APPENDIX

CHRISTOPHE CORNUT

1. Introduction

Let $M = \text{Sh}_K(G, \mathcal{X})$ be a Shimura variety. An analytic subvariety $Y$ of $\mathcal{X}$ is \textit{special} if it is a connected component of $Y$ for some sub Shimura datum $(H, Y)$ of $(G, \mathcal{X})$. An algebraic subvariety $Z$ of $M$ is \textit{special} if there exists an element $g \in G(A_f)$ and a special subvariety $Y$ of $\mathcal{X}$ such that $Z$ is the image of $gK \times Y$ in $M(\mathbb{C}) = G(\mathbb{Q}) \setminus (G(A_f)/K \times \mathcal{X})$.

The special subvarieties of dimension zero are the special points, which correspond to sub Shimura datum $(H, Y)$ for which $H$ is a $\mathbb{Q}$-subtorus of $G$. The following proposition is well known, see for instance [8, Chapter 13].

\textbf{Proposition 1.1.} Any special subvariety contains a dense set of special points.

\textbf{Conjecture 1.2.} (André-Oort) Any irreducible subvariety $Z$ of $M$ which contains a Zariski dense set of special points is special.

This conjecture has been proven under GRH [7], as well as unconditionally in many special cases [1, 5, 6, 7]. For various applications [2, 3], one has to list, or at least parametrize in a more explicit way the set of all special subvarieties of a given Shimura variety. We give such a parametrization for $M$ a product of Shimura curves, following Bas Edixhoven’s treatment of products of modular curves [5].

2. Statement

Let $F$ be totally real number field, $B$ a quaternion algebra over $F$ and $G = \text{Res}_{F/\mathbb{Q}} \text{GL}_1(B)$. Thus $G$ is a reductive group over $\mathbb{Q}$ with center $Z = \text{Res}_{F/\mathbb{Q}} \text{Gm,F}$. Let $d = [F : \mathbb{Q}]$ and denote by $\{\tau_1, \ldots, \tau_d\}$ the embeddings $F \hookrightarrow \mathbb{R}$. We assume that $B_j = B \otimes_{F, \tau_j} \mathbb{R}$ splits only for $j = 1$. Put $S = \text{Res}_{C/R} \text{Gm,C}$ and let $\mathcal{X}$ be the $G(\mathbb{R})$-conjugacy class of morphisms $S \to G(\mathbb{R}) \simeq \prod_{j=1}^{d} \text{GL}_1(B_j)$ whose projection to $\text{GL}_1(B_j)$ is trivial for $j \neq 1$, and whose projection $S \to \text{GL}_1(B_1)$ is induced by an embedding of $\mathbb{R}$-algebras $C \hookrightarrow B_1 \simeq M_2(\mathbb{R})$. Then $(G, \mathcal{X})$ is a Shimura datum with reflex field $\tau_1(F) \subset C$ and its associated Shimura varieties have dimension one: these are the Shimura curves. The classification of all special subvarieties of a given product of Shimura curves, as sketched without proof in [4], is easily derived from the following proposition.

\textbf{Proposition 2.1.} Let $I \neq \emptyset$ be a finite set. For any special subvariety $Y$ of $\mathcal{X}^I$, there exists

1. a partition $I = J' \bigsqcup J$, $J = \bigsqcup_{\alpha \in A} J(\alpha)$,
2. for each $j \in J'$, a special point $y_j \in \mathcal{X}$,
3. for each $j \in J'$, an element $\sigma_j \in G(\mathbb{Q})$,
4. for each $\alpha \in A$, a connected component $\mathcal{X}_\alpha$ of $\mathcal{X}$
such that
\[ Y = \{(y_j)_{j \in J}\} \times \prod_{\alpha \in A} \Delta_{\alpha}(X_\alpha) \text{ in } \mathcal{X}' = \mathcal{X}'^{J} \times \prod_{\alpha \in A} \mathcal{X}'^{J(\alpha)}, \]
where \( \Delta_{\alpha} : \mathcal{X} \to \mathcal{X}'^{J(\alpha)} \) maps \( x \) to \( (\sigma_j x)_{j \in J(\alpha)} \).

3. Preliminaries

In the sequel, we view \( F \) as a subfield of \( \mathbb{R} \subset \mathbb{C} \) through \( \tau_1 : F \hookrightarrow \mathbb{R} \). For a subfield \( L \) of the algebraic closure \( \overline{Q} \) of \( Q \) in \( C \), we put \( \overline{\Gamma}_L = \text{Gal}(Q/L) \). For a torus \( T \) over a subfield of \( C \), we denote by \( X_\ast(T) \) (resp. \( X_\ast^*(T) \)) the group of cocharacters \( \mathbf{G}_{m,C} \to T_C \) (resp. characters \( T_C \to \mathbf{G}_{m,C} \)).

If \( T \) is defined over \( L \), \( \Gamma_L \) acts on these groups and
\[ X_\ast(\text{Res}_L/Q T) = \text{Ind}_{\Gamma_L}^{\Gamma} X_\ast(T) \quad X_\ast^*(\text{Res}_L/Q T) = \text{Ind}_{\Gamma_L}^{\Gamma} X_\ast^*(T). \]

**Lemma 3.1.** The maximal \( Q \)-subtori of \( G \) are in bijection with the quadratic (maximal) \( F \)-algebras \( K \) inside \( B \) via \( K \hookrightarrow T_K = \text{Res}_F/Q GL_1(K) \).

**Proof.** Let \( T \) be a maximal \( Q \)-subtorus of \( G \). Then \( T \) is the centralizer in \( G \) of some regular element \( \gamma \in T(Q) \subset G(Q) = B^\times \). Let \( K \) be the commutant of \( \gamma \) in \( B \). Then \( K \) is a maximal (quadratic) commutative \( F \)-subalgebra of \( B \) and \( T = T_K \). \( \square \)

For any algebraic \( Q \)-subgroup \( H \) of \( G \), we denote by \( \mathcal{X}(H) \) the set of all \( x \in \mathcal{X} \) whose corresponding morphism \( h_x : S \to \mathbf{G}_R \) factors through \( H_R \).

**Proposition 3.2.** For a connected \( Q \)-subgroup \( H \) of \( G \), \( \mathcal{X}(H) \) contains a special point if and only if \( H = G \) or \( H = H_K \) for some totally imaginary quadratic extension \( K \) of \( F \) inside \( B \).

**Proof.** Let \( x \in \mathcal{X}(H) \) be a special point, \( T \) a maximal \( Q \)-subtorus of \( G \) such that \( h_x : S \to \mathbf{G}_R \) factors through \( T_R \) and \( S \) the connected component of \( T \cap H \). Thus \( S \) is a maximal \( Q \)-subtorus of \( H \) and \( h_x : S \to H_R \) factors through \( S_R \).

Write \( T = T_K \) for some quadratic \( F \)-subalgebra \( K \) inside \( B \). Put \( K_j = K \otimes_{F, \tau_j} R \), so that \( T_R \simeq \prod_{j=1}^n GL_1(K_j/R) \). For \( j \neq 1 \), the projection of \( h_x : S \to T_R \) to \( GL_1(K_j/R) \) is trivial. For \( j = 1 \), it is induced by an \( R \)-algebra isomorphism \( K_1 \simeq C \) (corresponding to an extension of \( \tau_1 \) to \( K \hookrightarrow C \)). Since also \( K_j \simeq C \) for \( j \neq 1 \) (because \( B_j \) does not split), \( K \) is a totally imaginary extension of \( F \).

Let \( \eta : \mathbf{G}_{m,R} \to \mathbf{G}_R \) be the restriction of \( h_x : S \to \mathbf{G}_R \) to \( \mathbf{G}_{m,R} \subset S \). Then \( \eta \) factors through \( Z_R \), and its Galois orbit \( \Gamma_Q \cdot \eta \) spans \( X_\ast(Z) \simeq \text{Ind}_{T}^{\mathbf{G}_R} Z \). Thus any \( Q \)-subtorus of \( G \) through which \( \eta \) factors, such as \( S \), contains \( Z \). Put \( \overline{S} = S/Z \), \( \overline{T} = T/Z \) and \( \overline{S} = \overline{S}/\mathbf{G}_{m,R} \). Then \( h_x : S \to T_R \) induces a morphism \( \overline{h}_x : \overline{S} \to \overline{T}_R \).

Since \( \overline{S}_C \cong \mathbf{G}_{m,C} \), we may view \( \overline{h}_x \) as a character of \( \overline{T}_R \). One checks again that its Galois orbit \( \Gamma_Q \cdot \overline{h}_x \) spans \( X_\ast(T) \simeq \text{Ind}_{T_R}^{\mathbf{G}_R} Z \). So any \( Q \)-subtorus of \( \overline{T} \) through which \( \overline{h}_x \) factors, such as \( \overline{S} = S/Z \), actually equals \( \overline{T} \). Thus \( S = T \).

Now \( h_x(i) \in H(R) \) yields a Cartan involution \( \text{Ad} h_x(i) \) of \( \mathbf{G}_R \), and thus also one of \( H_R \). Therefore \( H \) is a reductive group. As such, it is now entirely characterized by the set of roots of \( T \) in \( \text{Lie} H \), which has to be a Galois stable subset of the roots of \( T \) in \( \text{Lie} G \). Since \( \Gamma_Q \) acts transitively on the latter, \( H = T \) or \( H = G \). This finishes the proof of the non-obvious part of the proposition. \( \square \)

**Proposition 3.3.** Let \( \overline{G} = G/Z \) be the adjoint group of \( G \). Then

1. \( \overline{G} \) is \( Q \)-simple: any normal \( Q \)-subgroup of \( \overline{G} \) equals 1 or \( \overline{G} \).
(2) Any $\mathbb{Q}$-automorphism of $\overline{G}$ is inner.
(3) The projection $G(\mathbb{Q}) \to \overline{G}(\mathbb{Q})$ is surjective.

Proof. (1) The Dynkin diagram $D$ of $\overline{G}$ has no edges, and its $\Gamma_{\mathbb{Q}}$-set of vertices is isomorphic to $\Gamma_{\mathbb{Q}}/\Gamma_F$. Since $\Gamma_{\mathbb{Q}}$ acts transitively on $D$, $\overline{G}$ is almost $\mathbb{Q}$-simple: its connected normal $\mathbb{Q}$-subgroup are 1 and $\overline{G}$. Since $\overline{G}$ is also adject, it is $\mathbb{Q}$-simple.

(2) Any $\mathbb{Q}$-automorphism $\sigma$ of $\overline{G}$ induces an automorphism of $D$ which $(a)$ commutes with the transitive action of $\Gamma_{\mathbb{Q}}$ and $(b)$ fixes the node of $D$ which corresponds to the unique non-compact simple component of $\overline{G}_R$. Thus $\sigma$ acts trivially on $D$, and therefore is inner. (3) There is an exact sequence

$$1 \to Z(\mathbb{Q}) \to G(\mathbb{Q}) \to \overline{G}(\mathbb{Q}) \to H^1(\mathbb{Q}, Z).$$

By Shapiro’s lemma and Hilbert’s theorem 90, $H^1(\mathbb{Q}, Z) = H^1(F, G_m) = 0$. Thus $G(\mathbb{Q}) \to \overline{G}(\mathbb{Q})$ is indeed surjective. \qed

**Proposition 3.4.** Let $\overline{G}$ be a $\mathbb{Q}$-simple group. Let $I$ be a finite set and let $H$ be an algebraic $\mathbb{Q}$-subgroup of $\overline{G}^I$ all of whose projections to $\overline{G}$ are surjective. Then there exists a partition $I = \bigsqcup_{\alpha \in A} I(\alpha)$ and elements $\sigma_{\alpha} \in \text{Aut}_\mathbb{Q}(\overline{G})$ (for all $i \in I$) such that

$$H = \bigsqcup_{\alpha \in A} \Delta_{\alpha}(\overline{G}) \quad \text{in} \quad \overline{G}^I = \bigsqcup_{\alpha \in A} \overline{G}^{I(\alpha)},$$

where $\Delta_{\alpha} : \overline{G} \to \overline{G}^{I(\alpha)}$ maps $g$ to $(\sigma_{\alpha}(g))_{i \in I(\alpha)}$.

**Proof.** Call special the subgroups which are of this form. We argue by induction on $|I|$. If $|I| = 1$, there is nothing to prove. If $|I| = 2$, let $p_{1/2} : \overline{G}^2 \to \overline{G}$ be the projections and let $H_{2/1}$ be the kernel of the restriction of $p_{1/2}$ to $H$, so that $H_1 \times H_2$ is a normal subgroup of $H$ and $p_{2H}/p_{2H} \simeq H/H_1 \times H_2 \simeq p_{1H}/p_{1H}$.

Since $p_1H = p_2H = \overline{G}$, either $p_1H_1 = p_2H_2 = \overline{G}$ (in which case $H = \overline{G}^2$) or $p_1H_1 = p_2H_2 = 1$, in which case $H = \Delta(\overline{G})$ with $\Delta(g) = (g, \theta(g))$ for $\theta = (p_2|_H) \circ (p_1|_H)^{-1} \in \text{Aut}_\mathbb{Q}(\overline{G})$. In both cases, $H$ is indeed special.

Suppose now that $|I| \geq 3$. For any subset $J \neq \emptyset$ of $I$, let $p_J : G^J \to \overline{G}^J$ be the projection. By induction, we know already that $p_JH$ is special for every proper subset $J \neq \emptyset$ of $I$. Suppose that there is such a $J$ for which $p_JH \neq \overline{G}^J$. This means that there exists $i_1 \neq i_2 \in I$ and $\sigma_1, \sigma_2 \in \text{Aut}_\mathbb{Q}(\overline{G})$ such that $H$ is contained in the image of the morphism

$$\overline{G}^{J^{-\{i_1,i_2\}}} \times \overline{G} \to \overline{G}^J = \overline{G}^{J^{-\{i_1,i_2\}}} \times \overline{G}^{\{i_1,i_2\}}

((g_i), g) \mapsto ((g_i), (\sigma_1(g), \sigma_2(g)))$$

By induction, the preimage of $H$ under this morphism is special, and so is $H$.

Finally, suppose that for any proper subset $J \neq \emptyset$ of $I$, $p_JH = \overline{G}^J$. Then $H = \overline{G}^J$ (which is special). Indeed, it is sufficient to show that $H(\overline{Q})$ contains every factor $\overline{G}(\mathbb{Q})$ of $\overline{G}^J(\mathbb{Q})$. Thus, fix $j \in I$, $g \in \overline{G}(\mathbb{Q})$, and also pick two other elements $j_\pm \in I$ (recall that $|I| \geq 3$). Since $\overline{G}$ is semi-simple, there exists $g_\pm \in \overline{G}(\mathbb{Q})$ such that $g = [g_+, g_-]$. For $\epsilon \in \{\pm 1\}$, we know that $H(\mathbb{Q})$ contains an element $h_\epsilon$ with $p_j(h_\epsilon) = g_\epsilon$, and $p_j(h_\epsilon) = 1$ for all $i \notin \{j, j_\pm\}$. Then $p_j([h_+, h_-]) = g$ if $i = j$ and 1 otherwise. This proves our claim. \qed
4. Proof of Proposition 2.1

Let \((H, \mathcal{Y})\) be a sub Shimura datum of \((G^J, \mathcal{X}^J)\) such that \(Y\) is a connected component of \(\mathcal{Y}\). For any subset \(J \neq \emptyset\) of \(I\), let \(p_J : G^J \to G^J\) and \(q_J : \mathcal{X}^J \to \mathcal{X}^J\) be the projections. Define

\[ J = \{ i \in I \mid p_i H = G \} \quad \text{and} \quad J' = \{ i \in I \mid p_i H \neq G \}. \]

By Propositions 3.4 and 3.3, there exist a partition \(J = \bigcup_{\alpha \in A} J(\alpha)\) and elements \(\sigma_j \in G(\mathbb{Q})\) (for all \(j \in J\)) such that the two subgroups \(p_J H\) and \(H_J = \prod_{\alpha \in A} \Delta_\alpha(G)\) of \(G^J = \prod_{\alpha \in A} G^J(\alpha)\) have the same projection to \((G/Z)^J\), where

\[ \Delta_\alpha : G \to G^J(\alpha) \quad \text{maps} \quad g \to (\sigma_J g \sigma_{J}^{-1})_{j \in J(\alpha)}. \]

According to Proposition 1.1, there exists at least one special point \(y \in Y\). For all \(i \in I\), put \(y_i = q_i(y)\) and for each \(\alpha \in A\), denote by \(\mathcal{X}_\alpha\) the connected component of \(\mathcal{X}\) which contains \(x_\alpha = \sigma_{J(\alpha)}^{-1} \cdot y_{J(\alpha)}\) for some arbitrary (but fixed) choice of an element \(J(\alpha)\) in \(J(\alpha)\) (we will see that \(x_\alpha\) does not depend upon that choice).

For any \(i \in I\), \(y_i = q_i(y)\) is a special point of \(q_i(Y) \subset X(p_i H)\). Thus by Proposition 3.2, for any \(j \in J'\), \(p_J H\) is a (CM type) maximal subtorus of \(G\). Then \(X(p_J H)\) is a finite (two points) set and its connected subset \(q_J(Y)\) reduces to \(\{y_J\}\). This already shows that

\[ Y = \{(y_J)_{J \in J'}\} \times q_J(Y) \quad \text{in} \quad \mathcal{X}^J = \mathcal{X}^{J'} \times \mathcal{X}^J. \]

Put \(y_J = q_J(y)\) and \(y_\alpha = q_J(\alpha)(y)\). Since \(Y = H(\mathbb{R})^0 \cdot y\),

\[ q_J(Y) = p_J(H(\mathbb{R})^0) \cdot y_J = p_J(H)(\mathbb{R})^0 \cdot y_J = h_J(\mathbb{R})^0 \cdot y_J \]

because \(p_J(H(\mathbb{R})^0) = (p_J H)(\mathbb{R})^0\) and \(Z^J \cdot p_J H = Z^J \cdot H_J\). Therefore

\[ q_J(Y) = \prod_{\alpha \in A} q_J(\alpha)(Y) \quad \text{with} \quad q_J(\alpha)(Y) = \Delta_\alpha(G(\mathbb{R})^0) \cdot y_\alpha. \]

Now \(y_\alpha = q_J(\alpha)(y) \in \mathcal{X}^J(\alpha)\) corresponds to a morphism \(h_\alpha : S \to G^J(\alpha)\) which factors through \((p_J H)_{\mathbb{R}}\) and therefore also through

\[ (Z^J(\alpha) \cdot \Delta_\alpha(G))_{\mathbb{R}} = (Z^J(\alpha) \cdot p_J H)_{\mathbb{R}}. \]

Put \(Z_\alpha = \{1\} \times Z^J(\alpha) - \{(\alpha)\} \subset Z^J(\alpha)\), so that \(Z^J(\alpha) \cdot \Delta_\alpha(G) = Z_\alpha \times \Delta_\alpha(G)\) and \(h_\alpha = h^Z_\alpha \cdot h^\Delta_\alpha\) with \(h^Z_\alpha : S \to (Z_\alpha)_{\mathbb{R}}\) and \(h^\Delta_\alpha : S \to \Delta_\alpha(G)_{\mathbb{R}}\). Write \(h^\Delta_\alpha = \Delta_\alpha \circ h\) for some \(h : S \to G_{\mathbb{R}}\). Then

\[ \text{Ad}(\sigma_{J(\alpha)}) \circ h = p_J(\alpha) \circ h^\Delta_\alpha = q_J(\alpha) \circ h_\alpha = h_{y_J(\alpha)} = \text{Ad}(\sigma_{J(\alpha)}) \circ h_{x_\alpha}\]

so that \(h = h_{x_\alpha}\). For each \(j \in J(\alpha)\), choose \(g_j \in G(\mathbb{R})\) such that \(g_j \cdot y_j = \sigma_j \cdot x_\alpha\) in \(X\). Then for all \(s \in S\),

\[ h^\lambda_\alpha(s) = h_\alpha(s)h^\Delta_\alpha(s)^{-1} = (h(y_j(s) \cdot h_{y_J(\alpha)}(s)^{-1}j_{J(\alpha)}) = (h(y_j(s)g_jh_{y_J(\alpha)}(s)^{-1}g_j^{-1}j_{J(\alpha)}) \]

Thus \(h^\lambda_\alpha : S \to (Z_\alpha)_{\mathbb{R}} \hookrightarrow Z^{J(\alpha)}_{\mathbb{R}} \hookrightarrow G^{J(\alpha)}_{\mathbb{R}}\) factors through the derived group of \(G^{J(\alpha)}_{\mathbb{R}}\). Since the latter has finite intersection with \(Z^{J(\alpha)}_{\mathbb{R}}\) while \(S\) is connected, \(h^\lambda_\alpha = 1\) and \(h_\alpha = h^\Delta_\alpha \circ h_{x_\alpha}\). This simply means that \(y_\alpha = \Delta_\alpha(x_\alpha)\), so that

\[ q_J(\alpha)(Y) = \Delta_\alpha(G(\mathbb{R})^0) \cdot y_\alpha = \Delta_\alpha(G(\mathbb{R})^0) \cdot x_\alpha = \Delta_\alpha(X_\alpha). \]

Thus \(Y = \{(y_J)_{J \in J'}\} \times \prod_{\alpha \in A} \Delta_\alpha(X_\alpha)\), which proves the Proposition.
APPENDIX

REFERENCES


