ϑ-functions, dual pairs, $H^*(\Gamma\backslash G/K)$

$$\vartheta(\tau)=\sum_{n\in\mathbb{Z}} e^{\pi i n^2\tau} \qquad \tau\in\mathcal{H}=\{\tau\in\mathbb{C}\mid \mathrm{Im}(\tau)>0\}\simeq SL_2(\mathbb{R})/SO(2)$$

matrix version: on $G/K=\widetilde{G}/\widetilde{K}$, $\quad G=Sp(2n,\mathbb{R})$, $K=U(n)$

$$1\longrightarrow\{\pm1\}\longrightarrow\widetilde{G}\xrightarrow{\pi}G\longrightarrow 1$$

dual pairs in G and $\widetilde{G}$:

(1) $V,(\,,\,)$ orthogonal, $W,\langle\,,\,\rangle$ symplectic $\Big\}\Rightarrow V\otimes W$ symplectic

$O(V)\times Sp(W)\hookrightarrow Sp(V\otimes W)$

mutual centralisers

splits $\rightarrow$ $\pi^{-1}(O(V))\times\pi^{-1}(Sp(W))\hookrightarrow\widetilde{Sp(V\otimes W)}$

(2) $V,(\,,\,)$ hermitian, $W,\langle\,,\,\rangle$ skew-hermitian $\Big\}\Rightarrow V\otimes W$ symplectic

$U(V)\times U(W)\hookrightarrow Sp(V\otimes W)$

$\pi^{-1}(U(V))\times\pi^{-1}(U(W))\hookrightarrow\widetilde{Sp(V\otimes W)}$

symmetric spaces: $G=O(V)=O(p,q)\supset K=O(p)\times O(q)$

$G'=Sp(2n,\mathbb{R})\supset K'=U(n)$

$$G/K\times\widetilde{G'}/\widetilde{K'}\longrightarrow\widetilde{Sp(V\otimes W)}/(\text{max. cpt.})=\underline{\underline{\widetilde{G}}}/\underline{\underline{\widetilde{K}}}$$

ϑ-correspondence: general ϑ-fns on $\uparrow$ give rise to

integral operators

$$\Gamma(G/K,\mathcal{L})\longrightarrow\Gamma(\widetilde{G'}/\widetilde{K'},\mathcal{L}')$$

$\cup$ $\qquad$ bundles $\qquad$ $\cup$

Γ-invariant sections $\qquad$ $\widetilde{\Gamma'}$-invariant sections

automorphic forms

$\Gamma\subset G$

$\Gamma'\subset G'$

arithmetic subgroups

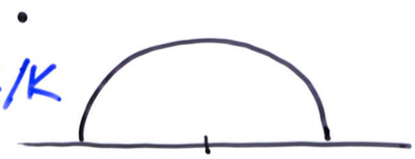
Kudla–Millson: constructed natural closed differential forms on G/K $\overset{\vartheta\text{-corr.}}{\rightsquigarrow}$ automorphic forms on $\widetilde{G'}/\widetilde{K'}$ with values in $H^*(\Gamma\backslash G/K)$ whose Fourier coefficients are cohomology classes of (comb. of) subvarieties $\Gamma_H\backslash H/K_H\hookrightarrow\Gamma\backslash G/K$, $H=O(p',q)\subset\underset{G}{\underline{O(p,q)}}$

$\Gamma_H=H\cap\Gamma$, $K_H=H\cap K$.

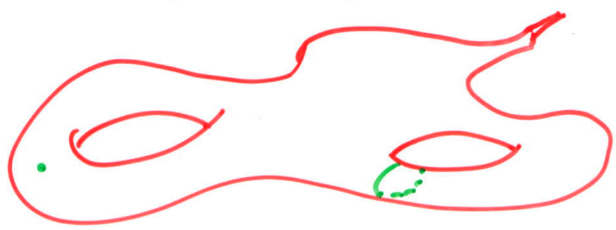
Ex.: $\mathcal{H}=SL_2(\mathbb{R})/SO(2)\simeq SO_0(1,2)/SO(2)\simeq O(1,2)/O(1)\times O(2)$

image of $O(0,2)$ = point, of $O(1,1)$ = geodesic

G/K



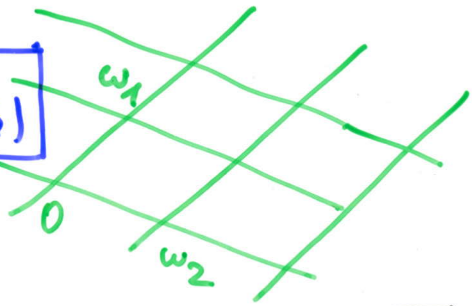
$\Gamma\backslash G/K$



Metaplectic group, oscillator (= Weil) representation

$$\boxed{\theta(z,\tau)=\sum_{n\in\mathbb{Z}} e^{\pi i n^2\tau+2\pi i n z}} \qquad (\tau\in\mathcal{H},\ z\in\mathbb{C})$$

$$\theta(z+1,\tau)=\theta(z,\tau),\qquad \theta(z+\tau,\tau)=e^{-2\pi i\left(z+\frac{\tau}{2}\right)}\,\theta(z,\tau)$$



Def. A θ-function w.r.t. lattice $L=\mathbb{Z}\omega_1+\mathbb{Z}\omega_2\subset\mathbb{C}$ is a holomorphic $f:\mathbb{C}\longrightarrow\mathbb{C}$ s.t.

$$\forall \omega\in L\ \exists \begin{pmatrix}a(\omega)\\ b(\omega)\end{pmatrix}\in\mathbb{C}^2\ \forall z\in\mathbb{C}\quad \boxed{f(z+\omega)=e^{a(\omega)z+b(\omega)}f(z)}$$

Note: $f\in\underbrace{\Gamma(\mathbb{C}/L,}_{\text{hol. sections}}\ \underbrace{\mathcal{O}(\mathcal{L})}_{\text{line bundle}})$

Transformation rules in z (τ and $L=\mathbb{Z}\tau+\mathbb{Z}$ fixed): action of the Heisenberg group (H. Weyl) – combines translations and multiplication by e^{az+b}.

Transformation rules in τ ($\mathbb{Z}\tau+\mathbb{Z}$ "fixed", basis changed)

$$\theta(\tau):=\theta(0,\tau)=\sum_{n\in\mathbb{Z}} e^{\pi i n^2\tau}$$

$$\boxed{\theta(\tau+2)=\theta(\tau),\quad \theta\left(-\tfrac{1}{\tau}\right)=\underbrace{\sqrt{\tfrac{\tau}{i}}}_{=1 \text{ at } \tau=i}\,\theta(\tau)\quad \forall\tau\in\mathcal{H}}$$

$\forall \begin{pmatrix}a&b\\ c&d\end{pmatrix}\in SL_2(\mathbb{Z})$, $2\mid ab, cd$ $\Longleftarrow$

$$\theta\left(\frac{a\tau+b}{c\tau+d}\right)=\zeta\,\underbrace{\sqrt{c\tau+d}}_{\text{branch of }\sqrt{\ }}\,\theta(\tau)\quad\forall\tau\in\mathcal{H},\qquad \zeta^8=1.$$

Action of the metaplectic group $\widetilde{SL_2(\mathbb{R})}$ (A. Weil) – two-fold cover of $SL_2(\mathbb{R})$.

Action of the semi-direct product (Heisenberg) $\rtimes\ \widetilde{SL_2(\mathbb{R})}$.

Key point: rigidity of unitary representations of (Heis) $\Rightarrow$ existence of $\widetilde{SL_2(\mathbb{R})}$, of θ, properties of θ...

Lie algebra representation (on $C^\infty(\mathbb{R})$): x coordinate on $\mathbb{R}$

$$\mathrm{Lie}(\mathrm{Heis})\xrightarrow{\sim}\mathrm{span}_{\mathbb{R}}\left(i,\ ix,\ \frac{d}{dx}\right)$$

$$\mathrm{Lie}(\widetilde{SL_2})=\mathrm{Lie}(SL_2)\xrightarrow{\sim}\mathrm{span}_{\mathbb{R}}\Big(\underset{\uparrow X}{\frac{ix^2}{2}},\ \underset{\uparrow Y}{\frac{i}{2}\left(\frac{d}{dx}\right)^2},\ \tfrac12\big(x\underset{\uparrow H}{\frac{d}{dx}}+\frac{d}{dx}x\big)\Big)$$

$[X,Y]=H$
$[H,X]=2X$
$[H,Y]=-2Y$

$$X=\begin{pmatrix}0&1\\0&0\end{pmatrix},\quad Y=\begin{pmatrix}0&0\\1&0\end{pmatrix},\quad H=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

Fourier, Poisson and θ: fix $\psi : (\mathbb{R},+) \twoheadrightarrow U(1) = \{z \in \mathbb{C} \mid |z| = 1\}$

$$\boxed{\psi(x) = e^{2\pi i x}}$$

(1) $\mathbb{Z} \subset \mathbb{R}$ is ψ-selfdual : $\mathbb{Z} = \{x \in \mathbb{R} \mid \psi(x\mathbb{Z}) = 1\}$

(2) Fourier transform $(\mathcal{F}f)(x) = \int \psi(xy) f(y)\, dy$ is self-dual: $\boxed{\mathcal{F} \circ \mathcal{F} = [-1]}$, $([r]f)(x) = f(rx)$, $\boxed{\mathcal{F} \circ [r] = |r|^{-1} [r^{-1}] \circ \mathcal{F} \quad 0 \neq r \in \mathbb{R}}$

(3) The gaussian $\varphi_0(x) = e^{-\pi x^2}$, $\boxed{\mathcal{F}\varphi_0 = \varphi_0}$

(4) Poisson formula: $\boxed{\sum_{n \in \mathbb{Z}} f(n) = \sum_{n \in \mathbb{Z}} (\mathcal{F}f)(n)}$ (f rapidly decreasing)

Back to $\theta(-\frac{1}{\tau})$: for $y \in \mathbb{R}_{>0}$ apply (4) to $f = \underbrace{[y^{1/2}]\varphi_0}_{e^{-\pi y x^2}}$

$$\theta(iy) = \sum_{n \in \mathbb{Z}} ([y^{1/2}]\varphi_0)(n) = y^{-1/2} \sum_{n \in \mathbb{Z}} ([y^{-1/2}]\varphi_0)(n) = y^{-1/2}\theta\left(-\frac{1}{iy}\right)$$

$f(\tau) := \theta(\tau) - \left(\frac{\tau}{i}\right)^{-1/2} \theta\left(-\frac{1}{\tau}\right)$ holomorphic in $\mathcal{H}$, $f|_{i\mathbb{R}_{>0}} = 0 \Rightarrow f = 0$.

Why oscillator representation?

classical mechanics (in dim = 1): space $= \mathbb{R} \xrightarrow{V} \mathbb{R}$, $x \mapsto V(x)$ potential energy

kinetic energy $T = \frac{p^2}{2m}$

$$\boxed{H = T + V = \frac{p^2}{2m} + V(x)}$$

x = coordinate, p = momentum, m = mass

quantum mechanics: operators $p \rightsquigarrow \hat{p} = \frac{\hbar}{i}\frac{d}{dx}$ $\qquad \hbar = \frac{h}{2\pi}$

$$H \rightsquigarrow \hat{H} = \frac{\hat{p}^2}{2m} + V(x) = -\frac{\hbar^2}{2m}\left(\frac{d}{dx}\right)^2 + V(x)$$

Schrödinger equation: $\hat{H}\psi = -\frac{\hbar}{i}\frac{\partial \psi}{\partial t}$ $\qquad \psi = \psi(x,t)$

try $\psi(x,t) = e^{-\frac{iE}{\hbar}t}\varphi(x)$ $\qquad \boxed{\hat{H}\varphi = E\varphi}$

Harmonic oscillator: $V(x) = \frac{ax^2}{2}$ $\quad (a > 0)$

$$\boxed{\left(\frac{h^2}{2m}\left(\frac{1}{2\pi i}\frac{d}{dx}\right)^2 + \frac{a}{2}x^2\right)\varphi(x) = E\varphi(x)}$$

Heisenberg Lie algebra: $\mathbb{R}\cdot P + \mathbb{R}\cdot Q + \mathbb{R}\cdot E$, $[P,Q] = E$, $[E,\cdot] = 0$

want: representations s.t. $E \longmapsto (2\pi i)\mathbb{1}$

Standard ("Schrödinger") representation:

$$P \longmapsto \hat{P} = \frac{d}{dx}, \quad Q \longmapsto \hat{Q} = 2\pi i x, \quad E \longmapsto \hat{E} = 2\pi i$$

$(\exp(t\hat{P})f)(x) = f(x+t)$, $\quad (\exp(t\hat{Q})f)(x) = e^{2\pi i t x} f(x)$

$(\exp(t\hat{E})f)(x) = e^{2\pi i t} f(x)$ $\qquad$ unitary operators on $L^2(\mathbb{R})$, $\|f\|^2 = \int |f|^2 dx$

Action on the gaussian $\varphi_0 = e^{-\pi x^2}$: $\hat{P} = \frac{d}{dx}$, $\hat{Q} = 2\pi i x$

$$\hat{P}\varphi_0 = -2\pi x\varphi_0 = i\hat{Q}\varphi_0 \implies \boxed{(\hat{P} - i\hat{Q})\varphi_0 = 0} \qquad \boxed{A_-\varphi_0 = 0}$$

$$A_- = \hat{P} - i\hat{Q} = \frac{d}{dx} + 2\pi x$$

$$A_+ = \hat{P} + i\hat{Q} = \frac{d}{dx} - 2\pi x$$

$$[A_+, A_-] = 4\pi \cdot 1$$

Representation of $sl(2)$:

$$\boxed{\begin{array}{ll} X \longmapsto \frac{1}{8\pi} A_+^2 & H \longmapsto -\frac{1}{8\pi}(A_+A_- + A_-A_+) = \\ Y \longmapsto -\frac{1}{8\pi} A_-^2 & = \frac{1}{2} - \frac{1}{4\pi} A_+A_- = \underline{\hat{H}} \end{array}}$$

Standard action of $sl(2)$ on $\mathbb{R}A_+ + \mathbb{R}A_-$:

$$[H, A_\pm] = \pm A_\pm, \quad [X, A_-] = A_+, \quad [Y, A_+] = A_-$$
$$[X, A_+] = 0, \quad [Y, A_-] = 0$$

Def: $\varphi_n = A_+^n \varphi_0$ $(n \geq 0)$

$$\varphi_0 = e^{-\pi x^2}, \quad \varphi_1 = (-4\pi x) e^{-\pi x^2}$$

$$\boxed{\begin{array}{l} Y\varphi_0 = Y\varphi_1 = 0 \\ H\varphi_0 = \frac{1}{2}\varphi_0, \quad H\varphi_1 = \frac{3}{2}\varphi_1 \end{array}}$$

lowest weight vectors for $sl(2)$

$$\begin{tikzcd} \varphi_n \arrow[r, bend left, "X"] & \varphi_{n+2} \arrow[l, bend left, "Y"] \end{tikzcd}$$

(up to a mult. const. $\neq 0$)

$L^2(\mathbb{R})$ has orthogonal basis $\varphi_0, \varphi_1, \varphi_2, \ldots$

$$\boxed{H\varphi_n = (n + \tfrac{1}{2})\varphi_n}$$

Formula for φ_n: $A_+ = e^{\pi x^2} \circ \frac{d}{dx} \circ e^{-\pi x^2}$

$$\varphi_n = \left(e^{\pi x^2} \circ \left(\frac{d}{dx}\right)^n \circ e^{-\pi x^2}\right)(e^{-\pi x^2}) = f_n(x) e^{-\pi x^2}$$

$$f_n(x) = e^{2\pi x^2}\left(\frac{d}{dx}\right)^n (e^{-2\pi x^2}) = (-1)^n (2\pi)^{n/2} H_n(x\sqrt{2\pi})$$

Hermite polynomials: $H_n(t) = (-1)^n e^{t^2}\left(\frac{d}{dt}\right)^n (e^{-t^2})$

Fourier transform on $L^2(\mathbb{R})$: $\mathcal{F} \circ \mathcal{F} = [-1]$, $([r]f)(x) = f(rx)$

$$(\mathcal{F}f)(x) = \int_{\mathbb{R}} e^{2\pi i x y} f(y)\, dy$$

$$\left.\begin{array}{l} \hat{P} \circ \mathcal{F} = \mathcal{F} \circ \hat{Q} \\ \hat{P} \circ [-1] = -[-1] \circ \hat{P} \\ \hat{Q} \circ [-1] = -[-1] \circ \hat{Q} \end{array}\right\} \implies \mathcal{F} \circ \hat{P} = -\hat{Q} \circ \mathcal{F} \implies \boxed{\mathcal{F} \circ A_\pm = \pm i A_\pm \circ \mathcal{F}}$$

$$\mathcal{F}\varphi_0 = \varphi_0 \implies \mathcal{F}\varphi_n = \mathcal{F}A_+^n\varphi_0 = i^n A_+^n \mathcal{F}\varphi_0 = i^n \varphi_n \qquad (\forall n \geq 0)$$

$$H\varphi_n = (n + \tfrac{1}{2})\varphi_n \implies e^{\pi i H/2}\varphi_n = i^n e^{\pi i/4}\varphi_n = e^{\pi i/4}\mathcal{F}\varphi_n$$

$$\boxed{\mathcal{F} = e^{-\pi i/4}\, e^{\pi i H/2}}$$

Next time: given $u : \mathbb{Z} \longrightarrow \mathbb{Z}/N\mathbb{Z} \longrightarrow \mathbb{C}$

$$\left.\begin{array}{ll} \theta_{1/2,u}(\tau) = \sum_{n\in\mathbb{Z}} u(n) e^{\pi i n^2 \tau}, & u(-n) = u(n) \\ \theta_{3/2,u}(\tau) = \sum_{n\in\mathbb{Z}} u(n)\, n\, e^{\pi i n^2 \tau}, & u(-n) = -u(n) \end{array}\right\} \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} (\mathrm{mod}\ 2N^2)$$

$$\theta_{\lambda,u}\left(\frac{a\tau + b}{c\tau + d}\right) = (\text{const.})(c\tau + d)^\lambda \theta_{\lambda,u}(\tau)$$

Question: $\lambda = \frac{5}{2}, \frac{7}{2}, \ldots$??

Interfwining operators $\quad \hat P = \frac{d}{dx}, \ \hat Q = 2\pi i x$

$$\begin{pmatrix}\hat P\\ \hat Q\end{pmatrix}\circ \mathcal F = \mathcal F\circ \begin{pmatrix}\hat Q\\ -\hat P\end{pmatrix} = \mathcal F\circ\begin{pmatrix}0&1\\-1&0\end{pmatrix}\begin{pmatrix}\hat P\\ \hat Q\end{pmatrix} \iff \boxed{\mathcal F^{-1}\circ\begin{pmatrix}\hat P\\ \hat Q\end{pmatrix}\circ\mathcal F = \begin{pmatrix}0&1\\-1&0\end{pmatrix}\begin{pmatrix}\hat P\\ \hat Q\end{pmatrix}}$$

Any $\begin{pmatrix}a&b\\c&d\end{pmatrix}\in M_2(\mathbb R)$: $\quad [a\hat P + b\hat Q,\ c\hat P + d\hat Q] = (ad-bc)\underbrace{[\hat P,\hat Q]}_{2\pi i}$

$\forall\ g = \begin{pmatrix}a&b\\c&d\end{pmatrix}\in SL_2(\mathbb R)$ $\quad \begin{pmatrix}P\\ Q\end{pmatrix}\longmapsto\begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}\hat P\\ \hat Q\end{pmatrix}$, $\quad E\longmapsto (2\pi i)\mathbb 1$ $\quad$ representation of the Heisenberg Lie algebra

Is it equivalent to the standard one $P\mapsto\hat P$, $Q\mapsto\hat Q$?

$\Updownarrow$

$$\exists\ F_g\in U(\mathcal H)\qquad \boxed{\begin{pmatrix}\hat P\\ \hat Q\end{pmatrix}\circ F_g = F_g\circ\begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}\hat P\\ \hat Q\end{pmatrix}}\ ? \qquad \mathcal H = L^2(\mathbb R)$$

(1) $g = \begin{pmatrix}a&0\\0&a^{-1}\end{pmatrix}$: $\quad F_g = (\text{const})\cdot[a]$: $\quad [a]\circ\hat Q = a\,\hat Q\circ[a]$, $\ \hat P\circ[a] = a[a]\circ\hat P$

$\boxed{F_g = |a|^{1/2}u_g\,[a],\ u_g\in U(1)}$ $\qquad \|[a]f\|^2 = |a|^{-1}\|f\|^2$

(2) $g = \begin{pmatrix}1&b\\0&1\end{pmatrix}$: $\quad f\in C^\infty(\mathbb R)\quad \hat Q\circ f = f\circ\hat Q,\ \ \hat P\circ f = f\circ\hat P + f'$

want : $f' = (2\pi i)xbf \implies f = (\text{const.})e^{\pi i b x^2}$, $\ \boxed{F_g = u_g e^{\pi i b x^2},\ u_g\in U(1)}$

(3) $g = \begin{pmatrix}a&b\\c&d\end{pmatrix}$, $c\neq 0$, $ad-bc=1$: $\quad (F_g f)(x) = \int_{\mathbb R} K(x,y)\,\underline{f(y)}\,dy$ $\quad$ rapidly decreasing

$$((\hat P\circ F_g)f)(x) = \int_{\mathbb R}\frac{\partial K}{\partial x}(x,y)f(y)\,dy$$

$$(F_g\circ\hat P f)(x) = \int_{\mathbb R}K(x,y)f'(y)\,dy = \overbrace{[K(x,y)f(y)]_{y=-\infty}^{y=+\infty}}^{0} - \int_{\mathbb R}\frac{\partial K}{\partial y}(x,y)f(y)\,dy$$

$$\left.\begin{aligned}\frac{\partial K}{\partial x} &= -a\frac{\partial K}{\partial y} + (2\pi i b)yK\\ 2\pi i x K &= -c\frac{\partial K}{\partial y} + (2\pi i d)yK\end{aligned}\right\}\quad \begin{aligned}\partial K/\partial y &= 2\pi i\left(\frac{-x+dy}{c}\right)K\\ \partial K/\partial x &= 2\pi i\left(\frac{ax-y}{c}\right)K\end{aligned}$$

$$\boxed{K = (\text{const.})\,e^{\pi i\left(\frac{ax^2-2xy+dy^2}{c}\right)},\quad (\text{const.}) = |c|^{-1/2}u_g,\quad u_g\in U(1)}$$

the Operators $F_g\in U(\mathcal H)$ are unique up to $U(1)$ $\Longrightarrow$

$$\begin{array}{ccc} SL_2(\mathbb R) & \longrightarrow & U(\mathcal H)/U(1) = PU(\mathcal H) \\ g & \longmapsto & F_g \pmod{U(1)} \\ & & \uparrow \\ & \dashrightarrow & U(\mathcal H)\end{array}$$

is a group homomorphism

does this lift ?

(to a group homomorphism) ?

$\Updownarrow$

$\exists\, u_g\in U(1)$ s.t. $\forall\, g,g'\in SL_2(\mathbb R)$ $\qquad F_g F_{g'} = F_{gg'}$? $\quad$ NO!

Projective representations and 2-cocycles

Given: $G \xrightarrow{r} PGL(V) = GL(V)/\mathbb{C}^* \cdot 1_V$, G group, V $\mathbb{C}$-v.space

$r =$ group homomorphism

Choose:
$$\begin{tikzcd} G \arrow[r, "r"] \arrow[dr, "\tilde{r}"'] & PGL(V) \\ & GL(V) \arrow[u, "pr"'] \end{tikzcd}$$

$\forall g_i \in G$
$$\tilde{r}(g_1)\tilde{r}(g_2) = c(g_1, g_2)\tilde{r}(g_1 g_2)$$
$c(g_1,g_2) \in \mathbb{C}^*$

$\tilde{r} =$ map s.t. $pr \circ \tilde{r} = r$

Associativity: $(\tilde{r}(g_1)\tilde{r}(g_2))\tilde{r}(g_3) = \tilde{r}(g_1)(\tilde{r}(g_2)\tilde{r}(g_3))$

$\Longrightarrow$
$$\boxed{\begin{gathered} c(g_1, g_2)\, c(g_1 g_2, g_3) = c(g_1, g_2 g_3)\, c(g_2, g_3) \\ \text{2-cocycle identity: } \quad c \in Z^2(G, \mathbb{C}^*) \end{gathered}}$$

Change of $\tilde{r}$: $\tilde{r} \rightsquigarrow \tilde{r}' = \tilde{r} \cdot d$, $\quad d : G \longrightarrow \mathbb{C}^*$ (map)
$$\boxed{c'(g_1, g_2) = c(g_1, g_2)\, \frac{d(g_1)\, d(g_2)}{d(g_1 g_2)}}$$

Assume: the values of c' lie in a subgroup $A \subset \mathbb{C}^*$

Def: group $\tilde{G} = \{(g, a) \mid g \in G, a \in A\}$
$$(g_1, a_1)(g_2, a_2) = (g_1 g_2,\ a_1 a_2 c'(g_1, g_2))$$

$a \longmapsto (e, a)$

$$\begin{tikzcd} 1 \arrow[r] & A \arrow[r, "\iota"] \arrow[d, hook] & \tilde{G} \arrow[r, "pr"] \arrow[d, "\rho"] & G \arrow[r] \arrow[d, "r"] & 1 \\ 1 \arrow[r] & \mathbb{C}^* \arrow[r] & GL(V) \arrow[r] & PGL(V) \arrow[r] & 1 \end{tikzcd}$$

$\iota(A) \subset Z(G)$
central extension

$$\rho((g, a)) = a\, \tilde{r}'(g)$$

Homogeneous vs non-homogeneous cocycles

Given: group G acting on a set X, map $f : X^3 \longrightarrow A$ (for an abelian group A) s.t. $f(gx_1, gx_2, gx_3) = f(x_1, x_2, x_3)$ $\forall g \in G$.

Fix $x \in X$.

Then: f satisfies $\forall x_0, x_1, x_2, x_3 \in X$ $\quad \sum_{i=0}^{3} (-1)^i f(x_0, \dots, \widehat{x_i}, \dots, x_3) = 0$

$\Downarrow$

$c : G \times G \longrightarrow A$, $\quad c(g_1, g_2) = f(x, g_1(x), g_1 g_2(x))$

satisfies
$$c(g_1, g_2) + c(g_1 g_2, g_3) = c(g_1, g_2 g_3) + c(g_2, g_3)$$

Higher-dimensional case

Given: W real v. sp., $\dim(W) \overset{=2n}{<\infty}$, $B: W\times W \longrightarrow \mathbb{R}$ symplectic (alternating, non-degenerate)

$\exists$ symplectic basis: $W = \bigoplus_{j=1}^{n} (\mathbb{R}P_j \oplus \mathbb{R}Q_j)$ $\qquad B(P_j, P_k) = B(Q_j, Q_k) = 0$, $\quad B(P_j, Q_k) = \delta_{jk}$

Heisenberg Lie algebra: $\mathfrak{heis} = W \oplus \mathbb{R}\cdot E$

$[E, \cdot\,] = 0$, $\quad \forall x, y \in W \quad [x,y] = B(x,y)E$

Heisenberg group: (1) simply connected: exponentiate $[\,,\,]$ using Campbell–Hausdorff:

$$\exp(X)\exp(Y) = \exp\left(X + Y + \tfrac{1}{2}[X,Y] + \text{higher } [[\,,\,],\,]\right)$$

$$\{\underbrace{\exp(x+tE)}_{\{x,t\}} \mid x \in W, t \in \mathbb{R}\} \qquad \{x,t\}\{x',t'\} = \{x+x', t+t' + \tfrac{1}{2}B(x,x')\}$$

(2) push-forward via $\exp(\mathbb{R}E) = \mathbb{R} \xrightarrow{\psi} U(1)$, $\quad t \longmapsto e^{2\pi i t}$

$$\mathrm{Heis} = \{(x,a) \mid x \in W,\ a \in U(1)\}$$

$$(x,a)(x',a') = (x+x', aa'\psi(\tfrac{1}{2}B(x,x')))$$

$$1 \longrightarrow U(1) \longrightarrow \mathrm{Heis} \xrightarrow{\pi} W \longrightarrow 0, \quad \pi(x,a) = x, \quad Z = U(1)$$

Schrödinger representation: on $C^\infty(\mathbb{R}^n)$ $\qquad x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$

$P_j \longmapsto \hat{P}_j = \dfrac{\partial}{\partial x_j}$, $\quad Q_j \longmapsto \hat{Q}_j = 2\pi i x_j$, $\quad u \in U(1) \longmapsto u$

$$\left(u \exp\left(\textstyle\sum a_j P_j + b_j Q_j\right) f\right)(x) = u\, e^{2\pi i\left({}^t b x + \frac{1}{2}{}^t b a\right)} f(x+a)$$

irreducible unitary (on $L^2(\mathbb{R}^n)$)

Key Thm (Stone, von Neumann)

An irreducible unitary representation $\rho: \mathrm{Heis} \longrightarrow U(\mathcal{H})$ ($\mathcal{H}$ = Hilbert space) s.t. $\rho(u) = u\cdot 1$ $\forall u \in U(1)$ is isomorphic to the above representation.

Construction of $\rho: \mathrm{Heis} \longrightarrow U(\mathcal{H})$, $\rho(u) = u \cdot 1$ $\forall u \in U(1)$:

induced from characters of abelian subgroups containing $U(1)$

$$\left\{\begin{array}{l}\text{abelian subgps of} \\ \text{Heis containing } U(1)\end{array}\right\} = \left\{ \pi^{-1}(L) \;\middle|\; \begin{array}{l} L \subset W \text{ subgroup such that} \\ \forall x,y \in L \quad B(x,y) \in \mathbb{Z} \end{array}\right\}$$

L closed $\Longrightarrow$

$$\begin{tikzcd}
1 \arrow[r] & U(1) \arrow[r] & \mathrm{Heis} \arrow[r] & W \arrow[r] & 0 \\
1 \arrow[r] & U(1) \arrow[u, equal] \arrow[r] & \pi^{-1}(L) \arrow[u, phantom, "\cup"] \arrow[r] \arrow[l, bend right, "\chi"'] & L \arrow[u, phantom, "\cup"] \arrow[r] & 0
\end{tikzcd}$$

$\exists$ cont. χ

Def: $\mathrm{Ind}_{\pi^{-1}(L)}^{\mathrm{Heis}}(\chi) = \left\{ \begin{array}{l} f: \mathrm{Heis} \to \mathbb{C},\ f(hg) = \chi(h) f(g) \quad \forall h \in \pi^{-1}(L), \forall g \in \mathrm{Heis} \\ f \text{ measurable}, \quad \int_{\pi^{-1}(L)\backslash \mathrm{Heis}} |f|^2 < \infty \end{array}\right\}$

$(g * f)(g') = f(g'g)$

Unitary representation of Heis, irreducible if L is maximal.

<u>Formulas</u>: choice of $\chi \iff$ choice of $\pi^{-1}(L) \underset{\longrightarrow}{\overset{\sigma}{\longleftarrow}} L$, $\chi \circ \sigma = 1$

$\boxed{\sigma(y) = (y, \alpha(y))}$, $\forall y, y' \in L$ $\sigma(y)\sigma(y') = \sigma(y+y') \Longrightarrow \alpha : L \to U(1)$

satisfies $\boxed{\alpha(y)\alpha(y')e^{\pi i B(y,y')} = \alpha(y+y')}$ $\forall y, y' \in L$

$f \in \mathrm{Ind}_{\pi^{-1}(L)}^{\mathrm{Heis}}(\chi) \Longrightarrow f((x,u)) = u f((x,1))$ $x \in W$, $u \in U(1)$

write $f(x) := f((x,1))$, $x \in W$

$f(hg) = \chi(h) f(g) \iff f(y+x) = \alpha(y)^{-1} e^{-\pi i B(y,x)} f(x)$ $\forall x \in W$ $\forall y \in L$

$\int_{\pi^{-1}(L)\backslash \mathrm{Heis}} |f|^2 < \infty \iff \int_{W/L} |f(x)|^2 \, d\dot{x} < \infty$, $d\dot{x}$ any invariant measure

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<u>Examples of maximal $L \subset W$ s.t. $B(L,L) \subseteq \mathbb{Z}$:</u>

(1) $L = \bigoplus_{j=1}^{n} (\mathbb{Z}P_j + \mathbb{Z}Q_j)$

(2) $L = \ell$ $\mathbb{R}$-v. subspace s.t. $\ell = \ell^{\perp}$ $\underbrace{\{x \in W \mid \forall y \in \ell \;\; B(x,y) = 0\}}$ (ℓ = lagrangian subspace)

$\ell = \bigoplus_{j=1}^{n} \mathbb{R}P_j$, $\bigoplus_{j=1}^{n} \mathbb{R}Q_j$ $(\alpha = 1)$ $\underline{\chi((x,u)) = u}$

For ℓ identify W/ℓ with $\ell' = \bigoplus_{j=1}^{n} \mathbb{R}Q_j$.

$f \in \mathrm{Ind}_{\pi^{-1}(\ell)}^{\mathrm{Heis}}$ is determined by $f|_{\ell'} \in L^2(\mathbb{R}^n) \overset{\text{get}}{\Longrightarrow}$ Schrödinger repr.

<u>Intertwining operators</u>

<u>Measures (invariant)</u>: $\ell \in \mathrm{Lagr}(W,B) = \{$lagrangian subspaces of $W\}$

$0 \neq e \in \Lambda^n \ell \Rightarrow$ measure on $W/\ell \simeq \ell^* \Longrightarrow H(\ell, e) = L^2(W/\ell)$

<u>Exercise</u>: (1) $\ell_1, \ell_2 \in \mathrm{Lagr}(W) \Longrightarrow ((\ell_1+\ell_2)/(\ell_1 \cap \ell_2), \overline{B})$ symplectic

$\Longrightarrow$ measure on $(\ell_1+\ell_2)/(\ell_1 \cap \ell_2)$ $\Big\}$ $\Longrightarrow$ measure on $\ell_2/(\ell_1 \cap \ell_2)$.

(2) Given $0 \neq e_i \in \Lambda^n \ell_i$

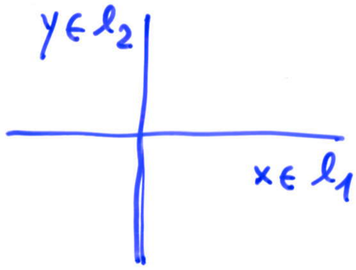
<u>Def</u>: $\ell_1, \ell_2 \in \mathrm{Lagr}(W)$, $0 \neq e_i \in \Lambda^n \ell_i$:

$$H(\ell_1, e_1) = \mathrm{Ind}_{\pi^{-1}(\ell_1)}^{\mathrm{Heis}}(\chi) \xrightarrow{F_{\ell_2,\ell_1}} H(\ell_2, e_2)$$

$$f \longmapsto \left(g \longmapsto \int_{\pi^{-1}(\ell_1 \cap \ell_2)\backslash \pi^{-1}(\ell_2) \simeq \ell_2/(\ell_1 \cap \ell_2)} \chi(h)^{-1} f(hg) \right)$$

<u>Heis-equivariant</u>

<u>Ex</u>: $\ell_1 \cap \ell_2 = \{0\}$, $H(\ell_1) = L^2(\ell_2) \to H(\ell_2) = L^2(\ell_1)$



$$(F_{\ell_2,\ell_1} f)(x) = \int_{\ell_2} f(\underbrace{(y,1)(x,1)}_{(0,\, e^{2\pi i B(y,x)})(x,1)(y,1)}) \, dy$$

$$= \int_{\ell_2} e^{2\pi i B(y,x)} f(y) \, dy$$

$F_{\ell_2,\ell_1} = \mathcal{F} \circ [-1] = \mathcal{F}^{-1}$

General case: $V = \ell_1 + \ell_2$, $B : (W/V)^* \simeq V^\perp = \ell_1 \cap \ell_2$

$$\begin{tikzcd}
 & & \ell_1 \arrow[dr, hook] & & \\
\{0\} \arrow[r, hook] & V^\perp = \ell_1 \cap \ell_2 \arrow[ur, hook] \arrow[dr, hook] & & V = \ell_1 + \ell_2 \arrow[r, hook] & W \\
 & & \ell_2 \arrow[ur, hook] & &
\end{tikzcd}$$

dim = $\quad m \quad\quad n \quad\quad 2n-m \quad\quad 2n$

$\ell_i / \ell_1 \cap \ell_2 \in \mathrm{Lagr}(V/V^\perp)$ transversal

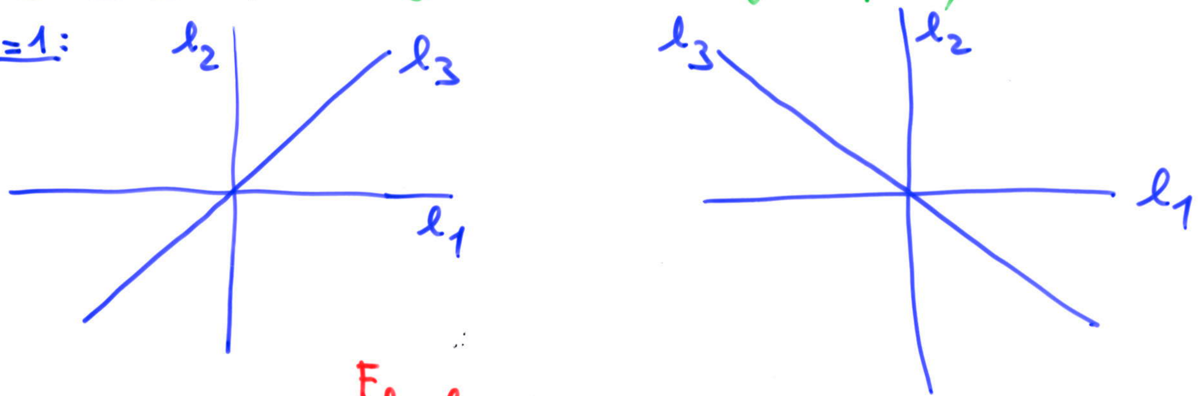
$\Rightarrow$ $F_{\ell_2,\ell_1} : H(\ell_1) \longrightarrow H(\ell_2)$ is a partial Fourier transform (in $n-m$ variables) $\Rightarrow$ $\boxed{F_{\ell_1,\ell_2} = F_{\ell_2,\ell_1}^{-1}}$

($\Rightarrow$ unitary)

Invariants of triples of lagrangian subspaces

Basic 2-cocycle: given $\ell_i \in \mathrm{Lagr}(W,B)$, $0 \neq e_i \in \Lambda^n \ell_i$ $(i = 1,2,3)$

n=1:



$$\begin{tikzcd}
H(\ell_1, e_1) \arrow[rr, "F_{\ell_2,\ell_1}"] & & H(\ell_2, e_2) \arrow[dl, "F_{\ell_3,\ell_2}"] \\
 & H(\ell_3, e_3) \arrow[ul, "F_{\ell_1,\ell_3}"] &
\end{tikzcd}$$

unitary, Heis-equiv.

independent of e_1, e_2, e_3

$$F_{\ell_1,\ell_3} \circ F_{\ell_3,\ell_2} \circ F_{\ell_2,\ell_1} = \underbrace{a(\ell_1,\ell_2,\ell_3)}_{\in U(1)} \cdot 1$$

Properties: (1) $a(\ell_1,\ell_2,\ell_3) = a(\ell_2,\ell_3,\ell_1)$

(2) a is a 2-cocycle: $\prod_{j=1}^{4} a(\ell_1, -, \hat{\ell}_j, -, \ell_4)^{(-1)^j} = 1$

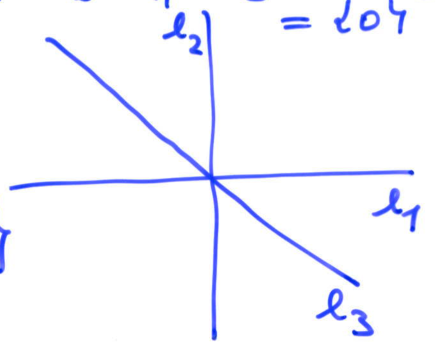
(3) $\forall \gamma \in G = Sp(W,B) = \{g \in GL(W) \mid B(gx, gy) \ \forall x,y \in W\}$

$a(\gamma\ell_1, \gamma\ell_2, \gamma\ell_3) = a(\ell_1,\ell_2,\ell_3)$

Pf of (3): G acts on Heis $\gamma(x,u) = (\gamma(x), u)$ and on $\{f : \mathrm{Heis} \to \mathbb{C}\}$, $({}^\gamma f)(g) = f(\gamma^{-1}(g))$, ${}^\gamma\chi = \chi$.

$H(\ell, e) \xrightarrow{\gamma} H(\gamma\ell, \gamma e)$, commutes with F_{ℓ_2,ℓ_1}.

Algebraic invariants: **transversal case** $\ell_1 \cap \ell_2 = \ell_1 \cap \ell_3 = \ell_2 \cap \ell_3 = \{0\}$



$B : \ell_2 \xrightarrow{\sim} \ell_1^* \implies W = \ell_1 \oplus \ell_1^*$

$B\left(\begin{pmatrix} x \\ x^* \end{pmatrix}, \begin{pmatrix} y \\ y^* \end{pmatrix}\right) = \langle y^*, x\rangle - \langle x^*, y\rangle$

$\ell_3 = \Gamma_T =$ graph of $T : \ell_1 \xrightarrow{\sim} \ell_1^* = \left\{ \begin{pmatrix} x \\ Tx \end{pmatrix} \mid x \in \ell_1 \right\}$

$\ell_3 \in \mathrm{Lagr} \iff \forall x, y \in \ell_1 \quad \underbrace{B\left(\begin{pmatrix} x \\ Tx \end{pmatrix}, \begin{pmatrix} y \\ Ty \end{pmatrix}\right)}_{\langle Ty, x\rangle - \langle Tx, y\rangle = \langle (T^* - T)x, y\rangle} = 0$

$\underbrace{T = T^*}_{\text{non-degenerate quadratic form on } \ell_1} \iff$

Action of $\{ g \in G \mid g(\ell_i) = \ell_i,\ i = 1, 2\} = \left\{ \begin{pmatrix} a^{-1} & 0 \\ 0 & a^* \end{pmatrix} \mid a \in GL(\ell_1) \right\}$:

$g(\Gamma_T) = \left\{ \begin{pmatrix} a^{-1}x \\ a^*Tx \end{pmatrix} \mid x \in \ell_1 \right\} = \left\{ \begin{pmatrix} y \\ (a^*Ta)y \end{pmatrix} \mid y \in \ell_1 \right\} = \Gamma_{a^*Ta}$

Cor: $G \backslash \left(\mathrm{Lagr}(W)^3\right)^{\text{transversal}} \xrightarrow{\sim} \left\{ \begin{matrix} \text{non-deg. quadratic} \\ \text{forms on } \mathbb{R}^n \end{matrix} \right\} / \text{Isom}$

General case: $\ell_i \in \mathrm{Lagr}(W)$

$Q_{\ell_1, \ell_2, \ell_3} : \ell_1 \oplus \ell_2 \oplus \ell_3 \longrightarrow \mathbb{R}$ quadratic form (on $\simeq \mathbb{R}^{3n}$)

$(x_1, x_2, x_3) \longmapsto B(x_1, x_2) + B(x_2, x_3) + B(x_3, x_1)$

Exercise: in the transversal case $Q \simeq T \perp S \perp (-S)$

Maslov index: $\mathrm{Maslov}(\ell_1, \ell_2, \ell_3) = \mathrm{sgn}\left(Q_{\ell_1, \ell_2, \ell_3}\right) \in \mathbb{Z}$

($= \mathrm{sgn}(T)$ in the transversal case).

Key Thm. (0) $\mathrm{Maslov}(\gamma\ell_1, \gamma\ell_2, \gamma\ell_3) = \mathrm{Maslov}(\ell_1, \ell_2, \ell_3) \quad \forall \gamma \in G$

(1) $\sum_{j=0}^{3} (-1)^j \,\mathrm{Maslov}(\ell_0, -, \widehat{\ell_j}, -, \ell_3) = 0$ (1-cocycle)

(2) $a(\ell_1, \ell_2, \ell_3) = \left(e^{-2\pi i/8}\right)^{\mathrm{Maslov}(\ell_1, \ell_2, \ell_3)}$

Pf: (1) WLOG ℓ_0 transversal to ℓ_1, ℓ_2, ℓ_3. Use T above.

(2) Reduce to the transversal case. Then $\ell_3 = \Gamma_T$,

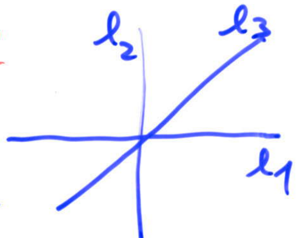
$T = \sum x_j^2 - \sum y_k^2 \implies$ direct sum of

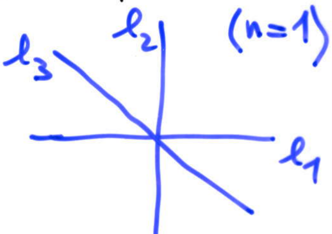
For $n = 1$, $a(\ell_1, \ell_2, \ell_3)$ combines

$\mathcal{F}$ and $e^{\pm \pi i x^2}$

Apply them to suitable $f_\tau = e^{\pi i \tau x^2} \implies$

(n=1)





$e^{-2\pi i/8}$ $\qquad$ $e^{2\pi i/8}$

$1 = \mathrm{Maslov} = -1$

Why "−"? For $\psi_t(x) = e^{2\pi i t x}$ $\begin{cases} \text{no change} & \text{if } t = r^2 > 0 \\ \text{complex conj.} & \text{if } t = -r^2 < 0 \end{cases}$

Why 8? Bott periodicity, Clifford algebras

F field (char $\neq 2$): Witt ring $WF = K_0(\text{quadr. forms}/F)/\langle x^2 - y^2\rangle$

$0 \to IF \to WF \xrightarrow{rk} \mathbb{Z}/2\mathbb{Z} \to 0$ sgn: $W\mathbb{R} \xrightarrow{\sim} \mathbb{Z}$

Q quadr. form (non-deg.) $\Longrightarrow$ Clifford algebra $C(Q)$, $\mathbb{Z}/2$-graded

$C(Q_1 \perp Q_2) \simeq C(Q_1) \hat{\otimes} C(Q_2)$

$C: (WF/I^3F, +) \xrightarrow{\sim} ((\text{Clifford algebras over } F)/\sim, \hat{\otimes})$

$I\mathbb{R} = 2\mathbb{Z}$, $W\mathbb{R}/I^3\mathbb{R} = \mathbb{Z}/8\mathbb{Z}$

Reduced (oriented) Maslov index

Thm: For $\tilde{\ell}_1, \tilde{\ell}_2, \tilde{\ell}_3 \in \widetilde{Lagr}(W)$ oriented lagrangian subspaces

let $m(\tilde{\ell}_1, \tilde{\ell}_2) = n - \dim(\ell_1 \cap \ell_2) + \begin{cases} 0 & \text{if } B: \ell_1/\ell_1\cap\ell_2 \xrightarrow{\sim} (\ell_2/\ell_1\cap\ell_2)^* \\ & \text{is compatible with orient.} \\ 2 & \text{if not} \end{cases}$

then: $\mathrm{Maslov}(\ell_1, \ell_2, \ell_3) \equiv m(\tilde{\ell}_1, \tilde{\ell}_2) - m(\tilde{\ell}_1, \tilde{\ell}_3) + m(\tilde{\ell}_2, \tilde{\ell}_3) \pmod 4$

Note: $\mathrm{sgn}(Q) \pmod 4 \Longleftrightarrow \mathrm{rk}(Q) \pmod 4$, $\det(Q) \in \mathbb{R}^*/\mathbb{R}^{*2}$

Def. $\widetilde{\mathrm{Maslov}}(\tilde{\ell}_1, \tilde{\ell}_2, \tilde{\ell}_3) := \underbrace{\mathrm{Maslov}(\ell_1, \ell_2, \ell_3) - m(\tilde{\ell}_1, \tilde{\ell}_2) + m(\tilde{\ell}_1, \tilde{\ell}_3) - m(\tilde{\ell}_2, \tilde{\ell}_3)}_{4\mathbb{Z}}$

$\tilde{F}_{\tilde{\ell}_2, \tilde{\ell}_1} := (e^{-2\pi i/8})^{m(\tilde{\ell}_1, \tilde{\ell}_2)} F_{\ell_2, \ell_1}$

Thm: $\tilde{F}_{\tilde{\ell}_3, \tilde{\ell}_2} \circ \tilde{F}_{\tilde{\ell}_2, \tilde{\ell}_1} = \underbrace{(e^{-2\pi i/8})^{\widetilde{\mathrm{Maslov}}(\tilde{\ell}_1, \tilde{\ell}_2, \tilde{\ell}_3)}}_{\pm 1} \tilde{F}_{\tilde{\ell}_3, \tilde{\ell}_1}$

Metaplectic group

Fix: $\tilde{\ell} \in \widetilde{Lagr}(W)$, $0 \neq e \in \Lambda^n \ell \Longrightarrow \forall g \in G = Sp(W)$ $g\tilde{\ell}, ge$

Def: $\mathcal{H} = H(\ell, e) \underset{F_{\ell, g\ell}}{\overset{g}{\rightleftarrows}} H(g\ell, ge)$ $R_\ell(g) := F_{\ell, g\ell} \circ g \in U(\mathcal{H})$

$R_\ell(g_1) R_\ell(g_2) = c(g_1, g_2) R_\ell(g_1 g_2)$, $c(g_1, g_2) = (e^{2\pi i/8})^{\mathrm{Maslov}(\ell, g_1\ell, g_1 g_2 \ell)}$

Def: $\tilde{R}_{\tilde{\ell}}(g) := (e^{2\pi i/8})^{m(\tilde{\ell}, g\tilde{\ell})} R_\ell(g) \in U(\mathcal{H})$

$\tilde{R}_{\tilde{\ell}}(g_1) \tilde{R}_{\tilde{\ell}}(g_2) = \underbrace{(e^{2\pi i/8})^{\widetilde{\mathrm{Maslov}}(\tilde{\ell}, g_1\tilde{\ell}, g_1 g_2 \tilde{\ell})}}_{\pm 1} \tilde{R}_{\tilde{\ell}}(g_1 g_2)$

Cor: $\{\tilde{R}_{\tilde{\ell}}(g)\}$ define a unitary representation of a central extension $1 \to \{\pm 1\} \to \widetilde{Sp}(W, B) \to Sp(W, B) \to 1$

Oscillator (= Weil) representation of the metaplectic group $Mp(2n, \mathbb{R}) = \widetilde{Sp(2n, \mathbb{R})}$ on $L^2(\mathbb{R}^n)$:

$(W, B) = \left(\mathbb{R}^{2n}, \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}\right)$, $\ell = \begin{pmatrix} \mathbb{R}^n \\ 0 \end{pmatrix}$, $1 \to \{\pm 1\} \to \underbrace{Mp(2n, \mathbb{R})}_{\widetilde{G}} \to \underbrace{Sp(2n, \mathbb{R})}_{G} \to 1$

Formulas: $f \in L^2(\mathbb{R}^n)$, $x \in \mathbb{R}^n$

$$0 < \det(a) \Rightarrow \left(\widetilde{R}\begin{pmatrix} a & 0 \\ 0 & {}^t a^{-1} \end{pmatrix} f\right)(x) = \det(a)^{1/2} f({}^t a x)$$

$${}^t b = b \Rightarrow \left(\widetilde{R}\begin{pmatrix} I & b \\ 0 & I \end{pmatrix} f\right)(x) = e^{\pi i\, {}^t x b x} f(x)$$

$$\left(\widetilde{R}\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} f\right)(x) = \left(e^{2\pi i/8}\right)^n \int_{\mathbb{R}^n} e^{2\pi i\, {}^t x y} f(y)\, dy$$

Action of $\mathrm{Lie}(\widetilde{G}) = \mathrm{Lie}(G) = \mathfrak{g}$: (1) $\boxed{n = 1}$

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \mapsto \pi i x^2, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mapsto x\frac{d}{dx} + \frac{1}{2}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \mapsto \frac{i}{4\pi}\left(\frac{d}{dx}\right)^2$$

$$G = SL_2(\mathbb{R}) \supset K = SO(2) = \left\{\begin{pmatrix} a & -c \\ c & a \end{pmatrix} \middle| a^2 + c^2 = 1\right\} \xrightarrow{\sim} U(1) \ni ci + a$$

action of $k = \mathrm{Lie}(\widetilde{K}) = \mathrm{Lie}(K) = \{c(-X+Y) \mid c \in \mathbb{R}\} \xrightarrow{\sim} \mathrm{Lie}(U(1)) \ni u = ci$

$\varphi_0 = e^{-\pi x^2}$, $(-X+Y)\varphi_0 = -\frac{i}{2}\varphi_0 \Leftrightarrow \boxed{u\varphi_0 = -\frac{u}{2}\varphi_0 \quad \forall u \in \mathrm{Lie}(U(1))}$

does NOT integrate to an action of K

$\Rightarrow \quad 1 \to \{\pm 1\} \to \widetilde{K} \to K \to 1$

does not split.