

Some remarks on the BSD Conjecture

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Arithmetic local constants (Mazur–Rubin 2007)

Formula of Mazur–Rubin:

$$\dim_{\mathbb{F}} H^1_{\mathcal{F}}(k, X) - \dim_{\mathbb{F}} H^1_{\mathcal{F}'}(k, X) \equiv \sum_{v \in S_f} d(\mathcal{F}_v, \mathcal{F}'_v) \pmod{2}$$

$[k : \mathbb{Q}] < \infty$, $[\mathbb{F} : \mathbb{F}_p] < \infty$, $p \neq 2$ $S \supset \{v | p \infty\}$ finite

$X = \mathbb{F}[G_{k,S}]$ -module (finite), $\mathcal{F}_v, \mathcal{F}'_v \subset H^1(k_v, X)$

$$\mathcal{F}_v = \mathcal{F}'_v = H^1_{\text{ur}}(k_v, X) \quad \forall v \notin S = \{v | \infty\} \cup S_f$$

$$H^1_{\mathcal{F}}(k, X) = \text{Ker}(H^1(G_{k,S}, X) \rightarrow \bigoplus_{v \in S_f} H^1(G_{k_v}, X) / \mathcal{F}_v)$$

X self-dual: $\langle , \rangle_X : X \times X \rightarrow \mathbb{F}(1)$ alternating

$H^1(k_v, X)$, $\cup : H^1(k_v, X) \times H^1(k_v, X) \rightarrow H^2(k_v, \mathbb{F}(1)) \cong \mathbb{F}$
is a quadratic space over \mathbb{F}

$\mathcal{F}, \mathcal{F}'$ self-dual: $\forall v \neq \infty \quad \mathcal{F}_v, \mathcal{F}'_v \subset H^1(k_v, X)$
are Lagrangian subspaces ($\mathcal{F}_v = \mathcal{F}_v^{\perp}$, $\mathcal{F}'_v = (\mathcal{F}'_v)^{\perp}$)

Def: (W, q) quadratic space over \mathbb{F}

$L, L' \subset W$ Lagrangian subspaces

$$\begin{aligned} d(L, L') &= \dim_{\mathbb{F}} L / (L \cap L') \pmod{2} \in \mathbb{Z}/2\mathbb{Z} \\ &= d(L', L) \end{aligned}$$

Cocycle identity (Klagsbrun–Mazur–Rubin 2013)

$$d(L_0, L_1) - d(L_0, L_2) + d(L_1, L_2) = 0$$

Ex: $\mathbb{F} = \mathbb{O}/\pi\mathbb{O}$, $\tau \in \mathbb{O} \subset \mathbb{K}$, $[\mathbb{K} : \mathbb{Q}_p] < \infty$

$$X = T/\pi T = \bar{T} \iff T \subset V = T \otimes_{\mathbb{O}} \mathbb{K} \xrightarrow{\exists} G_{L,S}$$

$\langle , \rangle_T : T \times T \rightarrow \mathcal{O}(1)$ alternating, $T \xrightarrow{\sim} T^*(1)$

Assume: $\forall v \mid p$ $V|_{G_{L,v}}$ is a de Rham representation

Bloch-Kato: Lagrangian Selmer structure

$$H_f^1(k_v, V) (= H_{ur}^1(k_v, V) \text{ if } v \nmid p) \subset H^1(k_v, V) \quad (v \neq \infty)$$

$$\text{propagation} \Rightarrow H_f^1(k_v, -) \subset H^1(k_v, -) \quad - = T_1 V / T_1 \bar{T}$$

$\mathbb{F}_v = H_f^1(k_v, \bar{T})$ is Lagrangian

$$\dim_{\mathbb{F}} H^0(k, \bar{T}) - \dim_{\mathbb{F}} H_f^1(k, \bar{T}) =$$

$$= \dim_{\mathbb{K}} H^0(k, V) - \dim_{\mathbb{F}} H_f^1(k, V/T) [\pi] \equiv$$

$$\equiv \underbrace{\dim_{\mathbb{K}} H^0(k, V) - \dim_{\mathbb{K}} H_f^1(k, V)}_{\chi_f(k, V)} \pmod{2}$$

Cassels-Tate-Flach pairing on $\frac{H_f^1(k, V/T)}{H_f^1(k, T) \otimes \mathbb{K}/\mathbb{O}}$

Formula of Mazur-Rubin: given also

$$\bar{T}' \iff T' \subset V', \quad \langle , \rangle_{T'} : T' \times T' \rightarrow \mathcal{O}(1)$$

$(\bar{T}, \langle , \rangle_{\bar{T}}) \simeq (\bar{T}', \langle , \rangle_{\bar{T}'}) = X$, then

$$\chi_f(k, V) - \chi_f(k, V') \equiv \sum_{v \in S_f} \frac{d(\mathbb{F}_v, \mathbb{F}'_v)}{d_v(T, T')} \pmod{2}$$

Local constants (= ε-factors) at ∞

$WD_v(V)$ = representation of the Weil-Deligne group of k_v attached to $\begin{cases} V|_{G_{k_v}} & v \nmid p \\ D_{\text{pst}}(V|_{G_{k_v}}) & v \mid p \end{cases}$

$\epsilon_v(V) = \epsilon(WD_v(V), \psi, dx_\psi) \in \{\pm 1\}$ independent of ψ = additive character of k_v ($dx_\psi = \psi$ -self dual measure)

Motivic situation: M = pure motive over k , with coefficients in L , $[L:\mathbb{Q}] < \infty$, $z: L \hookrightarrow \mathbb{C}$

$$L(M, s) = \sum_{n \geq 1} a_n n^{-s} \quad (a_n \in L), \quad L(2M, s) = \sum_{n \geq 1} z(a_n) n^{-s}$$

M self-dual: $M \xrightarrow{\sim} M^*(1)$ alternating

$\forall p \nmid p$ in L $V = M_p$ p -adic realisation ($\mathcal{X} = L$)

Conjectures: Block-Kato, Fontaine-Perrin-Riou

$$\text{ord}_{s=0} L(2M, s) \stackrel{?}{=} -\chi_f(k, V^*(1)) \quad (= -\chi_f(k, V))$$

Functional equation:

$$(L_\infty \cdot L)(2M, s) \stackrel{?}{=} a^s \epsilon(2M) (L_\infty \cdot L)(2M, -s)$$

$$\Downarrow \leftarrow \text{ord}_{s=0} L_\infty(2M, s) = 0$$

Parity conjecture for the p -Selmer group:

$$(-1)^{\chi_f(k, V)} \stackrel{?}{=} \epsilon(2M) = \underbrace{\epsilon_\infty(2M)}_{\text{at } \infty} \prod_{v \neq \infty} \epsilon_v(V)$$

depends only on {Hodge-Tate weights ($V|_{G_{k_v}}$)}
 $v \nmid p$

Expected local statement: rtoo in k
 If V, V' are representations of G_{k_v} (de Rham if rtp) such that $(\bar{T}, \langle , \rangle_{\bar{T}}) \simeq (\bar{T}', \langle , \rangle_{\bar{T}'})$ as symplectic $\mathbb{F}[G_{k_v}]$ -modules, then

$$(-1)^{\text{dr}(T, T')} \stackrel{(*)}{=} \epsilon_r(V)/\epsilon_r(V')$$

$$\begin{cases} \text{if rtp} \\ \text{if rtp and } \underbrace{\text{HT}(V)}_{\text{Hodge-Tate weights}} = \text{HT}(V') \end{cases}$$

Cheffy: $(*)$ holds in certain cases arising from abelian varieties in dihedral extensions of k .

If rtp and $\text{HT}(V) \neq \text{HT}(V')$, one needs to add some correction term to $(*)$.

Local results

Thm 1: $(*)$ holds if rtp [Compositio M., 2015]

Thm 2: $(*)$ holds if rtp and if V (resp. V') becomes Barsotti-Tate over a finite abelian tamely ramified extension of k_v .

Global results

Thm 3. V, V' representations of $G_{k,S}$,
 $(\bar{\tau}, \langle , \rangle_{\bar{\tau}}) \simeq (\bar{\tau}', \langle , \rangle_{\bar{\tau}'})$. If $\forall v \nmid p$ $V|_{G_{k_v}}$ (resp. $V'|_{G_{k_v}}$) becomes Barsotti-Tate over a finite abelian extension of k_v , then

$$(-1)^{x_f(k, V)} / \prod_{v \neq \infty} \varepsilon_v(V) = (-1)^{x_f(k, V')} / \prod_{v \neq \infty} \varepsilon_v(V')$$

Thm 4. If k is totally real, f = cuspidal Hilbert modular eigenform over k of parallel weight 2 with trivial character, $L = \mathbb{Q}$ (Hecke eigenvalues of f), $p \nmid p \neq 2$ in L , $V = V_p(f)(1)$, then

$$\dim_{L_p} H^1_f(k, V) \equiv \underbrace{\text{ord}_{s=1} L(f, s)}_{\in \mathbb{Z}} \pmod{2}$$

Thm 5. If k, L are totally real, A = abelian variety over k , $O_L \subset \text{End}(A)$, $\dim(A) = [L : \mathbb{Q}]$, $p \nmid p \neq 2$ in L , $\tau : L \hookrightarrow \mathbb{R}$, then

$$\underbrace{rk_{O_L} A(k) + \text{cork}_{O_L} \text{H}^1(A/k)[p^\infty]}_{\dim_{L_p} H^1_f(k, V_p(A))} \equiv \underbrace{\text{ord}_{s=1} L(2A/k, s)}_{\in \mathbb{Z} \text{ independent of } \tau} \pmod{2}$$

Thm 4, 5: proved with additional assumptions in [Algebra and Number Theory, 2013].

Pf of Thm 1: $\mathcal{F}_{\text{ur}} = H_{\text{ur}}^1(k_v, \bar{\mathbb{F}}) \subset H^1(k_v, \bar{\mathbb{F}})$ Lagrangian

Need to show: $(-1)^{d(\mathcal{F}_{\text{ur}}, \mathcal{F}_v)} \epsilon_v(V) \stackrel{?}{=} (-1)^{d(\mathcal{F}_{\text{ur}}, \mathcal{F}_v')} \epsilon_v(V')$

Key formula:

$$(-1)^{d(\mathcal{F}_{\text{ur}}, \mathcal{F}_v)} \epsilon_v(V) = \det(-F_{v,v} | \bar{\mathbb{F}}^{I_v}) \epsilon_{O_{v,v}}(V)^{-1} (\text{mod } \pi O)$$

$$\epsilon_{O_{v,v}}(V) = \epsilon_v(V) \det(-F_{v,v} | V^{I_v}) \text{ depends only on } \bar{\mathbb{F}} \text{ (Deligne)}$$

Key formula \Leftarrow duality + bilinear algebra (parity constraint on the sizes of Jordan blocks of unipotent elements of orthogonal groups).

Pf of Thm 2: if $T = T_p(g)$ for a Barsotti-Tate group with O -action over O_{k_v} , then $\mathcal{F}_v = H_{\text{ur}}^1(O_{k_v}, g[\pi]) =$

$$= \text{Ext}^1(\mathbb{F}, g[\pi])$$

finite flat \mathbb{F} -vs gp schemes/ O_{k_v}
 \Updownarrow equivalence of categories

Breuil-Kisin modules

In general, need to work with Breuil-Kisin modules with tame descent data of a special kind.

Pf of Thm 3: replace k by an abelian extension of degree p^N to get rid of wild ramification.

Apply Thm 1, 2 and the formula of Mazur-Rubin.

Pf of Thm 4: known if f comes from a Shimura curve.

If not, apply Thm 3 to $V_p(f), V_{p'}(f')$ with f' obtained by level raising (needs separate pf if $\bar{\mathbb{F}}$ reducible).

Thm 4 \Rightarrow Thm 5: potential modularity of A + Solomon's induction thm + Taylor's trick.

Generalised modular parameterisations of elliptic curves

$A = \text{elliptic curve over } F, F \text{ totally real}, [F:\mathbb{Q}] = r$

Expected analytic modularity:

$$L(A/F, s) = L(f, s)$$

$f = \text{cuspidal Hilbert modular form of weight 2 over } F$

(known potentially over some F'/F ; over F if $r \leq 2$)

Classical geometric modularity:

If $\pi_f \xleftarrow{\text{Jacquet-Langlands}} \pi_B$

automorphic representation
of $\mathrm{PGL}_2(\mathbb{A}_F)$ attached to f

automorphic representation
of B_A^\times / F_A^\times

B_F quaternion algebra ,

$$B_R \cong M_2(\mathbb{R}) \times \mathbb{H}^{r-1}$$



defined over F

$X = \text{Shimura curve attached to } B$,

— " —

$$h_1(X) \xrightarrow{\alpha} h_1(A)$$

CM points: $[K:F] = 2, K \text{ CM}$

$\alpha(X(\text{CM pts by } K)) \subset A(\text{ring class fields of } K)$

Generalisations of results of Kolyvagin, Gross-Zagier,
Bertolini-Darmon \Rightarrow weak χ -equivariant BSD

Conjecture for $L(A/K, \chi, s)$ if $\mathrm{ord}_{s=1} \leq 1$
($A \neq \text{CM}$) ring class character of K

Non-classical geometric modularity:

What if $L(A/F, s) = L(f, s)$, $\pi(f) \xleftarrow{J-L} \pi_B$
 π_B representation of B_A^\times / F_A^\times , $B_R \simeq M_2(\mathbb{R})^J \times \mathbb{H}^{J^c}$ 2
 $\overline{\mathbb{Q}} \subset \mathbb{C}$, $\{F \hookrightarrow \mathbb{R}\} = \{F \hookrightarrow \overline{\mathbb{Q}}\} = J \sqcup J^c$, $1 \leq t = |J| \leq r$

Quaternionic Shimura variety:

$X = Sh_U(B^\times, (\mathbb{P} \setminus \mathbb{R})^t)$, $\dim(X) = t$, $U \subset \widehat{B}^\times_{\text{open cpt}}$

Defined over its reflex field

$$E = \left\{ \sum_{\sigma \in J} \sigma(x) \mid x \in F \right\} \subset F^{\text{gal}} \quad (t=1 \Rightarrow E=F)$$

Expected geometric modularity (Oda, Yoshida):

$$\exists h_t(X) \xrightarrow{\alpha} \underset{\sigma \in J}{\bigotimes} h_1(A_\sigma) = \text{Ind}_J^\otimes(h_1(A))$$

partial tensor induction

$$\text{Ex: } B = M_2(F), t = r, E = \mathbb{Q}, J^c = \emptyset \quad (\text{Oda})$$

$X = \text{Hilbert modular variety}$

$$\exists h_r(X) \xrightarrow{\alpha} \underset{\sigma: F \hookrightarrow \overline{\mathbb{Q}}}{\bigotimes} h_1(A_\sigma) = \text{Ind}_{F/\mathbb{Q}}^\otimes(h_1(A)) \subset h_r(R_{F/\mathbb{Q}}(A))$$

full tensor induction

α should come from an algebraic cycle

$$Z \subset X \times R_{F/\mathbb{Q}}(A)$$

Darmon's ATR (almost totally real) points:

$B = M_2(F)$, $X = \text{Hilbert modular variety}$, $\dim(X) = ?$

Assume: Hodge-de Rham version of α exists
(Oda's period conjecture + ϵ)

$$\begin{array}{ccc} H_B^r(X) & \xleftarrow{\alpha} & H_B^1(A_\sigma) \\ \uparrow & & \sigma: F \hookrightarrow \mathbb{R} \\ H_{\text{dR}}^r(X) & \hookleftarrow T_\sigma = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \xrightarrow{\sigma} \\ & & \text{cplx conjugation} \end{array}$$

$$w_f = c_1 f(\tau_1, \dots, \tau_r) d\tau_1 \wedge \dots \wedge d\tau_r$$

$$w_f^+ = \prod_{j=1}^r (1 + T_{\sigma_j}) w_f$$

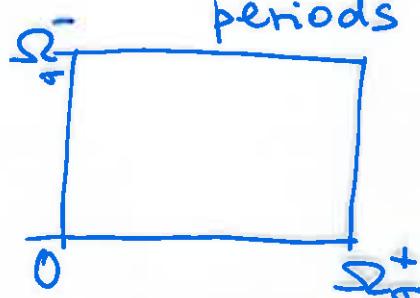
$$\sigma_1, \dots, \sigma_r: F \hookrightarrow \mathbb{R}$$

$$\text{Fix } 0 \neq \beta \in T(A, S^1_{A/F})$$

$$\text{periods } \mathbb{C}/(\mathbb{Z}\Omega_\sigma^+ + \mathbb{Z}\Omega_\sigma^-)$$

$$\downarrow \deg \leq 2$$

$$A_\sigma(\mathbb{C})$$

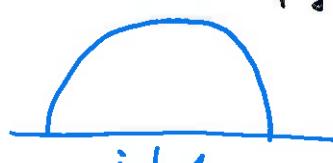


$$\{\text{periods of } c_2 w_f^+ \text{ on } X\} = \text{lattice} \subset (\mathbb{Z}\Omega_1^+ + \mathbb{Z}\Omega_1^-) \prod_{j=1}^r \mathbb{Z}\Omega_j^+$$

Exotic Abel-Jacobi map: integrate $c_2 w_f^+$ over \mathbb{S}^1 (real tori $(S^1)^{r-1}$ coming from $K^\times/F^\times \subset \text{PGL}_2(F) \subset \text{PGL}_2(F_R)$)

$$[K:F]=2, K=\text{ATR}: K \otimes_{F, \sigma_j} \mathbb{R} \simeq \begin{cases} \mathbb{C} & j=1 \\ \mathbb{R} \times \mathbb{R} & j \neq 1 \end{cases}$$

pt.

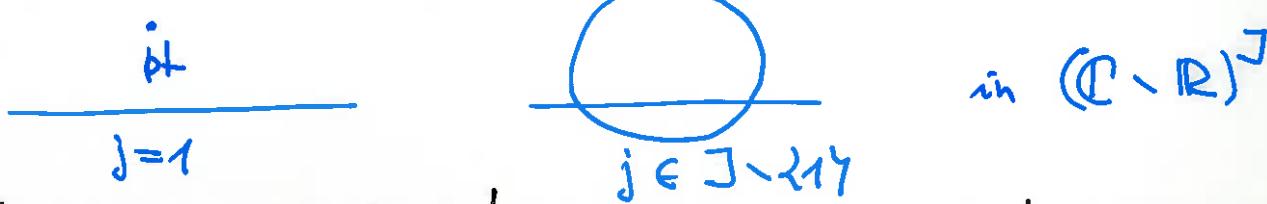


geodesic

$$\left(\prod_{j=1}^r \Omega_j^+ \right)^{-1} \int_{\mathbb{S}^1(\text{tori})} c_2 w_f^+ \in \mathbb{C}/(\mathbb{Z}\Omega_1^+ + \mathbb{Z}\Omega_1^-) \rightarrow A_{\sigma_1}(\mathbb{C}) \text{ should lie in } A \text{ (ring class field of } K/F \text{)}$$

Gärtner: $\pi(f) \leftarrow \pi_B^t$, $B_R \cong M_2(\mathbb{R}) \times \mathbb{H}^{\mathbb{C}}$, $1 \leq t = |\mathbb{J}| \leq r$
 $[K:F] = 2$, $K \otimes_{F_{\sigma_j}} R \cong \begin{cases} \mathbb{C} & j=1 \in \mathbb{J} \text{ or } j \in \mathbb{J}^c \\ \mathbb{R} \times \mathbb{R} & j \in \mathbb{J} \setminus \{1\} \end{cases}$

Considered orbits of non-connected "tori" K_R^\times / F_R^\times



Assuming Yoshida's period conjecture + ε:

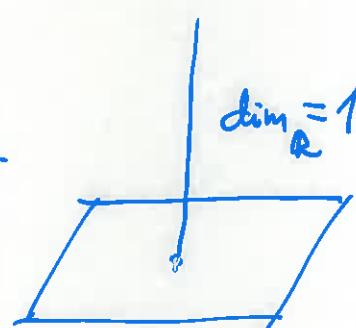
$$\left(\prod_{j \in \mathbb{J} \setminus \{1\}} \Omega_j^{\pm} \right)^{-1} \int_{\mathcal{A}^{-1}(\text{"tori"})} c_2 \underbrace{\omega_{f,B}}_{\text{no need for } \omega_f^+} \in \mathbb{C}/(\mathbb{Z}\Omega_1^+ + \mathbb{Z}\Omega_1^-) \rightarrow A_{\sigma_1}(\mathbb{C})$$

"toric points" should be
"secondary toric periods" in A (ring class fields of K/F)

If $F \neq$ totally real: $[F:\mathbb{Q}] = r_1 + 2r_2$, $[K:F] = 2$ $K \hookrightarrow B$
 torus $T = K^\times / F^\times \subset B^\times / F^\times$, B/F quaternion algebra, $F \subset C$

# {j}	1	t-1	r_1-t	r_2
F_{σ_j}	\mathbb{R}	\mathbb{R}	\mathbb{R}	\mathbb{C}
K_{σ_j}	\mathbb{C}	$\mathbb{R} \times \mathbb{R}$	\mathbb{C}	$\mathbb{C} \times \mathbb{C}$
B_{σ_j}	$M_2(\mathbb{R})$	$M_2(\mathbb{R})$	\mathbb{H}	$M_2(\mathbb{R})$
symm. space	$\mathbb{C} \setminus \mathbb{R}$	$\mathbb{C} \setminus \mathbb{R}$	$\mathbb{H}^{\text{pt}} \setminus \mathbb{H}$	\mathbb{H}_3
toric orbit	$\{\text{pt}\}$	$\{\text{pt}\}$	pt	geodesic or

$\dim_{\mathbb{R}} = 1$



cohomology degree 1 1 0 1 or 2

Guitart - Masden - Sengun: consider and geodesics

Plectic speculations (with A. J. Scholl)

F totally real, $[F:\mathbb{Q}] = r$

Plectic Conjecture: In the presence of real multiplication by F , motives have a canonical additional "plectic" structure.

Ex: X = quaternionic Shimura variety, $B \neq M_2(F)$

$$h_t(X)_{\overline{\mathbb{Q}}} \stackrel{?}{=} \bigoplus \left(\bigotimes_{\sigma \in \Sigma} h_1(A_\sigma)_{\overline{\mathbb{Q}}} \right)$$

A_F abelian variety of $GL(2)$ -type

Above, $h_1(A_\sigma)_{\overline{\mathbb{Q}}} =$ "arithmetic quark"

l -adic version: $X \rightarrow \text{Spec}(k)$ diagram

involving $\text{Sh}_{U^\circ}(G, \mathfrak{A})$, $G = R_{F/\mathbb{Q}}(H)$
 $k >$ all reflex fields $E(G, \mathfrak{A})$

Conj: $R\Gamma_{et}((X), \overline{\mathbb{Q}}_l) \in D^+(\overline{\mathbb{Q}}_l[\Gamma_k])$ has a canonical lift to
 $R\Gamma_{\text{plec}, et}((X), \overline{\mathbb{Q}}_l) \in D^+(\overline{\mathbb{Q}}_l[\Gamma_k^{\text{plec}}])$ ($\Gamma_k^{\text{plec}} \supset \Gamma_k = \text{Gal}(\overline{\mathbb{Q}}/k)$),
functorial w.r.t. Hecke correspondences.

\downarrow (for certain $m \in \mathbb{Z}$)

\exists plectic l -adic cohomology

$$R\Gamma_{\text{plec}, et}(X, \overline{\mathbb{Q}}_l(m)) = R\Gamma(\Gamma_k^{\text{plec}}, R\Gamma_{\text{plec}, et}((X), \overline{\mathbb{Q}}_l)(m))$$

and plectic Hochschild-Serre spectral sequence

$$E_2^{ij} = H^i(\Gamma_k^{\text{plec}}, H^j_{et}((X), \overline{\mathbb{Q}}_l)(m)) \Rightarrow H_{\text{plec}, et}^{i+j}(X, \overline{\mathbb{Q}}_l(m))$$

Sci-fi version:

$$\begin{array}{ccc}
 X_0 & \longrightarrow & X_0, \text{plec} \\
 \downarrow f & & \downarrow f_{\text{plec}} \\
 \underbrace{\text{Spec}(k)} & \longrightarrow & \underbrace{\text{Spec}(k)_{\text{plec}}}_{K(\Gamma_k^{\text{plec}}, 1)} \\
 & & K(\Gamma_k^{\text{plec}}, 1)
 \end{array}
 \quad R\Gamma_{\text{plec}, \text{et}}((X_0)_{\overline{k}}, \overline{\mathbb{Q}}_l) = R\Gamma_{\text{plec}}(\overline{\mathbb{Q}}_l)$$

$$H^i_{\text{plec}, \text{et}}(X_0, \overline{\mathbb{Q}}_l(m)) = H^i_{\text{et}}(X_0, \text{plec}, \overline{\mathbb{Q}}_l^{(m)})$$

[N, Scholl: Introduction to plectic cohomology]: possible applications include construction of generalisations of classical Euler systems (cyclotomic and elliptic units, Kato's Euler system) from \mathbb{Q} to F , when $\text{ord}_S L(M, s) = n$.

Do CM points have a plectic analogue?

Quaternionic case: B/F quaternion algebra,

$$B_R \simeq M_2(\mathbb{R})^{\mathbb{Z}} \times \mathbb{H}^{\mathbb{Z}^c}, \quad 1 \leq t = |\mathbb{Z}| \leq r = [F:\mathbb{Q}]$$

$$[K:F] = 2, \quad K \text{ CM}, \quad K \xrightarrow[F]{\subset} B$$

$$X_1 = \text{Sh}_{U_1}(B^\times, (\mathbb{C} \setminus R)^t) \quad \supset X_0 = \text{Sh}_{U_0}(K^\times, \langle pt \rangle)^{\dim=0}$$

Plectic reflex groups: uninteresting

$$(\Gamma_F^t \rtimes S_t) \times (\Gamma_F^{r-t} \rtimes S_{r-t}) \supset (\Gamma_K^t \rtimes S_t) \times (\Gamma_K^{r-t} \rtimes S_{r-t})$$

If the l -adic plectic conjecture holds for

$$[R\Gamma_{\text{et}}((X_1)_{\overline{Q}}, \mathbb{Q}_l) \rightarrow R\Gamma_{\text{et}}((X_1 - X_0)_{\overline{Q}}, \mathbb{Q}_l)] \quad \text{and} \quad R\Gamma_{\text{et}}((X_1)_{\overline{Q}}, \mathbb{Q}_l),$$

it gives rise to a plectic l -adic cycle class

$$\text{cl}_{\text{plec}, \text{et}}(X_0) \in H_{\text{plec}, \text{et}}^{2t}(X_1, \mathbb{Q}_l(t))$$

Modularity $L(A/F, s) = L(f, s)$, $\pi(f) \hookrightarrow \pi_B$ on B
 $\Rightarrow \exists$ l -adic version of α : $H_{et}^t((X_1)_{\overline{Q}_l}, \overline{Q}_l(t)) \xrightarrow{\alpha} \text{Ind}_{\mathbb{Z}_l}^{\otimes} V_l(A)$

Moreover, $\exists h \in \text{Hecke algebra}$ such that

$$\begin{aligned}
 h \cdot \alpha_{\text{plec}, et}(X_0) &\in h H_{\text{plec}, et}^{2t}(X_1, \overline{Q}_l(t)) \\
 &\quad \parallel \text{plectic Hochschild-Serre} \\
 h H^t(\Gamma_K^t \rtimes S_t, H_{et}^t((X_1)_{\overline{Q}_l}, \overline{Q}_l(t))) \\
 &\quad \downarrow \alpha_e \\
 H^t(\Gamma_K^t \rtimes S_t, V_l(A)^{\otimes t}) \\
 &\quad \parallel \\
 P_\alpha &\in \Lambda_{\overline{Q}_l}^t H^1(K, V_l(A)) \supset \Lambda_{\overline{Q}_l}^t A(K)_{\overline{Q}_l} \supset \Lambda_{\overline{Q}_l}^t A(K)_{\overline{Q}_l}
 \end{aligned}$$

plectic CM element.

Variant: If $(X_1)_{\text{plec}}$ and $(X_0)_{\text{plec}}$ exist and be have well,
then \exists plectic CM elements over ring class fields K' of K , lying in
 $H^t(\Gamma_{K'}^t \rtimes S_t, V_l(A)^{\otimes t} \otimes_{\overline{Q}_l} \overline{Q}_l[\text{Gal}(K'/K)]) = \Lambda_{\overline{Q}_l[\Gamma_{K'/K}]}^t H^1(K', V_l(A))_{\overline{Q}_l} =$
 $= \bigoplus_{X: \Gamma_{K'/K} \rightarrow \overline{Q}_l^\times} \Lambda_{\overline{Q}_l}^t (H^1(K', V_l(A))_{\overline{Q}_l}^{(x^{-1})}) \supset \bigoplus_X \Lambda_{\overline{Q}_l}^t (A(K')_{\overline{Q}_l}^{(x^{-1})}).$

(CM theory is plectic: [N, Momin's 70th birthday volume])

Classical case: $t=1$, $X = \text{Shimura curve over } F$,
 $\alpha: \text{Jac}(X) \rightarrow A$

$$P_\alpha = \alpha \circ \text{Tr}_{K'/K} ((\text{a CM pt on } X) - \text{Hodge class})$$

Does this purely formal conjectural construction of
 $P_{\alpha, \chi} \in \bigwedge_{\mathbb{Q}_\ell}^t H^1(K, V_\ell(A)) \xrightarrow{\chi^{-1}} \bigwedge_{\mathbb{Q}_\ell}^t (A(K) \otimes_{\mathbb{Q}_\ell}^{\chi^{-1}}) \xrightarrow{\sim} \bigwedge_{\mathbb{Q}}^t (A(K) \otimes_{\mathbb{Q}}^{\chi^{-1}})$
 $(\chi: \Gamma_{K'/K} \rightarrow \overline{\mathbb{Q}}^\times)$ make any sense?

Def: $\rho_\chi = \text{ord}_{s=1} L(A/K, \chi, s)$

Prop. Functoriality of the Plectic Conjecture implies that

$$P_{\alpha, \chi} \neq 0 \Rightarrow \left\{ \begin{array}{l} B \otimes \widehat{\mathbb{Q}} \text{ is determined by } A, K \text{ and } \chi \\ t \equiv \rho_\chi \pmod{2} \end{array} \right\}.$$

Note: $E(B^*, (\mathbb{C} \setminus \mathbb{R})^\times)$ (hence $\text{Sh}(B^*, (\mathbb{C} \setminus \mathbb{R})^\times)$) depends on $J \subset \{F \hookrightarrow R\}$, not just on $|J|=t$. However, $\text{Sh}(B^*, (\mathbb{C} \setminus \mathbb{R})^\times)$ _{plec} should depend only on t .

Basic plectic element: $\chi = 1$, $\rho = \text{ord}_{s=1} L(A/K, s)$

$$P_\alpha \in \bigwedge_{\mathbb{Q}_\ell}^t H^1(K, V_\ell(A)) \xrightarrow{\sim} \bigwedge_{\mathbb{Q}_\ell}^t A(K) \otimes_{\mathbb{Q}_\ell} \xrightarrow{\sim} \bigwedge_{\mathbb{Q}}^t A(K) \otimes_{\mathbb{Q}}.$$

Speculations: $r = [F : \mathbb{Q}] \geq t = \dim \text{Sh}(B^*) \geq 1$

In the best of all possible worlds:

(A) If $\boxed{t=\rho} \stackrel{?}{\Rightarrow} \exists \alpha \quad \bigwedge_{\mathbb{Q}}^{\rho} A(K) \otimes_{\mathbb{Q}} = \mathbb{Q} \cdot P_\alpha$
 $(1 \leq \rho \leq r)$ height(P_α) = (const $\neq 0$) $L^{(\rho)}(A/K, 1)$
 (plectic Kolyvagin and Gross-Zagier)

What if $\rho > r$? It is very likely that
 $\forall C > 0 \exists F'/F$ cyclic totally real, $[F':F] \geq C$,
 $\rho' = \text{ord}_{s=1} L(A/F'K, s) = \rho$.

If true, (A) for $A_{F'}$ would give, a plectic CM elt $P_{\alpha'}$ over FK
 such that $\bigwedge_{\mathbb{Q}}^{\rho'} A(K) \otimes_{\mathbb{Q}} = \bigwedge_{\mathbb{Q}}^{\rho'} A(F'K) \otimes_{\mathbb{Q}} = \mathbb{Q} \cdot P_{\alpha'}$.

(B) If $\rho > t \Rightarrow P_\alpha = 0$ (true if $t=1$)

(C) If $\rho < t$ - we are perplexed.

BSD Conjecture predicts that $\bigwedge^t_{\mathbb{Q}_\ell} A(K)_{\mathbb{Q}_\ell} = 0$.

Motivic meaning of $P_\alpha \in \bigwedge^t_{\mathbb{Q}_\ell} H^1(K, V_\ell(A))$ is unclear.

Something does not seem to be right...

Possible analogy

Grothendieck's Standard Conjectures



geometric explanation of Weil's Conjectures

Our plectic speculations either make no sense,
or give a geometric explanation of
the BSD Conjecture over ring class fields of K.

Question: are there plectic toric elements
if $[K:F] = 2$, $K \neq CM$?

HAPPY BIRTHDAY,
KARL !