The work of Kudla – Millson (et al.)

Positive semi-definite symmetric matrices parameterise **two** kinds of objects:

(1) Fourier coefficients of Siegel modular forms:

$$f(\tau) = \sum_{B \in \mathrm{Sym}_n(\frac{1}{N}\mathbb{Z})_{\geq 0}} e^{2\pi i \,\mathrm{Tr}(Bu)} \times \underbrace{(\text{function of } B, v)}_{a(B)\, e^{2\pi i \,\mathrm{Tr}(Biv)} \text{ if } f \text{ holomorphic}} \qquad \tau = u + iv \in \mathcal{H}_n$$

(2) special cycles in $D = \mathrm{Gr}_q^-(V) \simeq O(p,q)/O(p) \times O(q)$.

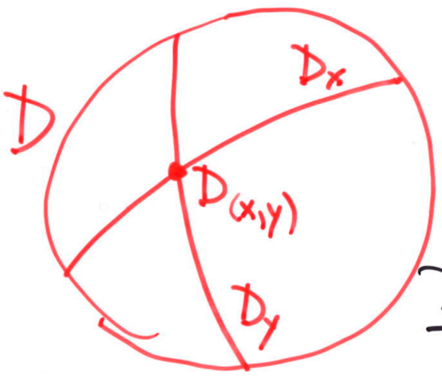
Special cycles in D

Recall: V, $(\,,\,)$ quadratic space over $\mathbb{R}$, $\mathrm{sgn}(V) = (p,q)$, $\boxed{p,q \geq 1}$, $D = \mathrm{Gr}_q^-(V) = \{ Z \subset V \mid \dim(Z) = q,\ (\,,\,)|_Z < 0 \} \ni \sigma$

$G = O(V)$ $(\simeq O(p,q))$, $K = G_\sigma$ $(\simeq O(p) \times O(q))$, $G/K \xrightarrow{\sim} D$

$\boxed{\dim_{\mathbb{R}}(D) = pq}$

Def: $\boxed{x \in V \Rightarrow D_x := \{ Z \in D \mid Z \perp x \}} \subset D$



(0) $(x,x) < 0 \Rightarrow D_x = \emptyset$

(1) $(x,x) > 0 \Rightarrow D_x = \mathrm{Gr}_q^-(\underbrace{x^\perp}_{\mathrm{sgn}(p-1,q)}) \Rightarrow \mathrm{codim}_{\mathbb{R}}(D_x) = q$

Def: $X = (x_1, \dots, x_n) \in V^n \Rightarrow \boxed{D_X := D_{x_1} \cap \dots \cap D_{x_n} \subset D}$

$\underline{\underline{X}} := \mathrm{span}_{\mathbb{R}}(x_1, \dots, x_n) \subset V$, $(X,X) = ((x_j, x_k)) \in \mathrm{Sym}_n(\mathbb{R})$

(2) If $(X,X) > 0$ (pos. def.) $\Rightarrow D_X = \mathrm{Gr}_q^-(\underbrace{\underline{\underline{X}}^\perp}_{\mathrm{sgn}(p-n,q)})$ $\quad \mathrm{codim}_{\mathbb{R}}(D_X) = nq$

(3) If $\left.\begin{array}{l} \dim_{\mathbb{R}}(\underline{\underline{X}}) = t \leq n \\ (\,,\,)|_{\underline{\underline{X}}} > 0 \end{array}\right\} \Rightarrow D_X = \mathrm{Gr}_q^-(\underbrace{\underline{\underline{X}}^\perp}_{\mathrm{sgn}(p-t,q)}) \Rightarrow \mathrm{codim}_{\mathbb{R}}(D_X) = tq$

Special cycles in $\Gamma\backslash D$

Arithmetic subgroups $\Gamma \subset G = O(V)$:

(1) if $\exists$ lattice $L \subset V$ s.t. $(L,L) \subseteq \mathbb{Z} \Rightarrow$
$\Gamma(L) = \{g \in G \mid gL \subseteq L\} \subset G$ discrete, $\mathrm{vol}(\Gamma\backslash D) < \infty$

$\Gamma(L)\backslash D$ <u>not</u> compact $\Leftrightarrow \exists\, 0 \neq x \in L\ (x,x) = 0 \Leftarrow p+q > 4$.

(2) Can get $\Gamma \subset G$ with $\Gamma\backslash D$ compact from $(\,,\,)$ over a totally real number field $F \neq \mathbb{Q}$ when $\mathrm{sgn}((\,,\,)) = (p,q) \times (p+q,0)^{[F:\mathbb{Q}]-1}$.

Assume: $\exists L$; fix $\Gamma \subset \Gamma(L)$ congruence subgroup.

Consider: $X \in L^n$ as in (3) above.

Def: $G_X = \{g \in G \mid g(\underline{\underline{X}}) = \underline{\underline{X}}\}$ (depends only on $\underline{\underline{X}} = \mathrm{span}_{\mathbb{R}}(\{x_i\})$)
$\Gamma_X = \Gamma \cap G_X$. Then $G_X \simeq O(p-t,q)$ acts on $D_X \simeq \dfrac{O(p-t,q)}{O(p-t)\times O(q)}$.

Basic fact: if $\Gamma \subset$ suitable Γ', then:

(a) $\Gamma_X\backslash D_X$ is <u>orientable</u>

(b) $i_X : \Gamma_X\backslash D_X \longrightarrow \Gamma\backslash D$ is <u>injective</u>. $(\mathrm{codim}_{\mathbb{R}} = tq)$

Fundamental class:

$$\boxed{[\Gamma_X\backslash D_X] \in H^{tq}(\Gamma\backslash D, \mathbb{R})}$$

(depends only on $\Gamma\cdot X$)

Euler class: tautological $rk = q$ bundle (oriented, G-equivariant) on $D \Rightarrow$ Euler class $\boxed{e_q \in H^q(\Gamma\backslash D, \mathbb{R})}$

<u>$2 \nmid q \Rightarrow e_q = 0$</u>

Main Theorem of Kudla – Millson (Publ. IHES, 1990)

Fix $N \geq 1$, $v \in L$; set $\mathcal{L} = N \cdot L + v$. Then:

$$f(T) = \sum_{B \in \mathrm{Sym}_n(\mathbb{Z})_{\geq 0}} \left(\sum_{\substack{X \in \Gamma \backslash \mathcal{L}^n \\ \frac{(X,X)}{2} = B}} [\Gamma_X \backslash D_X] \cup e_q^{\,n - rk(B)} \right) e^{2\pi i \, Tr(BT)} \qquad (T \in \mathcal{H}_n)$$

$\Gamma \backslash \mathcal{L}^n$: finite set

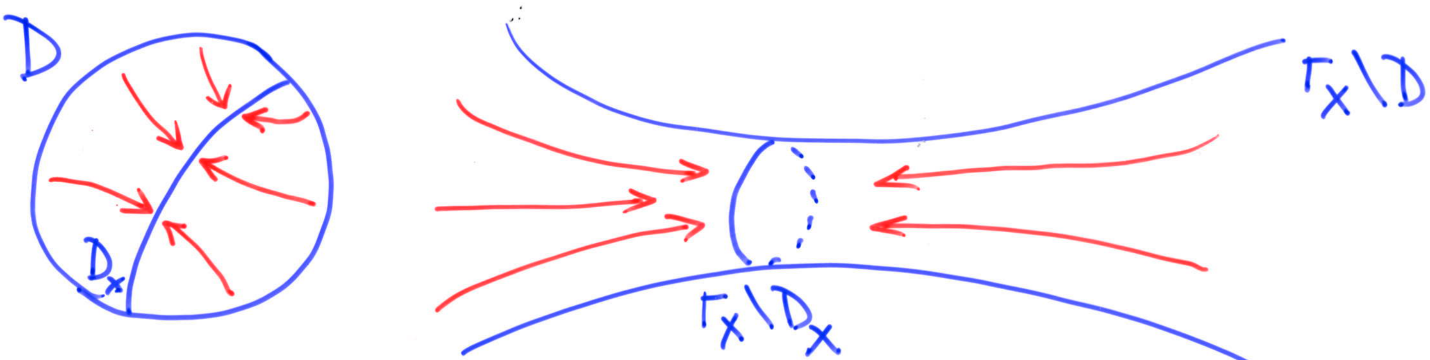
is a <u>holomorphic Siegel modular form</u> of weight $\frac{p+q}{2}$ with coefficients in $H^{nq}(\Gamma \backslash D, \mathbb{R})$

$\Rightarrow \exists\ \Gamma \subset Sp(2n, \mathbb{Z})$ congruence subgroup s.t.

$$\forall \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma \quad f\left((AT+B)(CT+D)^{-1}\right) = (\text{const.}) \det(CT+D)^{\frac{p+q}{2}} f(T) \qquad (T \in \mathcal{H}_n)$$

<u>Main ideas of proof</u>:

<u>Geometry</u>: "geodesic flow"



get $\Gamma_X \backslash D \simeq$ the normal bundle of $\Gamma_X \backslash D_X$

oriented real vector bundle of $rk = tq$

$$\begin{tikzcd} \Gamma_X \backslash D \arrow[dr, "p"] \arrow[d, shift right, "pr_X"'] & \\ \Gamma_X \backslash D_X \arrow[u, shift right, "i"'] \arrow[r, hook, "i_X"] & \Gamma \backslash D \end{tikzcd}$$

Fundamental class of the 0-section:

$$\begin{array}{c} E \\ \pi \downarrow \uparrow i \\ B \end{array}$$ oriented $rk_{\mathbb{R}} = n$ bundle (C^∞)

Thom form: $\omega \in A^n(E)$, $d\omega = 0$, cpt supp along the fibres, $\underbrace{\pi_* \omega}_{\text{integral along fibres.}} = 1$

(so $\forall b \in B$

$$H^n_c(\pi^{-1}(b), \mathbb{R}) \xrightarrow[\sim]{\int} \mathbb{R}$$
$$\omega|_{\pi^{-1}(b)} \longmapsto 1$$

Then: $\forall \eta \in A_c^{\dim(B)}(E)$, $d\eta = 0$

$$\int_E \omega \wedge \eta = \int_B (\pi_* \omega) \wedge i^*(\eta) = \int_{i(B)} \eta \implies [\omega] = [i(B)] \in H^n(E, \mathbb{R})$$

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(1) One needs a more general version of this for

$$\begin{array}{c} \Gamma_X \backslash D \\ \downarrow pr_X \\ \Gamma_X \backslash D_X \end{array}$$ with ω only "rapidly decreasing".

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(2) Construction of $\varphi \in \underbrace{(\mathcal{S}(V^n) \otimes A^{nq}(G/K))^G}_{(\mathcal{S}(V^n) \otimes \Lambda^{nq} p^*)^K}$, $d\varphi = 0$

$\mathfrak{g} = Lie(G) = k \oplus p$

$p = T_o D \simeq V_+ \otimes V_-^*$, $\dim_{\mathbb{R}} p = pq$

at o $V = V_+ \oplus V_-$ $\quad \varphi = \varphi_{[nq]} = \overbrace{\varphi_{[q]} \wedge \cdots \wedge \varphi_{[q]}}^{n \text{ times}}$

$n=1$: $\varphi_{[q]} \in (\mathcal{S}(V) \otimes \Lambda^q p^*)^K$

At o: majorant of $S = (x,x) = \sum_1^p x_j^2 - \sum_{p+1}^{p+q} x_k^2$ is $\tilde{S} = \sum_1^{p+q} x_j^2$

$\varphi^+ = e^{-\pi \tilde{S}}$

Ex: (1) $sgn(V) = (1,q)$: $\varphi_{[q]} = \underbrace{\left(x_1 - \frac{1}{2\pi}\frac{\partial}{\partial x_1}\right)^q \varphi^+}_{\text{Hermite pol.} \times \varphi^+} \otimes (\text{basis of } \Lambda^{max} p^*)$

(up to a constant)

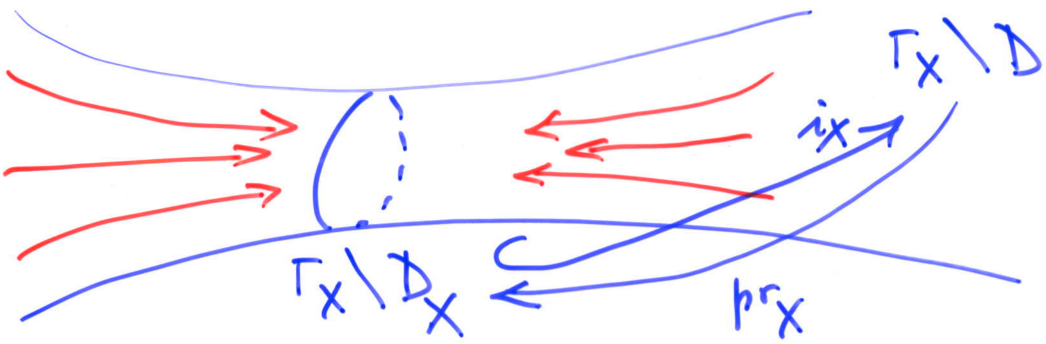
(2) $sgn(V) = (p,1)$: $\varphi_{[1]} = \sum_{\alpha=1}^p \underbrace{\left(x_\alpha - \frac{1}{2\pi}\frac{\partial}{\partial x_\alpha}\right)\varphi^+}_{2x_\alpha \varphi^+} \otimes \omega_\alpha$ } basis of p^* dual to x_α

Properties of $\varphi \in \left(\mathcal{S}(V^n)\otimes A^{nq}(G/K)\right)^G$:

value at $X\in V^n$: $\varphi_X \in A^{nq}(G/K)^{G_X}$, $d\varphi_X = 0$

$\forall g\in G \quad g^*\varphi_X = \varphi_{g^{-1}X}$ $\quad \boxed{\text{From now on: } \begin{matrix} X\in L^n \\ (X,X)>0 \end{matrix}}$

(3) Integrals along fibres:



$$\boxed{(pr_X)_*\varphi_X = 1}$$

Cor: $\varphi_{\Gamma\cdot X} := \sum_{Y\in\Gamma\cdot X}\varphi_Y = \sum_{\gamma\in\Gamma/\Gamma_X}\gamma^*(\varphi_X)$ satisfies

$$[\varphi_{\Gamma\cdot X}] = [\Gamma_X\backslash D_X] \in H^{nq}(\Gamma\backslash D)$$

Pf: $\forall\, \eta\in A^{(p-n)q}(\Gamma\backslash D)$, $d\eta = 0$, η bounded

$$\int_{\Gamma\backslash D}\varphi_{\Gamma\cdot X}\wedge\eta = \int_{\Gamma_X\backslash D}\varphi_X\wedge\eta \overset{(1)}{=} \int_{\Gamma_X\backslash D_X}(pr_X)_*\varphi_X\wedge i_X^*(\eta) \overset{(3)}{=} \int_{\Gamma_X\backslash D_X} i_X^*(\eta)$$

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(4) θ – machinery: $\widetilde{G'} = Mp(2n,\mathbb{R}) \supset \widetilde{K'}$

$G\times\widetilde{G'} \subset Mp(V\otimes \underbrace{W}_{\mathbb{R}^{2n}})$ acts on $L^2(V^n)\supset\mathcal{S}(V^n)$

by the Weil representation.

Facts: (a) $\widetilde{K'}$ acts on φ by $\det^{-(p+2)/2}$.

(b) $\lambda_{\mathcal{L}^n} = \sum_{X\in\mathcal{L}^n}\delta_X : \mathcal{S}(V^n)\longrightarrow\mathbb{C}$ is $\widetilde{\Gamma'}$–invariant

(for suitable $\Gamma'\subset Sp(2n,\mathbb{Z})$)

Cor: the function $\underset{\substack{\uparrow\\ \tilde{G}'}}{g'} \mapsto \lambda_{\mathcal{L}^n}(\tilde{R}(g')\cdot\varphi) = \sum_{X\in\mathcal{L}^n} (\tilde{R}(g')\varphi)_X$

non-holomorphic

gives rise to a function $F_\varphi(T)$ $(T\in\mathcal{H}_n)$ with values in $A_{cl}^{nq}(G/K)$ s.t.

$$\forall \begin{pmatrix} A & B \\ C & D\end{pmatrix} \in \Gamma' \quad F_\varphi((AT+B)(CT+D)^{-1}) = (\text{cnst.}) \det(CT+D)^{\frac{p+q}{2}} F_\varphi(T)$$

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(5) As $(\tilde{R}(\overbrace{\begin{pmatrix} I & u \\ 0 & I\end{pmatrix}}^{g'})\varphi)_X = e^{\pi i \,\mathrm{Tr}(u(X,X))}\varphi_X$, $\boxed{u\in \mathrm{Sym}_n(\mathbb{R})}$

we get that the cohomology class

$$\boxed{F_\varphi(u+iI)} = \sum_{X\in\mathcal{L}^n} e^{\pi i \,\mathrm{Tr}(u(X,X))}[\varphi_X] = \sum_{\substack{X\in\mathcal{L}^n\\ (X,X)>0}} + \sum_{\text{rest}} =$$

$$= \sum_{\substack{X\in\Gamma\backslash\mathcal{L}^n \text{ finite}\\ B=\frac{(X,X)}{2}\in\mathrm{Sym}_n(\mathbb{Z})_{>0}}} \underbrace{[\varphi_{\Gamma\cdot X}]}_{[\Gamma_X\backslash D_X]} e^{2\pi i\,\mathrm{Tr}(uB)} + \text{rest}$$

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(6) $\underbrace{\mathrm{Lie}(\tilde{G}')}_{sp(2n)} = k'\oplus p'$, $p'_{\mathbb{C}} = p'_+\oplus p'_-$.

Fact: $p'_-\varphi \in \mathcal{S}(V^n)\otimes dA^{nq-1}(G/K)$ is exact.

Cor: $[F_\varphi(T)]$ is a holomorphic Siegel modular form of weight $\frac{p+q}{2}$ with values in $H^{nq}(G/K)$.

$\Downarrow$ (5)

$\forall B\in\mathrm{Sym}_n(\mathbb{Z})_{>0}$ the B-th Fourier coefficient of $[F_\varphi(T)]$ is equal to $\sum_{\substack{X\in\Gamma\backslash\mathcal{L}^n\\ (X,X)=B}} [\Gamma_X\backslash D_X]$.

(7) More work is needed to treat $X \in \mathcal{L}^n$ with $(X, X) \geq 0$ but not $(X, X) > 0$
(those with (X, X) not ≥ 0 do not contribute to $[F_\varphi(T)]$: for $n > 1$ by the Koecher principle, for $n = 1$ since $[F_\varphi(T)]$ has polynomial growth).

"QED".

Refinements

(a) similar result for $G \times G' = U(p, q) \times U(n, n)$
(K-M, Publ. IHES 1990)

(b) cohomology with coefficients
(Funke - Millson)

(c) Bergeron - Millson - Moeglin:
$\forall\ n < \frac{1}{2}\left(m - \left[\frac{m}{2}\right] - 1\right)$ $\quad (m = p+q)$
$H^{nq}(\underbrace{\Gamma \backslash D}, \mathbb{C})$ is generated by $[\Gamma_X \backslash D_X]$
if compact (even with coefficients)

(d) $\mathrm{sgn}(V) = (p, 2)$: D is hermitian,
$\Gamma \backslash D$ is a Shimura variety
(quasi-projective algebraic variety, smooth if Γ is small, defined over a small number field).

Thm (Bergeron – Millson – Moeglin)
If $\mathrm{sgn}(V) = (p, 2)$ and $\Gamma\backslash D$ is compact, then the <u>Hodge conjecture</u> holds for $\underbrace{H^{n,n}(\Gamma\backslash D) \cap H^{2n}(\Gamma\backslash D, \mathbb{Q})}$ if $n < \frac{1}{2}\left[\frac{p+1}{2}\right]$

any element is a $\mathbb{Q}$-linear combination of classes of algebraic subvarieties of the Shimura variety $\Gamma\backslash D$ (of $\mathrm{codim}_{\mathbb{C}} = n$).

<u>Kudla's programme (arithmetic case)</u>

If $\Gamma\backslash D$ is a <u>Shimura variety</u> ($G = O(p, 2)$ or $U(p, q)$) there should be an arithmetic refinement of the main result of Kudla–Millson, with $H^{\bullet}(\underbrace{\Gamma\backslash D}_{Y})$ replaced by

- <u>Chow groups</u> $CH^r(Y)$ ($CH^1(Y)$ = divisor class group)
- <u>Arakelov Chow groups</u> $\widehat{CH}^r(Y_{\mathbb{Z}})$

of a model $Y_{\mathbb{Z}}$ of Y over $\mathbb{Z}$.

Green's currents appear!
Solutions of $dd^cG - \delta_{\Gamma_X\backslash D_X} = \varphi_{\text{Kudla-Millson}}$
Related to "Mock θ-functions".

References:

- Mumford – Tata lectures on Theta, III
- Lion, Vergne – The Weil representation, Maslov index and theta series
- Kudla, Millson – Publ. IHES (1990)
- Kudla – Séminaire Bourbaki No 876, 1999–2000
  – in MSRI, vol. 49