MOTIVES

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Motives = "building blocks of (cohomology groups of) algebraic varieties

<u>A. Grothendieck</u>: a motive has:

— a <u>zeta-function</u>

— various <u>realizations</u>

$$
\begin{tikzcd}
 & \text{Betti cohomology} \\
\begin{array}{c}\text{alg. variety}\\ \text{(motive)}\end{array} \arrow[ur, "\mathrm{TOP}"] \arrow[r, "\mathrm{AN}"] \arrow[dr, "\mathrm{ALG}"'] & \text{de Rham coh.} \\
 & \text{étale coh.}
\end{tikzcd}
$$

History : 3 stages

(1) Grothendieck (1960's) k = field

"pure motives over k" $\longleftrightarrow$ "universal (geometric) cohomology for smooth projective alg. varieties over k"

Definition (+ properties) conditional on "standard conjectures" on algebraic cycles.

(2) Beilinson, Deligne (1980's)

"mixed motives over k" $\longleftrightarrow$ univ. (geom.) coh. for all alg. var./k (+ their diagrams)

motivic cohomology $\longleftrightarrow$ universal absolute (= arithmetic) cohomology

<u>motivic cohomology was defined unconditionally</u>:

- $H^*_M(\quad)\otimes\mathbb{Q}$ = piece of alg. K-theory
  (Beilinson)
- $H^*_M(\quad)$ = higher $\underbrace{\text{Chow groups}}_{\text{generalization}}$ of divisor class group
  (S. Bloch)

(3) <u>Voevodsky, Morel</u> (1990's)

- motivic homotopy theory
- triangulated category of mixed motives with correct $H^*_M(\quad)$

  (sheafifying higher Chow gps)

Examples of pure motives (+ realizations)

$\underbrace{M \subset H^n(X)}_{\text{"pure of weight } n\text{"}}$, X smooth proj.

<u>Ex 0</u>: $\underline{\mathbb{Z}(0) = h_0(\text{point})}$ $(= h^0(pt)^\vee)$ pure, wt $= 0$

<u>Ex 1</u>: $\underline{\mathbb{Z}(1) = h_1(\mathbb{G}_m)}$ $\mathbb{G}_m = \mathbb{P}^1 - \{0, \infty\}$

the <u>Tate motive</u> (wt $= -2$) <u>not</u> proj.

$\mathbb{Z}(1)^\vee = \mathbb{Z}(-1) = h^1(\mathbb{G}_m) = h^2(\mathbb{P}^1)$ (wt $= 2$)

<u>Betti realization:</u> $(k \subset \mathbb{C})$ $\mathbb{G}_m(\mathbb{C}) = \mathbb{C}^* = \mathbb{C} - \{0\}$

$\mathbb{Z}(1)_B = H_1(\mathbb{C}^*, \mathbb{Z}) = \mathbb{Z} \cdot \gamma$

γ = unit circle

<u>de Rham realization</u>: (rather, its dual)

$\mathbb{Z}(1)^\vee_{dR} = \mathbb{Z}(-1)_{dR} = \mathbb{Z} \cdot \frac{dz}{z}$ z = standard coordinate

$\frac{dz}{z} \in \Omega^1(\mathbb{C}^*)$, log-singularities at $\infty, 0$

<u>Étale realizations</u>: (i) over $\mathbb{C}$: $n \geq 1$

$\mathbb{Z}/n\mathbb{Z}(1) = H_1(\mathbb{C}^*, \mathbb{Z}/n\mathbb{Z}) = \mu_n(\mathbb{C})$ n-th roots of unity in $\mathbb{C}$

(ii) over k s.t. $n \cdot 1 \neq 0$ in k:

$\mathbb{Z}/n\mathbb{Z}(1) = \mu_n(\bar{k})$ n-th roots of unity in $\bar{k}$ ($\bar{k}$ = sep. closure of k)

This is a <u>Galois representation</u>

(analogue of "<u>monodromy representation</u>")

Recall: Galois gps $\longleftrightarrow$ covering (= deck) transformations

topology: $p : Y \longrightarrow X$ unram. covering

$$G = \mathrm{Deck}(Y/X) := \left\{ \begin{array}{l} g : Y \longrightarrow Y \text{ cont.} \\ p \circ g = p \end{array} \right\}$$

p is **regular** $:=$ G acts transitively on $p^{-1}(x)$

$$\Updownarrow$$

$$X = G \backslash Y$$

$$\pi_1(X) = \mathrm{Deck}(\widetilde{X}/X) \qquad \widetilde{X} = \text{univ. cov.}$$

algebra: $K \subset L$ fields, $\dim_K L < \infty$

$$G = \mathrm{Aut}(L/K) := \left\{ \begin{array}{l} g : L \xrightarrow{\sim} L \text{ preserves } +, \times \\ \text{fixes } K \end{array} \right\}$$

L/K is **Galois** $:=$ $|G| \geq \dim_K L$

$$\Updownarrow$$

$$|G| = \dim_K L$$

$$\Updownarrow$$

$$K = L^G$$

Notation: $G = G(L/K)$

G_K = "limit" of $G(L/K)$ over all Galois L/K $= G(\overline{K}/K)$

Back to $\mu_n = \{n\text{-th roots of } 1\}$: $\quad (n\cdot 1 \neq 0$ in $k)$

$$G(k(\mu_n)/k) \xhookrightarrow{\chi_n} \mathrm{Aut}(\mu_n) = GL_1(\mathbb{Z}/n\mathbb{Z})$$
$$\sigma \longmapsto a$$

$(\forall \xi \in \mu_n) \qquad \sigma(\xi) = \xi^a$

Periods: $\quad \mathbb{Z}(1)_B \times \mathbb{Z}(1)_{dR}^\vee \xrightarrow{\text{integration}} \mathbb{C}$

$$\gamma \;,\; \frac{dz}{z} \longmapsto \int_\gamma \frac{dz}{z} = 2\pi i$$

"Comparison isomorphism" ("de Rham"):

$$\mathbb{Z}(1)_B \otimes \mathbb{C} \xrightarrow{\sim} \mathbb{Z}(1)_{dR} \otimes \mathbb{C}$$
$$\gamma \otimes 1 \longmapsto \left(\frac{dz}{z}\right)^* \otimes \frac{1}{2\pi i}$$

Hodge types: $\quad H^2(\mathbb{P}^1(\mathbb{C}), \mathbb{C}) \simeq H^1(\mathbb{P}^1(\mathbb{C}), \Omega^1)$

Hodge type $(1,1)$

$\mathbb{Z}(1)_B \quad$ Hodge type $(-1,-1)$

Ex 2 : $M = h_1(E)$, $E_{/k}$ elliptic curve

E = smooth proj. curve of genus 1; $O_E \in E(k)$ fixed
(pure of wt $= -1$)

Betti realization: $(k \subset \mathbb{C})$

$E(\mathbb{C})$

$$M_B = H_1(E(\mathbb{C}), \mathbb{Z}) = \\ = \mathbb{Z}\gamma_1 \oplus \mathbb{Z}\gamma_2$$

(dual to $H^1(\dots)$)

Weierstrass model: (if $6 \cdot 1 \neq 0$ in k)

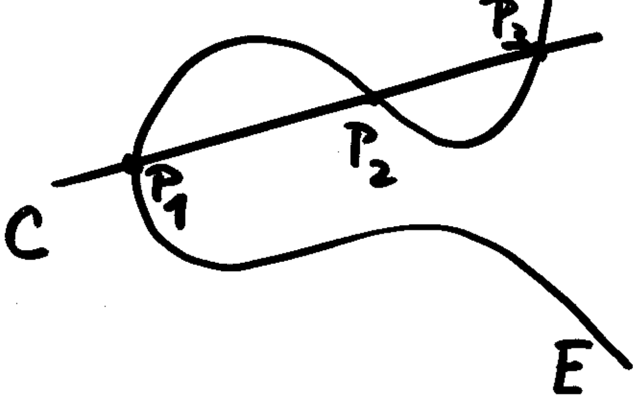
$E : y^2 = f(x)$ (+ one point O_E at infinity)

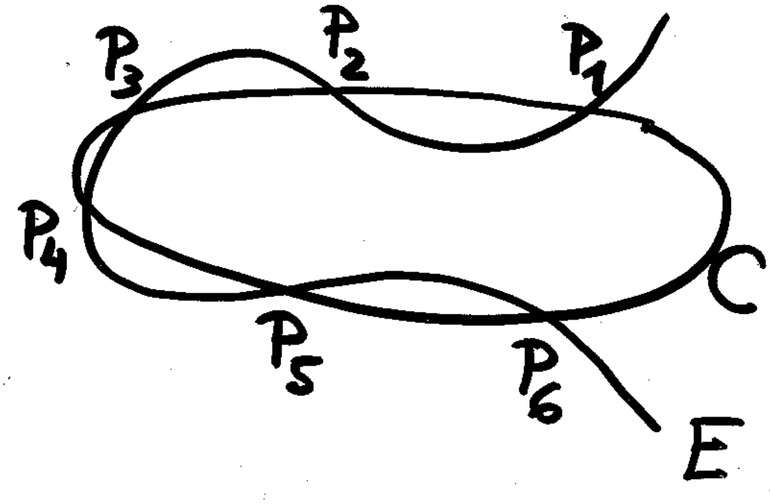
$\deg(f) = 3$, $\operatorname{disc}(f) \neq 0$
f monic

Abelian group law on $E(K)$: $(K \supset k$ field$)$

$\forall$ proj. curve $C \subset \mathbb{P}^2_K$ $\quad O_E$ = zero
of $\deg(C) = d$ s.t. $\underbrace{(C \cap E)(\overline{K})}_{\{P_1, \dots, P_{3d}\}}$ is finite

$$P_1 \boxplus \cdots \boxplus P_{3d} = O_E$$





de Rham realization:

$$M_{dR}^{\vee} = \mathbb{Z}\omega \oplus \mathbb{Z}\eta , \quad \omega = \frac{dx}{y} \in \Omega^1(E)$$

$$\eta = \frac{x\,dx}{y} \in \Omega^1(E - \{O_E\})$$

$$\mathrm{res}_{O_E}(\eta) = 0$$

periods: $(k = \mathbb{C})$

$$\omega_j = \int_{\gamma_j} \omega , \quad \eta_j = \int_{\gamma_j} \eta \quad (j = 1, 2)$$

integration: $M_B \otimes \mathbb{C} \xleftrightarrow{\sim} M_{dR} \otimes \mathbb{C}$

has matrix $\begin{pmatrix} \omega_1 & \eta_1 \\ \omega_2 & \eta_2 \end{pmatrix}$

Legendre: $\det(\text{—}\,''\,\text{—}) = 2\pi i$

Complex uniformization: $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 \subset \mathbb{C}$ lattice

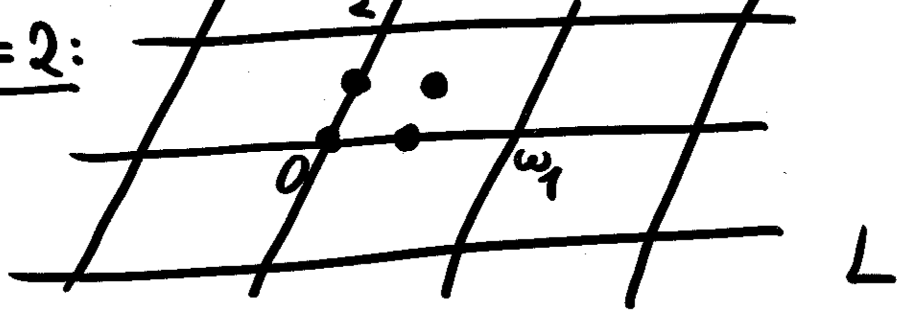
$$\begin{array}{ccc} \mathbb{C}/L & \xrightarrow{\sim} & E(\mathbb{C}) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ z & \longmapsto & \left(\tfrac{1}{2}\wp(z, L), \wp'(z, L)\right) \end{array}$$

Étale realizations: $n \geq 1$, $n \cdot 1 \neq 0$ in k

(i) $k = \mathbb{C}$

$$H_1(E(\mathbb{C}), \mathbb{Z}/n\mathbb{Z}) \simeq \tfrac{1}{n}L/L \simeq \underbrace{E(\mathbb{C})_n}_{\text{pts of order } n}$$

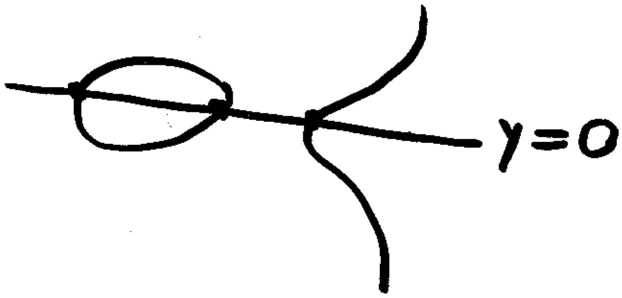
$n = 2$:



(ii) Over any k: $\quad (n \cdot 1 \neq 0$ in $k)$

$$E_n := E_n(\bar{k}) \simeq (\mathbb{Z}/n\mathbb{Z})^2$$

$n=2$: $\quad E_2 = \{O_E\} \cup (E \cap \{y=0\})$



$n=3$: $\quad E_3 = \{$inflexion pts of $E\}$

$k(E_n)/k$ is a Galois extension

Galois repr.:

$$\rho_{E,n} : G(k(E_n)/k) \hookrightarrow \mathrm{Aut}(E_n) \simeq GL_2(\mathbb{Z}/n\mathbb{Z})$$

A. Weil: $\chi_n = \det(\rho_{E,n}) (\Leftrightarrow \Lambda^2 E_n \xrightarrow{\sim} \mu_n)$ ([redacted])
(as repr. of $G(k(E_n)/k)$)

pairing
$$E_n \times E_n \longrightarrow \mu_n$$
$$P, Q \longmapsto f_P(Q)$$

$$\mathrm{div}(f_P) = n(P) - n(O_E) \qquad f_P \in k(E)^*$$

Motivically:

$$\Lambda^2 h_1(E) = \mathbb{Z}(1)$$

Betti, de Rham $\nearrow$ Legendre's relation

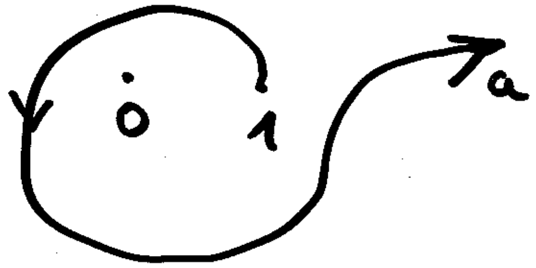
étale $\searrow$ Weil pairing

Examples of mixed motives

Ex 1: $a \in k^* \rightsquigarrow$ Kummer motive $\mathcal{K}_a$

$\mathcal{K}_a \subset h_1(\mathbb{G}_m, \{1,a\})$ (relative coh.)

consists of paths $(-)$ s.t. $\partial(-) \in \mathbb{Z}([a]-[1])$



$\mathbb{G}_m = \mathbb{P}^1 - \{0,\infty\}$

Betti realization: $(k \subset \mathbb{C})$

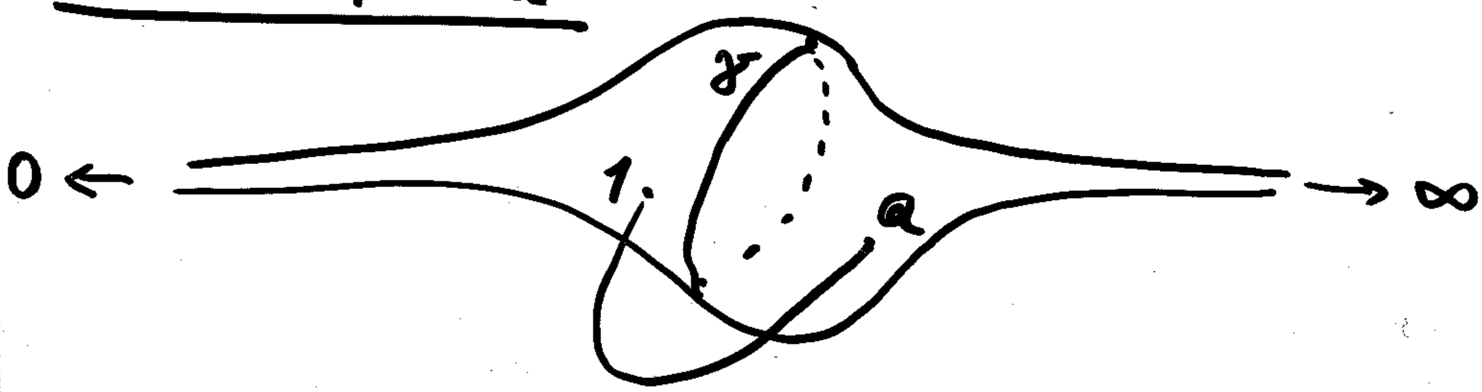
$$\begin{tikzcd}
0 \arrow[r] & H_1(\mathbb{C}^*, \mathbb{Z}) \arrow[r] \arrow[d, equal] & H_1(\mathbb{C}^*, \{1,a\}, \mathbb{Z}) \arrow[r] & \underbrace{H_0(\{1,a\}, \mathbb{Z})}_{\mathbb{Z}[a]\oplus\mathbb{Z}[1]} \arrow[r, "\Sigma"] & \underbrace{H_0(\mathbb{C}^*)}_{\mathbb{Z}} \arrow[r] & 0 \\
0 \arrow[r] & \mathbb{Z}\cdot\gamma \arrow[r] & (\mathcal{K}_a)_B \arrow[r] \arrow[u, hook] & \mathbb{Z}([a]-[1]) \arrow[r] \arrow[u, hook] & 0 &
\end{tikzcd}$$

Periods of $\mathcal{K}_a$ determine $\log(a)$ ($\Rightarrow$ also a):

"Abel-Jacobi map": $\log : \mathbb{C}^* \xrightarrow{\sim} \mathbb{C}/2\pi i\mathbb{Z}$

$$a \longmapsto \int_1^a \frac{dz}{z} \left(\mathrm{mod} \int_{\mathbb{Z}\gamma} \frac{dz}{z} \right)$$

picture of $\mathcal{K}_a$:



Étale realizations: $n \geq 1$

(i) $k \subseteq \mathbb{C}$:

$$\begin{tikzcd}[column sep=small]
 & & & H_0(\mathbb{C}^*, \mathbb{Z}/n\mathbb{Z}) \arrow[r] & 0 \\
0 \arrow[r] & H_1(\mathbb{C}^*, \mathbb{Z}/n\mathbb{Z}) \arrow[r] \arrow[d, equal] & H_1(\mathbb{C}^*, \{1,a\}, \mathbb{Z}/n\mathbb{Z}) \arrow[r] \arrow[d, phantom, "\cup"] & H_0(\{1,a\}, \mathbb{Z}/n\mathbb{Z}) \arrow[u] & \\
0 \arrow[r] & \underbrace{\mu_n}_{\text{elts of order } n \text{ in } \bar{k}^*} \arrow[r] & \underbrace{\underbrace{(\mathcal{K}_a)_{\mathbb{Z}/n\mathbb{Z}}}_{\text{elts of order } n \text{ in } \bar{k}^*/a^{\mathbb{Z}}}}_{\text{generated by } \mu_n,\ \sqrt[n]{a}} \arrow[r] & (\mathbb{Z}/n\mathbb{Z})\cdot([a]-[1]) \arrow[r] \arrow[u, hook] & 0
\end{tikzcd}$$

$(*)_a$

(ii) k arbitrary, ($n \cdot 1 \neq 0$ in k)

$\zeta_n \in \mu_n$ (a fixed generator)

$\sqrt[n]{a} \in \bar{k}$ (a fixed solution of $x^n - a = 0$)

$$\sigma \in G(k(\mu_n, \sqrt[n]{a})/k), \quad \sigma : \zeta_n \longmapsto \zeta_n^{\chi_n(\sigma)}$$
$$\sqrt[n]{a} \longmapsto \sqrt[n]{a} \cdot \zeta_n^{c(\sigma)}$$

$$\downarrow \rho_a$$

$$\begin{pmatrix} \chi_n(\sigma) & \boxed{c(\sigma)} \\ 0 & 1 \end{pmatrix} \in \begin{pmatrix} (\mathbb{Z}/n\mathbb{Z})^* & \mathbb{Z}/n\mathbb{Z} \\ 0 & 1 \end{pmatrix} \subset GL_2(\mathbb{Z}/n\mathbb{Z})$$

is a <u>group homomorphism</u>.

$(*)_a$ is an exact sequence of repr. of G_k

$$0 \to \mu_n \to ? \to \mathbb{Z}/n\mathbb{Z} \to 0$$

Its class lies in

$$H^1(G_k, \mu_n) := \mathrm{Ext}^1_{\mathbb{Z}/n\mathbb{Z}[G_k]}(\mathbb{Z}/n\mathbb{Z}, \mu_n)$$

and is given by the class of $\boxed{c(\sigma)}$.

__Kummer theory__ :

$$k^* \otimes \mathbb{Z}/n\mathbb{Z} = k^*/k^{*n} \xrightarrow{\sim} H^1(G_k, \mu_n)$$

$$a \otimes 1 \longmapsto \text{class of } \rho_a \ (= \text{of } c(\sigma))$$

is an __isomorphism__.

__Motivically__ :

$$0 \to \mathbb{Z}(1) \to \mathcal{K}_a \to \mathbb{Z}(0) \to 0$$

__Expected__ : $MM_k = \{\text{mixed motives over } k\}$

$$\mathrm{Ext}^1_{MM_k}(\mathbb{Z}(0), \mathbb{Z}(1)) \xleftarrow{\sim} k^*$$

$$\text{class of } \mathcal{K}_a \longleftarrow a$$

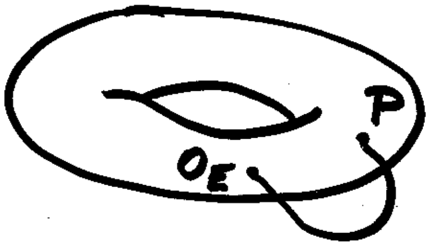
Betti / de Rham: class of $\log(a) \in \mathbb{C}/2\pi i$

étale: class of $a \otimes 1$ in $k^* \otimes \mathbb{Z}/n\mathbb{Z}$ $(n \cdot 1 \neq 0 \text{ in } k)$

Ex 2: $E_{/k}$ elliptic curve, $P \in E(k)$

$M(P) \subset h_1(E, \{O_E, P\})$ consisting of paths (with $\partial(-) = [P] - [O_E]$)

$M = h_1(E)$



$E(\mathbb{C})$

As before:

$$0 \to M \to M(P) \to \mathbb{Z}(0) \to 0$$

in various realizations.

$k \subset \mathbb{C}$: periods of $M(P)$ determine $AJ(P)$ ($\Rightarrow P$):

Abel-Jacobi map

$$AJ : E(\mathbb{C}) \xrightarrow{\sim} \mathbb{C}/(\mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2)$$

$$P \longmapsto \int_{O_E}^{P} \omega \ \Big(\text{mod} \underbrace{\int \omega}_{\text{closed paths}}\Big)$$

Étale realizations: ($n \cdot 1 \neq 0$ in k)

$$0 \to \underbrace{E_n}_{\substack{\text{elts of order } n \\ \text{in } E(\bar{k})}} \longrightarrow \underbrace{M(P)_{\mathbb{Z}/n\mathbb{Z}}}_{\substack{\text{elts of order } n \\ \text{in } E(\bar{k})/\mathbb{Z}\cdot P}} \longrightarrow \mathbb{Z}/n\mathbb{Z} \to 0$$

elts of order n in $E(\bar{k})/\mathbb{Z}\cdot P$: gen. by E_n and $\frac{1}{n}P$

$Q := \frac{1}{n}P \in E(\overline{k})$ — fixed solution of $\underbrace{[n]}_{\text{mult. on } E(\overline{k})} Q = P$

$\updownarrow$

analogue of $\sqrt[n]{a}$

$$\sigma \in G(k(E_n, \tfrac{1}{n}P)/k) \xrightarrow{\text{homom.}} \left\{ \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & 1 \end{pmatrix} \right\} \subset GL_3(\mathbb{Z}/n\mathbb{Z})$$

$$\sigma \mapsto \begin{pmatrix} \boxed{\rho_{E,n}(\sigma)} & \overbrace{\begin{matrix} * \\ * \end{matrix}}^{z(\sigma)} \\ 0 \quad 0 & 1 \end{pmatrix}$$

σ acts on $E_n \cong (\mathbb{Z}/n\mathbb{Z})^2$ by $\rho_{E,n}(\sigma) \in GL_2(\mathbb{Z}/n\mathbb{Z})$

$\sigma(\frac{1}{n}P) = \frac{1}{n}P \boxplus z(\sigma)$, $z(\sigma) \in E_n$

$$\underbrace{E(k) \otimes \mathbb{Z}/n\mathbb{Z}}_{E(k)/nE(k)} \hookrightarrow \underbrace{\mathrm{Ext}^1_{\mathbb{Z}/n\mathbb{Z}[G_k]}(\mathbb{Z}/n\mathbb{Z}, E_n)}_{H^1(G_k, E_n)}$$

$$P \otimes 1 \longmapsto \text{class of } z(\sigma)$$

Unfortunately, this map is <u>not</u> surjective ($\Longrightarrow$ it is <u>very hard</u> to determine $E(k)/nE(k)$ if k is a number field, e.g. $k = \mathbb{Q}$)

Motivically:

$$0 \longrightarrow h_1(E) \longrightarrow M(P) \longrightarrow \mathbb{Z}(0) \longrightarrow 0$$

Expected:

$$\begin{array}{ccc} E(k) & \xrightarrow{\sim} & \mathrm{Ext}^1_{MM_k}(\mathbb{Z}(0), h_1(E)) \\ \psi & & \psi \\ P & \longmapsto & \text{class of } M(P) \end{array}$$

Betti / de Rham $\rightsquigarrow$ Abel–Jacobi image of P

étale $\rightsquigarrow$ class of $P \otimes 1$ in $E(k) \otimes \mathbb{Z}/n\mathbb{Z}$ ($n \cdot 1 \neq 0$ in k)

ZETA FUNCTIONS

<u>Ex</u>: $\zeta(s) = \sum_{n\geq 1} n^{-s} \underset{\substack{\uparrow\\ \text{unique fact.}}}{=} \prod_{p \text{ prime}} (1+p^{-s}+p^{-2s}+\dots) = \prod_{p} (1-p^{-s})^{-1}$

p prime $\Longleftrightarrow$ $(p) = p\mathbb{Z}$ maximal ideal of $\mathbb{Z}$

$|\mathbb{Z}/p\mathbb{Z}| = |\mathbb{F}_p| = p$

<u>General case</u>: $A = \mathbb{Z}$-algebra of finite type

$A = \mathbb{Z}[T_1, \dots, T_N]/(f_1, \dots, f_M)$ $\qquad (f_j \in \mathbb{Z}[T_1, \dots, T_N])$

ideal $m \subset A$ is <u>maximal</u> $\Longleftrightarrow$ A/m is a <u>finite field</u>

$N(m) := |A/m|$

<u>Def</u>: $\zeta_A(s) = \prod_{m \in Max(A)} (1 - N(m)^{-s})^{-1}$ $\left(\begin{array}{l}\text{abs. conv. if}\\ Re(s) > \dim(A)\end{array}\right.$

<u>Ex</u>: (0) $\zeta_{\mathbb{F}_p}(s) = (1-p^{-s})^{-1}$

(1) $\zeta_{\mathbb{Z}}(s) = \prod_p \zeta_{\mathbb{F}_p}(s) = \zeta(s)$

(2) $\zeta_{\mathbb{Z}[i]}(s) = \frac{1}{4} \sum'_{(m,n)\in\mathbb{Z}^2} (m^2+n^2)^{-s} = \zeta(s) L(s)$

$$L(s) = \sum_{\substack{n\geq 1\\ 2\nmid n}} (-1)^{(n-1)/2} n^{-s} = \prod_{p\neq 2} \left(1-(-1)^{\frac{p-1}{2}} p^{-s}\right)^{-1}$$

(3) $A = \mathbb{F}_p[T]$: $\quad \zeta_{\mathbb{F}_p[T]}(s) = (1-p^{1-s})^{-1}$

<u>Two proofs</u>:

Easy method:

$$\mathrm{Max}(\mathbb{F}_p[T]) = \{ (f) \mid f \in \mathbb{F}_p[T],\ f \text{ monic, irred.} \}$$

$$\mathbb{F}_p[T]/(f) \simeq \mathbb{F}_{p^{\deg(f)}} \Longrightarrow \zeta_{\mathbb{F}_p[T]}(s) = Z(p^{-s}),$$

$$Z(t) = \prod_{\substack{f \in \mathbb{F}_p[T] \\ \text{monic irred.}}} (1 - t^{\deg(f)})^{-1} \underset{\text{U.F.}}{=} \sum_{\substack{g \in \mathbb{F}_p[T] \\ \text{monic}}} t^{\deg(g)} = \sum_{n \geq 0} p^n t^n = (1-pt)^{-1}$$

Complicated method:

$$\mathbb{F}_{p^d} \subset \mathbb{F}_{p^n} \Longleftrightarrow d \mid n$$

$\mathbb{F}_{p^n} = \{ x \in \overline{\mathbb{F}_p} \mid x^{p^n} = x \} \Longrightarrow$ in $\mathbb{F}_p[T]$, we have

$$T^{p^n} - T = \prod_{d \mid n} \underbrace{\prod_{f \in \mathbb{F}_p[T],\ \text{monic, irred.},\ \deg(f) = d}}_{f \in \mathrm{Irr}_d} f \Longrightarrow \sum_{d \mid n} d \cdot |\mathrm{Irr}_d| = p^n$$

As $\displaystyle -\log(1-x) = \sum_{k \geq 1} \frac{x^k}{k}$,

$$\log Z(t) = -\sum_{d \geq 1} |\mathrm{Irr}_d| \log(1 - t^d) = \sum_{d, k \geq 1} |\mathrm{Irr}_d| \frac{t^{kd}}{k} \overset{n = kd}{=} \sum_{n \geq 1} \underbrace{\Big(\sum_{d \mid n} d |\mathrm{Irr}_d| \Big)}_{p^n} \frac{t^n}{n} = -\log(1 - pt)$$

General case: geometric language

$\mathbb{Z}[T_1, \dots, T_N] =$ functions on $\mathbb{A}^N_{\mathbb{Z}}$ (affine space over $\mathbb{Z}$ of dim $= N$)

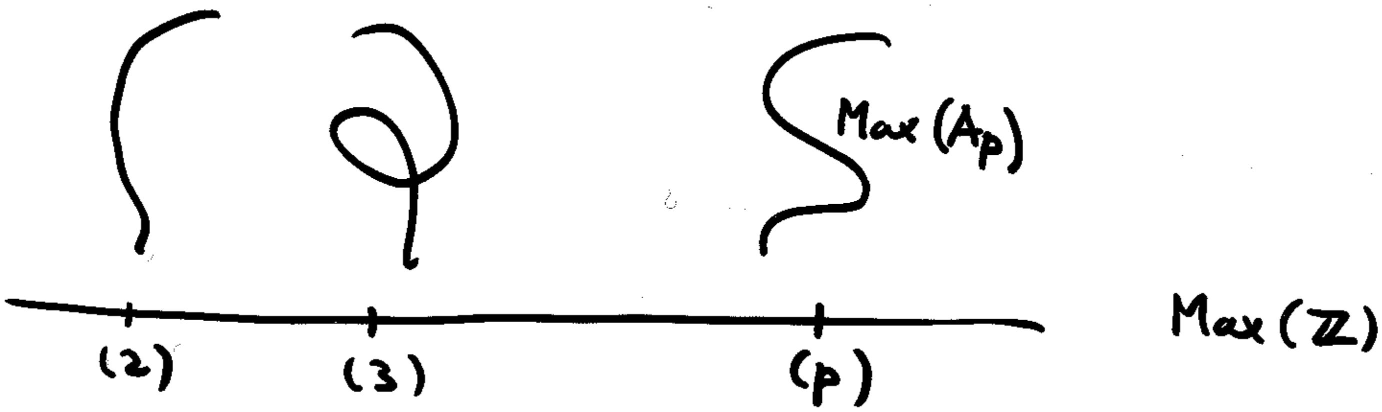
$A = \mathbb{Z}[T_1, \dots, T_N]/(f_1, \dots, f_M) =$ functions on X

$X \subset \mathbb{A}^N_{\mathbb{Z}}$ given by the equations $f_1 = \dots = f_M = 0$

("$X = \mathrm{Spec}(A)$")

picture: p prime, $A_p = A/pA$

$X_p = \mathrm{Spec}(A_p) \subset \mathbb{A}^N_{\mathbb{F}_p}$ given by equations $f_j \ (\mathrm{mod}\, p) = 0$



Prop: $$|X_p(\mathbb{F}_{p^n})| = \sum_{d|n} d \left(\sum_{\substack{m \in \mathrm{Max}(A_p) \\ N(m) = p^d}} 1 \right)$$

(gen. of $p^n = \sum_{d|n} d \cdot |\mathrm{Irr}_d|$).

Pf: $\forall R$ $\mathbb{F}_p$-algebra

$$X(R) = \{ x = (x_1, \dots, x_N) \in R^N \mid \forall j \ f_j(x_1, \dots, x_N) = 0 \in R \}$$
$$\xrightarrow{\sim} \mathrm{Hom}_{\mathbb{F}_p\text{-alg}}(A_p, R)$$
$$x \longmapsto (\text{class of } T_i \longmapsto x_i)$$

$$\left.\begin{array}{l} m \in \mathrm{Max}(A_p) \\ N(m) = p^d \end{array}\right\} \quad A_p \twoheadrightarrow \underbrace{A_p/m}_{\text{isomorphic to } \mathbb{F}_{p^d}}$$

defines d conjugate points of $X(\mathbb{F}_{p^d})$

<u>Cor</u>: $\zeta_{X_p}(s) := \zeta_{A_p}(s) = Z_{A_p}(p^{-s}) = Z_{X_p}(p^{-s})$, where

$$\log Z_{X_p}(t) = \sum_{n \geq 1} |X_p(\mathbb{F}_{p^n})| \frac{t^n}{n}$$

<u>Ex</u>: <u>Affine spaces</u>: $X = \mathbb{A}^N_{\mathbb{Z}}$, $A = \mathbb{Z}[T_1, \dots, T_N]$

$X_p = \mathbb{A}^N_{\mathbb{F}_p}$, $A_p = \mathbb{F}_p[T_1, \dots, T_N]$, $|X_p(\mathbb{F}_{p^n})| = p^{Nn}$

$$\zeta_{\mathbb{A}^N_{\mathbb{F}_p}}(s) = \exp\left(\sum_{n \geq 1} p^{nN} \frac{p^{-ns}}{n}\right) = (1 - p^{N-s})^{-1}$$

$$\zeta_{\mathbb{A}^N_{\mathbb{Z}}}(s) = \prod_p (1 - p^{N-s})^{-1} = \zeta(s - N)$$

<u>Schemes of finite type over $\mathbb{Z}$</u>:

$X = \bigcup_{i=1}^{r} \mathrm{Spec}(A_i)$, open covering

$A_i = \mathbb{Z}$-alg. of finite type

<u>Ex</u>: $\mathbb{P}^N_{\mathbb{Z}} = \bigcup_{i=0}^{N}$ (standard affine subspaces $\mathbb{A}^N_{\mathbb{Z}}$)

<u>Closed points of X</u>: $|X| = \bigcup_{i=1}^{r} \mathrm{Max}(A_i)$

$x \longleftrightarrow m_i \in \mathrm{Max}(A_i)$

$N(x) := N(m_i)$

<u>Def</u>: $\zeta_X(s) := \prod_{x \in |X|} (1 - N(x)^{-s})^{-1}$

Again: $\zeta_X(s) = \prod_p \zeta_{X_p}(s)$, $\zeta_{X_p}(s) = Z_{X_p}(p^{-s})$,

$$Z_{X_p}(t) = \exp\Big(\sum_{n\geq 1} |X_p(\mathbb{F}_{p^n})| \frac{t^n}{n}\Big) \qquad (*)$$

Ex: Projective spaces: F field

$$\mathbb{P}^N(F) = \mathbb{A}^N(F) \amalg \mathbb{P}^{N-1}(F) = \coprod_{j=0}^{N} \mathbb{A}^j(F)$$

(1) $F = \mathbb{F}_{p^n}$: $(*) \Rightarrow \zeta_{\mathbb{P}^N_{\mathbb{F}_p}}(s) = \prod_{j=0}^{N} \zeta_{\mathbb{A}^j_{\mathbb{F}_p}}(s) = \prod_{j=0}^{N} (1-p^{j-s})^{-1}$

$$\zeta_{\mathbb{P}^N_{\mathbb{Z}}}(s) = \prod_{j=0}^{N} \zeta_{\mathbb{A}^j_{\mathbb{Z}}}(s) = \prod_{j=0}^{N} \zeta(s-j)$$

(2) $F = \mathbb{C}$: $\mathbb{P}^N(\mathbb{C}) = \coprod_{j=0}^{N} \mathbb{C}^j$ cell decomposition

$$H_k(\mathbb{P}^N(\mathbb{C}), \mathbb{Z}) = \begin{cases} \mathbb{Z}(\text{closure of } \mathbb{C}^j), & k = 2j,\ 0 \leq j \leq k \\ 0, & \text{otherwise} \end{cases}$$

Motivic origin of these decompositions:

$$h(\mathbb{P}^N) = \bigoplus_{j=0}^{N} \underbrace{h^{2j}(\mathbb{P}^N)}_{h^2(\mathbb{P}^N)^{\otimes j} = \mathbb{Z}(-j)}, \quad h^2(\mathbb{P}^N) = h^2(\mathbb{P}^1) = \mathbb{Z}(-1)$$

$\zeta(s-j)$ = zeta function of $\mathbb{Z}(-j)$
(base field $= \mathbb{Q}$)

Zeta functions of smooth projective curves

$X_{/\mathbb{F}_p}$ smooth irred. proj. curve, $\mathbb{F}_q = \Gamma(X, \mathcal{O}_X)$ field of constants

g = genus of $X = \dim_{\mathbb{F}_q} \Gamma(X, \Omega_{X/\mathbb{F}_q})$

$\zeta_X(s) = Z_X(q^{-s})$, $Z_X(t) = \exp\left(\sum_{n\geq 1} |X(\mathbb{F}_{q^n})| \frac{t^n}{n}\right)$

<u>Thm</u> (E. Artin, F. K. Schmidt, H. Hasse, A. Weil)

(1) $Z_X(t) = \dfrac{P(t)}{(1-t)(1-qt)}$, $P \in \mathbb{Z}[t]$, $\deg(P) = 2g$

(2) $\zeta_X(1-s) = (q^{1/2-s})^{2-2g} \zeta_X(s)$ $\left(\Longleftrightarrow P(t) = \prod_{j=1}^{g} (1-\alpha_j t)(1-\frac{q}{\alpha_j} t)\right)$

(3) $\zeta_X(s) = 0 \Longrightarrow \mathrm{Re}(s) = \frac{1}{2}$ $\left(\Longleftrightarrow \forall j \quad |\alpha_j| = \sqrt{q}\right)$

"<u>Pf.</u>" (1), (2) $\exists$ additive formula for $Z_X(t)$ (as in the "easy method" for $X = \mathbb{A}^1$):

$Z_X(t) = \sum_{\substack{D \in \mathrm{Div}(X) \\ D \geq 0}} t^{\deg(D)}$ $\quad D = \sum_{x \in |X|} n_x (x)$, $n_x \in \mathbb{Z}$ (finite sum)

$q^{\deg(D)} = \prod_x N(x)^{n_x}$

$\mathrm{Cl}(X) = \mathrm{Div}(X)/\langle \mathrm{div}(f) \rangle$ $\quad (f \in \mathbb{F}_q(X)^*)$

$\mathcal{L}(D) = \{0\} \cup \{f \mid (f) + D \geq 0\}$, $\ell(D) = \dim_{\mathbb{F}_q} \mathcal{L}(D)$

$$Z_X(t) = \sum_{C \in \mathrm{Cl}(X)} t^{\deg(C)} \cdot \underbrace{|\{D \in C;\, D \geq 0\}|}_{|\mathbb{P}(\mathcal{L}(C))| = \frac{q^{\ell(C)}-1}{q-1}}$$

R.-R.: $\ell(C) - \ell(K_X - C) = \deg(C) - g + 1$

ε: $\exists D \in \mathrm{Div}(X)$ $\deg(D) = 1$

$\Longrightarrow$ (1), (2)

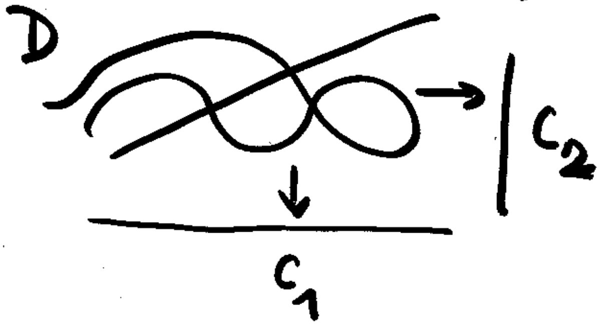
Proof of (3): requires a "positivity" property

Castelnuovo–Severi inequality: k field, C_i $(i=1,2)$ smooth irred. proj. curve over k, $X = C_1 \times C_2$

$D \in \mathrm{Div}(C_1 \times C_2)$ ($\mathbb{Z}$-linear combination of irred. curves on X)

$d_i = \deg(D \xrightarrow{\text{proj.}} C_i)$

$$\boxed{(\mathrm{CS}) \qquad \frac{D^2}{2} \leq d_1 d_2} \qquad D^2 = D \cdot D$$



Ex: $\varphi_1, \varphi_2 : C_1 \longrightarrow C_2$, Γ_{φ_i} = graph of $\varphi_i \subset C_1 \times C_2$

$g_i = \text{genus}(C_i)$ (Γ_{φ_i} smooth, isomorphic to C_1)

$D = s\Gamma_{\varphi_1} + t\Gamma_{\varphi_2}$ $(s,t \in \mathbb{Z})$

Lemma: $\Gamma_{\varphi_i}^2 = \Gamma_{\varphi_i} \cdot \Gamma_{\varphi_i} = \deg(\varphi_i)(2-2g_2)$

(CS) for D: $d_1 = s+t$, $d_2 = s\deg(\varphi_1) + t\deg(\varphi_2)$

$\Updownarrow$ Lemma

$$g_2 \deg(\varphi_1) s^2 + \left(\deg(\varphi_1) + \deg(\varphi_2) - \Gamma_{\varphi_1}\cdot\Gamma_{\varphi_2}\right) st + g_2 \deg(\varphi_2) t^2 \geq 0$$
$$\forall\, s,t \in \mathbb{Z}$$

$\Updownarrow$

$$(*) \qquad \left| \deg(\varphi_1) + \deg(\varphi_2) - \Gamma_{\varphi_1}\cdot\Gamma_{\varphi_2} \right| \leq 2 g_2 \sqrt{\deg(\varphi_1)\deg(\varphi_2)}$$

Special case: $k = \mathbb{F}_q$, $C_1 = C_2 = C$, $g = \text{genus}(C)$

$\varphi_1 = \mathrm{id}$, $\varphi_2 = \text{Frobenius}_{q^n}$: $x \longmapsto x^{q^n}$

$\deg(\varphi_1) = 1$, $\deg(\varphi_2) = q^n$, $\Gamma_{\varphi_1}\cdot\Gamma_{\varphi_2} = |C(\mathbb{F}_{q^n})|$

$(*) \Leftrightarrow \left| 1 + q^n - |C(\mathbb{F}_{q^n})| \right| \leq 2g\sqrt{q^n} \Longrightarrow (3)$ for C.

<u>Pf of Lemma:</u> X smooth proj. irred. surface
$\cup$
E —— " —— curve

<u>Adjunction:</u> $K_E = K_X + E|_E \Rightarrow \underline{2g_E - 2 = K_X\cdot E + E^2}$

$X = C_1 \times C_2$, $e_i = \deg(E \xrightarrow{\text{proj.}} C_i)$

$K_X = (K_{C_1}\times C_2) + (C_1 \times K_{C_2}) \Rightarrow K_X\cdot E = (2g_1-2)e_1 + (2g_2-2)e_2$

If $E = \Gamma_\varphi$ $(\varphi: C_1 \to C_2)$, $e_1 = 1$, $e_2 = \deg(\varphi)$, $E \xrightarrow{\sim} C_1$, $g_E = g_1$

$$\Downarrow$$

$$0 = (2g_2-2)\deg(\varphi) + \Gamma_\varphi^2 .$$

<u>Hodge index theorem:</u> k field (say, alg. closed)

X smooth irred. proj. surface; fix $X \hookrightarrow \mathbb{P}^N_k$

$H :=$ a hyperplane section of X $(H \in \mathrm{Div}(X))$

(HI) $(\forall E \in \mathrm{Div}(X)\otimes\mathbb{Q})\ (E\cdot H = 0 \Rightarrow E^2 \leq 0)$

<u>Prop.</u> (HI) $\Rightarrow$ (CS).

<u>Pf:</u> $X = C_1 \times C_2$: $h_1 := H\cdot(C_1 \times pt) \geq 1$
$h_2 := H\cdot(pt \times C_2) \geq 1$ $\qquad d := H\cdot D$

$D\cdot(pt \times C_2) = d_1$, $\quad D\cdot(C_1 \times pt) = d_2$

For $s, t \in \mathbb{Q}$, take $E := D + s(C_1 \times pt) + t(pt \times C_2)$

$E\cdot H = d + sh_1 + th_2$

$$\frac{E^2}{2} = \frac{D^2}{2} + sd_2 + td_1 + st = \left(\frac{D^2}{2} - d_1d_2\right) + (s+d_1)(t+d_2)$$

For $s := -d_1$, $t := \dfrac{-d - sh_1}{h_2} = \dfrac{d_1h_1 - d}{h_2}$, we have

$$E\cdot H = 0, \quad \frac{E^2}{2} = \frac{D^2}{2} - d_1d_2 \xRightarrow{(HI)} (CS).$$

General case:

$X \subset \mathbb{P}^N_{\mathbb{Z}}$ proj., $X_{\mathbb{Q}} \subset \mathbb{P}^N_{\mathbb{Q}}$ smooth irred., $d = \dim(X_{\mathbb{Q}})$

$\Rightarrow \exists$ finite set S_X of "bad primes" such that $(\forall p \notin S_X)$ $X_p \subset \mathbb{P}^N_{\mathbb{F}_p}$ is smooth irred., of $\dim = d$.

A. Weil's Conjectures (1948):

$(\forall p \notin S_X)$ $\quad \mathbb{F}_q := \Gamma(X_p, \mathcal{O}_{X_p})$ (field of constants)

(1) $\zeta_{X_p}(s) = \prod_{j=0}^{2d} P_j(q^{-s})^{(-1)^{j-1}}$, $\quad P_j \in \mathbb{Z}[t]$

(2) $\deg(P_j) = b_j := \dim_{\mathbb{Q}} H^j(X(\mathbb{C}), \mathbb{Q})$

(3) $P_j(t)$ is related to $P_{2d-j}(1/qt)$

(4) $P_j(t) = \prod_{k=1}^{b_j} (1 - \underbrace{\alpha_{j,k}}_{\text{algebraic integers}} t)$, $\quad |\alpha_{j,k}| = q^{j/2}$

$\left(P_j(q^{-s}) = 0 \Rightarrow \mathrm{Re}(s) = \frac{j}{2} \right)$

Earlier work:
- $\dim(X) = 1$
- explicit formulas for diagonal equations

$$X: \; a_0 X_0^n + \cdots + a_n X_N^n = 0.$$

<u>Weil's suggestion:</u> construct a (contravariant) cohomology theory $Y \longmapsto \underbrace{H^*(Y)}_{\text{vector space over a field } K \supset \mathbb{Q}}$
($Y/\mathbb{F}_q$ smooth proj.)

<u>axioms:</u> Künneth formula, Poincaré duality, $\exists$ cycle classes, ... ("Weil cohomology")

$\Downarrow$

<u>Lefschetz formula:</u> for $f: Y \longrightarrow Y$

$$\Delta \cdot \Gamma_f = \sum_{j=0}^{2d} (-1)^j \operatorname{Tr}(f^* \mid H^j(Y)) \qquad d = \dim(Y)$$

<u>Special case:</u> $f = \varphi_{q^n} : Y \longrightarrow Y$, $y \longmapsto y^{q^n}$

$$\Delta \cdot \Gamma_{\varphi_{q^n}} = |\{\text{fixed pts of } \varphi_{q^n} \text{ in } Y(\overline{\mathbb{F}}_q)\}| = |Y(\mathbb{F}_{q^n})|$$

As $\exp\left(\sum_{n \geq 1} \operatorname{Tr}(A^n) \frac{t^n}{n}\right) = \frac{1}{\det(1 - tA)}$,

$$\exp\left(\sum_{n \geq 1} |Y(\mathbb{F}_{q^n})| \frac{t^n}{n}\right) = \prod_{j=0}^{2d} (1 - t\varphi_q^* \mid H^j(Y))^{(-1)^{j-1}}$$

<u>So</u>: If $\exists$ Weil cohomology $\Longrightarrow$ (1) with

$$P_j(t) = \det(1 - t\varphi_q^* \mid H^j(X_p)) \in \underline{\underline{K[t]}}$$

Poincaré duality $\Longrightarrow$ (3)

Do Weil cohomologies exist?

Yes! Classical Weil cohomologies: étale
crystalline

Étale cohomology (for Y over $\mathbb{F}_p$):

coefficients in $\mathbb{Z}/n\mathbb{Z}$ $(p \nmid n)$

passage to a limit: $\ell \neq p$ prime

coeff. in $\mathbb{Z}_\ell = \varprojlim_m \mathbb{Z}/\ell^m\mathbb{Z}$ (ℓ-adic integers)

—" —— $\mathbb{Q}_\ell = \mathbb{Z}_\ell[\frac{1}{\ell}]$ (field of ℓ-adic numbers)

$$\mathbb{Z}_\ell = \left\{ x = \sum_{i=0}^{\infty} x_i \ell^i \mid x_i \in \{0, 1, \dots, \ell-1\} \right\}$$

$$\mathbb{Q}_\ell = \left\{ x = \sum_{i \geq i_0(x) \in \mathbb{Z}}^{\infty} x_i \ell^i \mid \text{—"—} \right\}$$

$$x = \dots\dots x_2 x_1 x_0 . x_{-1} x_{-2} \dots x_{i_0} 00000\dots$$

Ex: $\ell = 3$ $\qquad x = 1 + 3 + 3^2 + 3^3 + \dots \in \mathbb{Z}_3$

$$\begin{aligned} x &= \dots . 11111.00000 \dots \\ 2x &= \dots . 22222.00000 \dots \\ 2x+1 &= \dots . 00000.00000 \dots = 0 \end{aligned}$$

$$x = \frac{1}{1-3} = -\frac{1}{2}$$

Notation: H^*_ℓ = étale coh. with coeff in $\mathbb{Q}_\ell$ $(\ell \neq p)$

(crystalline coh. has coeff. in a finite extension of $\mathbb{Q}_p$)

H^*_ℓ is a Weil cohomology

$\Downarrow$

(1) holds with $P_{j,\ell}(t) = \det(1 - t\varphi_q^* \mid H^j_\ell(Y))$

$\uparrow$

$\mathbb{Z}_\ell[t]$

<u>Questions:</u> • is $P_{j,\ell} \in \mathbb{Q}[t]$?

• is $P_{j,\ell}$ independent of ℓ ?

• is there a generalization of (HI) implying (4) ?

<u>In char $= 0$</u>: for varieties over $k \subset \mathbb{C}$,

étale coh. with coeff. in $\mathbb{Z}/n\mathbb{Z} = H^*(X(\mathbb{C}), \mathbb{Z}/n\mathbb{Z})$

$\Downarrow$

($\forall \ell$ prime) $H^*_\ell(X) = H^*(X(\mathbb{C}), \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$

<u>Question (char $= p > 0$)</u>: for Y over $\mathbb{F}_{p^r}$ as before,

is there a Weil cohomology $H^*_?$

with coeff. in $\mathbb{Q}$ s.t. $H^*_\ell(Y) = H^*_?(Y) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$?

<u>Answer</u>: NO! (Serre)

$\exists\, Y$ s.t. (a) $\dim_{\mathbb{Q}_\ell} H^1_\ell(Y) = 2$

(b) $\exists\ \alpha, \beta : Y \longrightarrow Y$ s.t. $A = \alpha^* \in \mathrm{End}(H^1_\ell)$, $B = \beta^* \in$

satisfy $AB = -BA$, $A^2, B^2 \in \mathbb{Z}_{<0}$

(the $\mathbb{Q}$-algebra generated by A, B has no representation over $\mathbb{Q}$ of dim $= 2$)

($Y =$ ell. curve [redacted] which is "supersingular" [redacted])

Is there any motive behind this narrative?

$X \subset \mathbb{P}^N_{\mathbb{Z}}$ as before, $p \notin S_X = S$ some Weil coh.

$$\zeta_{X_p}(s) = \prod_{j=0}^{2d} \underbrace{\det(1 - \varphi_q^* q^{-s} \mid H^j(X_p))^{(-1)^{j-1}}}_{\text{"}\zeta(h^j(X_p), s)\text{"}}$$

$$\zeta_{\{S\}}(h^j(X_{\mathbb{Q}}), s) := \prod_{p \notin S} \zeta(h^j(X_p), s) \qquad X_{\mathbb{Q}} \subset \mathbb{P}^N_{\mathbb{Q}}$$

Expected: (1) $h^j(X_{\mathbb{Q}}) = M_1 \oplus \dots \oplus M_r$ (submotives)

$$?? \Updownarrow$$

$$\zeta_{\{S\}}(h^j(X_{\mathbb{Q}}), s) = \prod_{i=1}^{r} \zeta_{\{S\}}(M_i, s)$$

(2) $M = N \overset{??}{\Longleftrightarrow} \zeta_{\{S\}}(M, s) = \zeta_{\{S\}}(N, s)$

pure motives over $\mathbb{Q}$

$$\Updownarrow$$

$\forall \ell \quad M_\ell^{ss} = N_\ell^{ss}$

Ex: $C: X_1^4 + X_2^4 = X_0^4 \subset \mathbb{P}^2_{\mathbb{Z}}$ $\quad S_C = \{2\}$

(affine equation: $x^4 + y^4 = 1$) $\quad$ genus$(C_{\mathbb{Q}}) = 3$

$E: u^4 + v^2 = 1$ (+ 2 pts at ∞)

$E': 1 + a^4 = b^2$ (— " —) $\quad$ genus = 1

$$\boxed{h^1(C) = h^1(E)^{\oplus 2} \oplus h^1(E')}$$

Why is this true?

$$\begin{array}{ccccc} & & C & & \\ & \swarrow^{f_0} & \downarrow f_1 & \searrow^{f_2} & \\ E' & & E & & E \end{array}$$

$$\begin{array}{ccccc} & & (x,y) & & \\ & \swarrow & \downarrow & \searrow & \\ (a,b)=\left(\frac{y}{x},\frac{1}{x^2}\right) & & (u,v)=(x,y^2) & & (u,v)=(y,x^2) \end{array}$$

de Rham realizations: over k, $2\cdot 1\neq 0$ in k

$$H^0(C,\Omega_{C/k}) = (k\cdot 1 \oplus k\cdot x \oplus k\cdot y)\frac{dx}{y^3}$$

$$H^0(E,\Omega_{E/k}) = k\cdot\frac{du}{v}, \quad H^0(E',\Omega_{E'/k}) = k\cdot\frac{da}{b}$$

$$f_0^*\left(\frac{da}{b}\right) = -\frac{dx}{y^3}, \quad f_1^*\left(\frac{du}{v}\right) = y\frac{dx}{y^3}, \quad f_2^*\left(\frac{du}{v}\right) = -x\frac{dx}{y^3}$$

periods: of E: $\displaystyle\int_0^1 \frac{du}{\sqrt{1-u^4}} \sim B(\tfrac14,\tfrac12)$

of E': $\displaystyle\int_0^\infty \frac{da}{\sqrt{1+a^4}} \sim B(\tfrac14,\tfrac14)$

of C: $\displaystyle\int_0^1 \frac{dx}{\sqrt[4]{(1-x^4)^3}}\left\{\begin{matrix}1\\ x\\ \sqrt[4]{1-x^4}\end{matrix}\right\} \sim \left\{\begin{matrix}B(\tfrac14,\tfrac14)\\ B(\tfrac12,\tfrac14)\\ B(\tfrac14,\tfrac12)\end{matrix}\right.$

zeta functions ($\Leftrightarrow$ étale realizations): $p\neq 2$

$$p^n+1-|C(\mathbb{F}_{p^n})| \stackrel{?}{=} 2\,(p^n+1-|E(\mathbb{F}_{p^n})|) + (p^n+1-|E'(\mathbb{F}_{p^n})|)$$

$$\Uparrow$$

amusing calculation with "Jacobi sums" (= finite field analogues of $B(a,b)$).

In fact:

$$\mathrm{Jac}(C) \overset{\text{isogeneous}}{\sim} E\times E\times E'$$

Positivity in Hodge theory

$Y \subset \mathbb{P}^N(\mathbb{C})$ complex mfd, $d = \dim_{\mathbb{C}}(Y)$

$H \in H^2(Y, \mathbb{Q}) \cap H^{1,1}$ class of hyperplane section

Lefschetz operator: $L : H^k(Y, \mathbb{Q}) \xrightarrow{\cup H} H^{k+2}(Y, \mathbb{Q})$

Strong Lefschetz theorem: $(\forall\, k \leq d)$

$$L^{d-k} : H^k(Y, \mathbb{Q}) \xrightarrow{\sim} H^{2d-k}(Y, \mathbb{Q}) \quad \text{isomorphism}$$

$$\Updownarrow$$

$\exists$ $\mathfrak{sl}_2$-triple $(L, \Lambda, h = [L, \Lambda])$

$$H^k(Y, \mathbb{Q}) \xleftarrow{\Lambda} H^{k+2}(Y, \mathbb{Q}), \quad h|_{H^k(Y,\mathbb{Q})} = (k-d)\cdot 1$$

$$\Downarrow$$

Primitive decomposition:

$$H^k(Y, \mathbb{Q}) = \bigoplus_{j \geq 0} L^j P^{k-2j}, \quad P^r = H^r(Y, \mathbb{Q}) \cap \operatorname{Ker}(\Lambda)$$

Intersection pairing:

$$\begin{array}{ccc} H^k(Y, \mathbb{Q}) \times H^{2d-k}(Y, \mathbb{Q}) & \longrightarrow & H^{2d}(Y, \mathbb{Q}) \xrightarrow{\sim} \mathbb{Q} \\ x \;,\; y & \longmapsto & x \cdot y \end{array}$$

Hodge index theorem: $(\forall\, k \leq r \leq \frac{d}{2})$

$\exists$ explicit sign $\varepsilon_{k,r} \in \{\pm 1\}$ s.t.

$\forall\, x \in (H^{2r}(Y, \mathbb{R}) \cap H^{r,r}) \cap L^k P^{2r-2k}$, $x \neq 0$

$$\varepsilon_{k,r}(x \cdot L^{d-2r} x) > 0$$

Cycle classes: k = base field, $K \supset \mathbb{Q}$ coeff. field

$H^*: (\mathrm{SmProj}_{/k})^{op} \longrightarrow (\mathrm{Vect}_K)$ Weil cohomology

$X \in \mathrm{SmProj}_{/k}$, $Z^j(X) = \left\{ \sum_{i=0}^{N} \underset{\mathbb{Z}}{\underset{\uparrow}{n_i}} Y_i \;\middle|\; \begin{array}{l} Y_i \subset X \text{ irred.,} \\ \text{def. over } k, \\ \text{codim} = j \end{array} \right\}$

$$Z^j_{hom}(X)_{\mathbb{Q}} := \mathrm{Im}\left(Z^j(X) \otimes \mathbb{Q} \longrightarrow H^{2j}(X)\right)$$

(Some of) Grothendieck's Standard Conjectures:

$X_{/k}$ smooth proj. irred., $d = \dim(X)$

(Lef) $Z^r_{hom}(X \times X)_{\mathbb{Q}} \xrightarrow{L^{2d-2r}} Z^{2d-r}_{hom}(X \times X)_{\mathbb{Q}}$

is an isomorphism $(\forall\, r \le d)$

($\Rightarrow$ primitive decomposition of $Z^r_{hom}(X \times X)_{\mathbb{Q}}$)

(Index) $(\forall\, k \le r \le d)$

$(\forall\, x \in Z^r_{hom}(X \times X)_{\mathbb{Q}} \cap L^k P Z^{r-k}_{hom}(X \times X)_{\mathbb{Q}},\ x \ne 0)$

$$\varepsilon_{k,r}\left(x \cdot L^{2d-2r} x\right) > 0.$$

$k \subset \mathbb{C}$: usual Hodge index thm. $\Rightarrow$ (Index)

$k = \bar{k} \subset \mathbb{C}$: Hodge conjecture for $X_{\mathbb{C}} \times X_{\mathbb{C}}$

$\left(Z^j_{hom}(X_{\mathbb{C}} \times X_{\mathbb{C}})_{\mathbb{Q}} \overset{?}{=} H^{2j}((X \times X)(\mathbb{C}), \mathbb{Q}) \cap H^{j,j} \right) \Rightarrow$ (Lef).

$k = \mathbb{F}_q$: (Lef) + (Index) $\Longrightarrow$ the remaining Weil's conjectures

$\left(P_j(t) \in \mathbb{Q}[t], \quad |\alpha_{j,k}| = q^{j/2} \right)$

Motivic Galois groups (examples)

k field, $M_{/k}$ motive $(M \subset h(X))$ [redacted]

Want: $\underbrace{G_{mot}(M)}_{\text{alg. group over } \mathbb{Q}}$ "symmetry group of M"

alg. group over $\mathbb{Q}$ (coeff. field)

Ex 1. $M = \mathbb{Z}(1)$, $G_{mot}(M) = GL_{1/\mathbb{Q}}$, $\dim(-)=1$

realization with $\mathbb{Z}/n\mathbb{Z}$-coeff: $\mu_n = \{\sqrt[n]{1}\}$

$$\underbrace{G(k(\mu_n)/k) \xhookrightarrow{\chi_n} GL_1(\mathbb{Z}/n\mathbb{Z})}_{\text{bounded index}} \quad \text{if } [k:\mathbb{Q}]<\infty$$

periods: $2\pi i$ $\qquad$ tr.deg. $\mathbb{Q}(2\pi i)/\mathbb{Q} = 1$

Ex 2: $M = \mathcal{K}_a$, $a \in k^*$: $0 \to \mathbb{Z}(1) \to \mathcal{K}_a \to \mathbb{Z}(0) \to 0$

$$G(k(\mu_n, \sqrt[n]{a})/k) \hookrightarrow \left\{\begin{pmatrix} * & ** \\ 0 & 1 \end{pmatrix}\right\} = (GL_1 \ltimes \mathbb{G}_a)(\mathbb{Z}/n\mathbb{Z})$$

$* = \chi_n$

$**$ given by $a \pmod{k^{*n}} \in k^*/k^{*n}$

$\mu(k) := k^*_{tors} = \{\text{roots of unity in } k\}$

Assume $[k:\mathbb{Q}]<\infty$:

$**$ has bounded $\begin{Bmatrix}\text{order} \\ \text{index}\end{Bmatrix}$ in $\mathbb{Z}/n\mathbb{Z} \Longleftrightarrow \begin{cases} a \in \mu(k) \\ a \notin \mu(k) \end{cases}$

$$G_{mot}(\mathcal{K}_a) = \begin{cases} \left\{\begin{pmatrix} * & 0 \\ 0 & 1 \end{pmatrix}\right\} \simeq GL_1, & a \in \mu(k) \quad \dim = 1 \\ \left\{\begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix}\right\} \simeq GL_1 \ltimes \mathbb{G}_a, & a \notin \mu(k) \quad \dim = 2 \end{cases}$$

periods: $2\pi i$, $\log(a)$

$$\text{tr.deg. } \mathbb{Q}(2\pi i, \log(a))/\mathbb{Q} = \begin{cases} 1 & a \in \mu(k) \\ \text{conjecturally } 2 & a \notin \mu(k) \end{cases}$$

Ex 3. $\underline{M = h_1(E)}$, $E_{/k}$ elliptic curve

realization with $\mathbb{Z}/n\mathbb{Z}$-coeff.: $E_n = \{\tfrac{1}{n} O_E\}$

Endomorphism ring: $(n \cdot 1 \neq 0 \text{ in } k)$

$$\mathrm{End}_k(E) = \{\alpha : E \longrightarrow E,\ \alpha(O_E) = O_E\} \supset \mathbb{Z}$$

$(\Rightarrow \alpha$ commutes with $\boxplus)$

Assume $k \subset \mathbb{C}$: $E(\mathbb{C}) = \mathbb{C}/L$, $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$

$$\mathrm{End}_{\mathbb{C}}(E) = \{\alpha \in \mathbb{C} \mid \alpha L \subseteq L\} \qquad \alpha : z \longmapsto \alpha z$$

(a) $\underline{\mathrm{End}_{\mathbb{C}}(E) = \mathbb{Z}}$: $G(k(E_n)/k) \hookrightarrow GL_2(\mathbb{Z}/n\mathbb{Z})$

bounded index if $[k:\mathbb{Q}] < \infty$

(Serre)

$$\underline{G_{mot}(E) = \underset{\substack{\uparrow \\ \dim = 4}}{GL_{2/\mathbb{Q}}}}$$

periods: $\begin{pmatrix} \omega_1 & \eta_1 \\ \omega_2 & \eta_2 \end{pmatrix}$, $\begin{vmatrix} \omega_1 & \eta_1 \\ \omega_2 & \eta_2 \end{vmatrix} = 2\pi i$

conjecturally: $\mathrm{tr.deg.}\ \mathbb{Q}(\omega_1, \omega_2, \eta_1, \eta_2)/\mathbb{Q} \overset{?}{=} 4$

(b) $\underbrace{\mathrm{End}_{\mathbb{C}}(E)}_{\mathbb{Z} + \mathbb{Z}\alpha} \neq \mathbb{Z}$: case of complex multiplication (CM)

$$\alpha^2 + b\alpha + c = 0$$

$b, c \in \mathbb{Z}$, $D := b^2 - 4c < 0$

$$K := \mathrm{End}_{\mathbb{C}}(E) \otimes \mathbb{Q} = \mathbb{Q}(\sqrt{D})$$

(Sub) Ex : $E : y^2 = 1 - x^4$ (+ 2 pts at ∞)

$E(\mathbb{C}) = \mathbb{C}/\Omega \cdot \mathbb{Z}[i]$, $k \subset \mathbb{C}$ $\quad O_E = (0,1)$

(i) If $i \in k$:

$$\underset{\wr}{i} : (x,y) \longmapsto (ix, y)$$

$\mathbb{Z} + \mathbb{Z}i = \mathbb{Z}[i] = \mathrm{End}_k(E)$, $i^2 = -1$

$E_n \simeq \mathbb{Z}[i]/n\,\mathbb{Z}[i]$ (as $\mathbb{Z}[i]$-modules)

$$\underbrace{G(k(E_n)/k)}_{\substack{\text{elts commute} \\ \text{with } \mathrm{End}_k(E) = \mathbb{Z}[i]}} \hookrightarrow GL_1(\mathbb{Z}[i]/n\mathbb{Z}[i]) \subset GL_2(\mathbb{Z}/n\mathbb{Z})$$

$$GL_1(\mathbb{Z}[i]/n\mathbb{Z}[i]) = C, \qquad C \ni a+bi \longmapsto \underbrace{\begin{pmatrix} a & -b \\ b & a \end{pmatrix}}_{M_{a+bi}} \in GL_2(\mathbb{Z}/n\mathbb{Z})$$

bounded index if $[k:\mathbb{Q}] < \infty$

$G_{mot}(h_1(E)) = GL_{1/\mathbb{Q}(i)} \hookrightarrow GL_{2/\mathbb{Q}}$ $\begin{pmatrix} \text{non-split} \\ \text{Cartan sbgp} \end{pmatrix}$

(ii) If $i \notin k$:

$$\underbrace{G(k(E_n)/k)}_{\substack{\text{elts act on } \mathrm{End}_k(E) = \mathbb{Z}[i] \\ \text{(by conjug.) as } a+bi \mapsto a \pm bi}} \hookrightarrow N(C) = C \cup \sigma C \subset GL_2(\mathbb{Z}/n\mathbb{Z})$$

$(\forall g \in C)\ g M_{a+ib} g^{-1} = M_{a+ib}$

$\sigma M_{a+ib} \sigma^{-1} = M_{a-ib}$

(e.g. $\sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$)

$G_{mot}(h_1(E)) = GL_{1/\mathbb{Q}(i)} \rtimes \{1, \sigma\} \subset GL_{2/\mathbb{Q}}$

(normalizer of a non-split Cartan subgroup)

General CM case : similar

Summary: $k \subset \mathbb{C}$, $E_{/k}$ ell. curve

$$K := \mathrm{End}_{\mathbb{C}}(E) \otimes \mathbb{Q}$$

no CM: $K = \mathbb{Q}$, $\underline{G_{mot}(h_1(E)) = GL_{2/\mathbb{Q}}}$, $\dim = 4$

CM: $K = \mathbb{Q}(\sqrt{D})$
$(D<0)$

$$\underline{G_{mot}(h_1(E))} = \begin{cases} GL_{1/K} & \mathrm{End}_k(E)\otimes\mathbb{Q} = K \\ GL_{1/K} \rtimes \underbrace{\{1,\sigma\}}_{G(K/\mathbb{Q})} & \text{— " —} = \mathbb{Q} \end{cases}$$

$\dim = 2$

Periods in the CM case: $(k \subset \mathbb{C})$

$$\mathbb{Q}(\omega_1, \omega_2, \eta_1, \eta_2) = \mathbb{Q}(2\pi i, \omega_1)$$

$$\text{tr. deg. } \mathbb{Q}(2\pi i, \omega_1)/\mathbb{Q} = 2$$

$\exists$ __linear constraints__ on $\left\{ \begin{array}{l} \text{periods} \\ G(k(E_n)/k) \end{array} \right\}$

$$\Uparrow$$

$\exists$ $\underline{\text{non-trivial cycle } \Gamma_\alpha \in Z^1(E\times E)}$
$(\alpha \in \mathrm{End}_k(E),\ \alpha \notin \mathbb{Z})$

Fact: $E_{/k}$ ell. curve, $k \supset \mathbb{Q}$

$$Z^1_{hom}(E\times E)_{\mathbb{Q}} = \begin{cases} \langle E\times pt, pt\times E, \Delta \rangle & \mathrm{End}_k(E) = \mathbb{Z} \\ \langle \text{— " ———}, \Gamma_\alpha \rangle & \alpha \in \mathrm{End}_k(E),\ \alpha \notin \mathbb{Z} \end{cases}$$

<u>General case:</u> $k \subset \mathbb{C}$, $X, Y \in \mathrm{SmProj}/k$

(1) non-zero $Z \in \mathcal{Z}^r_{\mathrm{hom}}(X \times Y)_{\mathbb{Q}}$ (for $H = H_B$, the usual cohomology)

$\Downarrow$

linear relation between periods of X and Y

— " — Galois gps associated to étale realizations

(2) non-zero $Z \in \mathcal{Z}^r_{\mathrm{hom}}(X \times X \times \dots \times X)_{\mathbb{Q}}$

$\Downarrow$

polynomial relation between periods of X

— " — for Galois gps ....

<u>Conjecture</u> (Grothendieck, ...) If $[k:\mathbb{Q}] < \infty$, all [redacted] polynomial relations between periods of X (.... for Galois gps attached to étale cohomology of X) are obtained in this way.

<u>Reformulation</u>: $H_B(X) := \bigoplus_{j \geq 0} H^j(X(\mathbb{C}), \mathbb{Q})$

$X_{/k}$ smooth proj., $k \subset \mathbb{C}$

$\Downarrow$

$G_{\mathrm{mot}}(\underbrace{h(X)}_{\text{"}\bigoplus_{j\geq 0} h^j(X)\text{"}})$ = subgp of elts of $GL(H_B(X)) \times GL_{1/\mathbb{Q}}$ which fix all classes in $\mathcal{Z}^r_{\mathrm{hom}}(X^n)_{\mathbb{Q}} \subset H_B(X)^{\otimes n}$ $(\forall r, n \geq 0)$

($t \in GL_1$ acts on $\mathcal{Z}^r_{\mathrm{hom}}(X^n)_{\mathbb{Q}}$ by t^r; this technicality reflects the fact that the cycle class map really goes from $\mathcal{Z}^r(Y)$ to $\underbrace{H^{2r}(Y) \otimes (H^2(\mathbb{P}^1)^\vee)^{\otimes r}}_{H^{2r}(Y)(r)}$)

__Grothendieck's conjecture on periods:__

$[k:\mathbb{Q}]<\infty$, $X \in \mathrm{SmProj}_{/k}$

tr. deg. $\mathbb{Q}(\text{periods of } h(X))/_{\mathbb{Q}} \overset{?}{=} \dim G_{mot}(h(X))$

(inequality "$\leq$" always holds)

__Generalization to mixed motives:__

$[k:\mathbb{Q}]<\infty$, M mixed motive over k

tr. deg. $\mathbb{Q}(\text{periods of } M)/_{\mathbb{Q}} \overset{?}{=} \dim G_{mot}(M)$

(again, "$\leq$" holds").

__Ex__: multiple zeta values

$$\zeta(n_1, \dots, n_r) = \sum_{1 \leq k_1 < \dots < k_r} \frac{1}{k_1^{n_1} \cdots k_r^{n_r}} \qquad (n_j \in \mathbb{N})$$

are periods of certain mixed motives over $\mathbb{Q}$ occurring inside $\pi_1^{nilp}(\mathbb{P}^1 - \{\infty, 0, 1\})$.

We expect (after Grothendieck): k field

(1) $\{$pure motives over $k\}$ (with coeff. in $\mathbb{Q}$) = semi-simple $\mathbb{Q}$-linear abelian category

(2) If $k \subset \mathbb{C}$: M pure motive$/k \Rightarrow G_{mot}(M)_{/\mathbb{Q}}$ is reductive and

$$\operatorname{Rep}_{\mathbb{Q}}(G_{mot}(M)) \xleftrightarrow{\text{equiv.}} \langle M\rangle^{\otimes} = \left\{ \begin{array}{l}\text{subcategory of pure} \\ \text{motives}/k \text{ generated} \\ \text{by direct summands} \\ \text{of all } M^{\otimes m} \otimes (M^{\vee})^{\otimes n}\end{array}\right\}$$

$$\left(\text{alg. repr. } G_{mot}(M) \to GL_{N/\mathbb{Q}}\right)$$

(3) If $[k:\mathbb{Q}] < \infty$: M pure motive$/k$

direct summands of $M \xleftrightarrow{1-1}$ "factors" of $\zeta(M,s)$

Ex: $k \subset \mathbb{C}$, $E_{/k}$ ell. curve, $M = h^1(E) = h_1(E)^{\vee}$

$G = G_{mot}(M) \subset GL_{2/\mathbb{Q}}$

no CM: $M \leftrightarrow$ standard 2-dim. repr. of G "st"

$G = GL_{2/\mathbb{Q}}$

$$(\forall n \geq 1) \quad \underbrace{S^n M}_{\text{simple}} \leftrightarrow \underbrace{S^n(st)}_{\text{irreducible}}$$

simple $\Leftarrow$ irreducible

CM: $G = GL_{1/K}$ or $GL_{1/K} \rtimes G(K/\mathbb{Q})$

$(K = \operatorname{End}_{\mathbb{C}}(E) \otimes \mathbb{Q})$

$M \leftrightarrow$ standard 1-dim. repr V over K of $GL_{1/K}$

$$S^2 M \leftrightarrow S^2_{\mathbb{Q}} V = S^2_K V \oplus \Lambda^2_{\mathbb{Q}} V$$

<u>Tensor product</u>: M, N pure motives over $k = \mathbb{Q}$
S = finite set of bad primes for M, N (coeff. in $\mathbb{Q}$)

$$\zeta_{\{S\}}(M,s) = \prod_{p \notin S} \prod_{i=1}^{m} (1-\alpha_i(p)p^{-s})^{-1}$$

$$\prod_{i=1}^{m}(1-\alpha_i(p)t) = \det(1-\underbrace{Fr_p}_{\text{"Frobenius"}} t \mid \underbrace{M_\ell}_{\ell\text{-adic repr. of } G_{\mathbb{Q}}\ (\ell \neq p)}) \in \mathbb{Q}[t] \quad (p \notin S)$$

It should be possible to add correct factors also for $p \in S$ $\rightsquigarrow$ function $\zeta(M,s)$ with meromorphic continuation to $\mathbb{C}$, finitely many poles and a functional equation relating $\zeta(M,s)$ to $\zeta(M^\vee, 1-s)$.

If $\zeta_{\{S\}}(N,s) = \prod_{p \notin S} \prod_{j=1}^{N}(1-\beta_j(p)p^{-s})^{-1}$, then

$$\zeta_{\{S\}}(M \otimes N, s) = \prod_{p \notin S} \prod_{i,j} (1-\alpha_i(p)\beta_j(p)p^{-s})^{-1}.$$

<u>Ex</u>: $M = h^1(E)$, $E/\mathbb{Q}$ elliptic curve

$$\zeta_{\{S\}}(M,s) = \prod_{p \notin S} \left[(1-\alpha_p p^{-s})(1-\beta_p p^{-s})\right]^{-1}$$

$$\zeta_{\{S\}}(S^2M,s) = \prod_{p \notin S} \left[(1-\alpha_p^2 p^{-s})(1-\underbrace{\alpha_p\beta_p}_{p} p^{-s})(1-\beta_p^2 p^{-s})\right]^{-1}$$

$$= \underbrace{\zeta_{\{S\}}(s-1)}_{\zeta_{\{S\}}(\mathbb{Q}(-1),s)} \underbrace{\prod_{p \notin S} \left[(1-\alpha_p^2 p^{-s})(1-\beta_p^2 p^{-s})\right]^{-1}}_{F(s)}$$

<u>E has CM</u> $\Rightarrow S^2M = \mathbb{Q}(-1) \oplus N$, $F(s) = \zeta_{\{S\}}(N,s)$ hol.

<u>E has no CM</u> $\Rightarrow S^2M$ irred., $F(s)$ has ∞ many poles

Tannakian formalism

$F \supset \mathbb{Q}$ field, $G \subset GL_{N/F}$ affine alg. group over F

Fact: G can be reconstructed from

$(\mathrm{Rep}_F(G), \otimes, \omega_F)$ $\quad \omega_F : \mathrm{Rep}_F(G) \longrightarrow \mathrm{Vect}_F$ forgetful functor

More precisely: given

$\mathcal{T}$ = abelian F-linear category with tensor product $\otimes$, duals, unit object $\mathbb{1}$ with $\mathrm{End}(\mathbb{1}) = F$

$\omega : \mathcal{T} \longrightarrow \mathrm{Vect}_F$ exact faithful $\otimes$-functor ("fibre functor over F")

Then: ω lifts to a $\otimes$-equivalence of categories $\mathcal{T} \xrightarrow{\approx} \mathrm{Rep}_F(G)$, where $G = \underline{\mathrm{Aut}}^{\otimes}(\omega)$

(projective limit of affine alg. groups $/F$).

If $\mathcal{T}$ is $\otimes$-generated by one object M, then $G \subset GL(\omega(M))$.

$\mathcal{T}$ semi-simple $\Rightarrow$ G [redacted] reductive.

[redacted]

[redacted]

Grothendieck's conditional definition of $G_{mot}(M)$

$k \subset \mathbb{C}$, $X \in \mathrm{SmProj}_k$

Goal: construct $\underbrace{(\langle h(X)\rangle^{\otimes}, \dot{\otimes})}_{} \xrightarrow{H_B} \mathrm{Vect}_{\mathbb{Q}}$

$\mathcal{T} =$ abelian semi-simple Tannakian over $\mathbb{Q}$

$G_{mot}(h(X)) := \underline{\mathrm{Aut}}^{\otimes}(\text{this functor})$

Objects of $\mathcal{T}$: (Finite unions of) (X^n, p, r) $\quad n \geq 0$, $\underline{r \in \mathbb{Z}}$ Tate twist

$p \in Z^{n\dim(X)}_{hom}(X^n \times X^n)_{\mathbb{Q}}$, $p^2 = p$

$H_B((X^n, p, r)) = p\, H_B(X^n)(r)$ $\leftarrow$ shift in degree by $2r$

$\mathrm{Hom}_{\mathcal{T}}((X^n, p, r), (X^{n'}, p', r')) = p' Z^{n\dim(X) - r + r'}_{hom}(X^{n+n'})p$

$(X^n, p, r) \otimes (X^{n'}, p', r') = (X^n \times X^{n'}, p \times p', r + r')$

$(X^n, p, r)^{\vee} = (X^n, {}^t p, n\dim(X) - r)$

$\mathbb{1} = (\underbrace{X^0}_{\mathrm{Spec}(k)}, 1, 0)$

Problems: (1) Is $\mathcal{T}$ abelian semi-simple?

(2) $H_B : (\langle h(X)\rangle, \otimes) \longrightarrow (s\mathrm{Vect}_{\mathbb{Q}}, \otimes)$ has values in super vector spaces, as the products $H^i \otimes H^j \longrightarrow H^j \otimes H^i$ are super-commutative. $x \otimes y \longmapsto (-1)^{ij} y \otimes x$

Standard conjecture:

(Num) $(\forall r, n \geq 1)$ $Z^r_{hom}(X^n)_{\mathbb{Q}} \times Z^{n\dim(X) - r}_{hom}(X^n)_{\mathbb{Q}} \longrightarrow \mathbb{Q}$ is non-degenerate.

<u>Facts</u>: (a) (Lef) + (Index) $\Longrightarrow$ (Num)

(b) (Jannsen) (Num) $\Longrightarrow$ $\langle h(X)\rangle$ is semi-simple abelian

(c) (Num) $\Longrightarrow$ all projections $H_B(X^n) \twoheadrightarrow H_B^j(X^n)$ are defined by elements of $\underbrace{}_{p_{n,j}}$ $Z_{hom}^{n\dim}(X^n \times X^n)_{\mathbb{Q}}$

($\Rightarrow$ $h^j(X^n) := (X^n, p_{n,j}, 0)$ is defined)

$\Rightarrow$ one can change $\otimes$-isomorphisms $M\otimes N \xrightarrow{\sim} N\otimes M$ in $\mathcal{T}$ (defined by interchanging the factors) by adding suitable signs so that H_B becomes a $\otimes$-functor

$$H_B^{new} : (\langle h(X)\rangle, \otimes_{new}) \longrightarrow \mathrm{Vect}_{\mathbb{Q}} .$$

<u>Def</u>: $G = G_{mot}(h(X)) := \underline{\mathrm{Aut}}^{\otimes}(H_B^{new})$.

($\Rightarrow$ $\mathrm{Rep}_{\mathbb{Q}}(G) \approx (\langle h(X)\rangle, \otimes_{new})$)

<u>What if $k \supset \mathbb{F}_p$?</u> Even if we assume (Num) (for a fixed l-adic H_l^*), the semi-simple $\mathbb{Q}$-linear Tannakian category $(\langle h(X)\rangle, \otimes_{new})$ does <u>not</u> admit a fibre functor into $\mathrm{Vect}_{\mathbb{Q}}$. The fibre functors form a "gerbe" for a suitable Grothendieck topology on affine $\mathbb{Q}$-schemes.