathcal{O}_K^\times\cap\mathcal{O}_K[1/p]^\times_{cycl}\\
\cup & & \cup\\
\mathcal{O}_{K^+}^\times & \supset & \mathcal{O}_{K^+,cycl}^\times:=\mathcal{O}_{K^+}^\times\cap\mathcal{O}_K[1/p]^\times_{cycl}
\end{array}$$

these are finitely generated abelian groups, of respective ranks $\frac{p-1}{2}$ (the 1st row) and $\frac{p-3}{2}$ (the 2nd and 3rd rows). Their torsion subgroups are μ_{2p} (the first and 2nd rows) and $\{\pm1\}$ (the 3rd row). Moreover, $\mathcal{O}_K^\times=\mathcal{O}_{K^+}^\times\cdot\mu_p$ (Kummer), and $\mathcal{O}_K[1/p]^\times=\mathcal{O}_K^\times\cdot\mathcal{O}_K[1/p]^\times_{cycl}$.

The class number formula for K and K^* implies that

$$\boxed{|Cl_{K^+}|=[\mathcal{O}_K^\times:\mathcal{O}_{K,cycl}^\times]=[\mathcal{O}_{K^+}^\times:\mathcal{O}_{K^+,cycl}^\times]}$$

<u>p-parts and Δ-eigenspaces</u>: $A:=Cl_{K^+}\otimes\mathbb{Z}_p=(Cl_K\otimes\mathbb{Z}_p)^{c=1}=\bigoplus_{\chi(c)=1}A^{(\chi)}$ $(A^{(1)}=0)$

$$\begin{array}{ccl}
U':=\mathcal{O}_K[1/p]^\times\otimes\mathbb{Z}_p & \supset & U'_{cycl}:=\mathcal{O}_K[1/p]^\times_{cycl}\otimes\mathbb{Z}_p=\text{the } \mathbb{Z}_p\text{-submodule of } K^\times\otimes\mathbb{Z}_p \text{ generated by } \mu_p \text{ and } \mathbb{Z}_p[\Delta](1-\zeta_p).\\
\cup & & \cup\\
U:=\mathcal{O}_K^\times\otimes\mathbb{Z}_p & \supset & U_{cycl}:=\mathcal{O}_{K,cycl}^\times\otimes\mathbb{Z}_p=U\cap U'_{cycl}\\
\cup & & \cup\\
U^+:=\mathcal{O}_{K^+}^\times\otimes\mathbb{Z}_p & \supset & U^+_{cycl}:=\mathcal{O}_{K^+,cycl}^\times\otimes\mathbb{Z}_p
\end{array}$$

$$\mu_p\xrightarrow{\sim}U/U^+=U_{cycl}/U^+_{cycl}$$

Prop. Let $\chi : \Delta \longrightarrow \mathbb{Z}_p^\times$. (1) If $\underline{\chi(c) = -1}$, then $(U')^{(\chi)} = \begin{cases} \mu_p, & \chi = \omega \\ 0, & \chi \neq \omega. \end{cases}$

(2) If $\chi = 1$, then $U^{(\chi)} = 0$,

$(U')^{(\chi)} = (U'_{cycl})^{(\chi)} =$ the $\mathbb{Z}_p$-module generated by $p \otimes 1 \in K^\times \otimes \mathbb{Z}_p$.

(3) If $\underline{\chi(c) = 1}$, $\underline{\chi \neq 1}$, then $(U')^{(\chi)} = U^{(\chi)} = (U^+)^\chi$,

$(U'_{cycl})^{(\chi)} = (U_{cycl})^{(\chi)} = (U^+_{cycl})^{(\chi)} =$ the $\mathbb{Z}_p$-module generated by $\underbrace{e_\chi(1-\zeta_p) \otimes 1}_{\neq 0}$.

All ~~these~~ $\mathbb{Z}_p$-modules are free of rank $= 1$.

Pf. Exercise.

Statement of the first main result

Thm 1 (Mazur – Wiles) If $\chi : \Delta \longrightarrow \mathbb{Z}_p^\times$, $\chi(c) = 1$, $\chi \neq 1$, then

$$|A^{(\chi)}| = [U^{(\chi)} : U^{(\chi)}_{cycl}] = [U^{(\chi)} : \mathbb{Z}_p\, e_\chi(1-\zeta_p)]$$

Because of the class number formula $\prod_{\substack{\chi(c)=1 \\ \chi \neq 1}} |A^{(\chi)}| / [U^{(\chi)} : U^{(\chi)}_{cycl}] = 1$, thm 1 follows from

Thm 2. If $\chi(c) = 1$, $\chi \neq 1$, then $|A^{(\chi)}|$ divides $[U^{(\chi)} : U^{(\chi)}_{cycl}]$.

[Equivalent formulation: if $\boxed{M = p^k \text{ and } k \gg 0}$, then

$$\underbrace{|(Cl_K \otimes \mathbb{Z}/M)^{(\chi)}| \text{ divides } [(\mathcal{O}_K^\times \otimes \mathbb{Z}/M)^{(\chi)} : (\mathbb{Z}/M) \cdot e_\chi(1-\zeta_p) \otimes 1]}$$]

Main ideas of an Euler system proof of Thm 2:

(1) Let $\kappa(1) := e_\chi(1-\zeta_p) \otimes 1 \in (\mathcal{O}_K^\times \otimes \mathbb{Z}/M)^{(\chi)}$ (the basic cyclotomic unit)

$\kappa(1) \neq 0$ if $k \gg 0$

(2) For suitable $n > 1$, namely, $n = l_1 \cdots l_r$, $l_i \equiv 1 \pmod{M}$ distinct primes,

$$\begin{tikzcd} e_\chi(K^\times \otimes \mathbb{Z}/M) \arrow[r, "\sim"] & e_\chi(K(\mu_n)^\times \otimes \mathbb{Z}/M)^{G(n)}, \\ \kappa(n) \arrow[u, phantom, sloped, "\in"] & e_\chi D_n(1-\zeta_{pn}) \otimes 1 \arrow[u, phantom, sloped, "\in"] \arrow[l, mapsto] \end{tikzcd}$$

$$\begin{tikzcd} G(n) = \mathrm{Gal}(K(\mu_n)/K) \arrow[d, "\chi_{cycl}", "\wr"'] \\ (\mathbb{Z}/n\mathbb{Z})^\times \end{tikzcd}$$

$\underbrace{\kappa(n)}$ Kolyvagin's derivative elements

for suitable Kolyvagin's derivative operators $\underline{D_n \in \mathbb{Z}[G(n)]}$

$\{ (1-\zeta_{pm}) \mid \text{all } m \geq 1\}$ is an Euler system

(3) <u>Factorisation of $\kappa(n) \in (K^\times \otimes \mathbb{Z}/M)^{(\chi)}$</u> : $n = \ell_1 \cdots \ell_r$, $\ell_i \equiv 1 \pmod{M}$ prime

(a) $$\operatorname{div}(\kappa(n)) \subset \bigoplus_{\ell | n} \underbrace{\Big(\bigoplus_{\lambda | \ell} (\mathbb{Z}/M) \cdot \lambda \Big)^{(\chi)}}_{(\mathbb{Z}/M)\cdot\lambda \text{ for one fixed } \lambda|\ell \text{ in } K} \subset (\operatorname{Div}(\mathcal{O}_K) \otimes \mathbb{Z}/M)^{(\chi)}$$

($\ell = \ell_i$ splits completely in $\mathbb{Q}(\mu_M) = \mathbb{Q}(\mu_{p^k}) \supset K = \mathbb{Q}(\mu_p) \Rightarrow K_\lambda = \mathbb{Q}_\ell$)

(b) <u>Inductive property (Kolyvagin)</u>: if $\lambda | \ell | n$, then $\operatorname{ord}_\lambda \kappa(n) \in \mathbb{Z}/M$ is determined by the image $\kappa(n/\ell)_\lambda$ of $\kappa(n/\ell)$ in $$\mathcal{O}_{K_\lambda}^\times \otimes \mathbb{Z}/M = \mathbb{Z}_\ell^\times \otimes \mathbb{Z}/M \xrightarrow[\sim]{\text{red}} \underbrace{\mathbb{F}_\ell^\times \otimes \mathbb{Z}/M}_{\text{free of } rk = 1 \text{ over } \mathbb{Z}/M}$$

(4) The value of $\kappa(n/\ell)_\lambda$ is determined by

$$\operatorname{Fr}_{L/K}(\lambda_L) \in \underset{\mu_M}{\underset{\cap}{\operatorname{Gal}(L/\mathbb{Q}(\mu_M))}} \subset \operatorname{Gal}(L/K),$$

$$\begin{tikzcd} L := \mathbb{Q}(\mu_M, \kappa(n/\ell)^{1/M}) \arrow[d, dash] & \lambda_L \arrow[dd, dash] \\ \mathbb{Q}(\mu_M) \arrow[d, dash] & \\ K = \mathbb{Q}(\mu_p) \arrow[d, dash] & \lambda \arrow[d, dash] \\ \mathbb{Q} & \ell \end{tikzcd}$$

This is <u>very important</u>: one translates various conditions involving $\ell = \ell_i$ in terms of Fr (some prime above ℓ) in a suitable (finite) extension (e.g., $\ell \equiv 1 \pmod{M} \Leftrightarrow \ell$ splits completely in $\mathbb{Q}(\mu_M)/\mathbb{Q} \Leftrightarrow \operatorname{Fr}_{\mathbb{Q}(\mu_M)/\mathbb{Q}}(\ell) = 1$).

(5) <u>Čebotarev's Thm</u> $\Rightarrow$ one can control $\operatorname{Fr}_{L/K}(\lambda_L)$ in (4) to build inductively $n = \ell_1, \ell_1\ell_2, \ldots, \ell_1 \cdots \ell_d$ with enough information above $\operatorname{div}(\kappa(\ell_1 \cdots \ell_i))$ to prove Thm 2.

A more precise version (Kolyvagin, Euler Systems, Thm 7) then gives a <u>full structure</u> of $(Cl_K \otimes \mathbb{Z}/p^k)^{(\chi)} = A^{(\chi)}$ $(k \gg 0)$ in terms of $\{\kappa(n)\}$ (a weak Kolyvagin system).

We are going to prove in detail a slight variant of this structure theorem.

Properties of $1-\xi_m \in \mathbb{Q}(\mu_m)^\times$ $(m>1)$

(1) $m=\ell^k$, ℓ prime $\Longrightarrow 1-\xi_m \in \mathcal{O}_{\mathbb{Q}(\mu_m)}[1/\ell]^\times$

(2) $m \neq \ell^k$, — " — $\Longrightarrow 1-\xi_m \in \mathcal{O}^\times_{\mathbb{Q}(\mu_m)}$

(3) ℓ prime $\Longrightarrow N_{\mathbb{Q}(\mu_{m\ell})/\mathbb{Q}(\mu_m)}(1-\xi_{m\ell}) = \begin{cases} 1-\xi_m, & \ell \mid m \\ \underbrace{(1-\mathrm{Fr}(\ell)(1-\xi_m)}_{\frac{1-\xi_m}{1-\xi_m^{\ell^{-1}}}}, & \ell \nmid m \end{cases}$

$\boxed{\text{Norm Relation}}$

Notation:

$$\begin{tikzcd} \mathrm{Fr}(\ell) := \mathrm{Fr}_{geom}(\ell) \arrow[d, mapsto] & \in & \mathrm{Gal}(\mathbb{Q}(\mu_m)/\mathbb{Q}) \arrow[d, "\chi_{cycl}", "\wr"'] \\ (\ell \,(\mathrm{mod}\, m))^{-1} & \in & (\mathbb{Z}/m\mathbb{Z})^\times \end{tikzcd}$$

(4) $\ell \nmid m$, ℓ prime, $\lambda \mid \ell$ in $\mathbb{Q}(\mu_{m\ell}) \Longrightarrow 1-\xi_{m\ell} \equiv \underbrace{\mathrm{Fr}(\ell)(1-\xi_m)}_{1-\xi_m^{\ell^{-1}}} \pmod{\lambda}$

$\boxed{\text{Congruence Relation}}$

Kolyvagin's derivative operators: fix $M=p^k$ $(k \gg 0)$

$\mathcal{P}_k := \{\text{primes } \ell \equiv 1 \,(\mathrm{mod}\, M)\}$, $\mathcal{N}_k := \{n=\ell_1 \cdots \ell_r \mid r \geq 0,\ \ell_i \in \mathcal{P}_k \text{ distinct}\}$

Galois groups: $n \in \mathcal{N}_k$

$$\begin{tikzcd} \mathbb{Q}(\mu_n) \arrow[r, dash, "\Delta"] \arrow[d, dash, "G(n)"'] & K(\mu_n) = \mathbb{Q}(\mu_{np}) \arrow[d, dash, "G(n)"] \\ \mathbb{Q} \arrow[r, dash, "\Delta"'] & K = \mathbb{Q}(\mu_p) \end{tikzcd}$$

$$\begin{tikzcd} G(n) \arrow[d, "\chi_{cycl}"', "\wr"] & = & \prod_{\ell \mid n} G(\ell) \arrow[d, "\chi_{cycl}", "\wr"'] \\ (\mathbb{Z}/n\mathbb{Z})^\times & & (\mathbb{Z}/\ell\mathbb{Z})^\times \end{tikzcd}$$

Fix: $\forall \ell \in \mathcal{P}_k$ a generator $\sigma_\ell \in G(\ell) \xrightarrow{\sim} (\mathbb{Z}/\ell\mathbb{Z})^\times$

Def. For $\ell \in \mathcal{P}_k$: $N_\ell := \sum_{\sigma \in G(\ell)} \sigma = \sum_{i=0}^{\ell-2} \sigma_\ell^i$, $D_\ell := \sum_{i=0}^{\ell-2} i\sigma_\ell^i \in \mathbb{Z}[G(\ell)]$

$$\boxed{(\sigma_\ell - 1) D_\ell = (\ell-1) - N_\ell}$$

for $n \in \mathcal{N}_k$: $D_n := \prod_{\ell \mid n} D_\ell \in \mathbb{Z}[G(n)]$

Euler system: for $m \geq 1$, $p \nmid m$,

$$c(m) := e_\chi((1-\xi_{pm}) \otimes 1) \in (K(\mu_m)^\times \otimes \mathbb{Z}/M)^{(\chi)}$$

Note: $c(m) \in (\mathcal{O}^\times_{K(\mu_m)} \otimes \mathbb{Z}/M)^{(\chi)}$ $(p \nmid m)$

Kolyvagin's derivative classes

Prop. (a) $\forall n \in \mathcal{N}_k \quad D_n c(n) \in e_\chi \left(K(\mu_n)^\times \otimes \mathbb{Z}/M \right)^{G(n)}$

(b) — " — the map $K^\times \otimes \mathbb{Z}/M \xrightarrow{res} \left(K(\mu_n)^\times \otimes \mathbb{Z}/M \right)^{G(n)}$

satisfies $\mathrm{Ker}(res) = \mathrm{Ker}(res)^{(\omega)}$, $\mathrm{Coker}(res) = \mathrm{Coker}(res)^{(\omega)}$.

Cor. $\forall n \in \mathcal{N}_k \quad \exists! \ \boxed{\kappa(n) \in (K^\times \otimes \mathbb{Z}/M)^{(\chi)}}$ such that $res(\kappa(n)) = D_n c(n)$.

$[\ \kappa(1) = c(1) = e_\chi \ (1 - \zeta_p) \otimes 1\ ]$

Pf of (a). OK if $n = 1$. If $n = \ell m \quad (\ell \in P_k)$, then

$$(\sigma_\ell - 1) D_n c(n) = (\underbrace{(\ell - 1)}_{\equiv 0 \, (\mathrm{mod}\, M)} - N_\ell) D_{n/\ell} c(n) = - D_{n/\ell} N_\ell c(n) =$$

$$= (\underset{G(n/\ell)}{\underset{\cap}{\mathrm{Fr}(\ell)}} - 1) \underbrace{D_{n/\ell} c(n/\ell)}_{\text{fixed by } G(n/\ell), \text{ by induction}} = 0 .$$

Therefore $D_n c(n)$ is fixed by all σ_ℓ $(\ell | n)$ $\Rightarrow$ by $G(n)$.

Pf of (b). $H^1(K, \mu_M) \xrightarrow{res} H^1(K(\mu_n), \mu_M)^{G(n)}$

$\mathrm{Ker}(res) = H^1(G(n), \underbrace{\mu_M(K(\mu_n))}_{\mu_p})$, $\mathrm{Coker}(res) \subset H^2(G(n), \mu_p)$

$\mathrm{Gal}(K(\mu_n)/\mathbb{Q})$ is abelian $\Rightarrow \Delta$ acts on $H^i(G(n), \mu_p)$ via its action on μ_p) but $\mu_p = \mu_p^{(\omega)}$.

Localisation / factorisation of $x \in K^\times \otimes \mathbb{Z}/M$

Fix $v \nmid \infty$ prime of K,

$$\begin{tikzcd}
 & & K^\times \otimes \mathbb{Z}/M \ni x \arrow[d, "\mathrm{loc}_v"] \arrow[ddd, bend left] & & \\
0 \arrow[r] & \mathcal{O}^\times_{K_v} \otimes \mathbb{Z}/M \arrow[r] \arrow[d, "\wr"] & K_v^\times \otimes \mathbb{Z}/M \arrow[r, "\mathrm{ord}_v"] \arrow[d, "\wr\ \delta"] & \mathbb{Z}/M \arrow[r] \arrow[d, "\wr"] & 0 \\
0 \arrow[r] & H^1_f(K_v, \mu_M) \arrow[r] \arrow[d, equal] & H^1(K_v, \mu_M) \arrow[r] \arrow[d, equal] & H^1_{/f}(K_v, \mu_M) \arrow[r] \arrow[d, equal] & 0 \\
 & H^1_f & H^1 \ni x_v \arrow[r, mapsto] & x_v^s \in H^1_s = H^1/H^1_f &
\end{tikzcd}$$

$$\left(\mathrm{ord}_v(x) = \underset{\mathbb{Z}/M}{\overset{\uparrow}{x_v^s}} \right)$$

the "finite part" $\qquad\qquad$ the "singular part" of H^1

The finite-singular isomorphism for $\lambda | \ell \in P_k$

If $\ell \in P_k \Longrightarrow \ell$ splits completely in $K/\mathbb{Q}$; let $\lambda | \ell$ in K $(\Longrightarrow K_\lambda = \mathbb{Q}_\ell)$.

Canonical isomorphisms :

① $H^1_f(K_\lambda, \mu_M) \xleftarrow[\sim]{\delta} \mathcal{O}^\times_{K_\lambda} \otimes \mathbb{Z}/M \; (\Longleftarrow \mathbb{Z}_\ell^\times \otimes \mathbb{Z}/M) \xrightarrow[\sim]{\mathrm{red}_\lambda} \mathbb{F}_\ell^\times \otimes \mathbb{Z}/M \xleftarrow[\sim]{\chi_{\mathrm{cycl}}} \underbrace{G(\ell) \otimes \mathbb{Z}/M}_{\text{free of rk}=1 \text{ over } \mathbb{Z}/}$

$H^1_s(K_\lambda, \mu_M) := \frac{H^1}{H^1_f}(K_\lambda, \mu_M) \xleftarrow[\sim]{\delta} \mathbb{Z}/M$

$\Longrightarrow \underline{\phi_\lambda : H^1_f(K_\lambda, \mu_M) \xrightarrow{\sim} H^1_s(K_\lambda, \mu_M) \otimes G(\ell}$

② Cohomological version :

Convention : if F is a nonarchimedean local field, let Fr_F be the geometric Frobenius element and let the reciprocity map $\mathrm{rec}_F : F^\times \longrightarrow \mathrm{Gal}(F^{ab}/F) = G_F^{ab}$ be normalised by letting a uniformiser $\pi \in \mathcal{O}_F$ correspond to Fr_F ($\Longrightarrow$ sign change w.r.t. the classical definition, such as in [Cassels–Fröhlich]).

Ex :
$$\begin{tikzcd}
\mathbb{Z}_\ell^\times \arrow[r, equal] & \mathbb{Z}_\ell^\times \\
\mathbb{Q}_\ell^\times \arrow[u, phantom, "\cup"] \arrow[r, "\mathrm{rec}_{\mathbb{Q}_\ell}"] & G^{ab}_{\mathbb{Q}_\ell} \arrow[u, "\chi_{\mathrm{cycl}}"]
\end{tikzcd}$$
with our convention

$$H^1_f(K_\lambda, \mu_M) \underset{\substack{\uparrow \\ \lambda \nmid p}}{=} H^1_{ur}(K_\lambda, \mu_M) = \mathrm{Hom}_{\mathrm{cont}}(\langle \mathrm{Fr}(\lambda) \rangle, \mu_M(K_\lambda)) \xrightarrow[\sim]{ev_{\mathrm{Fr}(\lambda)}} \mu_M(K_\lambda) = \mu_M(\mathbb{Z}_\ell)$$
$$\wr\downarrow \mathrm{red}_\ell$$
$$\boxed{\mathbb{F}_\ell^\times[M]}$$

Compatibility with (1) : $\alpha \in \mathbb{Z}_\ell^\times$ $\quad \frac{\mathrm{Fr}(\lambda)(a^{1/M})}{a^{1/M}} \equiv a^{\frac{1-\ell}{\ell M}} \equiv a^{\frac{1-\ell}{M}} (\mathrm{mod}\, \ell)$

So
$$\begin{tikzcd}
 & \mathbb{F}_\ell^\times \otimes \mathbb{Z}/M \ni \alpha \arrow[d, "\wr"] \\
H^1_f \arrow[ur, "(1)", "\sim"'] \arrow[r, "(2)", "\sim"'] & \mathbb{F}_\ell^\times[M] \ni \alpha^{\frac{1-\ell}{M}}
\end{tikzcd}$$

$$H^1_s(K_\lambda, \mu_M) = H^1(I_\lambda, \mu_M)^{\langle \mathrm{Fr}(\lambda) \rangle} = \mathrm{Hom}_{\mathrm{cont}}(I_\lambda, \mu_M(K_\lambda))^{\langle \mathrm{Fr}(\lambda) \rangle} = \mathrm{Hom}(G(\ell), \mu_M(\mathbb{Z}_\ell))$$
$$\wr\downarrow \mathrm{red}_\ell$$
$$\mathrm{Hom}(G(\ell), \mathbb{F}_\ell^\times[M])$$
$$\downarrow\wr\ ev_{\sigma_\ell}$$
$$\mathbb{F}_\ell^\times[M]$$

Get again $\phi_\lambda : H^1_f \xrightarrow[\sim]{ev_{\mathrm{Fr}(\lambda)}} \mathbb{F}_\ell^\times[M] \xleftarrow{\sim} G(\ell) \otimes H^1_f$

Prime Factorisation of $\kappa(n)$

Goal: for $n \in \mathcal{N}_k$, determine the image

$$\begin{tikzcd}(K^\times \otimes \mathbb{Z}/M)^{(\chi)} \arrow[r] & (\mathrm{Div}(\mathcal{O}_K) \otimes \mathbb{Z}/M)^{(\chi)} \\ \kappa(n) \arrow[u, phantom, "\in" rotate=90] \arrow[r, mapsto] & \sum_v \mathrm{ord}_v(\kappa(n)) v \arrow[u, phantom, "\in" rotate=90]\end{tikzcd}$$

Prop 1. $\forall n \in \mathcal{N}_k$, $\forall v \nmid n$ in K $\quad \mathrm{ord}_v(\kappa(n)) = 0 \in \mathbb{Z}/M$.

Pf. If $w | v$ in $K(\mu_n)$, then $K(\mu_n)_w / K_v$ is unramified, hence

$\mathrm{ord}_v \, \kappa(n) = \mathrm{ord}_w \, D_n c(n) = 0 \in \mathbb{Z}/M$

$(\; c(n) \in \mathcal{O}^\times_{K(\mu_n)} \otimes \mathbb{Z}/M).$

Cor. $\forall n \in \mathcal{N}_k$ $\quad \mathrm{div}(\kappa(n)) \in \bigoplus_{\ell | n} \Big(\bigoplus_{\substack{\lambda | \ell \\ \text{in } K}} (\mathbb{Z}/M) \cdot \lambda \Big)^{(\chi)} \xleftarrow{\sim} \bigoplus_{\ell | n} (\mathbb{Z}/M) \cdot \lambda$

$(\ell \in \mathcal{P}_k)$

(for any choice of $\lambda | \ell$, since Δ acts simply transitively on $\{\lambda | \ell\}$)

Recall: $\forall \lambda | \ell | n$, $\quad \phi_\lambda : \underbrace{H^1_f(K_\lambda, \mu_M)}_{\mathcal{O}^\times_{K_\lambda} \otimes \mathbb{Z}/M} \xrightarrow{\sim} G(\ell) \otimes \underbrace{H^1_s(K_\lambda, \mu_M)}_{\mathbb{Z}/M} \quad (K_\lambda = \mathbb{Q}_\ell)$

Key Proposition (Kolyvagin): $\forall \lambda | \ell | n \in \mathcal{N}_k$, ~~$\sigma_\ell \otimes \mathrm{ord}_\lambda(\kappa(n)) = \phi_\lambda(\kappa(n/\ell))$~~

$$\boxed{\phi_\lambda(\kappa(n/\ell)_\lambda) = \pm \sigma_\ell \otimes \mathrm{ord}_\lambda(\kappa(n))}$$

(I am not completely sure about the sign with our definitions)

Pf: Later, in an abstract cohomological setting.

Rmks: (1) $\kappa(n/\ell)_\lambda \in H^1_f = \mathcal{O}^\times_{K_\lambda} \otimes \mathbb{Z}/M$, of Cor. above.

(2) **Reformulation**: for $x \in (K^\times \otimes \mathbb{Z}/M)^{(\chi)}$, let $x_\ell = (x_\lambda)_{\lambda | \ell}$,

$x^s_\ell = (x^s_\lambda)_\lambda$. Then $\quad \boxed{\phi_\ell(\kappa(n/\ell)_\ell) = \pm \sigma_\ell \otimes \kappa(n)^s_\ell}$

The transversal local condition (*)

Prop-Def. Let $\ell \in \mathcal{P}_k$, $\lambda | \ell$ in K. The transversal subgroup

$$H^1_{tr} := \mathrm{Ker}\big(\underbrace{K_\lambda^\times \otimes \mathbb{Z}/M}_{H^1(\underbrace{K_\lambda}_{\mathbb{Q}_\ell}, \mu_M)} \longrightarrow \underbrace{K_\lambda(\mu_\ell)^\times \otimes \mathbb{Z}/M}_{H^1(\underbrace{K_\lambda(\mu_\ell)}_{\mathbb{Q}_\ell(\mu_\ell)}, \mu_M)} \big) = H^1(G(\ell), \mu_M(\mathbb{Q}_\ell))$$

$\cap$

$H^1 := K_\lambda^\times \otimes \mathbb{Z}/M$

is free of rank 1 over $\mathbb{Z}/M$ and it defines a splitting of

$$\begin{tikzcd} 0 \arrow[r] & H^1_f \arrow[r] & H^1 \arrow[r] & H^1_s \arrow[r] & 0 \\ & & H^1_{tr} \arrow[u, hook] \arrow[ur, "\sim"] & & \end{tikzcd}$$

(*) Needed for the proof of (a variant of) Kolyvagin's structure thm for $A^{(\chi)}$, but not for the proof of Thm 2.

Pf :

$$\begin{tikzcd} H^1_{tr} \cap H^1_f = \mathrm{Ker}\big(\mathbb{Z}_\ell^\times \otimes \mathbb{Z}/M \arrow[r] \arrow[d, "\wr"', "red_\ell"] & \mathbb{Z}_\ell[\mu_\ell]^\times \otimes \mathbb{Z}/M\big) = 0 \arrow[d, "\wr"', "red_{1-\zeta_\ell}"] \\ \mathbb{F}_\ell^\times \otimes \mathbb{Z}/M \arrow[r, "id"] & \mathbb{F}_\ell^\times \otimes \mathbb{Z}/M \end{tikzcd}$$

$H^1_{tr} = \mathrm{Hom}(G(\ell), \mu_M(\mathbb{Q}_\ell))$ is free of $rk = 1$ over $\mathbb{Z}/M$.

Prop. $\forall \ell \in P_k$, $\forall \lambda | \ell$ $\boxed{\kappa(\ell)_\lambda \in H^1_{tr}(K_\lambda, \mu_M)}$ transversality property

Pf. The image of $\kappa(\ell)_\lambda$ in $H^1(\underbrace{K_\lambda(\mu_\ell)}_{K(\mu_\ell)_{\lambda'}}, \mu_M)$ is equal to

$$D_\ell \underbrace{c(\ell)}_{\substack{\uparrow \\ \mathcal{O}^\times_{K(\mu_\ell)} \otimes \mathbb{Z}/M}}{}_{\lambda'} \in \mathbb{Z}_\ell[\mu_\ell]^\times \otimes \mathbb{Z}/M \ \Big\downarrow \wr\ red_{1-\zeta_\ell} \ \mathbb{F}_\ell^\times \otimes \mathbb{Z}/M$$

($\lambda' | \lambda$ the unique prime above λ in $K(\mu_\ell)$), but σ_ℓ acts trivially on $\mathbb{F}_\ell^\times \otimes \mathbb{Z}/M$

$\Downarrow$

D_ℓ acts by $\sum_{i=0}^{\ell-2} i = \frac{(\ell-2)(\ell-1)}{2} \equiv 0 \,(\mathrm{mod}$

$\Longrightarrow D_\ell c(\ell)_{\lambda'} = 0$.

Why do we care about the transversality property?

Sometimes we want to know whether the localisation $x_\lambda \in H^1 = H^1(K_\lambda, \mu_M)$ of $x \in H^1(K, \mu_M)$ is zero, but we only know that its image $x_\lambda^s \in H^1/H^1_f = H^1_s$ is zero (e.g., if $x = \kappa(n)$ with $\ell | n$ and $\kappa(n/\ell)_\lambda = 0$).

If we knew ~~that~~ a priori $x_\lambda \in H^1_{tr}$, we would have

$$[\, x_\lambda = 0 \iff x_\lambda^s = 0 \,].$$

Problem: If $r \geq 2$, $\kappa(\ell_1 \cdots \ell_r)_{\ell_i}$ need not be in $H^1_{tr}(K_{\lambda_i}, \mu_M)$.

Remedy: (Mazur–Rubin, Kolyvagin Systems, App. A)

$\exists$ modified classes $\kappa'(n)$ $(n \in \mathcal{N}_k)$ s.t.

(1) $\kappa'(n) = \kappa(n)$ if $n = 1$ or $n = \ell \in P_k$

(2) $\kappa'(n) = \kappa(n) + \sum_{\substack{m|n \\ m \neq n}} \underset{\substack{\uparrow \\ \mathbb{Z}}}{A_{m,n}} \kappa(m)$ (explicit)

(3) $\forall \lambda | n$ $\kappa'(n)_\lambda \in H^1_{tr}(K_\lambda, \mu_M)$

(4) $\forall \ell \| n$ $\phi_\ell(\kappa'(n/\ell)) = \sigma_\ell \otimes \kappa(n)_\ell^s$

"**Pf**": **Step 1**: compute the f-projections $\kappa(n)_{\ell,f} \in H^1_f(K_\lambda, \mu_M)$ $(\ell \| n)$ along H^1_{tr}.

Step 2: cooking up $\kappa'(n)$ inductively using the formulas from Step 1.

Terminology: $\{\kappa(n)\}$ weak Kolyvagin system

$\{\kappa'(n)\}$ Kolyvagin system

We'll need this!!

Back to Thm 2

Recall: $p \neq 2$ prime, $M = p^k$ $(k >> 0)$, $\Delta = \mathrm{Gal}(K/\mathbb{Q}) \xrightarrow{\chi} \mathbb{Z}_p^\times$, $\chi \neq 1$, $\chi(c) = 1$

$K = \mathbb{Q}(\mu_p)$, $A = Cl_K \otimes \mathbb{Z}/M$ $(= Cl_K \otimes \mathbb{Z}_p$ if $k >> 0)$

$c(1) = e_\chi((1-\zeta_p) \otimes 1) \in (\mathcal{O}_K^\times \otimes \mathbb{Z}_p)^{(\chi)}$ free of $rk = 1$ over $\mathbb{Z}_p$

$\underline{c(1) \neq 0}$

Def: $p^{m_0} := [(\mathcal{O}_K^\times \otimes \mathbb{Z}_p)^{(\chi)} : \mathbb{Z}_p c(1)]$ $(m_0 \geq 0)$

Thm 2. $|A^{(\chi)}|$ divides p^{m_0}.

Notation: X $\mathbb{Z}_p$-module, $x \in X$, $\underline{\mathrm{ind}(x; X) := \sup\{t \geq 0 \mid x \in p^t X\}}$ $(t, s \in \mathbb{Z})$

$\underline{\exp(x; X) := \inf\{s \geq 0 \mid p^s x = 0\}}$, $\underline{\exp(X) := \sup \exp(x; X)}$

Ex: $X = (\mathbb{Z}/p^N\mathbb{Z})x_0$,

x_0

$p x_0$

$\vdots$

$p^N x_0 = 0$

$\underline{\mathrm{ind}(up^m x_0) = m}$

$\underline{\exp(up^m x_0) = N - m}$

$\begin{pmatrix} u \in (\mathbb{Z}/p^N)^\times \\ 0 \leq m \leq N \end{pmatrix}$

Above: $m_0 = \mathrm{ind}(c(1); (K^\times \otimes \mathbb{Z}_p)^{(\chi)}) = \mathrm{ind}(\kappa(1); (K^\times \otimes \mathbb{Z}/M)^{(\chi)})$

$= - \; '' \; — ; \; K^\times \otimes \mathbb{Z}/M)$

Main tool: reformulation of Kolyvagin's f-s property

$\forall n \in \mathcal{N}_k$, $n = l_1 \cdots l_r$

$$\boxed{\mathrm{div}(\kappa(n)) = \sum_{i=1}^{r} u_i p^{t_i}(e_\chi \lambda_i) \in (\mathrm{Div}(\mathcal{O}_K) \otimes \mathbb{Z}/M)^{(\chi)}, \quad u_i \in (\mathbb{Z}/M)^\times}$$

$$\boxed{t_i = \mathrm{ind}\,(\kappa(n/l_i)_{\lambda_i} ; \mathcal{O}_{K_{\lambda_i}}^\times \otimes \mathbb{Z}/M)}$$

$\lambda_i \mid l_i$ in K

We need to be able to control the values of t_i. This is done by a combination of Kummer theory and the Čebotarev density thm. This will be done inductively and will give relations of the form

$$\forall i \leq r \quad \mathrm{div}\,\kappa(l_1 \cdots l_i) = u_i p^{t_i}(e_\chi \lambda_i) + \sum_{1 \leq j < i} c_{ji}(e_\chi \lambda_j) \in (\mathrm{Div} \otimes \mathbb{Z}/M)^{(\chi)},$$

with $\underline{t_i = \mathrm{ind}(\kappa(l_1 \cdots l_{i-1})_{\lambda_i})}$. These relations will be divided by $p^{m(l_1 \cdots l_i)}$, where $m(n) = \mathrm{ind}\,\kappa(n)$. One will try to minimize the values of t_i and $m(l_1 \cdots l_i)$.

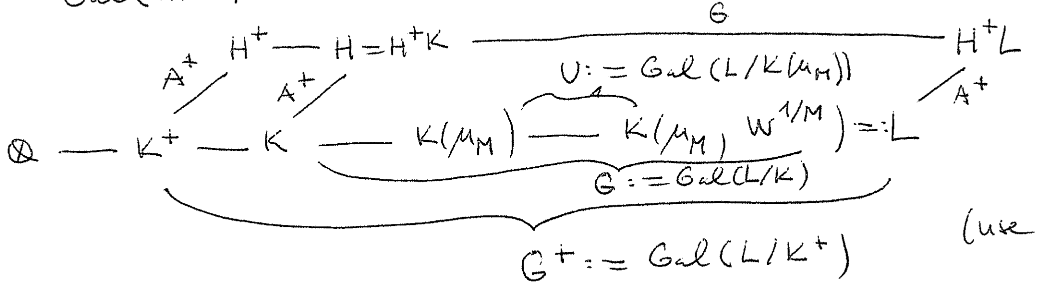
Kummer theory and Čebotarev

Data: $W \subset (K^+)^\times \otimes \mathbb{Z}/M \subset K^\times \otimes \mathbb{Z}/M$ finite subgroup.

$M = p^k$,

Field extensions: $H^+ :=$ the p-Hilbert class field of K^+, $H := KH^+$

$$\mathrm{Gal}(H/K) = \mathrm{Gal}(H^+/K^+) \simeq \underset{G}{Cl_{K^+} \otimes \mathbb{Z}_p} = (Cl_K \otimes \mathbb{Z}_p)^{c=1} = A^{c=1} =: A^+$$



$\mathrm{Gal}(H^+L/K^+) =$
$= G^+ \times A^+$

(use Lemma below)

(a) **Kummer pairing**: $U \times W \longrightarrow \mu_M$

$$(g, \alpha) \longmapsto g(\alpha^{1/M})/\alpha^{1/M} \quad (= (\delta\alpha)(g))$$

$$\delta : K^\times \otimes \mathbb{Z}/M \xrightarrow{\sim} H^1(K, \mu_M)$$

Induces an ~~[illegible]~~ isomorphism $\boxed{U \xrightarrow{\sim} \mathrm{Hom}(W, \mu_M)}$

(b) **cyclotomic character**:

$$\begin{tikzcd} G^+/U \arrow[r, "\chi_{cycl}", "\sim"'] & \{a \in (\mathbb{Z}/M)^\times \mid a \equiv \pm 1 \pmod p\} \\ G/U \arrow[u, hook] \arrow[r, "\sim"] & \{ \text{— " —} \mid a \equiv 1 \pmod p\} \end{tikzcd}$$

(c) $\rho : G^+ (\simeq U \rtimes G^+/U) \xrightarrow{\sim} \left\{ \begin{pmatrix} a & b \\ 0 & id_W \end{pmatrix} \middle| \begin{array}{l} a \in (\mathbb{Z}/M)^\times,\ a \equiv \cancel{\pm} 1 \pmod p \\ b \in \mathrm{Hom}(W, \mu_M) \simeq \mathrm{Hom}(W, \mathbb{Z}/M) \\ (\text{fix } \mu_M \simeq \mathbb{Z}/M) \end{array} \right.$

$$g \longmapsto \begin{pmatrix} \chi_{cycl}(g) & \alpha \mapsto \frac{g(\alpha^{1/M})}{\alpha^{1/M}} \\ 0 & id_W \end{pmatrix}$$

Lemma: $L \cap (K^+)^{ab} = K(\mu_M)$ $\left(\xRightarrow{\text{ramification}} \underline{L \cap H = K}\right)$

Pf: $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}^{-1}\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -2b/a \\ 0 & 1 \end{pmatrix} \Longrightarrow U \subset [G^+, G^+]$ $(p \neq 2$

Prop. $\forall\, C \in (Cl_K \otimes \mathbb{Z}_p)^{c=1} = Cl_{K^+} \otimes \mathbb{Z}_p$ $(= Cl_{K^+} \otimes \mathbb{Z}/M$ if $k \gg 0)$

$\forall\, \alpha \in \mathrm{Hom}(W, \mu_M) \simeq U$ $\quad (\mu_M = \mu_M(K(\mu_M)))$

$\exists$ infinitely many primes λ of K such that

(1) $\lambda \mid \ell$, ℓ prime number that splits completely in $K(\mu_M)/\mathbb{Q}$ $(\Leftrightarrow \ell \in P_k)$ and is unramified in $L/\mathbb{Q}$

(2) $C = [\lambda]$ (the ideal class of λ)

(3) $\exists\, \lambda' \mid \lambda$ in L such that $\mathrm{Fr}_{L/\mathbb{Q}}(\lambda') \in U \subset \mathrm{Gal}(L/\mathbb{Q})$

"is" α under the Kummer map

$$\begin{tikzcd} \mathrm{Fr}_{L/\mathbb{Q}}(\lambda') \arrow[d, mapsto] & \in U \arrow[d, "\wr"] \\ \alpha & \in \mathrm{Hom}(W, \mu_M) \end{tikzcd}$$

$$(\Longleftrightarrow \forall w \in W \quad \underbrace{(\delta w)(\mathrm{Fr}_{L/\mathbb{Q}}(\lambda'))}_{ev_{Fr(\lambda')}(\delta w)} = \alpha(w))$$

$$\begin{array}{c} \mathrm{Gal}(LH^+/K \\ \cup \\ U \times A^+ \end{array}$$

Pf. Čebotarev for $H^+L/\mathbb{Q} \Longrightarrow \exists_\infty\ \tilde{\lambda}$ in H^+L s.t. $\mathrm{Fr}_{LH^+/\mathbb{Q}}(\tilde{\lambda}) = (\alpha, C)$

Take $\lambda = \tilde{\lambda}|_K$.

Cor. Given: (a) $a_1, \dots, a_m \in (K^+)^\times \otimes \mathbb{Z}/M$ ~~such~~ "independent" in the following sence: the subgroups $W_i := (\mathbb{Z}/M)\cdot a_i \subset (K^+)^\times \otimes \mathbb{Z}/M$ satisfy $W := \sum_1^m W_i = \bigoplus_1^m W_i$;

(b) integers $s_1, \dots, s_m \geq 0$; (c) $C \in Cl_{K^+} \otimes \mathbb{Z}/M = (Cl_K \otimes \mathbb{Z}/M)^{c=1}$ $(k \gg 0)$,

then $\exists \infty$ prime numbers $\underbrace{\ell \in P_k}_{(\ell \text{ splits complety in } K(\mu_M)/\mathbb{Q})}$ unramified in $K(\mu_M, \sqrt[M]{a_1}, \dots, \sqrt[M]{a_m})$ such that

- $\exists \lambda \mid \ell$ in K such that $C = [\lambda]$
- $\forall i = 1, \dots, m$ $\quad \operatorname{ind}((a_i)_\lambda ; \mathcal{O}_{K_\lambda}^\times \otimes \mathbb{Z}/M) = \operatorname{ind}(a_i ; (K^+)^\times \otimes \mathbb{Z}/M) + s_i$.

Pf: (a) $\Rightarrow \exists\, \alpha^{\#} : W \longrightarrow \mu_M$ s.t. $\operatorname{ind}(\alpha^{\#}(a_i)) = \operatorname{ind}(a_i) + s_i \quad \forall i$.

~~$\mathbb{Z}/M$ is injective as a $\mathbb{Z}/M$-module $\Rightarrow \exists\, \alpha$ extending $\alpha^{\#}$~~

Applg Prop. above to this α.

Warm up – annihilation results for $A^{(\chi)} (= (Cl_K \otimes \mathbb{Z}/M)^{(\chi)}$, $\ell M = p^k$, $k \gg 0$)

Prop 1. $\boxed{p^{m_0} A^{(\chi)} = 0}$ $\quad (\Longrightarrow \exp(A^{(\chi)}) \leq m_0)$.

Pf. Let $C \in A^{(\chi)} := (Cl_K \otimes \mathbb{Z}_p)^{(\chi)}$. Čebotarev for $\mathbb{Q} \subset K(\mu_M, \kappa(1)^{1/M})^{H^+}$

$$\Longrightarrow \exists \ell \in P_k \ \exists \lambda | \ell \text{ in } K \text{ such that } \left\{ \begin{array}{l} e_\chi[\lambda] = C \\ \operatorname{ind} \kappa(1)_\ell = \underbrace{\operatorname{ind} \kappa(1)}_{m_0} \end{array} \right\}.$$

$$\Longrightarrow \operatorname{ind} \kappa(\ell)^s_\ell = \operatorname{ind} \kappa(1)_\ell = m_0$$

$$\Longrightarrow \operatorname{div} \kappa(\ell) = u p^{m_0} (e_\chi \lambda) \in (\operatorname{Div}(O_K) \otimes \mathbb{Z}/M)^{(\chi)}, \quad u \in (\mathbb{Z}/M)^\times$$

$$\longrightarrow p^{m_0} C = p^{m_0} e_\chi[\lambda] = 0 \in (Cl(O_K) \otimes \mathbb{Z}/M)^{(\chi)}, \text{ hence}$$

$p^{m_0} (Cl(O_K) \otimes \mathbb{Z}/M)^{(\chi)} = 0$. If $M = p^k$, $k > m_0$

$\Longrightarrow p^{m_0} A^{(\chi)} = 0$ (since $A^{(\chi)}$ is finite).

Rmk. If we didn't know a priori that $|(Cl(O_K) \otimes \mathbb{Z}_p)^{(\chi)}| < \infty$ but only that $A^{(\chi)} = (Cl(O_K) \otimes_{\mathbb{Z}} \mathbb{Z}_p)^{(\chi)}$ is a $\mathbb{Z}_p$-module of finite type ($\simeq (\mathbb{Z}_p^r \oplus (\text{finite}))$), the previous argument shows that

$\forall k \gg 0 \quad p^{m_0} (A^{(\chi)} \otimes \mathbb{Z}/p^k) = 0 \Longrightarrow r = 0 \Rightarrow |A^{(\chi)}| < \infty$.

Def. $[\ell]$ $(\ell \in P_k)$ has good localisation if for one $\lambda | \ell$ in K ($\Longleftrightarrow$ for all $\lambda | \ell$) $\quad \operatorname{ind} \kappa(1)_\lambda = \underbrace{\operatorname{ind} \kappa(1)}_{m_0}$.

Prop 2. If $[\ell]$ $(\ell \in P_k)$ has good localisation, then $m(\ell) := \operatorname{ind} \kappa(\ell)$ $\left\{\begin{array}{l}\text{in } (K^\times \otimes \mathbb{Z}/M)^{(\chi)} \\ \text{or in } K^\times \otimes \mathbb{Z}/M\end{array}\right.$ satisfies $m(\ell) \leq m_0$ and

$p^{m_0 - m(\ell)} e_\chi[\lambda] = 0 \in A^{(\chi)}$ (if $k \gg 0$).

Pf. $m(\ell) \leq \operatorname{ind} \kappa(\ell)^s_\lambda = \operatorname{ind} \kappa(1)_\lambda = \operatorname{ind} \kappa(1) = m_0$

Fix $y \in (K^\times \otimes \mathbb{Z}/M)^{(\chi)}$ such that $y^{p^{m(\ell)}} = \kappa(\ell)$; its image in $(K^\times \otimes \mathbb{Z}/M')^{(\chi)}$ $(M' = Mp^{-m_0})$ (also denoted by y) will then be canonical.

We have $\operatorname{div}(y) = u p^{m_0 - m(\ell)} (e_\chi \lambda) \in (\operatorname{Div}(O_K) \otimes \mathbb{Z}/M')^{(\chi)} \Longrightarrow$ result

---

Cor-Def. $m_1 := \inf\{ m(\ell) \mid \ell \in P_k \ (k \gg 0 \text{ fixed}),\ \operatorname{ind} \kappa(1)_\lambda = \operatorname{ind} \kappa(1) \}$ is well-defined (a priori it may depend on k), $m_1 \leq m_0$ and $\quad p^{m_0 - m_1} A^{(\chi)} = 0. \quad (\Longrightarrow \exp(A^{(\chi)}) = m_0 - m_1)$.

Pf: Čebotarev: $\forall C \in A^{(\chi)} \ \exists \ell \in P_k$ such that $[\ell]$ has good loc. and $\exists \lambda | \ell \ C = e_\chi[\lambda]$. ~~Prop~~ Then

$$p^{m_0 - m_1} C = p^{m(\ell) - m_1} \cdot \underbrace{p^{m_0 - m(\ell)} e_\chi[\chi]}_{0} = 0 \in \underbrace{A^{(\chi)} \otimes \mathbb{Z}/M'}_{A^{(\chi)} \text{ if } k \geq 2m_0}.$$

Improved version – determining the exponent of $A^{(\chi)}$.

the Proof will use the transversal subgroup !

Prop 3. If $[\ell]$ $(\ell \in P_k, k \gg 0)$ has good localisation, then
$\exp(e_\chi[\lambda]; A^{(\chi)}) = m_0 - m(\ell)$.

Cor. $\exp(A^{(\chi)}) = m_0 - m_1$ $(\Longrightarrow m_1$ does not depend on $k \gg 0)$.

Pf. Again, use Čebotarev for $\mathbb{Q} \subset K(\mu_M, \kappa(1)^{1/M})H^+$.

Pf of Prop 3. Notation (for any $N \geq 1$) : $K^\times \otimes \mathbb{Z}/p^N \longrightarrow K^\times \otimes \mathbb{F}_p$
$a \longmapsto \overline{a}$

We know that $\underline{\operatorname{div}(y_1) = u\, p^{m_0 - m(\ell)}(e_\chi \lambda)} \in (\operatorname{Div}(O_K) \otimes \mathbb{Z}/M')^{(\chi)}$, $M' = \frac{M}{p^{m_0}}$
$y_1^{p^{m(\ell)}} = \kappa(\ell)$ $\qquad u \in (\mathbb{Z}/M')^\times$

If $\exp(e_\chi[\lambda], A^{(\chi)}) < m_0 - m(\ell)$, then $\operatorname{div}(y_1) = p \operatorname{div}(z)$ for some z

$\Longrightarrow$ $\underline{\overline{y_1} \in (O_K^\times \otimes \mathbb{F}_p)^{(\chi)} \subset (K^\times \otimes \mathbb{F}_p)^{(\chi)}}$

However, if $y_0^{p^{m_0}} = \kappa(1)$, then $(O_K^\times \otimes \mathbb{F}_p)^{(\chi)} = \mathbb{F}_p \cdot \overline{y_0}$. So it is enough to prove ($\overline{y_0} \neq 0 \in (O_K^\times \otimes \mathbb{F}_p)^{(\chi)}$).

Lemma: If $[\ell]$ has good localisation $(\ell \in P_k, k \gg 0)$, then
(★1) $\overline{y_0}, \overline{y_1}$ are linearly independent over $\mathbb{F}_p$ in $(K^\times \otimes \mathbb{F}_p)^{(\chi)}$.

Pf of Lemma. (fix $\lambda | \ell$ in K) Use $H^1 := \underbrace{H^1(K_\lambda, \mu_M)}_{K_\lambda^\times \otimes \mathbb{Z}/M} = \underbrace{H^1_f}_{O_{K_\lambda}^\times \otimes \mathbb{Z}/M} \oplus H^1_{tr}$, $\underbrace{H^1_f, H^1_{tr}}_{\text{free of } rk=1 \text{ over } \mathbb{Z}/M}$

We know : (α) $\operatorname{ind}((y_0)_\lambda, H^1_f) = \operatorname{ind} \kappa(1)_\lambda - m_0 = 0 \Longrightarrow (\overline{y_0})_\lambda \neq 0 \in H^1_f/p \simeq \mathbb{F}_p$

(b) $\underline{\kappa(\ell)_\lambda \in H^1_{tr}} \Longrightarrow (\overline{y_1})_\lambda \in H^1_{tr}/p \simeq \mathbb{F}_p$, (c) $\underline{H^1_f/p \cap H^1_{tr}/p = 0}$

So, if $\overline{y_0}^a\, \overline{y_1}^b = 1 \in (K^\times \otimes \mathbb{F}_p)^{(\chi)}$ $(a, b \in \mathbb{F}_p) \Longrightarrow (\overline{y_0})_\lambda^a (\overline{y_1})_\lambda^b = 1$

$\overset{(c)}{\Longrightarrow}$ $(\overline{y_0})_\lambda^a = (\overline{y_1})_\lambda^{-b} = 1 \overset{(a)}{\Longrightarrow} \underline{a = 0 \in \mathbb{F}_p} \Longrightarrow \overline{y_1}^b = 1 \in (K^\times \otimes \mathbb{F}_p)^{(\chi)}$
$\Downarrow$ $\overline{y_1} \neq 0$ by def.
$\underline{b = 0 \in \mathbb{F}_p}$.

Rmk : the notation in the proof is a mess.
One writes the groups involved both additively and multiplicatively, with 0 becoming 1 ...

Proof of Thm 2 ($|A^{(\chi)}|$ divides p^{m_0})

Fix $M = p^k,\ k \gg 0$ (? $k > 2m_0$?)

Def. For $n \in \mathcal{N}_k$, let $m(n) := \mathrm{ind}\,(\kappa(n), (K^\times \otimes \mathbb{Z}/M)^{(\chi)})$

Def. A sequence $[l_1, \dots, l_r]$ ($r \geq 0$, $l_i \in P_k$ distinct) has good localisations if $\forall i = 0, \dots, r-1$ $\mathrm{ind}\,(\kappa(l_1 \cdots l_i)_{l_{i+1}}) = \underbrace{\mathrm{ind}\,(\kappa(l_1 \cdots l_i))}_{m(l_1 \cdots l_i)}$

Lemma. Given $[l_1, \dots, l_r]$ ($r \geq 0$) with good localisations and $C \in A^{(\chi}$ $\exists$ infinitely many $l_{r+1} \in P_k$ such that $[l_1, \dots, l_{r+1}]$ has good loc. and $\exists\ \lambda_{r+1} | l_{r+1}$ in K $C = e_\chi[\lambda_{r+1}]$.

Pf. Čebotarev for $\mathbb{Q} \subset K(\mu_M, \kappa(l_1 \cdots l_r)^{1/M})$ H^+.

Properties of $[l_1, \dots, l_r]$ ($r \geq 1$) with good localisations:

(1) $m(l_1 \cdots l_r) = \mathrm{ind}\,\kappa(l_1 \cdots l_r) \leq \mathrm{ind}\,\kappa(l_1 \cdots l_r)^s_{l_r} = \mathrm{ind}\,\kappa(l_1 \cdots l_{r-1})_{l_r} =$

$= \mathrm{ind}\,\kappa(l_1 \cdots l_{r-1}) = m(l_1 \cdots l_{r-1}) \leq \dots \leq m(l_1 l_2) \dots \leq m(l_1) \leq$

(2) $\forall i = 0, \dots, r$ get $y_i \in (K^\times \otimes \mathbb{Z}/M')^{(\chi)}$, $y_i^{p^{m(l_1 \cdots l_i)}} = \kappa(l_1 \cdots l_i)$ $(M' = \frac{M}{p^{m_0}}$

In $(\mathrm{Div}(\mathcal{O}_K) \otimes \mathbb{Z}/M')^{(\chi)}$, $\forall i \geq 1$

$$\mathrm{div}(y_i) = \underset{\substack{\uparrow \\ (\mathbb{Z}/M')^\times}}{u_i} \underbrace{p^{\overbrace{\mathrm{ind}\,\kappa(l_1 \cdots l_i)^s_{l_i}}^{m(l_1 \cdots l_{i-1})} - m(l_1 \cdots l_i)}}_{p^{n_i'}} (e_\chi \lambda_i) + \sum_{1 \leq j < i} \underset{\substack{\uparrow \\ \mathbb{Z}/M'}}{e_{ij}} (e_\chi \lambda_j)$$

(3) Def: $X_i :=$ the subgroup of $\underbrace{A^{(\chi)} \otimes \mathbb{Z}/M'}$ generated by $e_\chi[\lambda_j]$, $1 \leq j \leq i$

then $\forall i = 1, \dots, r$ $X_i / X_{i-1} =$ gen. by the image of $e_\chi[\lambda_i]$

(2) $\Longrightarrow$ $X_i / X_{i-1} \simeq \mathbb{Z}/p^{n_i}\mathbb{Z}$, $n_i \leq n_i' = m(l_1 \cdots l_{i-1}) - m(l_1 \cdots l_i)$

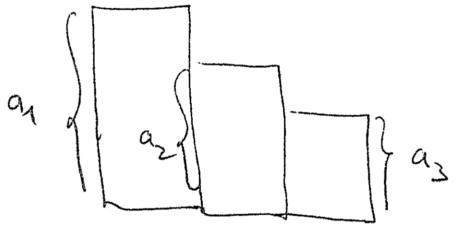
(4) $|X_r| = p^{n_1 + \dots + n_r}$ divides $p^{n_1' + \dots + n_r'} = p^{m_0 - m(l_1 \cdots l_r)}$ $\Longrightarrow$ divides p^{m_0}.

Pf of Thm 2: Lemma above $\Longrightarrow$ $\exists\ [l_1, \dots, l_r]$ with good loc. such that $X_r = A^{(\chi)} \otimes \mathbb{Z}/M' \overset{(4)}{\Longrightarrow} |\underbrace{A^{(\chi)} \otimes \mathbb{Z}/M'}_{A^{(\chi)} \text{ if } k > 2m_0}|$ divides p^{m_0}.

Rmk: Without using the finiteness of $A = Cl_K \otimes \mathbb{Z}_p$, the above arguments show that $\forall k \gg 0$ every finitely generated subgroup of $A^{(\chi)} \otimes \mathbb{Z}/p^k$ has order dividing p^{m_0} $\Longrightarrow$ $\forall k \geq 0$ $|A^{(\chi)} \otimes \mathbb{Z}/p^k|$ divides p^{m_0}.

$$\boxed{\text{Structure of } A^{(\chi)} \simeq \bigoplus_{i=1}^{d} (\mathbb{Z}/p^{a_i}\mathbb{Z})\, c_i \quad , \quad a_1 \geq \cdots \geq a_d > 0}$$

Goal:



etc.

$c_1 = e_\chi[\lambda_1]$ $\;c_2\;$ $c_3 = e_\chi[\lambda_3] + a_{31} e_\chi[\lambda_1] + a_{32} e_\chi[\lambda_2]$

Building $A^{(\chi)} \otimes \mathbb{Z}/M'$ from below

Reformulation of the proof of Thm 2: assume that $[\ell_1, \ldots, \ell_r]$ $(r \geq 1)$ $(\ell_i \in P_k)$ has good localisations

$\kappa(\ell_1 \cdots \ell_i) = y_i^{p^{m(\ell_1 \cdots \ell_i)}}$, $\quad m(\ell_1 \cdots \ell_i) = \operatorname{ind} \kappa(\ell_1 \cdots \ell_i)$

$y_i \in (K^\times \otimes \mathbb{Z}/M')^{(\chi)}$, $\quad M' = M/p^{m_0} = p^{k-m_0}$. Diagram:

$$\begin{tikzcd}
\langle y_0, \ldots, y_r \rangle \arrow[r, "\alpha_r"] \arrow[d, phantom, "\cap"] & \bigoplus_{i=1}^{r} (\mathbb{Z}/M') \cdot (e_\chi \lambda_i) \arrow[r] \arrow[d, phantom, "\cap"] & \operatorname{Coker}(\alpha_r) \arrow[r] \arrow[d] & 0 \\
(K^\times \otimes \mathbb{Z}/M')^{(\chi)} \arrow[r, "\operatorname{div}"] & (\operatorname{Div}(O_K) \otimes \mathbb{Z}/M')^{(\chi)} \arrow[r] & A^{(\chi)} \otimes \mathbb{Z}/M' \arrow[r] & 0
\end{tikzcd}$$

Recall: $(O_K^\times \otimes \mathbb{Z}/M')^{(\chi)} = (\mathbb{Z}/M') \cdot y_0$ $\quad (\Longrightarrow \overline{y_0} \neq 0$ in $(K^\times \otimes \mathbb{F}_p)^{(\chi)})$

Equivalent conditions (which may or may not be satisfied)

$\bigstar_r$

$\overline{y_0}, \ldots, \overline{y_r}$ are linearly independent over $\mathbb{F}_p$ in $(K^\times \otimes \mathbb{F}_p)^{(\chi)}$

$\Updownarrow$

$\overline{y_1}, \ldots, \overline{y_r}$ ———— " ———— in $((K^\times/O_K^\times) \otimes \mathbb{F}_p)^{(\chi)}$

$\Updownarrow$

the left square in the diagram is cartesian

$\Updownarrow$

$\operatorname{Coker}(\alpha_r) \longrightarrow A^{(\chi)} \otimes \mathbb{Z}/M'$ is injective.

Properties: (0) Def. $\begin{matrix} Y_i \subset \operatorname{Coker}(\alpha_r) \\ X_i \subset A^{(\chi)} \otimes \mathbb{Z}/M' \end{matrix} \Big\} :=$ the subgroup generated by the classes of $(e_\chi \lambda_j)$, $1 \leq j \leq i$

$(0 \leq i \leq r)$

(1) surjections $Y_i \twoheadrightarrow X_i$, $\;Y_i/Y_{i-1} \twoheadrightarrow X_i/X_{i-1}$
(isomorphisms if $\bigstar_r$ holds)

(2) $Y_i/Y_{i-1} \simeq (\mathbb{Z}/p^{n_i'}\mathbb{Z})(e_\chi \lambda_i)$, $\quad n_i' = m(\ell_1 \cdots \ell_{i-1}) - m(\ell_1 \cdots \ell_i)$

We have proved $(\star_1)$ using $\underline{\kappa(\ell)_\lambda \in H^1_{tr}}$.

<u>We want</u>: impose conditions implying $(\star_r)$ (to get the structure of all X_i/X_{i-1}, not just of Y_i/Y_{i-1}).

<u>I do not know how to do this if $r>1$.</u>

<u>Modified version</u>: use Mazur – Rubin, Kolyvagin Systems, App. A to replace $\{\kappa(n)\}$ by $\{\kappa'(n)\}$ such that

- $\kappa'(n) = \kappa(n)$ if $n=1$ or $n=\ell \in \mathcal{P}_k$
- $\forall \lambda \nmid n$ $\kappa'(n)_\lambda \in H^1_{tr}(K_\lambda, \mu_M)$
- $\forall \lambda | \ell | n$ $\phi_\lambda(\kappa'(n/\ell)_\lambda) = \pm \sigma_\ell \otimes \kappa'(n)^s_\lambda$
- $\kappa'(n) = \kappa(n) + \sum\limits_{\substack{m|n \\ m\neq n}} \underset{\substack{\wedge \\ \mathbb{Z}/M'}}{A_{n,m}} \kappa(m)$

Replace in the previous discussion $\kappa(n)$ by $\kappa'(n)$: obtain the notion of <u>good' localisations</u> of ~~a~~ $[\ell_1, \dots, \ell_r]$, $m'(n) = \operatorname{ind} \kappa'(n)$, $(y_i')^{p^{m'(\ell_1\cdots\ell_i)}} = \kappa'(\ell_1\cdots\ell_i)$, α_r', $X_i' = X_i$, Y_i'.

<u>Def</u>. $[\ell_1, \dots, \ell_r]$ ($r \geq 0$, $\ell_i \in \mathcal{P}_k$ distinct, $k \gg 0$) has <u>very good' localisations</u> if

(1) $\operatorname{ind} \kappa'(\ell_1\cdots\ell_i)_{\ell_{i+1}} = \operatorname{ind} \kappa'(\ell_1\cdots\ell_i)$ $(0 \leq i < r)$ (<u>good' loc.</u>)

(2) $\operatorname{ind} \kappa'(\ell_1\cdots\ell_j)_{\ell_{i+1}} > \operatorname{ind} \kappa'(\ell_1\cdots\ell_j)$ $(0 \leq j < i < r)$

<u>Prop</u>. Assume $r \geq 0$ and that $[\ell_1, \dots, \ell_r]$ has very good' localisations.

~~(1) $[\ell_1, \dots, \ell_r]$ has very good' loc. $\Rightarrow$ $(\bar{Y}_j)_{\ell_{i+1}} \begin{cases} \neq 0 \\ = 0 \end{cases}$~~

(1) If $0 \leq j \leq i < r$, then $(\overline{Y}_j')_{\ell_{i+1}} \begin{cases} \neq 0 & j=i \\ = 0 & j<i \end{cases}$ in $H^1_f(K_{\lambda_{i+1}}, \mu_p)$

(2) If $0 \leq i < j \leq r$, then $(Y_j')_{\ell_{i+1}} \in \underbrace{H^1_{tr}(K_{\lambda_{i+1}}, \mu_p)}_{\simeq \mathbb{F}_p}$

(3) $\overline{Y}_0', \dots, \overline{Y}_r'$ are linearly independent ~~over~~ $\mathbb{F}_p$ in $(K^\times \otimes \mathbb{F}_p)^{(\chi)}$ $(\star_r')$

<u>Pf</u>: (1), (2) by def. (3) <u>$r=0$</u>: we know that $\overline{Y}_0' = \overline{Y}_0 \neq 0$. If $r>0$, assume $(\star_{r-1}')$.

Enough to consider $\overline{Y}_r' = \prod\limits_{i=0}^{r-1} (\overline{Y}_i')^{b_i}$, $b_i \in \mathbb{F}_p$.

<u>localise at ~~λ_r~~ ℓ_r</u>: $\underset{H^1_{tr}}{\ni} (\overline{Y}_r')_{\ell_r} \overset{(1)}{=\!=} (\overline{Y}_{r-1}')^{b_{r-1}}_{\ell_r} \in H^1_f \Rightarrow (\overline{Y}_{r-1}')^{b_{r-1}}_{\ell_r} = 1 \overset{(1)}{\Longrightarrow} b_{r-1} = 0 \in \mathbb{F}_p$

<u>localise at ℓ_{r-1}</u> $\Rightarrow b_{r-2} = 0 \in \mathbb{F}_p$, etc. $\Longrightarrow$ $\forall i$ $b_i = 0$ $\Longrightarrow$ $\underbrace{\overline{Y}_r' = 1 \in (K^\times \otimes \mathbb{F}_p)^{(\chi)}}_{\text{false}}$.

Cor. 1. If $[l_1, \dots, l_r]$ $(r \geq 0)$ has very good' loc., then

$0 = X_0 \subset X_1 \subset \cdots \subset X_r \subset A^{(x)} \otimes \mathbb{Z}/M'$ satisfy $X_i/X_{i-1} \simeq \mathbb{Z}/p^{m'(l_1\cdots l_{i-1})-m(l_1\cdots l_i)}\mathbb{Z}$ $(1 \leq i \leq r)$

In order to get full $A^{(x)} \otimes \mathbb{Z}/M'$, we need to impose minimality condition on each $m'(l_1 \cdots l_i)$.

Cor. 2. If $[l_1, \dots, l_r]$ $(r \geq 0)$ has very good' loc. ~~then~~
if $c \in A^{(x)}$ $\Big\}$ $\Longrightarrow$

$\exists$ infinitely many $l_{r+1} \in P_k$ such that $[l_1, \dots, l_{r+1}]$ has very good' loc and $\exists\ \lambda_{r+1} | l_{r+1}$ in K $c = e_x[\lambda_{r+1}]$

Pf. $\textcircled{\star'_r} \Longrightarrow \sum_{i=0}^{r} \langle y'_i \rangle = \bigoplus_{i=0}^{r} \langle y'_i \rangle \subset (K^\times \otimes \mathbb{Z}/M')^{(x)}$

$\Longrightarrow$ idem for $\sum_{i=0}^{r} \langle k'(l_1 \cdots l_i) \rangle = \bigoplus_{i=0}^{r} \langle k'(l_1 \cdots l_i) \rangle \subset (K^\times \otimes \mathbb{Z}/M)^{(x)}$.

Apply Čebotarev for $\mathbb{Q} \subset K(\mu_M, k'(l_1 \cdots l_i)^{1/M}\ (0 \leq i \leq r))H^+$.

Def. $[l_1, \dots, l_r]$ is admissible'
- always if $r = 0$
- $(r > 0)$ if $[l_1, \dots, l_r]$ has very good' loc., $[l_1, \dots, l_{r-1}]$ is admissible' and $m'(l_1 \cdots l_r) = \underbrace{\inf\{m'(l'_1 \cdots l'_r) \mid [l'_1, \dots, l'_r] \text{ satisfies }\}}_{m'_r}$

Ex: (1) $[l]$ is admissible' $\Longleftrightarrow$ $[l]$ has (very) good' loc.
AND
$m'(l) = \underbrace{\inf\{m'(l'_1) \mid [l'_1] \text{ has (very) good' loc.}\}}_{m'_1}$

(2) $[l_1, l_2]$ is admissible' $\Longleftrightarrow$ $[l_1, l_2]$ has very good' loc., $m'(l_1) = m'_1$
$m'(l_1 l_2) = \underbrace{\inf\{m'(l'_1 l'_2) \mid [l'_1, l'_2] \text{ satisfy }\}}_{m'_2}$

Prop. $\forall r \geq 0$ admissible' $[l_1, \dots, l_r]$ exist $\Longrightarrow$ $\forall r \geq 0$ m'_r is defined, (~~$\Longrightarrow$~~ $m'_0 \geq m'_1 \geq m'_2 \geq \cdots$, as before).

Pf. Apply Cor. 2 above.

Kolyvagin's structure Thm (slightly modified – he stated it (Euler systems, Thm 7) with $m_0 \geq m_1 \geq \ldots$ and $\{\kappa(n)\}$, but I do not know how to prove $\textcircled{$\star_r$}$ in this case for $r > 1$).

(1) If $m'_0 > m'_1 > \cdots > m'_d = m'_{d+1}$, then $m'_d = m'_{d+1} = m'_{d+2} = \cdots =: m'_\infty$.

(2) $\forall$ admissible' $[\ell_1, \ldots, \ell_d]$ $0 = X_0 \subset X_1 \subset \cdots \subset X_d$
($X_i = \langle e_\chi[\lambda_1], \ldots, e_\chi[\lambda_i] \rangle$) are <u>direct summands</u> in $A^{(\chi)} \otimes \mathbb{Z}/M'$,
$X_d = A^{(\chi)} \otimes \mathbb{Z}/M'$ and $X_i / X_{i-1} \simeq \mathbb{Z}/p^{m'_{i-1} - m'_i}\mathbb{Z}$.

(3) $A^{(\chi)} \simeq \bigoplus_{i=1}^{d} \mathbb{Z}/p^{a_i}\mathbb{Z}$, $a_1 \geq a_2 \geq \cdots \geq a_d > a_{d+1} = \cdots = 0$
$\forall i \geq 1$ $\boxed{a_i = m'_{i-1} - m'_i}$

(4) $|A^{(\chi)}| = p^{m'_0 - m'_\infty} = p^{m_0 - m'_\infty}$.

<u>Pf</u>. Enough to show

<u>Prop.</u> $\forall r \geq 0$ we have

(A_r) $\forall i = 1, \ldots, r$ $a_i = m'_{i-1} - m'_i$.

(B_r) $\forall$ admissible' $[\ell_1, \ldots, \ell_r]$ $X_1 \subset \cdots \subset X_r$ are direct summands in $A^{(\chi)} \otimes \mathbb{Z}/M'$ and $\underbrace{(A^{(\chi)} \otimes \mathbb{Z}/M')/X_r}$ $\simeq \bigoplus_{j > r} \mathbb{Z}/p^{a_j}\mathbb{Z}$.

(C_r) $\forall$ admissible' $[\ell_1, \ldots, \ell_r]$ $\exp(\qquad) \leq m'_r - m'_{r+1}$.

<u>Pf</u>. Induction on r. (A_0), (B_0) hold automatically.
Assume $r \geq 0$ and (A_r) holds. If $[\ell_1, \ldots, \ell_r]$ is admissible',
we know, that (Cor 1 above) $X_i / X_{i-1} \simeq \mathbb{Z}/\underbrace{p^{m'_{i-1} - m'_i}}_{p^{a_i}}\mathbb{Z}$ $(1 \leq i \leq r)$, by (A_r)
$\Longrightarrow$ (B_r), by standard linear algebra.

$\textcircled{$\star'_r$}$ + Čebotarev $\Longrightarrow$ $\forall\, c \in A^{(\chi)}$ $\exists\, \ell_{r+1}$ $c = e_\chi[\lambda_{r+1}]$ and $[\ell_1, \ldots, \ell_{r+1}]$ has very good' loc. $\Longrightarrow \exp\left(c; \frac{A^{(\chi)} \otimes \mathbb{Z}/M'}{X_r}\right) \leq$
$\leq \underbrace{m'(\ell_1 \cdots \ell_r)}_{m'_r} - \underbrace{m'(\ell_1 \cdots \ell_{r+1})}_{\geq m'_{r+1}} \leq m'_r - m'_{r+1} \Longrightarrow (C_r)$.

If $[\ell'_1, \ldots, \ell'_{r+1}]$ is admissible' $\xRightarrow{(B_r)}$ $X'_1 \subset X'_2 \subset \cdots \subset X'_r$ are direct summands, $\exp(\underbrace{(A^{(\chi)} \otimes \mathbb{Z}/M')/X'_r}) = a_{r+1}$
But $\exp(e_\chi[\lambda_{r+1}];\qquad) = m'(\ell'_1 \cdots \ell'_r) - m'(\ell'_1 \cdots \ell'_{r+1}) = m'_r - m'_{r+1}$

$\Longrightarrow (A_{r+1})$. $\boxed{\text{Thm is proved}}$.

$\boxed{p \neq 2 \text{ fixed prime}}$

Kolyvagin's results on the structure of $(Ш, \mathrm{Sel})[p^\infty]$

$E/\mathbb{Q}$ elliptic curve of conductor N, $\alpha: X_0(N) \longrightarrow E$ modular parameterisation
$i\infty \longmapsto O$

$K = \mathbb{Q}(\sqrt{D})$, $D = D_K < 0$ imaginary quadratic field.

Assume: the Heegner hypothesis (Heeg): every prime $\ell | N$ splits in $K/\mathbb{Q}$

Fix an ideal $\mathcal{N} \subset \mathcal{O}_K$ such that $\mathcal{O}_K/\mathcal{N} \simeq \mathbb{Z}/N$.

Ring class fields of K: for $n \geq 1$, let $\mathcal{O}_n := \mathbb{Z} + n\mathcal{O}_K \subset \mathcal{O}_K$ Gorenstein ring (order of conductor n). Let $\mathrm{rec}_K : \mathbb{A}_K^\times / K^\times \longrightarrow G_K^{ab}$ be the reciprocity map. The ring class field of K of conductor n is the abelian extension $K[n]/K$ such that rec_K induces an isomorphism

$$\mathbb{A}_K^\times / (K\otimes\mathbb{R})^\times \cdot \mathbb{A}_\mathbb{Q}^\times \cdot K^\times \cdot (\mathcal{O}_n \otimes \hat{\mathbb{Z}})^\times \xrightarrow{\sim} \mathrm{Gal}(K[n]/K)$$

Facts: (a) $K[1]$ = the Hilbert class field of K

(b) $\exists$ exact sequence

$$\mathcal{O}_K^\times/\mathbb{Z}^\times \longrightarrow \frac{(\mathcal{O}_K \otimes \mathbb{Z}/n)^\times}{(\mathbb{Z}/n)^\times} \longrightarrow \mathrm{Gal}(K[n]/K) \longrightarrow \underbrace{\mathrm{Gal}(K[1]/K)}_{Cl_K} \longrightarrow 0$$

(c) $\underbrace{\mathbb{Q} - \underbrace{K - K[n]}_{\text{abelian}}}_{\text{Galois}}$, $\underbrace{\mathrm{Gal}(K[n]/\mathbb{Q})}_{\substack{\text{generalised}\\ \text{dihedral group}}} \xrightarrow{\sim} \underbrace{\mathrm{Gal}(K[n]/K)}_{\substack{c \text{ acts by}\\ g \mapsto g^{-1}}} \rtimes \{1, \underbrace{c}_{\substack{\text{cplx}\\ \text{conj.}}}\}$

Heegner points of conductor n, where $(n,N)=1$:

$\mathcal{N}_n := \mathcal{N} \cap \mathcal{O}_n$ is an $\underbrace{}_{\text{invertible}}$ ideal of $\mathcal{O}_n$, $\mathcal{N}_n^{-1}/\mathcal{O}_n \simeq \mathbb{Z}/N$, $\mathcal{O}_n/\mathcal{N}_n \simeq \mathbb{Z}/N$

$$\begin{tikzcd} x_n := [\mathbb{C}/\mathcal{O}_n \longrightarrow \mathbb{C}/\mathcal{N}_n^{-1}] \arrow[d] & \in & X_0(N)(K[n]) \arrow[d, "\alpha"] \\ y_n & \in & E(K[n]) \end{tikzcd}$$

, by CM theory

Norm relation: if $\ell \nmid n$ is a prime inert in $K/\mathbb{Q}$, then

(if $D \neq -3,-4$) $\underbrace{N_{K[n\ell]/K[n]}}_{N_\ell}(x_{n\ell}) = \underbrace{T(\ell)}_{\text{the Hecke operator}} x_n$ in $\mathrm{Jac}(X_0(N))(K[n])$

$$\begin{tikzcd} J_0(N) & \ni & 0 \\ X_0(N) \arrow[u] & \ni & i\infty \arrow[u] \end{tikzcd}$$

$\Longrightarrow$ $N_\ell\, y_{n\ell} = a_\ell\, y_n$, if

$$f(\tau) = \sum_{m\geq 1} a_m q^m \in S_2(\Gamma_0(N)) \ (a_1 = 1), \quad L(E/\mathbb{Q}, s) = \sum_{m \geq 1} a_m m^{-s} = L(f,s)$$

Assume (for simplicity only)

(1) $K \neq \mathbb{Q}(\sqrt{-1}), \mathbb{Q}(\sqrt{-3})$ $\quad (\Leftrightarrow \mathcal{O}_K^\times = \mathbb{Z}^\times)$

(2) $G_\mathbb{Q} \longrightarrow \mathrm{Aut}_{\mathbb{F}_p}(E[p]) \simeq GL_2(\mathbb{F}_p)$ is surjective $\quad (\Rightarrow E(K)[p]=0)$

Fix $M = p^k \quad (k \gg 0)$

Def. $\mathcal{P}_k^! := \{\ell$ prime number $\mid \ell$ inert in $K/\mathbb{Q}$, $a_\ell \equiv \ell+1 \equiv 0 \pmod M\}$

$\Updownarrow$ $\mathrm{Fr}_{\mathbb{Q}(E[M])/\mathbb{Q}}(\ell) =$ the conjugacy class of c

$\mathcal{N}_k^! = \{\ell_1 \cdots \ell_r \mid r \geq 0,\ \ell_i \in \mathcal{P}_k^!$ distinct$\}$

Facts: (a) $n = \ell_1 \cdots \ell_r \in \mathcal{N}_k^! \Rightarrow G(n) := \mathrm{Gal}(K[n]/K[1]) \simeq \prod_{i=1}^r G(\ell_i)$,

$G(\ell_i)$ cyclic of order $\ell_i + 1$; ~~fix~~

(b) $\forall \ell \in \mathcal{P}_k^!$ fix a generator σ_ℓ of $G(\ell)$; let $D_\ell := \sum_{i=0}^{\ell} i\sigma_\ell^i \in \mathbb{Z}[G(\ell)]$

$N_\ell := \sum_{\sigma \in G(\ell)} \sigma \quad \Rightarrow \quad (\sigma_\ell - 1)D_\ell = \ell + 1 - N_\ell$; $\quad D_{\ell_1 \cdots \ell_r} := \prod_{i=1}^r D_{\ell_i}$

(c) $\forall n \in \mathcal{N}_k^! \quad E(K(n))[p] = 0 \quad (\Leftarrow E(K)[p] = 0$ + ramification argument$)$

Prop. (1) $\forall n \in \mathcal{N}_k^! \quad (D_n y_n \otimes 1) \in (E(K[n]) \otimes \mathbb{Z}/M)^{G(n)} \xhookrightarrow{\delta} H^1(K[n], E[M])^{G(n)}$

~~(2) res: $E(K[1]) \otimes \mathbb{Z}/M \to (E(K[n]) \otimes \mathbb{Z}/M)^{G(n)}$~~

Pf. (1) $n=1$ OK; if $\ell \mid n$, $(\sigma_\ell - 1)(D_n y_n \otimes 1) = \underbrace{\underbrace{(\ell+1)}_{\equiv 0 \pmod M}(D_{n/\ell}\, y_n \otimes 1) - \underbrace{a_\ell}_{\equiv 0 \pmod M} D_{n/\ell}\, y_{n/\ell} \otimes 1}_{0}$

$\Rightarrow (D_n y_n \otimes 1)$ is fixed by all σ_ℓ, $\ell \mid n$.

~~(2) Ker(res) = $H^1(G(n), E(K[n]$~~

(2) res: $H^1(K[1], E[M]) \xrightarrow{\sim} H^1(K[n], E[M])^{G(n)}$.

Pf: $\mathrm{Ker}(\mathrm{res}) = H^1(G(n), \underbrace{E(K[n])[M]}_{=0}) = 0$

$\mathrm{Coker}(\mathrm{res}) \subset H^2($ — " — , — " — $) = 0$.

Def. For $n \in \mathcal{N}_k^!$, let $\kappa(n) := \mathrm{cor}_{K[1]/K}(\mathrm{res}^{-1}(D_n y_n \otimes 1)) \in H^1(K, E[M])$

$$\begin{tikzcd} H^1(K, E[M]) \\ E(K) \otimes \mathbb{Z}/M \arrow[u, hook, "\delta"] \end{tikzcd}$$

Note: $\kappa(1) = \delta(y_K \otimes 1)$, $\quad y_K = \mathrm{Tr}_{K[1]/K}(y_1) \in E(K)$

Def. For $n \in \mathcal{N}_k^!$, let $m(n) := \mathrm{ind}(\kappa(n); H^1(K, E[M])) \in \mathbb{N} \cup \{+\infty\}$.

Note: If $y_K \notin E(K)_{\mathrm{tors}}$ ~~[illegible]~~, then ~~[illegible]~~

$m(1) = m_0 := \mathrm{ind}(y_K \otimes 1, E(K) \otimes \mathbb{Z}_p) < \infty$.

Fact: $c(y_K) = -\varepsilon y_K$, where $\varepsilon = \varepsilon(E/\mathbb{Q}) \in \{\pm 1\}$

is the sign in the functional equation of $L(E/\mathbb{Q}, s) = L(f, s)$

Def. For $r \geq 0$, let $m_r := \inf\{ m(\ell_1 \cdots \ell_r) \mid \ell_1 \cdots \ell_r \in \mathcal{L}'_k \} \in \mathbb{N} \cup \{+\infty\}$
$(k \gg 0)$

Kolyvagin's structure thm 1 for $Ш(E/\mathbb{Q})[p^\infty]$ $(p \neq 2,\ G_{\mathbb{Q}} \twoheadrightarrow Gl_2(\mathbb{F}_p))$

If $y_K \notin E(K)_{tors} \Rightarrow$ (1) $\infty > m_0 \geq m_1 \geq m_2 \geq \cdots \geq m_\infty := \min\limits_{r \geq 0}(m_r) = \lim\limits_{r \to \infty} m_r$

$$(2)\quad \left(Ш(E/K)[p^\infty]\right)^{c=\varepsilon} = \bigoplus_{\substack{r \geq 0 \\ 2 \mid r}} \left(\mathbb{Z}/p^{m_r - m_{r+1}}\right)^{\oplus 2}$$

$$\left(Ш(E/K)[p^\infty]\right)^{c=-\varepsilon} = \bigoplus_{\substack{r \geq 1 \\ 2 \nmid r}} \left(\text{—} \ {}'' \ \text{—}\right)^{\oplus 2}$$

$$\Rightarrow \boxed{\left|Ш(E/K)[p^\infty]\right|^{1/2} = p^{m_0 - m_\infty}}$$

What if $y_K \in E(K)_{tors}$?

Kolyvagin's Conjecture : $\boxed{m_\infty < \infty}$ (proved in some cases by Wei Zhang)

Kolyvagin's structure thm 2 for $S_M = Sel(E/K, M) \subset H^1(K, E[M])$ $\left(\begin{matrix} M = p^k, \\ k \gg 0 \end{matrix}\right)$

$(p \neq 2,\ G_{\mathbb{Q}} \twoheadrightarrow Gl_2(\mathbb{F}_p))$

Assume that $m_\infty < \infty$; define $f := \min\{r \geq 0 \mid m_r < \infty\}$. then

(1) $\infty = m_0 = m_1 = \cdots = m_{f-1} > m_f \geq \cdots \geq m_\infty = \lim\limits_{r \to +\infty} m_r$

$$(2)\quad (S_M)^{c=\varepsilon} = \bigoplus_{i=1}^{f} (\text{unknown } \mathbb{Z}/p^?\mathbb{Z}) \oplus \bigoplus_{\substack{k \geq 0 \\ 2 \mid k}} \left(\mathbb{Z}/p^{m_{f+k} - m_{f+k+1}}\right)^{\oplus 2}$$

$$(S_M)^{c=-\varepsilon} = (\mathbb{Z}/M)^{\oplus(f+1)} \oplus \bigoplus_{\substack{k \geq 1 \\ 2 \nmid k}} \left(\text{—} \ {}'' \ \text{—}\right)^{\oplus 2}$$

Cor. If $m_\infty < \infty$, then $s_{-\varepsilon} := rk_{\mathbb{Z}_p} H^1_f(K, T_p(E))^{c=-\varepsilon} = f+1$

$$\underbrace{s_\varepsilon := rk_{\mathbb{Z}_p} H^1_f(K, T_p(E))^{c=\varepsilon}}_{\equiv f \ (\text{mod } 2)} \leq f \Leftarrow$$

Use the Cassels(-Tate) pairing $\Rightarrow \varinjlim\limits_{k} S_{p^k}^{\pm\varepsilon} \cong (\mathbb{Q}_p/\mathbb{Z}_p)^{s_{\pm\varepsilon}} \oplus (\text{sth})^{\oplus 2}$

Rmk: In "Euler systems", Kolyvagin proved that

$y_K \notin E(K)_{tors} \Rightarrow \left|Ш(E/K)[p^\infty]\right|^{1/2}$ divides p^{m_0}

(if $p \neq 2$ and $G_{\mathbb{Q}} \twoheadrightarrow Gl_2(\mathbb{F}_p)$).

In "Kolyvagin Systems" (and "KS of higher rank"), Mazur and Rubin developed an abstract theory which formalises (and extends) Kolyvagin's results on the structure of ideal class groups. The corresponding theory in a dihedral context (generalising Kolyvagin's structure theorems for Ш / Selmer groups of E) was developed by Howard.

Under the <u>Heegner</u> hypothesis, $\underline{\mathrm{ord}_{s=1} L(E/K, s) \equiv 1 \pmod 2}$.

Bertolini and Darmon developed a somewhat different method for proving, among other things, the following result.

<u>Thm</u>. If $E_{/\mathbb{Q}}$ is a modular elliptic curve and K is an imaginary quadratic field such that $\underline{L(E/K, 1) \neq 0}$, then $\underline{E(K) \text{ is finite}}$ (they needed a few additional assumptions, but those had been subsequently removed).

The main idea is to use level raising congruences (mod p^k between $f = f_E \in S_2(\Gamma_0(N))$ and suitable $g_\ell \in S_2(\Gamma_0(N\ell))$. The eigenform g_ℓ can be transferred to a certain quaternion algebra; it gives rise to an abelian variety $A_{/\mathbb{Q}}$ with <u>real multiplication</u> and an isomorphism $E[p^k] \xrightarrow{\sim} A[P^k]$ ($P | p$ in the field of). Analogues of Heegner points on A (= quotient of Jac(a Shimura curve)) define classes in

$$\begin{tikzcd} H^1_f(K, A[P^k]) \arrow[r, phantom, "\subset"] & H^1(K, A[P^k]) \arrow[d, phantom, "\cong"] \\ & H^1(K, E[p^k]) \end{tikzcd}$$

If $L(E/K, 1) \neq 0$, then these classes are <u>ramified</u> at ℓ (for $k \gg 0$) and give rise to appropriate annihilation relations for localisations x_ℓ of global elements $x \in H^1_f(K, E[p^k])$.

$\underbrace{\text{(Cyclotomic)}}$ cohomological Euler systems (Kato, $\overbrace{\text{Rubin}}^{\text{has more general theory}}$, Perrin-Riou)

base field $= \mathbb{Q}$, ES lives in $\mathbb{Q}^{ab}$ (Ex: cyclotomic elements $1-\zeta_m$, Kato's Euler system)

Elliptic units: very similar, base field $K = \mathbb{Q}(\sqrt{D})$, $D<0$; ES lives in K^{ab}

Heegner points: different! ES lives in $K^{\text{anticycl}} = \bigcup$ ring class fields of K

$$\underbrace{\mathbb{Q} - \underbrace{K - K^{\text{anticycl}}}_{\text{abelian}}}_{\text{generalised dihedral}} - K^{ab}$$

Recall: $\forall m>1$, $\forall$ prime ℓ

(1) $$N_{\mathbb{Q}(\mu_{m\ell})/\mathbb{Q}(\mu_m)}(1-\zeta_{m\ell}) = \begin{cases} 1-\zeta_m & \ell \mid m \\ \dfrac{1-\zeta_m}{1-\zeta_m^{\ell^{-1}}} = (1-\mathrm{Fr}(\ell))(1-\zeta_m), & \ell \nmid \end{cases}$$

$\mathrm{Fr}(\ell) = \mathrm{Fr}_{geom}(\ell) \in G(m) := \mathrm{Gal}(\mathbb{Q}(\mu_m)/\mathbb{Q})$

(2) $1-\zeta_{m\ell} \equiv \mathrm{Fr}(\ell)(1-\zeta_m)$ $\quad(\ell \nmid m)$

inside $(\mathbb{Z}[\mu_m]\otimes\mathbb{F}_\ell)^\times = \bigoplus_{\lambda \mid \ell \text{ in } \mathbb{Q}(\mu_m)} (\mathbb{Z}[\mu_m]/\lambda)^\times$

Cohomology: $T := \mathbb{Z}_p(1) = \varprojlim_k \mu_{p^k}$, $\quad T^*(1) := \mathrm{Hom}_{\mathbb{Z}_p}(T, \mathbb{Z}_p)\otimes_{\mathbb{Z}_p}\mathbb{Z}_p(1) = \mathbb{Z}_p$

$T^D = \mathbb{Q}_p/\mathbb{Z}_p$

$$\begin{tikzcd}
 & 1-\zeta_m \in \mathbb{Q}(\mu_m)^\times \arrow[r] & \mathbb{Q}(\mu_m)^\times \hat{\otimes} \mathbb{Z}_p \arrow[r, "\delta", "\sim"'] & H^1(\mathbb{Q}(\mu_m), \mathbb{Z}_p(1)) & \\
 & \cup & \cup & \cup & \\
\text{if } m \neq \ell^r \arrow[ur, "\in"] & \mathbb{Z}[\mu_m]^\times \arrow[r] & \mathbb{Z}[\mu_m]^\times \otimes \mathbb{Z}_p \arrow[r, "\delta", "\sim"'] & H^1_f(\mathbb{Q}(\mu_m), \mathbb{Z}_p(1)) & \mathcal{F} = \mathcal{F}_{BK} = \mathcal{F}_f \text{ Bloch-Kato local cond.} \arrow[l, bend left]
\end{tikzcd}$$

$H^1_{\mathcal{F}^D}(\mathbb{Q}(\mu_m), \mathbb{Q}_p/\mathbb{Z}_p) = \mathrm{Hom}(Cl(\mathbb{Z}[\mu_m]), \mathbb{Q}_p/\mathbb{Z}_p)$, $\quad \mathcal{F}^D = \mathcal{F}_{BK} = \mathcal{F}_f$

$\forall m \geq 1$ $\quad G(m) = \mathrm{Gal}(\mathbb{Q}(\mu_m)/\mathbb{Q}) \xrightarrow[\sim]{\chi_{cycl}} (\mathbb{Z}/m\mathbb{Z})^\times$ $\quad (G(p) = \Delta)$

We considered: $\chi: \underbrace{G(p)}_{\simeq} \longrightarrow \mathbb{Z}_p^\times$, $\chi(c) = 1$, $\chi \neq 1$

$\forall m \geq 1$

$$\begin{tikzcd}
 & \mathbb{Q}(\mu_{pm}) \arrow[dl, dash, "H"'] \arrow[dr, dash, "G(p)"] \arrow[dd, dash, "H\times G(p)"] & \\
\mathbb{Q}(\mu_p) \arrow[dr, dash, "G(p)"'] & & \mathbb{Q}(\mu_{pm})^{G(p)} \arrow[dl, dash, "H"] \\
 & \mathbb{Q} &
\end{tikzcd}$$

$(G(p) \xrightarrow{\omega} \mathbb{Z}_p^\times = \mathrm{Gal}(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q}))$

$(= \mathbb{Q}(\mu_m)$ if $p \nmid m)$

$$\begin{tikzcd}
e_\chi\, \delta(1-\zeta_{pm})\otimes 1 \in H^1_{\mathcal{F}}(\mathbb{Q}(\mu_{pm}), \mathbb{Z}_p(1))^{\cancel{(\chi)}} \arrow[r, "\simeq"] & H^1_{\mathcal{F}}(\mathbb{Q}(\mu_{pm}), \mathbb{Z}_p(1)\otimes\chi^{-1})^{G(p)} \\
 & H^1_{\mathcal{F}}(\mathbb{Q}(\mu_{pm})^{G(p)}, \mathbb{Z}_p(1)\otimes\chi^{-1}) \arrow[u, "\text{res}", "\wr"']
\end{tikzcd}$$

$(\mathcal{F} = \mathcal{F}_{BK}$

$$H^1_{\mathcal{F}^D}(\mathbb{Q}(\mu_{pm}), \mathbb{Q}_p/\mathbb{Z}_p)^{(\chi^{-1})} \simeq H^1_{\mathcal{F}^D}(\mathbb{Q}(\mu_{pm})^{G(p)}, (\mathbb{Q}_p/\mathbb{Z}_p)\otimes\chi) \simeq \mathrm{Hom}((Cl(\mathbb{Z}[\mu_{pm}])\otimes\mathbb{Z}_p)^{(\chi)}, \mathbb{Q}_p/\mathbb{Z}_p)$$

these elements live in the subfields $\mathbb{Q}(\mu_{pm})^{G(p)}$ of $(\mathbb{Q}^{ab})^{G(p)}$.
In the ES arguments above we only needed $e_\chi\, \delta(1-\zeta_{pm})\otimes 1$ for $m = \ell_1 \cdots \ell_r$, $\ell_i \equiv 1 \pmod{p^k}$ distinct primes $(k >> 0)$.
However, it is useful to know that they exist for all $m \geq 1$ (in particular, for all $m'p^n$ $(n \geq 0)$, m' fixed).

1st Version of this Euler system : work over $\mathbb{Q}(\mu_{pm})$, with $H^1(-, \mathbb{Z}_p(1))^{(\chi)}$.
2nd Version : —"— $\mathbb{Q}(\mu_{pm})^{G(p)}$, with $H^1(-, \mathbb{Z}_p(1) \otimes \chi^{-1})$

Where does $\left\{\begin{matrix} 1-Fr(\ell) & \text{in the norm relation} \\ Fr(\ell) & \text{—"— congruence —"—} \end{matrix}\right\}$ come from?

- $1 - X = \det(1 - X\, Fr(\ell) \mid T^*(1)) =: P_\ell(X)$ Euler factor
- $X = \dfrac{P_\ell(X) - P_\ell(\ell X)}{\ell - 1}$

Axiomatic version(s)

Data:
- $p \neq 2$ prime (many things work also for $p = 2$)
- $[\Phi : \mathbb{Q}_p] < \infty$ (Φ = the coefficient field), $\mathcal{O} := \mathcal{O}_\Phi$
- $N = N'p^r$, $r \geq 1$, $p \nmid N'$
- T = free $\mathcal{O}$-module of finite rank, with a continuous $\mathcal{O}$-linear action of $G_{\mathbb{Q}, \{\infty\} \cup \{\ell \mid N\}}$ (i.e., $\forall\, \ell \nmid N$ $T = T^{I_\ell}$)

Def: (1) $V := T \otimes_{\mathcal{O}} \Phi$, $T^*(1) := \mathrm{Hom}_{\mathcal{O}}(T, \mathcal{O} \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(1))$, $V^*(1) := T^*(1) \otimes_{\mathcal{O}} \Phi$
(2) $\forall$ prime $\ell \nmid N$ $P_\ell(X) := \det(1 - X\, \underset{Fr_{geom}(\ell)}{\underline{Fr(\ell)}} \mid V^*(1)) \in \mathcal{O}[X]$

($P_\ell(X) = 1 - X$ if $T = \mathbb{Z}_p(1)$, $N = p$)

Def. A cyclotomic Euler system for T is:
(1st Version) $\{ c(m) \in H^1(\mathbb{Q}(\mu_m), T) \mid$ for all $m \geq 1$ such that $(m, N') = 1$
satisfying the norm relation (ℓ prime, $(\ell m, N') = 1$)

(Norm) $$\mathrm{cor}_{\mathbb{Q}(\mu_{\ell m})/\mathbb{Q}(\mu_m)}\, c(m\ell) = \begin{cases} c(m) & \text{if } \ell \mid m \\ P_\ell(\underset{\substack{\uparrow \\ G(m)}}{Fr(\ell)})\, c(m) & \text{if } \ell \nmid mp \end{cases}$$

(2nd Version) Fix a field $\mathcal{K}$ such that

$$\underset{\text{which is pro-}p\text{ over }\mathbb{Q}}{\text{the maximal subfield of the R.H.S.}} \subset \mathcal{K} \subset \bigcup_{(m,N')=1} \mathbb{Q}(\mu_m)$$

(ex: $\displaystyle\bigcup_{(m,N')=1} \mathbb{Q}(\mu_m)^+ \supset \bigcup_{\substack{(m,N')=1\\ p|m}} \mathbb{Q}(\mu_m)^{G(p)}$)

ES = A system $\{c_F \in H^1(F,T) \mid \underbrace{\mathbb{Q} \xhookrightarrow[f]{} F}_{\text{finite degree}} \subset \mathcal{K}\}$ satisfying

(Norm) $\forall\ \mathbb{Q} \xhookrightarrow[f]{} F \xhookrightarrow[f]{} F' \subset \mathcal{K}$ $\quad \mathrm{cor}_{F'/F}(c_{F'}) = \displaystyle\prod_{\substack{\ell \in \mathrm{Ram}(F'/\mathbb{Q}) \setminus \mathrm{Ram}(F/\mathbb{Q})\\ \ell \neq p}} P_\ell(\overset{\mathrm{Gal}(F/\mathbb{Q})}{\overset{\Psi}{\mathrm{Fr}(\ell)}})\, c_F$

Rmks : (1) $\exists$ other versions : e.g. one can impose $\underset{\text{(Kato)}}{N|m}$ & $(N_0, m) = 1$

(2) As we shall see, (Norm) for $\left\{\begin{array}{l} \ell = p \mid m \\ F'/F \text{ ramified at } p \end{array}\right\}$ implies

(a) $\left\{\begin{array}{ll} c(m) \in H^1_{\mathcal{F}_{ur}\{p\}}(\mathbb{Q}(\mu_m), T) & \text{if } p|m \\ c_F \in H^1_{\mathcal{F}_{ur}\{p\}}(F, T) & \forall F \end{array}\right\}$

(b) the congruence relation (Congr) (see below).

(3) If one imposes only a weak version of (Norm), namely, for $\left\{\begin{array}{l} \ell \neq p \\ F'/F \text{ unramified at } p \end{array}\right\}$, one must add (Congr) as another axiom.

A local digression

If F is a nonarchimedean local field ($[F:\mathbb{Q}_\ell] < \infty$), let $\mathrm{Fr}_F \in G_F/I_F$ be the geometric Frobenius (and $I_F \subset G_F$ the inertia subgroup).

Fact : If M is a discrete or profinite G_F-module, then

$$H^1_{ur}(F,M) \simeq H^1_{\mathrm{cont}}(\underbrace{G_F/I_F}_{\mathrm{Fr}_F^{\widehat{\mathbb{Z}}}}, M^{I_F}) \underset{\sim}{\xrightarrow{\mathrm{ev}_{\mathrm{Fr}_F}}} M^{I_F}/(\mathrm{Fr}_F - 1) M^{I_F}.$$

$\uparrow$ evaluating a given cocycle at Fr_F.

Notation: for K'/K separable algebraic field extension, M = profinite topological G_K-module the Iwasawa cohomology of M in K'/K is defined as

$$H^i_{Iw}(K'/K, M) := \varprojlim_{K \overset{f}{\hookrightarrow} L \subset K'} H^i(L, M) \qquad (\text{transition maps} = \mathrm{cor}_{L'/L})$$

Prop. If $\ell \neq p$, $[F:\mathbb{Q}_\ell] < \infty$, F_∞/F = the unique $\mathbb{Z}_p$-extension of F ($F_\infty = \bigcup F_n$), T = a free $\mathbb{Z}_p$-module of finite rank with a cont. $\mathbb{Z}_p$-linear action of G_F, then

(1) $H^1_{Iw}(F_\infty/F, T) = H^1_{Iw,ur}(F_\infty/F, T) := \varprojlim_{n, \mathrm{cor}} H^1_{ur}(F_n, T)$

(2) $H^1_{Iw,ur}(F_\infty/F, T)$ is isomorphic to a quotient of T^{I_F} with no p-torsion.

Pf. (1) F_∞/F is unramified $\Rightarrow \forall n \; I_{F_n} = I_F$. In the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & H^1_{ur}(F_{n+1}, T) \arrow[r] \arrow[d] & H^1(F_{n+1}, T) \arrow[r] \arrow[d, "\mathrm{cor}"] & H^1(I_{F_{n+1}}, T) = H^1(I_F, T) \arrow[d, "p"] \\
0 \arrow[r] & H^1_{ur}(F_n, T) \arrow[r] & H^1(F_n, T) \arrow[r] & H^1(I_{F_n}, T) = H^1(I_F, T)
\end{tikzcd}$$

;

$H^1(I_F, T) \xrightarrow{\sim} H^1(I_F^{tame}, T^{I_F^{wild}}) \underset{ev}{\xrightarrow{\sim}} (T^{I_F^{wild}})_{I_F^{tame}}$ is a $\mathbb{Z}_p$-module of finite type

top-generator of I_F^{tame}

$\Rightarrow \varprojlim_n (3^{rd}$ column$) = 0 \Longrightarrow (1)$.

---

(2) $H^1_{Iw,ur}(T) \simeq \varprojlim_n T^{I_F}/(Fr_F^{p^n} - 1)T^{I_F} = T^{I_F}/\underbrace{\bigcap_{n\geq 1}(Fr_F^{p^n}-1)T^{I_F}}_{X}$ is a quotient of T^{I_F}

As in Hida theory, $e := \lim_{r \to +\infty} (Fr_F - 1)^{r!} \in \mathrm{End}_{\mathbb{Z}_p}(T^{I_F})$ is a projector ($e^2 = e$), $T^{I_F} = eT^{I_F} \oplus (1-e)T^{I_F}$.

$$\left.\begin{array}{l}\text{On } eT^{I_F},\ Fr_F - 1 \ (\text{hence } Fr_F^{p^n} - 1 \ (n \geq 1)) \text{ is invertible} \\ \text{On } (1-e)T^{I_F},\ Fr_F - 1 \text{ is topologically nilpotent}\end{array}\right\} \Longrightarrow X = eT^{I_F}$$

$\Longrightarrow H^1_{Iw,ur}(T) \simeq (1-e)T^{I_F}$ has no p-torsion.

Cor. (Norm) $\Longrightarrow \begin{cases} \text{if } (\ell m, N') = 1,\ \ell \neq p,\ p \mid m & c(m)_\ell \in H^1_{ur}(\mathbb{Q}(\mu_m) \otimes \mathbb{Q}_\ell, T) \\ \text{if } \ell \nmid N,\ \forall\, \mathbb{Q} \overset{f}{\hookrightarrow} F \subset \mathcal{K} & (c_F)_\ell \in H^1_{ur}(F \otimes \mathbb{Q}_\ell, T) \end{cases}$

Pf: $(c(mp^n)_{n\geq 0}) \in H^1_{Iw}(\mathbb{Q}(\mu_{mp^\infty})/\mathbb{Q}(\mu_m), T)$

The decomposition subgroup of ℓ in $\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q} \underset{\chi_{cycl}}{\xrightarrow{\simeq}}$ the subgroup of $\mathbb{Z}_p^\times$ topologically gen. by ℓ has finite index in $\mathbb{Z}_p^\times \Longrightarrow$ for every $\lambda \mid \ell$ in $\mathbb{Q}(\mu_{mp^\infty})$, $\mathbb{Q}(\mu_{mp^\infty})_\lambda$ contains the unique $\mathbb{Z}_p$-extension of $\mathbb{Q}_\ell$. Apply (1) of the Prop. above.

The Norm relation $\Longrightarrow$ the Congruence relation

Prop. If $p \mid m$, $(m, N') = 1$, $\ell \nmid mN$ prime, then

$$\begin{tikzcd}
c(m\ell)_\ell \in H^1_{ur}(\mathbb{Q}(\mu_{m\ell}) \otimes \mathbb{Q}_\ell, T) = \bigoplus\limits_{\substack{\lambda' | \ell \\ \text{in } \mathbb{Q}(\mu_{m\ell})}} H^1_{ur}(\mathbb{Q}(\mu_{m\ell})_{\lambda'}, T) = & \bigoplus\limits_{\lambda' | \ell} H^1(\underbrace{k(\lambda')}_{\mathbb{Z}[\mu_{m\ell}]/\lambda'}, T) \arrow[d, equal] \\
H^1_{ur}(\mathbb{Q}(\mu_m) \otimes \mathbb{Q}_\ell, T) \arrow[r, equal] & \bigoplus\limits_{\substack{\lambda | \ell \\ \text{in } \mathbb{Q}(\mu_m)}} H^1(k(\lambda), T)
\end{tikzcd}$$

is equal to

(Congr) $\left(\dfrac{P_\ell(\mathrm{Fr}(\ell)) - P_\ell(\ell \mathrm{Fr}(\ell))}{\ell - 1} \right) c(m)_\ell$. $(\mathrm{Fr}(\ell) \in G(m))$.

Pf: (a) $((\ell-1)\, c(mp^n\ell)_\ell)_{n \geq 0}$ and $(P_\ell(\mathrm{Fr}(\ell))\, c(mp^n)_\ell)_{n \geq 0}$

have the same images in $H^1_{Iw, ur}(\mathbb{Q}(\mu_{mp^\infty}) \otimes \mathbb{Q}_\ell / \mathbb{Q}(\mu_m) \otimes \mathbb{Q}_\ell, T)$, since (for each $m' = mp^n$)

$$\begin{tikzcd}
H^1_{ur}(\mathbb{Q}(\mu_{m'\ell}) \otimes \mathbb{Q}_\ell, T) \arrow[r, phantom, "="] \arrow[d, "cor"] & H^1(\mathbb{Z}[\mu_m] \otimes \mathbb{F}_\ell, T) \arrow[d, "\ell-1"] \\
H^1_{ur}(\mathbb{Q}(\mu_{m'}) \otimes \mathbb{Q}_\ell, T) \arrow[r, phantom, "="] & (\text{---}\ '' \ \text{---})
\end{tikzcd}$$

(b) $(P_\ell(\ell \mathrm{Fr}(\ell))\, c(mp^n)_\ell)_{n \geq 0} = 0$ (Cayley–Hamilton)

(c) $\dfrac{P_\ell(X) - P_\ell(\ell X)}{\ell - 1} \in \mathcal{O}[X]$

(d) $H^1_{Iw, ur}$ above is a free $\mathbb{Z}_p$-module of finite rank with no p-torsion $\Longrightarrow$ one can divide the relation obtained by subtracting (b) from (a) by $\ell - 1$.

KOLYVAGIN DERIVATIVE CLASSES

Below — a brute force approach. Rubin has a more general and more precise version (using the "universal Euler system"), with no assumptions (and no M_1, M_2 below)

Recall: we have $\{ c_{F'} \in H^1(F', T) \mid \mathbb{Q} \underset{f}{\hookrightarrow} F' \hookrightarrow \mathcal{K} \}$

Notation: • fix an ideal $(0) \neq M \subset \mathcal{O}$; define $P'_M := \{ \ell \text{ prime number} \mid \underline{\ell \nmid N,\ M \mid (\ell - 1),\ M \mid P_\ell(1)} \}$

$\mathcal{N}'_M := \{ \ell_1 \cdots \ell_r \mid r \geq 0,\ \ell_i \in P'_M \text{ distinct} \}$; $\forall \ell \in P'_M$ fix a generator $\sigma_\ell \in G(\ell) = Gal(\mathbb{Q}(\mu_\ell)/\mathbb{Q})$ and define, as before, $D_\ell := \sum_{i=0}^{\ell-2} i \sigma_\ell^i$, $D_{\ell_1 \cdots \ell_r} = \prod D_{\ell_j} \in \mathbb{Z}[G(\ell_1 \cdots \ell_r)]$

• let $\mathbb{Q}_\infty/\mathbb{Q}$ be the cyclotomic $\mathbb{Z}_p$-extension ($\Rightarrow \mathbb{Q}_\infty \subset \mathcal{K}$), $\chi_{cycl}: Gal(\mathbb{Q}_\infty/\mathbb{Q}) \overset{\sim}{\to} 1 + p\mathbb{Z}_p$

below, $\boxed{F \text{ is a fixed subfield } \mathbb{Q} \underset{f}{\hookrightarrow} F \hookrightarrow \mathbb{Q}_\infty}$

(used only for proving results in Iwasawa theory (over $\mathbb{Q}_\infty$); in order to prove results over $\mathbb{Q}$ we only need $F = \mathbb{Q}$).

• $c_{F'} (\text{mod } M) :=$ the image of $c_{F'}$ in $H^1(F', T/M)$

Prop. $\forall n \in \mathcal{N}'_M$ (1) $D_n(c_{F(\mu_n)} (\text{mod} M)) \in H^1(F(\mu_n), T/M)^{G(n)}$

(2) $res : H^1(F, T/M) \longrightarrow H^1(F(\mu_n), T/M)^{G(n)}$ satisfies

$Ker(res) = H^1(G(n), (T/M)^{G_F})$, $Coker(res) \subset H^2(G(n), (T/M)^{G_F})$.

($G(n)$ acts trivially on $\curvearrowleft$)

$$\begin{tikzcd} \mathbb{Q}(\mu_n) \arrow[r, dash] & F(\mu_n) \arrow[d, dash, "G(n)"] \\ \mathbb{Q} \arrow[u, dash, "G(n)"] \arrow[r, dash] & F \end{tikzcd}$$

Pf. (1) $n = 1$ OK; if $n > 1$, $\ell \mid n$, then $(\sigma_\ell - 1) D_n (c_{F(\mu_n)} (\text{mod } M)) = (\underbrace{(\ell-1)}_{\equiv 0 (\text{mod} M)} - N_\ell) D_{n/\ell} (c_{F(\mu_n)} \text{mod} M) =$

$= - \overset{\uparrow}{P_\ell(Fr(\ell))} \underbrace{D_{n/\ell}(c_{F(\mu_{n/\ell})} (\text{mod } M))}_{\text{fixed by } G(n/\ell) \text{ by induction}} = - \underbrace{P_\ell(1)}_{\equiv 0 (\text{mod} M)} (\text{— " —}) = 0.$

$G(n/\ell)$

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(2) $Ker(res) = H^1(G(n), (T/M)^{G_{F(\mu_n)}})$, but $(T/M)^{G_{F(\mu_n)}} = (T/M)^{G_F}$, since T is
$Coker(res) \subset H^2(\text{— " —})$ unramified outside N, and $\mathbb{Q}_{\{N\infty\}} \cap F(\mu_n) = F$

In order to get well-defined classes in $H^1(F, T/M)$ mapping to $D_n(c_{F(\mu_n)} \text{mod} M)$ we impose one of the following assumptions:

$(\text{Ass } 1)_{\mathbb{Q}}$ $V^{G_\mathbb{Q}} = 0$ (enough for $F = \mathbb{Q}$)

$(\text{Ass } 1)_{\mathbb{Q}_\infty}$ $V^{G_{\mathbb{Q}_\infty}} = 0$ (enough for all $\mathbb{Q} \underset{f}{\hookrightarrow} F \hookrightarrow \mathbb{Q}_\infty$)

Note: (a) $V^{G_\mathbb{Q}} \subset V^{G_{\mathbb{Q}_\infty}} \subset V$ are $G_\mathbb{Q}$-stable subspaces, so:

(b) If V is an irreducible $\Phi[G_\mathbb{Q}]$-module, then:

$(\text{Ass } 1)_\mathbb{Q}$ does not hold $\Longleftrightarrow$ $\dim_\Phi V = 1$ and V is the trivial repr. of $G_\mathbb{Q}$

$(\text{Ass } 1)_{\mathbb{Q}_\infty}$ (— " —) $\Longleftrightarrow$ $\dim_\Phi V = 1$ and $G_\mathbb{Q}$ acts on V through a cont. character $\chi : Gal(\mathbb{Q}_\infty/\mathbb{Q}) \longrightarrow \Phi^\times$

$\wr\wr$

$1 + p\mathbb{Z}_p$

Def. Assume $V^{G_{\mathbb{Q}_\infty}}=0$ ($\Longrightarrow \exists$ ideal $(0)\neq M_1 \subset \mathcal{O}$ s.t. $M_1\cdot (V/T)^{G_{\mathbb{Q}_\infty}}=0$).
For each $(0)\neq M' \subset \mathcal{O}$ divisible by MM_1^2 ~~and~~ $\forall n \in \mathcal{N}^1_{M'}$, define $\underline{\kappa_{F,M}(n) \in H^1(F,T/M)}$
$\forall\, \mathbb{Q} \underset{f}{\hookrightarrow} F \hookrightarrow \mathbb{Q}_\infty$
by the diagram

$$\begin{tikzcd}
D_n(c_{F(\mu_n)} \pmod{M'}) \arrow[dd] & & \in H^1(F(\mu_n),T/M')^{G(n)} \arrow[r] \arrow[d, "can"] \arrow[ddl, "\text{well-defined}"] & H^2(G(n),(T/M')^{G_F}) \arrow[d, "\boxed{can=0}"] \\
& H^1(F,T/(M'/M_1)) \arrow[r, "res"] \arrow[d, "can"] & H^1(F(\mu_n),T/(M'/M_1))^{G(n)} \arrow[r] & H^2(G(n),(T/(M'/M_1))^{G_F}) \\
\kappa_{F,M}(n) & \in H^1(F,T/M) & &
\end{tikzcd}$$

(for $F=\mathbb{Q}$, we only need $M_1\cdot(V/T)^{G_{\mathbb{Q}}}=0$ ($\Longleftarrow V^{G_{\mathbb{Q}}}=0$)).

Facts: (1) $\kappa_{F,M}(n)$ ($n\in\mathcal{N}^1_{M'}$) does not depend on M' divisible by MM_1^2

(2) $\mathrm{res}_{F(\mu_n)/F}\ \kappa_{F,M}(n) = D_n(c_{F(\mu_n)} \pmod M)$

(3) $\kappa_{F,M}(1) = c_F \pmod M$

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local behaviour of $\kappa_{F,M}(n)$

Outside np: **Prop.** If $v\nmid np\infty$ in F, then $\kappa_{F,M}(n)_v \in \underbrace{H^1_{ur}(F_v,T/M)}_{(=H^1_f(F_v,T/M)\text{ if } v\nmid N)}$
$\forall n\in\mathcal{N}^1_{M'}$, $MM_1^2 \mid M'$

Pf. If $v'\mid v$ in $F(\mu_n)/F$, then $e(v'|v)=1 \Rightarrow I_{v'}=I_v$

$$\begin{tikzcd}
& & 0 \\
D_n(c_{F(\mu_n)} (\mathrm{mod}\, M)) \arrow[urr, bend left, mapsto] & \in H^1(F(\mu_n),T/M) \arrow[r, "res"] & H^1(I_{v'},T/M) \arrow[u, phantom, "\rotatebox{90}{$\in$}"] \arrow[d, equal] \\
& H^1(F,T/M) \arrow[u, "res"] \arrow[r, "res"] & H^1(I_v,T/M) \\
\kappa_{F,M}(n) \arrow[uu, mapsto] \arrow[ur, phantom, "\in"] \arrow[rr, mapsto] & & \kappa_{F,M}(n)_v^s \arrow[u, phantom, "\rotatebox{90}{$\in$}"]
\end{tikzcd}$$

, since ~~pr~~ $\underline{(c_{F(\mu_n)})_{v'} \in H^1_{ur}(F(\mu_n)_{v'},T/M)}$

Rmk (the Bloch–Kato local condition at $v\nmid p$) If $v\nmid p\infty$ in F, then
$\mathcal{F}_{BK,v}=H^1_f$ is defined as follows: $H^1_f(F_v,V):=H^1_{ur}(F_v,V)$

Via

$$\begin{tikzcd}
T \arrow[r, hook] \arrow[d, two heads] & V \arrow[r, two heads] & V/T \\
T/M & & \frac{1}{M}T/T \arrow[u, hook]
\end{tikzcd}$$

, this induces $H^1_f \supset H^1_{ur}$ for T
$H^1_f \subset H^1_{ur}$ for V/T

and H^1_f for T/M, $\frac{1}{M}T/T$, compatible via $\frac{1}{M}T/T \xrightarrow{\sim} T/M$.
Moreover, $\mathcal{F}^D_{BK,v}=\mathcal{F}_{BK,v}$. If $V^{I_v}=V$, then $H^1_f=H^1_{ur}$ for $T, V/T, T/M$

Prop. $\exists\ (0)\neq M_2\subset\mathcal{O}$ s.t. $\forall n\in\mathcal{N}^1_{M'}$ with $MM_1^2M_2 \mid M'$
$\forall F$ $\forall v\nmid np\infty$ $\kappa_{F,M}(n)_v \in H^1_f(F_v,T/M)$

(Rubin showed this for $M_2=(1)$, but the proof is harder)

Pf: Enough for $v\mid N'$: $\ell_{\mathcal{O}}(H^1_f(F_v,T)/H^1_{ur}(F_v,T))$ is ~~bounded~~ given by the local Tamagawa factor (for T), and these are bounded in unramified extensions (for varying F).

localisation of $\kappa_{F,M}(n)$ at $\lambda | \ell | n$

Def. $\mathcal{P}^1_{M,F} := \{ \ell \in \mathcal{P}^1_M \mid \ell$ splits completely in $F/\mathbb{Q} \}$

$\mathcal{N}^1_{M,F} := \{ \ell_1 \cdots \ell_r \mid r \geq 0, \ \ell_i \in \mathcal{P}^1_{M,F}$ distinct $\}$

Def. For a prime $\ell \nmid N$, let $Q_\ell(X) := \dfrac{P_\ell(X) - P_\ell(1)}{X-1} \in \mathcal{O}[X]$.

Fact: $\ell \nmid N \implies P_\ell(\ell \, Fr(\ell))|_V = 0$ (Cayley–Hamilton)

Finite–singular morphism: assume $\ell \in \mathcal{P}^1_M$; fix a lift $\tilde{\sigma}_\ell$ of $\sigma_\ell \in G(\ell)$ under $I_{\mathbb{Q}_\ell} = I_\ell \xrightarrow{can} G(\ell)$. $\left.\begin{matrix}\text{Cayley–Hamilton} \\ M | P_\ell(1)\end{matrix}\right\} \implies \underline{(Fr(\ell)-1) Q_\ell(Fr(\ell))|_{T/M} = 0}$

$\implies$ the maps below are well-defined:

(a) $H^1_f = H^1_{ur}(\mathbb{Q}_\ell, T/M) \underset{\sim}{\xrightarrow{ev_{Fr(\ell)}}} (T/M)/(Fr(\ell)-1) \xrightarrow{Q_\ell(Fr(\ell))} (T/M)^{Fr(\ell)=1}$

(b) $H^1_s = H^1(\mathbb{Q}_\ell, T/M)/H^1_f = H^1(I_\ell, T/M)^{Fr(\ell)=1} = \mathrm{Hom}_{cont}(I_\ell^{tame}, T/M)^{Fr(\ell)=1} =$

$= \mathrm{Hom}(G(\ell), T/M)^{Fr(\ell)=1} \underbrace{=}_{|G(\ell)| \text{ divides } \ell-1} \mathrm{Hom}(G(\ell), (T/M)^{Fr(\ell)=1})$

$$\implies \boxed{\phi_\ell : H^1_f \longrightarrow (T/M)^{Fr(\ell)=1} \underset{\sim}{\xleftarrow{can}} G(\ell) \otimes \mathrm{Hom}(G(\ell), (T/M)^{Fr(\ell)=1}) = G(\ell) \otimes H^1_s}$$

~~Key~~ **Similarly**: if $\ell \in \mathcal{P}^1_{M,F}$ and $\lambda | \ell$ in F, then $F_\lambda = \mathbb{Q}_\ell$, $Fr(\lambda) = Fr(\ell)$ and we obtain $\phi_\lambda : H^1_f = H^1_{ur}(F_\lambda, T/M) \longrightarrow G(\ell) \otimes (H^1(F_\lambda, T/M)/H^1_f)$.

Key Property: if $\lambda | \ell | n \in \mathcal{N}^1_{M,F}$, $M M_1^2 | M'$, then

$$\boxed{\pm \phi_\lambda(\kappa_{F,M}(n/\ell)_\lambda) = \sigma_\ell \otimes \kappa_{F,M}(n)^s_\lambda}$$

$(\implies \mathrm{ind}_p(\kappa_{F,M}(n)^s_\lambda) = \mathrm{ind}_p(\kappa_{F,M}(n/\ell)_\lambda)$, $p \in \mathcal{O}$ uniformiser$)$

We need: ϕ_ℓ to be close to an isomorphism ($\iff$ idem for the map "$Q_\ell(Fr(\ell))$")

Ex: Assume $a \in \mathcal{O}$ is almost invertible ($\mathrm{ord}_p(a) \ll \mathrm{ord}_p(M)$); then the property holds if

$Fr(\ell) \equiv \begin{pmatrix} 1+a & 0 \\ 0 & 1 \end{pmatrix} \pmod{M}$ or $\equiv \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \pmod{M}$ or $\equiv \tau \pmod{M}$, $\dim_{\Phi}(V^{\tau=1}) = 1)$

but <u>not</u> if $Fr(\ell) \equiv \begin{pmatrix} 1+a & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \pmod{M}$.

Abstract annihilation results for $x \in H^1_{\mathcal{F}^D}(\mathbb{Q}, (T/M)^D)$

$\mathcal{F}$ = any Selmer structure ~~for~~ the $(\mathcal{O}/M)[G_{\mathbb{Q}}]$-module T/M

$\mathcal{F}^D$ = the dual — " — for $(T/M)^D \simeq T^*(1)/M$.

Local duality : $\forall \ell \neq \infty$

$$\underset{\mathcal{F}_\ell^D}{\underset{\cup}{H^1(\mathbb{Q}_\ell, (T/M)^D)}} \times \underset{\mathcal{F}_\ell}{\underset{\cup}{H^1(\mathbb{Q}_\ell, T/M)}} \xrightarrow{\cup} H^2(\mathbb{Q}_\ell, (\mathcal{O}/M)(1)) \xrightarrow[\sim]{inv_\ell} \mathcal{O}/M$$

induces, by definition of $\mathcal{F}^D$, a perfect pairing

$$\mathcal{F}_\ell^D \times \underbrace{H^1(\mathbb{Q}_\ell, T/M)/\mathcal{F}_\ell}_{H^1_s(\mathbb{Q}_\ell, T/M)} \xrightarrow{\cup} \mathcal{O}/M$$

Reciprocity law : $\forall x \in H^1(\mathbb{Q}, (T/M)^D)$, $\forall y \in H^1(\mathbb{Q}, T/M)$

$$\sum_{q \text{ prime number}} inv_q(x_q \cup y_q) = 0 \in \mathcal{O}/M$$

Annihilation relations : $\boxed{\forall x \in H^1_{\mathcal{F}^D}(\mathbb{Q}, (T/M)^D), \ \forall y \in H^1_{\underbrace{\mathcal{F}^{\{\ell\}}}_{\text{relaxed at one prime } \ell}}(\mathbb{Q}, T/M)}$

$\forall q \neq \ell \quad x_q \cup y_q = 0 \quad \overset{\text{rec-law}}{\Longrightarrow}$

$$\boxed{\underset{\mathcal{F}_\ell^D}{\underset{\uparrow}{x_\ell}} \cup \underset{H^1_s(\mathbb{Q}_\ell, T/M)}{\underset{\uparrow}{y_\ell^s}} = 0 \in \mathcal{O}/M}$$

One needs to find ~~many~~ y with $y_\ell^s \neq 0$ in order to get a usefull upper bound for $H^1_{\mathcal{F}^D}(\mathbb{Q}, (T/M)^D)$.

Below : the simplest general annihilation result for $H^1_{\underbrace{(\mathcal{F}_{BK})_{\{p\}}}_{\text{strict at } p}}(\mathbb{Q}, T^D)$ (only one ℓ and one y are required).

Thm (Rubin, Euler Systems, Thm 2.2.3) Assume that

(a) V is an irreducible $\Phi[G_{\mathbb{Q}}]$-module

(b) $V \neq$ the trivial representation of $G_{\mathbb{Q}}$

(c) $\exists\ \tau \in G_{\mathbb{Q}(\mu_{p^\infty})}$ such that $\dim_{\Phi}(V^{\tau=1}) = 1$

(d) $c_{\mathbb{Q}} \notin H^1(\mathbb{Q}, T)_{tors}$ $\left[c_{\mathbb{Q}} = \text{the bottom element of an Euler system } c_{F'} \in H^1(F', T),\ \mathbb{Q} \underset{f}{\subsetneq} F' \subset \mathcal{K}\right]$

Then $H^1_{\underbrace{(\mathcal{F}_{BK})_{\{p\}}}_{\text{the Bloch-Kato } \mathcal{F}_\ell \text{ for } \ell \neq p, \text{ with } \{0\} = \mathcal{F}_p}}(\mathbb{Q}, T^D)$ is finite.

Pf of Thm. Throughout the proof $C \in \mathbb{N}$ is a constant depending onf on T, but its actual value may change from one line to another (e.g., $2C$ will be replaced by C).

It is enough to show: $\forall M = p^k \ \forall x \in \underbrace{H^1_{\mathcal{F}^D}(\mathbb{Q}, (T/M)^D)}_{S_M}$ $\quad p^C x = 0$ $\quad \boxed{\text{Fix } M \text{ and } x}$

($\mathcal{F} = \mathcal{F}_{BK}^{\{p\}}$ Selmer structure on T ($\Rightarrow$ on T/M) given by the Bloch-Kato condition at all $\ell \neq p$, and by $\mathcal{F}_p = H^1(\mathbb{Q}_p, T)$

$\Rightarrow \mathcal{F}^D = (\mathcal{F}_{BK})_{\{p\}}$ for T^D ($\Rightarrow$ for T^D/M))

The Machinery of abstract annihilation relations discussed above implies that it is enough to find a prime $\ell \nmid Np$ such that:

$(1)_\ell$ $\exp(x_\ell; \mathcal{F}^D_\ell) \geq \exp(x; S_M) - C$

$(2)_\ell$ $\exists \ \mathcal{O}/M \longrightarrow \mathcal{F}^D_\ell$ with Ker, Coker killed by p^C ($\overset{\text{local duality}}{\Longrightarrow}$ idem for $H^1_s(\mathbb{Q}_\ell, T/M)$)

$(3)_\ell$ $\exists \ y \in H^1_{\mathcal{F}^{\{\ell\}}}(\mathbb{Q}, T/M)$ $\quad \mathrm{ind}_p(y^s_\ell; H^1_s(\mathbb{Q}_\ell, T/M)) \leq C$

Indeed, the annihilation relation

$$\begin{tikzcd}
x_\ell \otimes y^s_\ell \arrow[r, mapsto] & x_\ell \cup y^s_\ell = 0 \\
\mathcal{F}^D_\ell \times H^1_s(\mathbb{Q}_\ell, T/M) \arrow[r, "\cup"', "\text{perfect pairing}"'] & \mathcal{O}/M
\end{tikzcd}$$

almost isom. to $\mathcal{O}/M$, by $(2)_\ell$

implies (by $(3)_\ell$) that $p^C x_\ell = 0$

$\overset{(1)_\ell}{\Longrightarrow} p^C x = 0$.

Constructing ℓ and y: want to take $\boxed{y = \kappa_{\mathbb{Q},M}(\ell)}$ for well-chosen $\ell \in \mathcal{P}^1_{Mp^{k_0}}$ (k_0 = const., $M = p^k$, $k \gg 0$)

Consequences of the assumption (c):

- whenever $\ell \nmid Np$ is a prime satisfying $\mathrm{Fr}(\ell)\big|_{\mathbb{Q}(\mu_M, T/M)/\mathbb{Q}} = \tau\big|_{\mathbb{Q}(\mu_M, T/M)}$ $(*)$

$\Downarrow$ $M \mid (\ell - 1)$

$M \mid P_\ell(1)$

$\Rightarrow M \mid (\ell-1)$, $(2)_\ell$ holds

$\Rightarrow \ell \in \mathcal{P}^1_M$

Correction: replace here M by Mp^{k_0} to make $\kappa_{\mathbb{Q},M}(\ell)$ defined.

- in addition, $(2)_\ell \Longrightarrow \phi_\ell : \mathcal{F}_\ell = H^1_{ur}(\mathbb{Q}_\ell, T/M) \longrightarrow H^1_s = H^1(\mathbb{Q}_\ell, T/M)/\mathcal{F}_\ell$

(which is defined, since $\ell \in \mathcal{P}^1_M$) satisfies $p^C \mathrm{Ker}(\phi_\ell) = 0 = p^C \mathrm{Coker}(\phi_\ell)$

$\Longrightarrow$ $\boxed{\mathrm{ind}_p(y^s_\ell; H^1_s) \leq \mathrm{ind}_p(\kappa_{\mathbb{Q},M}(1)_\ell; \mathcal{F}_\ell) + C \ \pm \phi_\ell(\kappa_{\mathbb{Q},M}(1)_\ell)}$

Consequence of (d) : $\underline{\mathrm{ind}_p(\kappa_{\mathbb{Q},M}(1)) \leq C}$

As a result, $(3)_\ell$ will be a consequence of

$(1')_\ell$ $\exp(\kappa_{\mathbb{Q},M}(1)_\ell ; \mathcal{F}_\ell) \geq \exp(\kappa_{\mathbb{Q},M}(1); H^1_{\mathcal{F}}(\mathbb{Q}, T/M)) - C$

It remains to ~~choose~~ find $\ell \in P'_M$ satisfying $(1)_\ell$ and $(1')_\ell$.

Recall : $x \xmapsto{ev_{Fr(\ell)}} x_\ell \in (T/M)/(Fr(\ell)-1) = (T/M)/(\tau-1)$

Notation : $L_M := \mathbb{Q}(T/M, (T/M)^D) = \mathbb{Q}(T/M, \mu_M)$

$L := \bigcup_{k\geq 1} L_{p^k} = \mathbb{Q}(V, \mu_{p^\infty})$, $L' := \mathbb{Q}(V)$

(for $X \subset \overline{\mathbb{Q}}$, $\mathbb{Q}(X) := \overline{\mathbb{Q}}^{\{g \in G_{\mathbb{Q}} \mid \forall x \in X \ g(x) = x\}}$)

Lemma. Given $x \in H^1(\mathbb{Q}, T/M)$, $y \in H^1(\mathbb{Q}, (T/M)^D)$, then $\exists \gamma \in G_{L_M}$ such that

$\exp(x(\gamma\tau); (T/M)/(\tau-1)) \geq \exp(\mathrm{res}_{L_M/\mathbb{Q}, T/M}(x); H^1(L_M, T/M)) - C$

$\exp(y(\gamma\tau); (T/M)^D/(\tau-1)) \geq \exp(\mathrm{res}_{L_M/\mathbb{Q}, (T/M)^D}(y); H^1(L_M, (T/M)^D)) - C$

Pf : Linear algebra and the fact that

$\forall \gamma \in G_{L_M}$ $x(\gamma\tau) \equiv x(\gamma) + x(\tau) \pmod{(\tau-1)(T/M)}$ (and idem for y)

Next step : analysis of $\mathrm{res}_{L_M/\mathbb{Q}, X} : H^1(\mathbb{Q}, X) \longrightarrow H^1(L_M, X)$ $X = T/M$, $(T/M)^D$

Prop. (1) $[\exists C \ \forall M \ p^C \underbrace{\mathrm{Ker}(\mathrm{res}_{L_M/\mathbb{Q}, T/M})}_{H^1(L_M/\mathbb{Q}, T/M)} = 0 \iff H^1(L/\mathbb{Q}, V) = 0]$ (idem for $(T/M)^D$, $V^*(1)$)

(2) If V is a semisimple $\overline{\mathbb{Q}}[G_{\mathbb{Q}}]$-module, then $H^1(L'/\mathbb{Q}, V) = 0$.

(3) If V is an irreducible —— " —— and $V \neq$ trivial repr. (resp. and $V^*(1) \neq$ trivial repr.) $\Rightarrow H^1(L/\mathbb{Q}, V) = 0$ (resp. $H^1(L/\mathbb{Q}, V^*(1))$)

Combining $\underbrace{\text{Lemma}}_{\text{(for } x \text{ and } \kappa_{\mathbb{Q},M}(1) = y)}$ + Proposition, we see that $(1)_\ell$ and $(1')_\ell$ are satisfied for any $\ell \in \mathbb{N}$ such that $Fr_{E/\mathbb{Q}}(\ell) = \gamma\tau$, $E := \overline{\mathbb{Q}}^H$,

$H = \underbrace{\mathrm{Ker}(\mathrm{res}_{L_M/\mathbb{Q}, T/M}(x))}_{G_{L_M} \longrightarrow T/M} \cap \underbrace{\mathrm{Ker}(\mathrm{res}_{L_M/\mathbb{Q}, (T/M)^D}(y))}_{G_{L_M} \longrightarrow (T/M)^D}$,

provided Prop (3) applies, i.e., $V \neq$ trivial repr. (true by ass. (b))
$V^*(1) \neq$ trivial repr.

However, if $V^*(1) =$ trivial representation, then $T = \mathcal{O}(1)$ and $T^D = \mathbb{Q}_p/\mathbb{Z}_p \Rightarrow H^1_{(F_{BK})_{\{p\}}}(\mathbb{Q}, T^D) \subset H^1_{F_{BK}}(\mathbb{Q}, \mathbb{Q}_p/\mathbb{Z}_p) = \underbrace{\mathrm{Hom}(Cl_{\mathbb{Q}}, \mathbb{Q}_p/\mathbb{Z}_p)}_{\text{finite (in fact} = 0)}$

So it only remains to prove the Proposition above.

(1) is standard.

(2) let $G(V) := \mathrm{Gal}(L'/\mathbb{Q}) = \mathrm{Im}(G_{\mathbb{Q}} \longrightarrow \mathrm{Aut}_{\mathcal{O}}(T)) \subset GL_{\mathbb{Q}}(V)$.
this is a compact p-adic Lie group contained in $\uparrow$.
Results of Lazard ($H^i_{cont}(G(V), V)$ can be computed using $\mathbb{Q}_p$-analytic cochains) imply that it is enough to pass to the $\mathbb{Q}_p$-Lie algebra $\mathfrak{g} := \mathrm{Lie}\, G(V) \subset \mathfrak{gl}_{\mathbb{Q}_p}(V)$, which again acts semisimply on V ($\Longleftrightarrow$ $\mathfrak{g}$ is reductive and its centre acts semisimply on V). The required vanishing $H^1(\mathfrak{g}, V) = 0$ is a standard classical result.

(3) It follows from (2) that there is an exact sequence

$$0 \longrightarrow \underbrace{H^1(L'/\mathbb{Q}, V)}_{=0} \longrightarrow H^1(L/\mathbb{Q}, V) \xrightarrow{res} \underbrace{H^1(L/L', V)^{G(V)}}_{\mathrm{Hom}_{cont}(\underbrace{\mathrm{Gal}(L/L')}_{\Gamma = \mathrm{Gal}(L'(\mu_{p^\infty})/L') \xrightarrow[\chi_{cycl}]{} \mathbb{Z}_p^\times}, V)^{G(V)}} =$$

$$\overset{\uparrow}{=} \mathrm{Hom}_{cont}(\Gamma, V^{G(V)}).$$

the conjugation action of $G(V)$ on Γ is trivial.

However, $V^{G(V)} = V^{G_{\mathbb{Q}}} \subset V$ is $G_{\mathbb{Q}}$-stable, hence equal to

$$V^{G(V)} = \begin{cases} V & \text{if } V = \text{trivial representation} \\ 0 & \text{if not} \end{cases},$$ which finishes the proof of the Proposition and of the Theorem.

Rmk the assumption (b) ($V \neq$ trivial repr.) implies that $V^{G_{\mathbb{Q}}} = 0$, hence our brute force construction of $\kappa_{\mathbb{Q}, M}(n)$ applies.

Where do interesting $c_{F'} \in H^1(F', T)$ come from?

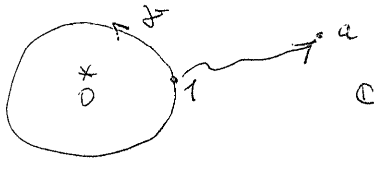
Recall: K field, $\mathrm{char}(K) \nmid n$, $\left.\begin{cases} \mathbb{G}_m = \text{mult. gp} \\ E = \text{elliptic curve} \end{cases}\right\}$ over K $\Rightarrow$ Kummer maps

$$\delta : K^\times \otimes \mathbb{Z}/n \xrightarrow{\sim} H^1(K, \mu_n) = \mathrm{Ext}^1_{\mathbb{Z}/n[G_K]}(\mathbb{Z}/n\mathbb{Z}, \mu_n)$$

$$\delta : E(K) \otimes \mathbb{Z}/n \hookrightarrow H^1(K, E[n]) = \mathrm{Ext}^1_{(\mathbb{Z}/n)[G_K]}(\mathbb{Z}/n\mathbb{Z}, E[n])$$

Topological version over $\mathbb{C}$: $\delta \longleftrightarrow$ logarithm / Abel-Jacobi map

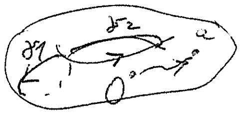
$\mathbb{G}_m$:



mixed Hodge structures with coeff. $= \mathbb{Z}$

$$\begin{array}{ccc} \mathbb{C}^\times = \mathbb{C} - \{0\} & \xrightarrow[\sim]{\log} & \mathbb{C}/2\pi i \mathbb{Z} = \mathrm{Ext}^1_{\mathbb{Z}\text{-MHS}}(\mathbb{Z}(0), H_1(\mathbb{C}^\times, \mathbb{Z})) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ a & \longmapsto & \int_1^a \frac{dz}{z} \ (\text{mod periods}) \end{array}$$

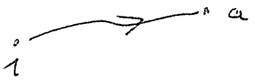
E :



$$\begin{array}{ccc} E(\mathbb{C}) & \xrightarrow[\sim]{A-J} & \mathbb{C}/(\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2) = \mathrm{Ext}^1_{\mathbb{Z}\text{-MHS}}(\mathbb{Z}(0), H_1(E(\mathbb{C}), \mathbb{Z})) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ a & \longmapsto & \int_0^a \omega \ (\text{mod periods}) \end{array}$$

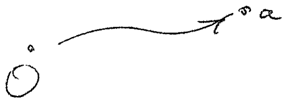
$(H^0(E(\mathbb{C}), \Omega^1) = \mathbb{C}\cdot\omega)$

$$0 \longrightarrow \underbrace{H_1(\mathbb{G}_m)}_{\mathbb{Z}\cdot\gamma} \longrightarrow H_1(\mathbb{G}_m; \{1, a\}) \longrightarrow \underbrace{H_0(\{1, a\})^{\deg = 0}}_{\mathbb{Z}([a] - [1])} \longrightarrow 0$$



$$0 \longrightarrow \underbrace{H_1(E)}_{\mathbb{Z}\cdot\gamma_1 \oplus \mathbb{Z}\cdot\gamma_2} \longrightarrow H_1(E; \{O, a\}) \longrightarrow H_0(\{O, a\})^{\deg = 0} \longrightarrow 0$$

$$\rotatebox{90}{$\in$} \qquad \mathbb{Z}([a] - [O])$$



This makes sense in every geometric cohomology theory.
For étale cohomology over K^{sep} with coefficients in $\mathbb{Z}/n\mathbb{Z}$
we recover δ.

Motivic version: $K^\times \xrightarrow{\sim} \mathrm{Ext}^1_{\underbrace{MM_K}_{\text{mixed motives over } K}}(\mathbb{Z}(0), \underbrace{H_1(\mathbb{G}_m)}_{\mathbb{Z}(1)})$

$$E(K) \xrightarrow{\sim} \mathrm{Ext}^1_{MM_K}(\mathbb{Z}(0), \underbrace{H_1(E)}_{\simeq H^1(E)(1)})$$

If $[K:\mathbb{Q}] < \infty$:

$$L(H_1(\mathbb{G}_m), s) = \zeta_K(s+1)$$
$$L(H_1(E), s) = L(E/K, s+1)$$

General picture: M motive over K, $[K:\mathbb{Q}] < \infty$

The leading term of $L(M, s)$ at $s = 0$ should be expressible in terms of $\mathrm{Ext}^i_{MM_{K,f}}(\mathbb{Z}(0), M) \subset \underbrace{\mathrm{Ext}^i_{MM_K}(\mathbb{Z}(0), M)}$ $(i = 0, 1)$

(Beilinson, Deligne, Bloch-Kato, Perrin-Riou, ...)

$\mathrm{Ext}^i_{MM_{K,f}} \longleftrightarrow$ something like $\mathcal{O}_K^\times$

$\mathrm{Ext}^i_{MM_K} \longleftrightarrow$ something like $K^\times$

Motives with coefficients : $[L:\mathbb{Q}]<\infty$, $\mathcal{O}_L \hookrightarrow \mathrm{End}(M)$, $M \in MM_K$

(let us pretend we can work $\underbrace{\text{with}}_{\text{integrally,}}$ coefficients in $\mathcal{O}_L$, rather than rationally, with coeff. in L)

Étale realisations : $\lambda \mid p$ in L, $\mathrm{char}(K) \neq p$

$$\begin{array}{ll} H_\lambda(M) & \text{continuous repr. of } G_K \text{ over } L_\lambda \quad (\dim_{L_\lambda} < \infty) \\ \cup & \\ T_\lambda(M) & G_K\text{-stable} \quad \mathcal{O}_{L_\lambda}\text{-lattice} \end{array}$$

Ex : $M = H^j(X)(m)$, X/K algebraic variety, $L=\mathbb{Q}$, $\lambda = p$

$$T_p(M) := H^j_{et}(X_{K^{sep}}, \mathbb{Z}_p)(m) \quad \Big(= \big(\varprojlim_n H^j_{et}(X_{K^{sep}}, \mathbb{Z}/p^n\mathbb{Z}) \otimes \mu_{p^n}^{\otimes m}\big) \Big)$$

$$H_p(M) = T_p(M) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \quad \big(=: H^j_{et}(X_{K^{sep}}, \mathbb{Q}_p)(m) \big).$$

λ-adic étale realisation :

$$\mathrm{Ext}^1_{MM_K}(\mathbb{Z}(0), M) \longrightarrow \mathrm{Ext}^1_{\mathbb{Z}_p[G_K]}(\mathbb{Z}_p, T_\lambda(M)) = H^1(G_K, T_\lambda(M)) \qquad \textcircled{*}$$

The Hochschild–Serre spectral sequence :

$$E_2^{i,j} = H^i(K, H^j_{et}(X_{K^{sep}}, \left\{ \begin{matrix} \mathbb{Z}_p \\ \mathbb{Q}_p \end{matrix} \right\})(m)) \Longrightarrow \underbrace{H^{i+j}_{et}(X, \left\{ \begin{matrix} \mathbb{Z}_p \\ \mathbb{Q}_p \end{matrix} \right\}(m))}_{\text{continuous étale coh.}}$$

(HS)

Induces

$$H^k_{et}(X, \left\{ \begin{matrix} \mathbb{Z}_p \\ \mathbb{Q}_p \end{matrix} \right\}(m))^0 := \mathrm{Ker}\Big(H^k_{et}(X, \left\{ \begin{matrix} \mathbb{Z}_p \\ \mathbb{Q}_p \end{matrix} \right\}(m)) \xrightarrow{\text{edge}} E_2^{0,k} \Big) \to H^1(K, \left\{ \begin{matrix} T_p \\ H_p \end{matrix} \right\})$$

for $T_p = H^{k-1}_{et}(X_{K^{sep}}, \mathbb{Z}_p)(m)$, $H_p = T_p \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$

Explicit geometric versions of $\textcircled{*}$ (possibly tensored with $\mathbb{Q}$ or $\mathbb{Z}[1/d!]$)

X smooth algebraic variety over K, $\dim(X) = d$, $p \neq \mathrm{char}(K)$

Étale cycle class: $\overbrace{CH^r(X)}^{\text{codim}=r \text{ cycles on } X / \text{rational equivalence}}$

$$\begin{tikzcd} CH^r(X) \arrow[r, "cl_X"] \arrow[d] & H^{2r}_{et}(X, \mathbb{Z}_p(r)) \arrow[d] \\ CH^r(X_{K^{sep}})^{G_K} \arrow[r, "cl_{X^{sep}}"] & E_2^{0,2r} = H^{2r}_{et}(X_{K^{sep}}, \mathbb{Z}_p(r))^{G_K} \end{tikzcd}$$

$$CH^r(X)_0 := \mathrm{Ker}(CH^r(X) \longrightarrow E^{0,2r}) \longrightarrow H^1(K, H^{2r-1}_{et}(X_{K^{sep}}, \mathbb{Z}_p)(r))$$

p-adic Abel–Jacobi map

Fact (SGA 6) $\quad K_0(X) \otimes \mathbb{Z}[1/d!] \xrightarrow{\sim} \underbrace{\bigoplus_{r=0}^{d} CH^r(X) \otimes \mathbb{Z}[1/d!]}$

(this should be true with $\mathbb{Z}[1/(d-1)!]$, I think)

(eigenspace for the Adams operations: $\forall k \geq 1$ $\quad \psi^k = k^r$ on $(K_0(X) \otimes \mathbb{Z}[1/d!])^{(r)}$)

Higher K-theory : $(i>0)$

$$K_i(X)\otimes\mathbb{Z}[1/d!] \xrightarrow{\sim} \bigoplus_{j=0}^{d} \underbrace{\left(K_i(X)\otimes\mathbb{Z}[1/(d+i)!]\right)^{(j)}}_{H^{2j-i}_{M}(X,\,\mathbb{Z}[1/(d+i)!](j))}$$

$\mathbb{Z}[1/(d+i-1)!]$ suffices if $i\geq 2$

motivic cohomology

Étale Chern classes : ~~étale~~

$$c_{i,j} : K_i(X) \longrightarrow H^{2j-i}_{et}(X,\mathbb{Z}_p(j))$$

p-adic regulator (= étale Chern character) : $(i>0)$

$$ch := \sum_{j\geq 1}^{d} \frac{c_{i,j}}{(-1)^{j-1}(j-1)!} : K_i(X) \longrightarrow \bigoplus_{j\geq 0}^{d} H^{2j-i}_{et}(X,\mathbb{Z}_p(j))\otimes\mathbb{Z}[1/(d+i)!]$$

$$\boxed{\text{Fact}: \quad ch(x\cup y) = ch(x)\cup ch(y)}$$

Rmk : if $[K:\mathbb{Q}]<\infty$ and X is smooth projective over K

$\Longrightarrow$ $H^{2j-i}_{et}(X_{\overline{K}},\mathbb{Q}_p(j))^{G_K}=0$ $\forall i>0$, cf Deligne's proof of the Weil conjectures

$$\Rightarrow \forall i>0 \quad c_{i,j} : K_i(X) \longrightarrow H^{2j-i}_{et}(X,\mathbb{Q}_p(j))^{0} \xrightarrow{HS} H^1(K, H^{2j-i-1}_{et}(X_{\overline{K}},\mathbb{Q}_p(j)))$$

Construction of $c_{i,j}$ $(i>0)$ for $X=\mathrm{Spec}(A)$, $A=\mathbb{Z}[1/p]$-algebra,

$\exists$ evaluation map $ev : \mathrm{Spec}(A)\times GL_N(A) \longrightarrow (GL_N)_{/A}$

$\Rightarrow$ the same for the simplicial scheme $B_\bullet(GL_N)_{/A}$

$GL_N \hookrightarrow GL_{N+1}$, ~~to~~ This gives $\qquad GL(A) = \varinjlim_N GL_N(A)$

$\left(g \longmapsto \begin{smallmatrix} g & 0 \\ 0 & 1\end{smallmatrix}\right)$

$$H^{2j}_{et}(B_\bullet(GL_N)_{/A},\mu_{p^k}^{\otimes j}) \xrightarrow{ev^*} H^{2j}_{et}(\underbrace{\mathrm{Spec}(A)}_{\text{scheme}}\times\underbrace{B_\bullet GL_N(A)}_{\text{simplicial set}},\mu_{p^k}^{\otimes j})$$

$$\downarrow \cap \text{ product}$$

$$\mathrm{Hom}_{\mathbb{Z}}(\underbrace{H_i(B_\bullet GL_N(A),\mathbb{Z})}_{H_i(GL_N(A),\mathbb{Z})}, H^{2j-i}_{et}(\mathrm{Spec}(A),\mu_{p^k}^{\otimes j}))$$

$$\Rightarrow \varprojlim_N H^{2j}_{et}(B_\bullet(GL_N)_{/A},\mu_{p^k}^{\otimes j}) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\underbrace{\varinjlim H_i(GL_N(A),\mathbb{Z})}_{H_i(GL(A),\mathbb{Z}) = H_i(BGL(A)^+,\mathbb{Z})}, \ \text{—}\ \text{''}\ \text{—})$$

$$\ni \quad (c_j((Std)_N - N\cdot\mathbb{1}))$$

$$\begin{tikzcd} H_i(GL(A),\mathbb{Z}) = H_i(BGL(A)^+,\mathbb{Z}) \\ K_i(A) = \pi_i(BGL(A)^+) \arrow[u, "\text{Hurewicz map}"'] \end{tikzcd}$$

Where do the terms $(-1)^{j-1}(j-1)!$ come from?

(1) The total Chern class $1+c_1+c_2+c_3+\cdots$ is not multiplicative w.r.t. the tensor product; the Chern character is:

$$e_{m+n}(E\otimes F) = \left(rk(E)rk(F) - \frac{(m+n-1)!}{(m-1)!(n-1)!}\right) c_m(E)c_n(F) + (\text{other terms})$$

(2) The ratio of the n-th coefficients of the mutually inverse series

$$e^T - 1 = \sum_{n\geq 1} \frac{T^n}{n!} \quad \text{and} \quad \log(1+T) = \sum_{n\geq 1} (-1)^{n-1}\frac{T^n}{n} \quad \text{is} \quad (-1)^{n-1}(n-1)!.$$

(3) If X is a smooth algebraic variety over K and $Z=\sum n_i Z_i$ is an algebraic cycle of codimension r on X ($n_i \in \mathbb{Z}$, $Z_i \subset \cancel{X}$ integral closed subscheme of codim $= r$), then the r-th Chern class of the class

$$\sum n_i[\mathcal{O}_{Z_i}] \in K_0'(X) = K_0(X) \quad \text{in } H^{2r}_{et}(X, \mathbb{Z}_p(r)) \quad (p \neq char(K))$$

is equal to $(-1)^{r-1}(r-1)!\, cl_X(Z)$ (possibly after $\otimes\, \mathbb{Z}\left[\frac{1}{(\dim(X)-1)!}\right]$)

(4) Newton's formulas $s_r - \sigma_1 s_{r-1} + \sigma_2 s_{r-2} - \cdots + (-1)^r r\sigma_r = 0$ relating the symmetric functions $\sigma_k = \sum_{i_1<\cdots<i_k} x_{i_1}\cdots x_{i_k}$ and $s_\ell = \sum x_i^\ell$

imply that

$$\left[s_1 = \cdots = s_{r-1} = 0 \Longrightarrow \underbrace{\frac{s_r}{r!}}_{\substack{\text{weight } r\\ \text{component}\\ \text{of the Chern character}}} = \frac{(-1)^{r-1}}{(r-1)!}\underbrace{\sigma_r}_{r\text{-th Chern class}} \right]$$

See also "The Riemann-Roch algebra" and SGA 6 for an ~~the~~ abstract treatment of the relation between

$$\begin{tikzcd} \text{K-theory} \arrow[r, shift left] & \text{cohomology} \arrow[l, shift left] \end{tikzcd}$$

$$\begin{tikzcd} \left\{\begin{array}{l}\text{special } \lambda\text{-rings}\\ \text{augmented over}\\ \text{a binomial } \lambda\text{-ring } \Lambda\end{array}\right\} \arrow[r, shift left] & \left\{\begin{array}{l}\text{augmented graded}\\ \Lambda\text{-algebras}\\ \text{(commutative in the nongraded sense)}\end{array}\right\} \arrow[l, shift left] \\ A \arrow[r, mapsto] & Gr^\bullet_\gamma(A) \\ \Lambda \times (1+A^+)^\wedge & A \arrow[l, mapsto] \end{tikzcd}$$

These functors are not quasi-inverse to each other. The compositions of them involves factors $(-1)^{r-1}(r-1)!$ in $\deg = r$.

The point is, of course, that the Chern character involves denominators.

L-functions

If $[K:\mathbb{Q}], [L:\mathbb{Q}] < \infty$ and if M is a motive over K with coefficients in $\mathbf{L}$, the characteristic polynomial of M at a prime $v \nmid \infty$ of K

$$\det_{L_p}(1 - \underbrace{Fr(v)}_{Fr_{geom}(v)} X \mid H_p(M)^{I_v}) \in L_p[X]$$

should $\overset{\boxed{\text{lie in } L[X] \text{ and}}}{}$ be independent of $p \nmid (Nv)$ in L; denote it by $P_v(X)$.

The L-function $L(M,s) := \prod_v \underbrace{P_v((Nv)^{-s})^{-1}}_{L_v(M,s)} = \sum_{n \geq 1} a_n n^{-s}$ will have coefficients $a_n \in L$. For each embedding $\iota : L \hookrightarrow \mathbb{C}$ the Dirichlet series $L(\iota M, s) := \sum_{n \geq 1} \frac{\iota(a_n)}{n^s}$ will have coeff. in $\mathbb{C}$

Expected properties: (a) $\exists$ functional equation relating

$$L(\iota M, s) \longleftrightarrow L(\iota M^*(1), -s) = L(\iota M^*, 1-s)$$

(if $M = H^i(X)(n)$ for a smooth projective variety X over K, then $M^*(1)$ is isomorphic to $M(i+1-2n)$ as a motive with coeff. in $\mathbb{Q}$)

(b) $$\underbrace{\text{ord}_{s=0} L(\iota M^*(1), -s)}_{r_M} \overset{?}{=} \dim_L \underbrace{\text{Ext}^1_{MM_K,f}(L(0), M)}_{H^1_{\mathcal{M},f}(K,M)} - \dim_L \underbrace{\text{Ext}^0_{MM_K}(L(0), M)}_{H^0_{\mathcal{M}}(K,M)}$$

(conj. of Beilinson, Deligne)

motivic extensions "extending to O_K"

Ex: $L = \mathbb{Q}$, $M = \mathbb{Q}(1)$:

$$H^1_{\mathcal{M},f} = \text{Ext}^1_{MM_K,f}(\mathbb{Q}(0), \mathbb{Q}(1)) = O_K^\times \otimes \mathbb{Q}$$

$$\cap \qquad\qquad \cap \qquad\qquad \cap$$

$$H^1_{\mathcal{M}} = \text{Ext}^1_{MM_K}(\mathbb{Q}(0), \mathbb{Q}(1)) = K^\times \otimes \mathbb{Q}$$

(c) $H^0_{\mathcal{M}}(K,M) \otimes_L L_p \xrightarrow{\sim} H^0(K, H_p(M))$ "generalised Tate's conjecture"

$H^1_{\mathcal{M},f}(K,M) \otimes_L L_p \xrightarrow{\sim} H^1_f(K, H_p(M))$ (conj. of Bloch–Kato)

(d) $\exists$ canonical motivic ζ-element $\zeta_{mot} \in \Lambda^{max} H^1_{\mathcal{M},f}(K,M)$ whose complex (Hodge) realisation (in Deligne cohomology $\Lambda^{max} H^{i+1}_{\mathcal{D}}(X_{\mathbb{R}}, \mathbb{R}(n))$ if $M = H^i(X)(n)$, X smooth projective over K) gives the leading term of $L(\iota M^*(1), -s)$ (hence also of $L(\iota M, s)$) at $s=0$ (conj. of Beilinson). The p-adic étale realisation of ζ_{mot} will land in $\Lambda^{max} \underbrace{H^1_f(K, H_p(M))}$

via H–S, related to $H^{i+1}_{et}(X, \mathbb{Q}_p(n))$ if $M = H^i(X)($

So, if M is pure of weight $w = wt(M) \neq 0$ (e.g., if $M = H^i(X)(n)$, X smooth projective, $w = i - 2n \neq 0$), then we should obtain canonical $\underset{\text{p-adic } \zeta-}{}$ elements in

$$\Lambda^{r_M} H^1_f(K, T_p(M)), \text{ for a suitable lattice } T_p(M) \subset H_p(M$$

If $M = H^i(X)(n)$, X smooth projective over K, $w = i - 2n < -1$, one should have $(L = \mathbb{Q})$

$$\begin{tikzcd}
H^1_{\mathcal{M}}(K, M) = \mathrm{Ext}^1_{MM_K}(\mathbb{Q}(0), M) = H^{i+1}_{\mathcal{M}}(X, \mathbb{Q}(n)) \\
\cup \qquad\qquad\qquad\qquad\qquad\qquad \parallel \\
H^1_{\mathcal{M},f}(K, M) = \text{image in} \nearrow (K_{2n-i-1}(X) \otimes \mathbb{Q})^{(n)} \\
\qquad\qquad\qquad\qquad \text{of } K_{2n-i-1}(\text{regular model } \mathcal{X}/\mathcal{O}_K \text{ of } X) \otimes \mathbb{Q} \arrow[u]
\end{tikzcd}$$

If $\underbrace{w}_{i-2n} = -1$ $\left(\Leftrightarrow s = 0 \text{ is the "central point" of the functional equation relating } L(M, s) = L(H^i(X), s+n) \updownarrow L(M^*(1), -s) = L(H^i(X), \underbrace{-s+i+1-n}_{-s+w+1+n}) \right)$

then one should have

$$H^1_{\mathcal{M}}(K, M) = H^{2n}_{\mathcal{M}}(X, \mathbb{Q}(n))^0 = \underbrace{((K_0(X) \otimes \mathbb{Q})^{(n)})^0}_{(CH^r(X) \otimes \mathbb{Q})^0 \text{ homologically trivial elements of } CH^r(X) \otimes \mathbb{Q}}$$

$$\parallel$$

$$H^1_{\mathcal{M},f}(K, M)$$

Twists

Motivic zeta elements for twists of M by characters χ of the Galois group $\mathrm{Gal}(F/K)$ of $K \underset{f}{\hookrightarrow} F$ $(\subset K^{ab})$ should give rise to element of $\Lambda^{r_M^{(\chi)}} H^1_f(K, H_p(M) \otimes \chi)$, with $r_M^{(\chi)}$ depending only on the signatures of χ at the real primes of K. These can be packaged into elements (for χ totally even) of $\Lambda^{r_M}_{\mathbb{Z}_p[\mathrm{Gal}(F/K)]} H^1_f(F, H_p(M))$ — an Euler system of rk $= r_M$.

Slightly different versions of <u>motivic cohomology</u> (even with integral coefficients): X smooth algebraic variety over K, $\dim(X) = d$

$X^{(i)} := \{\text{points } x \in X \mid \operatorname{codim}_X \overline{\{x\}} = i\}$

<u>Brown – Gersten – Quillen spectral sequence</u>:

$$E_1^{i,j}(X) = \bigoplus_{x \in X^{(i)}} K_{-i-j}(k(x)) \Longrightarrow K_{-i-j}(X)$$

For open $U \subset X$, the $\left(E_1^{i,j}(U), d_1^{i,j}\right)$: $\overbrace{\bigoplus_{x\in X^{(0)}} K_i(k(x))}^{E_1^{0,-i}} \to \cdots \to \bigoplus_{x \in X^{(i-1)}} \overbrace{K_1(k(x))}^{k(x)^\times} \xrightarrow{\text{div}} \overbrace{\bigoplus_{x\in X^{(i)}} \mathbb{Z}}^{E_1^{i,-i}}$

$\underbrace{\text{form a resolution of } \mathcal{K}_i}_{\text{Gersten's conjecture, proved in this case by Quillen.}}$ (= the sheafification of $U \mapsto K_i(U)$ in the Zariski topology)

This implies that $E_2^{i,j}(X) \cong H^i_{Zar}(X, \mathcal{K}_{-j})$

<u>Ex</u>: $E_2^{i,-i}(X) = CH^i(X) \xrightarrow{\sim} H^i_{Zar}(X, \mathcal{K}_i)$

$$E_2^{i,-i-1} = \frac{\operatorname{Ker}\left(\bigoplus_{x\in X^{(i)}} k(x)^\times \xrightarrow{\text{div}} \bigoplus_{x\in X^{(i+1)}} \mathbb{Z}\right)}{\operatorname{Im}\left(\bigoplus_{x\in X^{(i-1)}} K_2(k(x)) \xrightarrow[\text{symbol}]{\text{tame}} \bigoplus_{x\in X^{(i)}} k(x)^\times\right)} \xrightarrow{\sim} H^i_{Zar}(X, \mathcal{K}_{i+1})$$

<u>Voevodsky</u>: defined $\underbrace{\text{integral}}_{\text{coeff. in } \mathbb{Z}}$ motivic cohomology $H^i_{\mathcal{M}}(X, \mathbb{Z}(j))$

and showed that, for X as above, $H^i_{\mathcal{M}}(X, \mathbb{Z}(j)) \simeq \underbrace{CH^j(X, 2j-i)}_{\text{Bloch's higher Chow group}}$

<u>Ex:</u> (1) $CH^j(X, 0) = CH^j(X)$

(2) For $0 \le i \le j$ $\exists$ map $CH^j(X, j-i) \underbrace{\longrightarrow}_{\text{isomorphism if } j=i+1 \text{ (Landsburg)}} H^i_{Zar}(X, \mathcal{K}_j)$,

---

<u>p-adic étale realisations</u> ($p \ne \operatorname{char}(K)$) Bloch–Ogus theory gives rise to

$$\begin{tikzcd} H^i_{Zar}(X, \mathcal{K}_j) \arrow[r] & H^{i+j}_{et}(X, \mathbb{Z}_p(j)) \arrow[d, dashed, "H\text{-}S"'] \\ & H^1(K, H^{i+j-1}_{et}(X_{\overline{K}}, \mathbb{Z}_p(j))) \end{tikzcd}$$

if $H^{i+j}_{et}(X_{\overline{K}}, \mathbb{Z}_p(j))^{G_K} = 0$

Examples of interesting ~~elements~~ ζ-elements (motivic / p-adic):

① Y = open modular curve $\subset X = Y \cup \{\text{cusps}\}$, $K = \mathbb{Q}$

$g_1, g_2 \in \mathcal{O}(Y)^\times$ (modular units)

$$\begin{tikzcd}
g_1 \otimes g_2 \arrow[rrr, mapsto] \arrow[d, phantom, "\in"] \arrow[ddd, bend right=60, mapsto] & & & \text{Kato's elements} \arrow[d, phantom, "\in"] \\
\mathcal{O}(Y)^\times \otimes \mathcal{O}(Y)^\times \arrow[r, "\delta \otimes \delta"] \arrow[d] & H^1_{et}(Y, \mathbb{Z}_p(1))^{\otimes 2} \arrow[r, "\cup"] & H^2_{et}(Y, \mathbb{Z}_p(2)) \arrow[r, "H-S"] & H^1(\mathbb{Q}, \underbrace{H^1_{et}(Y_{\overline{\mathbb{Q}}}, \mathbb{Z}_p(2))}_{\text{contains Galois repr. attached to wt}=2\text{ modular forms}}) \\
K_1(Y) \otimes K_1(Y) \arrow[r, "\cup"] & K_2(Y) \arrow[ur, "ch_{2,2} = -c_{2,2}"'] & & \\
 & \{g_1, g_2\} \arrow[u, phantom, "\in"] & \text{Beilinson's elements} &
\end{tikzcd}$$

(2

Kato: considered these elements in the tower $\{Y(Np^r) \subset X(Np^r)\}_{r \geq 1}$; a non-abelian twist then gives elements corresponding to p-adic Galois representations attached to modular forms of wt > 2 (with suitable Tate twists)

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② $X = Y \cup \{\text{cusps}\}$ as in ①, $K = \mathbb{Q}$

$(X_\alpha, f_\alpha) = $ (another compact modular curve equipped with $X_\alpha \xrightarrow{i_\alpha} X \times X$
f_α = a rational function on X_α)

if $n_\alpha \in \mathbb{Z}$ and $\sum_\alpha n_\alpha (i_\alpha)_* \operatorname{div}(f_\alpha) = 0 \in Z^2(X \times X)$ then

$$\underbrace{\left[\sum n_\alpha (X_\alpha \xrightarrow{i_\alpha} X, f_\alpha)\right]}_{} \in H^1_{Zar}(X \times X, \mathcal{K}_2) \quad (\simeq CH^2(X \times X, 1))$$

first considered by Beilinson
(following a suggestion of Bloch)

$$\begin{tikzcd}
H^1_{Zar}(X \times X, \mathcal{K}_2) \arrow[d] \\
H^3_{et}(X \times X, \mathbb{Z}_p(2)) \arrow[d, "H-S"] \\
H^1(\mathbb{Q}, H^2_{et}(X \times X, \mathbb{Z}_p(2))) \arrow[d] \\
H^1(\mathbb{Q}, \operatorname{Sym}^2(T_p(\operatorname{Jac}(X)))(1))
\end{tikzcd}$$

- p-adic elements in $\uparrow$ were first studied by Fle
- Mildenhall (a student of Bloch) considered at the same time very similar objects.

More general elements:

Lei – Loeffler – Zerbes

Geometric constructions of extension classes (X/K smooth proj., $[K:\mathbb{Q}]<\infty$, $\overline{X} := X \otimes_K \overline{K}$)

(1) p-adic Abel-Jacobi map:

If Z is an algebraic cycle of codim $= r$ on X with trivial cohomology class in $H^{2r}(\overline{X}, \mathbb{Z}_p(r))^{G_K}$, the pull-back of

$$\begin{tikzcd}
0 \arrow[r] & \underbrace{H^{2r-1}_{et}(\overline{X},\mathbb{Z}_p)(r)}_{T} \arrow[r] & H^{2r-1}_{et}(\overline{X}-\underset{\text{support of } \overline{Z}}{|\overline{Z}|}, \mathbb{Z}_p(r)) \arrow[r] & H^{2r}_{et,|\overline{Z}|}(\overline{X},\mathbb{Z}_p)(r) \arrow[r] & H^{2r}_{et}(\overline{X},\mathbb{Z}_p(r)) \\
 & & & \mathbb{Z}_p \arrow[u, "cl_{\overline{Z}}"] \arrow[ur, "0"'] &
\end{tikzcd}$$

via $cl_{\overline{Z}}$ gives an exact sequence of $\mathbb{Z}_p[G_K]$-modules

$0 \to T \to E \to \mathbb{Z}_p \to 0$. Its class in $\mathrm{Ext}^1_{\mathbb{Z}_p[G_K]}(\mathbb{Z}_p, T) = H^1(K,T)$ coincides with the image of (the class of Z in $CH^r(X)_0$) by the p-adic A-J map.

A Similer construction works for higher Chow groups.

(2) If $z = \sum n_\alpha (Y_\alpha \xrightarrow{i_\alpha} X, f_\alpha \in k(Y_\alpha)^\times)$, $\sum n_\alpha (i_\alpha)_* \mathrm{div}(f_\alpha) = 0 \in Z^2(X)$, $(\mathrm{codim}_X(Y_\alpha) = 1)$, $[z] :=$ ~~the~~ class of z in $H^1_{Zar}(X, \mathcal{K}_2)$ maps to 0 in $H^3_{et}(\overline{X}, \mathbb{Q}_p(2))^{G_K} = 0$, so it has a well-defined image in $H^1(K, \underbrace{H^2_{et}(\overline{X}, \mathbb{Q}_p(2))}_{T'}) = \mathrm{Ext}^1_{\mathbb{Q}_p[G_K]}(\mathbb{Q}_p, T') \simeq H^1(K,T')$

If $Z = \bigcup |Y_\alpha|$, the exact sequence of $\mathbb{Q}_p[G_K]$-modules

$$\begin{tikzcd}
0 \arrow[r] & \dfrac{\left. H^2_{et}(\overline{X}, \mathbb{Q}_p(2)) \right\} T'}{\left. \mathrm{Im}\, H^2_{et,\overline{Z}}(\overline{X}, \mathbb{Q}_p(2)) \right\} T'_0} \arrow[r] & H^2_{et}(\overline{X}-\overline{Z}, \mathbb{Q}_p(2)) \arrow[r] & H^3_{et,\overline{Z}}(\overline{X}, \mathbb{Q}_p(2)) \arrow[r] & H^3_{et}(\overline{X}, \mathbb{Q}_p(2)) \\
 & & & \mathbb{Q}_p \arrow[u] \arrow[ur, "0"'] &
\end{tikzcd}$$

étale class of $[z]$ defined via the Bloch-Ogus theory

gives rise to an extension

$0 \to T'/T'_0 \to E' \to \mathbb{Q}_p \to 0$ with a class in $H^1(K, T'/T'_0)$.

These two constructions are compatible.

$(T'_0 =$ generated by (the cycle classes of the irreducible components of $\overline{Z}$)(1))

Twists in Iwasawa theory $([K:\mathbb{Q}]<\infty)$

If V is a finite-dimensional (cont.) representation of G_K with coefficients in Φ ~~$\mathbb{C}_p$~~ $([\Phi:\mathbb{Q}_p]<\infty)$ and $T\subset V$ is a G_K-stable $\mathcal{O}=\mathcal{O}_\Phi$-lattice, for every continuous character $\chi: G_K \longrightarrow \Phi^\times$ there is a canonical isomorphism of $\Phi[\mathrm{Gal}(K(\mu_{p^\infty})/K)]$-modules

$$\boxed{H^1_{Iw}(K(\mu_{p^\infty})/K, T)\otimes\chi \xrightarrow{\sim} H^1_{Iw}(K(\mu_{p^\infty})/K, T\otimes\chi)}$$

For $K=\mathbb{Q}(\mu_m)$, $T=\mathbb{Z}_p(1)$ and $\chi=\chi_{cycl}^k$ $(k\in\mathbb{Z}_{>0})$, these twists were introduced by Soulé, who related them to the Kubota-Leopoldt p-adic L-function. Soulé also considered analogous twists for elliptic curves with complex multiplication.

The general picture should be the following: if $V=H_p(M)$ of a motive M, consider ~~$(R=\mathcal{O}[[\mathrm{Gal}(K(\mu_{p^\infty}$~~

$(k\in\mathbb{Z})$

$$\begin{tikzcd}
\Lambda^{r_M}_R\left(H^1_{Iw}(K(\mu_{p^\infty})/K, T)\right)\otimes_{\mathbb{Z}_p}\mathbb{Z}_p(1)^{\otimes k r_M} \arrow[r] & \Lambda^{r_M}_R\left(H^1_{Iw}(T\otimes_{\mathbb{Z}_p}\mathbb{Z}_p(k))\right) \arrow[d] \\
\mathcal{O}[[\mathrm{Gal}(K(\mu_{p^\infty})/K)]] \arrow[u, equal] & \Lambda^{r_M}_{\mathcal{O}} H^1(K, \underbrace{T\otimes_{\mathbb{Z}_p}\mathbb{Z}_p(k)}_{T(k)}) \arrow[d] \\
\begin{pmatrix}\text{p-adic } \zeta\text{-elts of } T\\ \text{along } K(\mu_{p^\infty})/K\end{pmatrix}\otimes\begin{pmatrix}\text{fixed gen.}\\ \text{of } \mathbb{Z}_p(1)\end{pmatrix}^{\otimes k r_M} \arrow[uu, phantom, sloped, "\in"] \arrow[dr, mapsto, bend right] & \Lambda^{r_M}_{\mathcal{O}} H^1(K\otimes\mathbb{Q}_p, T(k)) \\
& z(k) \arrow[u, phantom, sloped, "\in"]
\end{tikzcd}$$

If $k>>0$ (possibly with fixed $k \pmod 2$), then $z(k)$ $\in\Lambda^{r_M}_{\mathcal{O}} H^1_f(K\otimes\mathbb{Q}_p, T(k))$ should be related to the "at k" value~~s~~ of "the cyclotomic p-adic L-function of M".

For other values of k, the image of $z(k)$ in $\Lambda^{r_M}_{\mathcal{O}}(H^1/H^1_f)(K\otimes\mathbb{Q}_p, T(k))$ should be related to the algebraic part~~s~~ of $L(M,k)$.

This is a very important part of the story. The corresponding p-adic computations rely on a heavy dose of p-adic Hodge theory. They involve, among other things, various explicit reciprocity laws.