(2) The general case:

$\boxed{G = Sp(2n,\mathbb{R})}$, $\mathfrak{g} = \left\{ \begin{pmatrix} A & B \\ C & -{}^tA \end{pmatrix} \middle| \ {}^tB = B,\ {}^tC = C \right\}$

$\begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} \longmapsto \pi i ({}^t x B x)$, $\begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix} \longmapsto \frac{i}{4\pi}\, {}^t\partial C \partial$, $\begin{pmatrix} A & 0 \\ 0 & -{}^tA \end{pmatrix} \longmapsto {}^t x A \partial + \frac{Tr(A)}{2}$

$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, $\partial = \begin{pmatrix} \partial/\partial x_1 \\ \vdots \\ \partial/\partial x_n \end{pmatrix}$, ${}^tB = B$, ${}^tC = C$

$\boxed{K = G \cap O(2n)} = \left\{ \begin{pmatrix} A & -C \\ C & A \end{pmatrix} \middle| \ Ci + A \in U(n) \right\} \xrightarrow{\sim} U(n) \ni U$

$\mathfrak{k} = Lie(K) = \left\{ \begin{pmatrix} A & -C \\ C & A \end{pmatrix} \middle| \ {}^tC = C,\ {}^tA = -A \right\} \xrightarrow{\sim} \left\{ U \in M_n(\mathbb{C}) \middle| \ {}^tU = -\overline{U} \right\} = Lie(U(n))$

$U = Ci + A \in Lie(U(n))$

Action on the gaussian

$\varphi_0 = e^{-\pi(x_1^2 + \dots + x_n^2)} = e^{-\pi\, {}^t x x}$ $\quad \boxed{U\varphi_0 = -\frac{i}{2} Tr(C)\varphi_0 = -\frac{Tr(U)}{2}\varphi_0}$

$\Longrightarrow 1 \to \{\pm 1\} \to \widetilde{K} \to K \to 1$ does not split.

Siegel upper half space

$G/K \xrightarrow{\sim} \mathcal{H}_n = \{ T \in M_n(\mathbb{C}) \mid {}^tT = T,\ Im(T) > 0 \}$

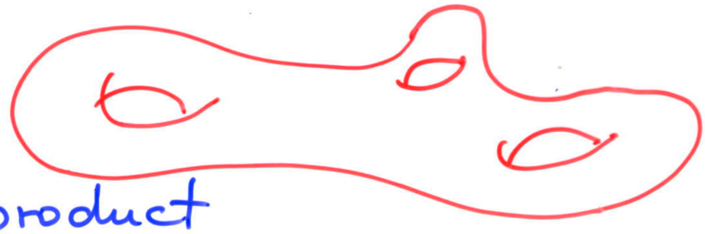
$gK \longmapsto g(i.I)$, $\quad \begin{pmatrix} A & B \\ C & D \end{pmatrix}(T) = (AT+B)(CT+D)^{-1}$

Coordinate-free description of $\mathcal{H}_n$:

$\{ \ell \in Lagr(W_{\mathbb{C}} = W \otimes_{\mathbb{R}} \mathbb{C}, B_{\mathbb{C}}) \mid \forall x \in \ell \ \ iB_{\mathbb{C}}(x, \overline{x}) > 0 \}$

Ex ("Hodge" theory):

X cpt Riemann surface



$W = H^1(X, \mathbb{R})$, $B = \cup$ product

$\ell = H^0(X, \Omega^1) \ni \omega, \omega'$ $\quad \int_X \omega \wedge \omega' = 0$, $i\int_X \omega \wedge \overline{\omega} > 0$

Basic θ-functions on $\mathcal{H}_n$:

$$\boxed{\Theta(T,Z)=\sum_{m\in\mathbb{Z}^n} e^{\pi i\,{}^tmTm+2\pi i\,{}^tmZ}} \qquad (T\in\mathcal{H}_n,\ Z\in\mathbb{C}^n)$$

Fix: $c:\mathbb{Z}^n\longrightarrow(\mathbb{Z}/M\mathbb{Z})^n\longrightarrow\mathbb{C}$

$$\theta_{c,\text{scalar}}(T)=\sum_{m\in\mathbb{Z}^n}c(m)e^{\pi i\,{}^tmTm},\quad \theta_{c,\text{vector}}(T)=\sum_{m\in\mathbb{Z}^n}c(m)m\,e^{\pi i\,{}^tmTm}\in\mathbb{C}^n$$

holomorphic on $\mathcal{H}_n$

Key property: $\exists$ congruence subgroup $\Gamma\subset Sp(2n,\mathbb{Z})$

$$\forall\underbrace{\begin{pmatrix}A&B\\C&D\end{pmatrix}}_{\gamma}\in\Gamma\quad \theta_{c,\text{scalar}}(\gamma(T))=(\text{const.})\sqrt{\det(CT+D)}\ \theta_{c,\text{scalar}}(T)$$

$$\theta_{c,\text{vector}}(\gamma(T))=(\text{const.})\sqrt{\det(CT+D)}\ (CT+D)\theta_{c,\text{vector}}(T)$$

Canonical automorphy factor:

$$J: G\times\mathcal{H}_n\longrightarrow GL_n(\mathbb{C})$$
$$g,\ T\longmapsto \boxed{CT+D=J(g,T)}\qquad g=\begin{pmatrix}A&B\\C&D\end{pmatrix}\in G$$

Properties: (1) $\forall\ k\in K\simeq U(n)$ $\quad\boxed{J(k,iI)=k\in U(n)}$

(2) 1-cocycle: $\boxed{J(gg',T)=J(g,g'(T))\,J(g',T)}$

(3) $\widetilde{G}\simeq\{[g\in G,\lambda:\mathcal{H}_n\to\mathbb{C}\text{ holomorphic s.t. }\lambda(T)^2=\det J(g,T)]\}$

$$[g,\lambda][g',\lambda']=[gg',(\lambda\circ g')\lambda']$$

the map $\widetilde{G}\longrightarrow\mathbb{C}^*$, $[g,\lambda]\longmapsto\lambda(iI)$ restricts to a group homomorphism "$\det^{1/2}$": $\widetilde{K}\longrightarrow U(1)$ whose square is equal to $\widetilde{K}\xrightarrow{pr}K=U(n)\xrightarrow{\det}U(1)$

Above: $\quad St$ = standard repr. of $U(n)$ on $\mathbb{C}^n$

$\left\{\begin{matrix}\theta_{c,\text{scalar}}\\ \theta_{c,\text{vector}}\end{matrix}\right\}$ is a holomorphic Siegel modular form of weight $\left\{\begin{matrix}\text{"}\det^{1/2}\text{"}\\ \text{"}\det^{1/2}\text{"}\otimes St\end{matrix}\right.$

Automorphy

$$\boxed{\text{on } G/K \longleftrightarrow \text{on } \Gamma\backslash G}$$

$G = Sp(2n,\mathbb{R})$
$K \cong U(n)$
$G/K \cong \mathcal{H}_n$
$gK \longmapsto g(iI)$

$$GL_n(\mathbb{C}) \ni \boxed{J\left(\begin{pmatrix} A & B \\ C & D \end{pmatrix}, T\right) = CT + D}$$

<u>Data</u>:
- $\Gamma \subset G$ discrete, $\mathrm{vol}(\Gamma\backslash G/K) < \infty$
- representation $\rho: GL_n(\mathbb{C}) \longrightarrow GL(V)$

---

(1) From G/K to $\Gamma\backslash G$:

if $\boxed{f: G/K \longrightarrow V}$ satisfies

$$\forall \gamma \in \Gamma \quad \boxed{f(\gamma(T)) = \rho(J(\gamma, T))\, f(T)} \quad (T \in G/K = \mathcal{H}_n)$$

$$\Downarrow$$

$$\boxed{\tilde{f}: G \longrightarrow V, \quad \tilde{f}(g) = \rho(J(g, iI))^{-1} f(g(iI))}$$

satisfies $\forall \gamma \in \Gamma \;\; \tilde{f}(\gamma g) = \tilde{f}(g) \iff \boxed{\tilde{f}: \Gamma\backslash G \longrightarrow V}$

and $\boxed{\forall k \in K \quad \tilde{f}(gk) = \rho(k)^{-1}\tilde{f}(g)}$ $\quad (g \in G)$

(2) From $\Gamma\backslash G$ to G/K: given $T = u + iv \in \mathcal{H}_n$

there is a positive definite $v^{1/2}$ s.t. $(v^{1/2})^2 = v$.

$$i \cdot I \xrightarrow{\begin{pmatrix} v^{1/2} & 0 \\ 0 & (v^{1/2})^{-1} \end{pmatrix}} iv \xrightarrow{\begin{pmatrix} I & u \\ 0 & I \end{pmatrix}} u + iv = T$$

$$i \cdot I \xrightarrow{g_T} T$$

$$\boxed{g_T := \begin{pmatrix} I & u \\ 0 & I \end{pmatrix}\begin{pmatrix} v^{1/2} & 0 \\ 0 & (v^{1/2})^{-1} \end{pmatrix} \in G}$$

$$\boxed{f(T) = f(g_T(iI)) = \underbrace{\rho(J(g_T, iI))}_{\rho(v^{1/2})^{-1}} \tilde{f}(g_T)}$$

What is an automorphic form?

<u>Def</u>: given • G (a covering of) a semisimple Lie group
• $K \subset G$ maximal compact subgroup
• $\Gamma \subset G$ discrete subgroup s.t. $\mathrm{vol}(\Gamma \backslash G) < \infty$
• V $\mathbb{C}$-v.space, $\dim(V) < \infty$.

A $\mathcal{C}^\infty$ function $\tilde{f} : G \longrightarrow V$ is an automorphic form if:

(1) $\tilde{f}(\gamma g) = \tilde{f}(g)$ $\forall \gamma \in \Gamma$ ($\Longleftrightarrow$ $\boxed{\tilde{f} : \Gamma \backslash G \longrightarrow V}$)

(2) $\boxed{\tilde{f}(gk) = \rho(k)^{-1} \tilde{f}(g)}$ $\forall k \in K$ for some repr. $\rho : K \longrightarrow GL(V)$
"$\tilde{f}$ has <u>weight ρ</u>"

(3) $\tilde{f}$ has "<u>polynomial growth</u>"

(4) $\tilde{f}$ satisfies a suitable system of linear differential equations: G acts on $\mathcal{C}^\infty(\Gamma \backslash G, V)$ by right translations $\Longrightarrow$ action of $U(\mathfrak{g})$.
We require $I \cdot \tilde{f} = 0$, $I \subset Z(\mathfrak{g})$ ideal of $\dim_{\mathbb{C}} Z(\mathfrak{g})/I < \infty$. "<u>$\tilde{f}$ is $Z(\mathfrak{g})$-finite</u>"
(<u>ex</u>: for a ring morphism $Z(\mathfrak{g}) \longrightarrow \mathbb{C}$,
$\mathrm{Ker}(\chi) \cdot \tilde{f} = 0 \Longleftrightarrow \forall z \in Z(\mathfrak{g})$ $z \cdot \tilde{f} = \chi(z) \tilde{f}$).

<u>Variant</u>: replace $\tilde{f}$ by $r \circ \tilde{f}$ for some linear form $V \xrightarrow{r} \mathbb{C}$. (2) is replaced by

(2′) Under the action of K on $\tilde{f} \in \mathcal{C}^\infty(\Gamma \backslash G, V)$ (by right multiplication), $\tilde{f}$ generates a finite-dimensional subspace of $\mathcal{C}^\infty(\Gamma \backslash G, V)$
"<u>f is K-finite</u>"

<u>Ex</u>: Given $G \longrightarrow GL(E)$, $\lambda \in \mathrm{Hom}(E, \mathbb{C})^\Gamma$, $e \in E$ s.t. $\dim_{\mathbb{C}}(\mathrm{span}\ K \cdot e) < \infty$, the matrix element $\tilde{f}(g) := \lambda(g \cdot e)$ satisfies (1), (2′).

Variant: $\lambda \in \mathrm{Hom}(E,\mathbb{C})^{\Gamma}$, $e \in (E\otimes V)^{K}$ $\Longrightarrow$ (diagonal K-action)

$\tilde f(g) = \lambda(g\cdot e) \in V$, $\tilde f : \Gamma\backslash G \longrightarrow V$, $\tilde f(gk) = \rho_V(k)^{-1}\tilde f(g)$

Weil's θ - distributions

$\tilde G = Mp(2n,\mathbb{R}) \supset \tilde K$

$\tilde G$ acts on $\mathcal{H} = L^2(\mathbb{R}^n) = L^2(W/\ell)$

$$\mathcal{H} \supset \mathcal{H}_\infty = \{ f\in \mathcal{H} \mid \underbrace{U(\mathrm{heis})f \in \mathcal{H}}_{\forall \alpha\ \forall \text{ polynomial } P} \}$$

$$\| $$

$$\mathcal{S}(\mathbb{R}^n) = \{ f \in C^\infty(\mathbb{R}^n) \mid \text{— " —}\ \ P(x)D^\alpha f(x) \longrightarrow 0 \text{ as } |x|\to\infty,\ P(x)D^\alpha f(x) \in L^2(\mathbb{R}^n) \}$$

Schwartz space

Ex: $A = {}^tA \in M_n(\mathbb{R})$ positive definite, P polynomial $\Longrightarrow$ $P(x)e^{-\pi\, {}^txAx} \in \mathcal{S}(\mathbb{R}^n)$.

Thm. There is a congruence subgroup $\Gamma \subset Sp(2n,\mathbb{Z})$ and $\beta : \tilde\Gamma \longrightarrow \mu_8$ such that the θ - distribution
$$\lambda = \sum_{u\in\mathbb{Z}^n} \delta_u : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathbb{C}$$
satisfies $\lambda(\gamma\cdot\varphi) = \beta(\gamma)\lambda(\varphi)$ $\forall \gamma\in\tilde\Gamma$, $\forall\varphi\in\mathcal{S}(\mathbb{R}^n)$.

Idem for $\lambda_{u_0} = \sum_{u\in\mathbb{Z}^n} \delta_{u+u_0}$ for fixed $u_0\in\mathbb{Q}^n$ (and smaller Γ).

Rmk. β disappears in the adelic version.

Idea of proof: $\mathcal{H} = \mathrm{Ind}_{\pi^{-1}(\ell)}^{\mathrm{Heis}}(\chi) \supset \mathcal{H}_\infty = \mathcal{S}(\mathbb{R}^n)$ ↺ $\tilde G$ acts

natural intertwiner for Heis $\longrightarrow$ Φ

$\mathcal{H}_{\mathbb{Z}} = \mathrm{Ind}_{\pi^{-1}(\mathbb{Z}^{2n})}^{\mathrm{Heis}}(\chi) \supset (\mathcal{H}_{\mathbb{Z}})_\infty$

$$\begin{tikzcd} \mathcal{H}_\infty = \mathcal{S}(\mathbb{R}^n) \arrow[d, "\Phi"] \arrow[dd, bend left=60, "\lambda_{u_0}"] \\ (\mathcal{H}_{\mathbb{Z}})_\infty \arrow[d, "\text{value at } u_0"] \\ \mathbb{C} \end{tikzcd}$$

Γ acts on $\mathbb{Z}^{2n}$

Stone-von Neumann $\Longrightarrow$ Φ intertwines the two $\tilde\Gamma$ - actions up to a scalar $\in U(1)$.

$$(\Phi f)(g) = \sum_{\pi^{-1}(L\cap\ell)\backslash\pi^{-1}(L)} \chi(h)^{-1} f(hg)$$

$L = \mathbb{Z}^{2n}$, $L\cap\ell \subset \ell$ lattice

Construction of automorphic forms

Data: $\rho : \tilde{K}\ (= 2\text{-fold cover of } U(n)) \longrightarrow GL(V)$

$\varphi \in (\mathcal{S}(\mathbb{R}^n) \otimes V)^{\tilde{K}}$, $\quad a \in \mathbb{Q}^n$

Define: for $g \in \tilde{G} = Mp(2n, \mathbb{R})$

$$\tilde{f}(g) = \lambda_a(g \cdot \varphi) \in V, \quad \tilde{f} : G \longrightarrow V$$

for $T = u + iv \in \mathcal{H}_n$, $\quad g_T = \begin{pmatrix} I & u \\ 0 & I \end{pmatrix} \begin{pmatrix} v^{1/2} & 0 \\ 0 & (v^{1/2})^{-1} \end{pmatrix}$

$$f(T) = \underbrace{\rho(J(g_T, iI))}_{\rho(v^{1/2})^{-1}} \tilde{f}(g_T) = \rho(v^{1/2})^{-1} \lambda_a(\underbrace{g_T \cdot \varphi}_{\text{action in the Weil representation}})$$

Then: $\exists$ congruence subgroup

$\Gamma_a \subset Sp(2n, \mathbb{Z})$ s.t. $\forall \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_a$ $\quad f((AT+B)(CT+D)^{-1}) =$

$= \zeta \rho(CT+D) f(T)$, $\quad \zeta^8 = 1$.

Ex: (1) $a = 0$, $\varphi = \varphi_0 = e^{-\pi\, {}^t x x} \xrightarrow{\begin{pmatrix} v^{1/2} & 0 \\ 0 & v^{-1/2} \end{pmatrix}} (\det v)^{1/4} e^{-\pi\, {}^t x v x}$

$V = \rho = \det^{1/2}$

$$\xrightarrow{\begin{pmatrix} I & u \\ 0 & I \end{pmatrix}} \underbrace{(\det v)^{1/4} e^{\pi i\, {}^t x T x}}_{g_T \cdot \varphi_0}$$

$$\boxed{f(T) = \lambda(e^{\pi i\, {}^t x T x}) = \sum_{u \in \mathbb{Z}^n} e^{\pi i\, {}^t u T u}}$$

(2) $\boxed{n=1}$, $\varphi = (\text{const.}) \underbrace{A_+^2 \varphi_0}_{\varphi_2} = \left(x^2 - \frac{1}{4\pi}\right)\varphi_0$, $\quad V = \rho = \det^{5/2}$

$\tau = u + iv \in \mathcal{H}$

$$g_\tau \cdot \varphi_2 = v^{1/4}\left(v x^2 - \frac{1}{4\pi}\right) e^{\pi i \tau x^2}$$

$$\boxed{f(\tau) = \sum_{n \in a + \mathbb{Z}} \left(n^2 - \frac{1}{4\pi v}\right) e^{\pi i n^2 \tau}}$$

NOT HOLOMORPHIC

$\forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_a \quad \underline{f\left(\frac{a\tau + b}{c\tau + d}\right) = \zeta (c\tau + d)^{5/2} f(\tau)}$

$\tilde{f}$ is Eigenvector for the Casimir $\Omega \in Z(\mathfrak{sl}_2)$: $\Omega \tilde{f} = \lambda \tilde{f}$

For which φ is f holomorphic?

$h : U(1) \longrightarrow K$, $h(e^{i\alpha}) = \begin{pmatrix} I\cdot\cos\alpha & -I\cdot\sin\alpha \\ I\cdot\sin\alpha & I\cdot\cos\alpha \end{pmatrix}$

$U(1) \searrow Z(K) \subset K$

complex structure J on $\mathfrak{g}/\mathfrak{k}$ is $\mathrm{Ad}\, h(e^{-2\pi i/8})$.

Formulas: $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$, $\mathfrak{p}_{\mathbb{C}} = \mathfrak{p}_+ + \mathfrak{p}_-$ $\quad (J = i, -i)$

$$\mathfrak{p}_\pm = \left\{ \begin{pmatrix} A & \pm iA \\ \pm iA & -A \end{pmatrix} \,\middle|\, A = {}^tA \in M_n(\mathbb{C}) \right\}$$

Properties: (1) $f : G \longrightarrow \mathbb{C}$ is of the form

$G \longrightarrow G/K \xrightarrow{\text{holomorphic}} \mathbb{C} \iff (\mathfrak{k} + \mathfrak{p}_-) f = 0.$

(2) $\quad \mathfrak{p}_-\, J(g, iI) = 0.$

Corollary: $\mathfrak{p}_- \varphi = 0 \Rightarrow f$ is holomorphic.

True for $\varphi = \varphi(x) = (\text{pol. of deg} \le 1)\, e^{-\pi x^2} \Longrightarrow$

$f = \theta_{c,\text{scalar}}, \ \theta_{c,\text{vector}}$.

θ-functions of quadratic forms

Ex: $\theta(\tau)^d = \sum_{u \in \mathbb{Z}^d} e^{\pi i \overbrace{(u_1^2 + \cdots + u_d^2)}^{S(u)} \tau}$, $\quad \theta\left(\frac{a'\tau + b'}{c'\tau + d'}\right)^d = \zeta\, (c'\tau + d')^{d/2}\, \theta(\tau)^d$

General case: (V, S) quadratic form (non-deg.), $d = \dim_{\mathbb{R}} V$

(W, B) as before $\Longrightarrow (V \otimes W, S \otimes B)$ symplectic

Dual pair: $O(V) \times Sp(W) \longrightarrow Sp(V \otimes W)$

$\qquad h, g \longmapsto h \otimes g$

Weil representation of $Mp(V \otimes W)$: $\boxed{\dim_{\mathbb{R}} W = 2n}$

Fix: $\ell \in \mathrm{Lagr}(W) \Longrightarrow V \otimes \ell \in \mathrm{Lagr}(V \otimes W)$

$\qquad O(V)$-stable

$\tilde{R}_{V \otimes \ell}$ acts on $L^2(V \otimes \underbrace{(W/\ell)}_{\ell^*}) \simeq L^2(V^n) \supset \mathcal{S}(V^n)$

$h \in O(V)$ acts by

$$\boxed{(\tilde{R}(h) f)(v) = \alpha(h)\, f(h^{-1} v)}$$

$v \in V^n$

$\alpha : O(V) \longrightarrow \{\pm 1\}$

R. Howe's idea: restriction of $\tilde{R}_{V\otimes \ell}$ to $O(V) \times Mp(W)$ should give a correspondence between some representations of $O(V)$, $Mp(W)$

Case of S positive definite: fix a basis of ℓ and of $V \Rightarrow$ coordinates x_{jp} on $V^n = V \otimes \ell^*$ $(1 \leq j \leq n,\ 1 \leq p \leq d)$

Def: a polynomial $P : (V^n)_{\mathbb{C}} \longrightarrow \mathbb{C}$ is pluriharmonic (w.r.t. S) if

$$\forall p, q \quad \sum_{j,k=1}^{n} \left(\frac{\partial}{\partial x_{jp}} \frac{\partial}{\partial x_{kq}} \right) (S^{-1})_{pq} P = 0$$

$[\ S = {}^tS \in M_d(\mathbb{R}) \quad \text{pos. def.}]$

Consider: $\varphi = P \cdot \varphi_0 \in \mathcal{S}(V^n)$, $\varphi_0 = e^{-\pi \sum_{j,p,q} x_{pj} S_{pq} x_{qj}}$

Proposition: P pluriharmonic $\Longleftrightarrow$ $\mu_- \varphi = 0$.

Fix: $O(V)$-stable subspace of pluriharmonic polynomials and its basis $P_1, \dots, P_M$.

Thm: For each $a \in M_{d\times n}(\mathbb{Q})$, $T \in \mathcal{H}_n$

$$\theta_a(T) = \sum_{X \in M_{d\times n}(\mathbb{Z}) + a} \begin{pmatrix} P_1 \\ \vdots \\ P_M \end{pmatrix} (X)\, e^{\pi i \operatorname{Tr}(T\, {}^tX S X)} \qquad 0 < S = {}^tS \in M_d(\mathbb{Q})$$

is a vector-valued holomorphic automorphic form on $Mp(2n, \mathbb{R})$ (in fact, on $Sp(2n, \mathbb{R})$ if $2 \mid d = \dim(V)$)

Ex : $\boxed{n=1}$ $P : V_{\mathbb{C}} \to \mathbb{C}$ is a harmonic polynomial

if $\underbrace{\sum_{p,q=1}^{d} (S^{-1})_{pq} \frac{\partial}{\partial x_p} \frac{\partial}{\partial x_q}}_{\text{(const.) action of } Y=\begin{pmatrix}0&0\\1&0\end{pmatrix} \in \mathrm{Lie}(\overbrace{SL_2(\mathbb{R})}^{Sp(W)})} P = 0.$

---

(1) $d=1$: $S(x)=x^2$, $P(x) = \boxed{1, x}$ are harmonic

$\Rightarrow$ get $\theta_{1/2}(\tau)$, $\theta_{3/2}(\tau)$.

(2) $d=2$: $S(x) = x_1^2+x_2^2$, $\boxed{(x_1 \pm i x_2)^M \quad (M \geq 0)}$ are harmonic

$\Rightarrow$ get $f(\tau) = \sum_{n_1,n_2 \in \mathbb{Z}+a} (n_1+in_2)^M e^{\pi i (n_1^2+n_2^2)\tau}$ $(a \in \mathbb{Q})$

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = (\text{const.})\,(c\tau+d)^{\overbrace{d/2+\deg(P)}^{M+1}} f(\tau)$$

$\forall \begin{pmatrix}a&b\\c&d\end{pmatrix} \in$ some congruence subgroup of $SL_2(\mathbb{Z})$

θ - fns and indefinite quadratic forms

What if S is indefinite ? $(S={}^tS \in M_d(\mathbb{Q}))$

$\sum_{n \in \mathbb{Z}^d} e^{\pi i \tau \underbrace{S(n)}_{{}^tnSn}}$ does not converge $(\tau \in \mathcal{H})$

1^st approach (Hecke, .., Rallis – Schiffman)

Consider $\sum_{\substack{n \in \mathbb{Z}^d \\ S(n)>0}} \psi(n) e^{\pi i \tau\, {}^tn S n}$ for suitable $\psi : \{x \in \mathbb{R}^d \mid S(x)>0\} \to \mathbb{C}$

2^{nd} approach (Siegel): use majorants $\tilde{S} > 0$

Ex: $S = \underbrace{x_1^2 + \cdots + x_p^2}_{S_+} + \underbrace{(-y_1^2 - \cdots - y_q^2)}_{S_-}$ on $V = \mathbb{R}^{p+q}$

majorant of S

$$\boxed{\tilde{S} = S_+ - S_- = x_1^2 + \cdots + x_p^2 + y_1^2 + \cdots + y_q^2}$$

Action of $\mathrm{Lie}(Mp(W)) = \mathfrak{sl}(2) \subset \mathrm{Lie}(Mp(V \otimes W))$ on $\mathcal{S}(V)$: $(x \in V)$ $(\dim W = 2)$

$X \mapsto \pi i \underbrace{({}^t x S x)}_{S}$, $Y \mapsto \frac{i}{4\pi}({}^t\partial S^{-1} \partial)$, $H \mapsto \frac{1}{2}({}^t x \partial + {}^t \partial x)$ (true for any S)

Gaussian of $\tilde{S}$: $\varphi = e^{-\pi \tilde{S}} \in \mathcal{S}(V)$ $(\,,\,)$ on V

$X\varphi = (\pi i S)\varphi$, $Y\varphi = ((\pi i S) + \frac{i}{2}(q-p))\varphi$, $H\varphi = (-2\pi\tilde{S} + \frac{p+q}{2})\varphi$

$$\Rightarrow \boxed{i(X-Y)\varphi = \frac{-p+q}{2}} \qquad K = SO(2) \subset SL_2(\mathbb{R}) = Sp(W)$$

$K \cong U(1)$, $\tilde{K} \to K$, $Mp(W) \to Sp(W)$, $\tilde{K} \subset Mp(W)$

$\Downarrow$

φ is $\tilde{K}$-finite, with $\tilde{K}$ acting by $\det^{-\frac{p-q}{2}}$.

General θ-machinery: $\boxed{\tau = u + iv} \in \mathcal{H}$, $a_0 \in \mathbb{Q}^{p+q}$

$$\varphi \xrightarrow{\begin{pmatrix} v^{1/2} & 0 \\ 0 & v^{-1/2}\end{pmatrix}} v^{\frac{p+q}{4}} e^{-\pi v \tilde{S}} \xrightarrow{\begin{pmatrix} 1 & u \\ 0 & 1\end{pmatrix}} v^{\frac{p+q}{4}} e^{\overbrace{\pi i u S - \pi v \tilde{S}}^{\pi i(\tau S_+ + \bar{\tau} S_-)}}$$

$g_\tau = \begin{pmatrix} 1 & u \\ 0 & 1\end{pmatrix}\begin{pmatrix} v^{1/2} & 0 \\ 0 & v^{-1/2}\end{pmatrix} : i \mapsto \tau$

$$\underbrace{(v^{-1/2})^{\frac{p-q}{2}} \lambda_{a_0}(g_\tau \cdot \varphi)}_{f(\tau)} = v^{q/2} \sum_{\binom{x}{y} \in \mathbb{Z}^{p+q} + a_0} e^{\pi i (\tau(x_1^2 + \cdots + x_p^2) - \bar{\tau}(y_1^2 + \cdots + y_q^2))}$$

$\forall \begin{pmatrix} a & b \\ c & d\end{pmatrix} \in \Gamma_{a_0} \subset SL_2(\mathbb{Z})$ $\quad f\left(\frac{a\tau+b}{c\tau+d}\right) = (\text{const.})(c\tau+d)^{\frac{p-q}{2}} f(\tau)$

Is f an automorphic form?

$F(\tau) = v^{-q/2} f(\tau)$ $\quad F\left(\frac{a\tau+b}{c\tau+d}\right) = (\cdot\cdot)(c\tau+d)^{p/2}(c\bar{\tau}+d)^{q/2} F(\tau)$

f is NOT $Z(sl_2)$ – finite $\Longrightarrow$ NOT an automorphic form! (if $p, q \geq 1$)

$\Omega = XY + YX + \frac{H^2}{2}$

$\forall n \geq 1 \quad \Omega^n \varphi = ((-8\pi^2 S_+ S_-)^n + \text{lower terms})\varphi$

Explanation: $\tilde{S}$ is NOT canonical.

$\{\text{majorants of } S\} \longleftrightarrow \underbrace{\{V = V_+ \perp V_-,\ S|_{V_+} > 0,\ S|_{V_-} < 0\}}_{\text{symmetric space of } O(V)}$

$Gr_q^-(V) = \left\{ Z \subset V \,\middle|\, \begin{array}{l} \dim(Z) = q \\ S|_Z < 0 \end{array} \right\} \simeq O(p,q)/O(p) \times O(q)$

$\operatorname{sgn}(S) = (p,q)$

$Z \in Gr_q^-(V)$

$$\boxed{\text{majorant} \quad \tilde{S}_Z(x) = S(pr_{Z^\perp}(x)) - S(pr_Z(x))}$$

Siegel: consider for each $Z \in Gr_q^-(V)$

$$f(\tau, Z) = f_Z(\tau) = (\operatorname{Im}\tau)^{q/2} \sum_{\binom{x}{y} \in \mathbb{Z}^{p+q} + a_0} e^{\pi i(\tau S(pr_{Z^\perp}(x)) + \bar{\tau} S(pr_Z(x)))}$$

and integrate over $Gr_q^-(V)$:

$$I(f)(\tau) = \int_{Gr_q^-(V)} f(\tau, Z)\, dZ$$ is an automorphic form.

Siegel's Formula: for $p+q > 4$

$I(f)$ is an Eisenstein series: linear combination of functions such as

$$\sum_{\binom{m}{n} \in \mathbb{Z}^2 + b_0}{}' \frac{\operatorname{Im}(\tau)^{q/2}}{(m\tau+n)^{p/2}(m\bar{\tau}+n)^{q/2}}.$$

Theta correspondence: consider

$$\tau \longmapsto \int_{Gr_q^-(V)} f(\tau, Z)\,(\text{automorphic form on } O(p,q))\, dZ$$