APPENDIX

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1. Introduction

Let $M = \mathrm{Sh}_K(G, \mathcal{X})$ be a Shimura variety. An analytic subvariety Y of $\mathcal{X}$ is *special* if it is a connected component of $\mathcal{Y}$ for some sub Shimura datum $(H, \mathcal{Y})$ of $(G, \mathcal{X})$. An algebraic subvariety Z of M is *special* if there exists an element $g \in G(\mathbf{A}_f)$ and a special subvariety Y of $\mathcal{X}$ such that Z is the image of $gK \times Y$ in

$$M(\mathbf{C}) = G(\mathbf{Q})\backslash \left(G(\mathbf{A}_f)/K \times \mathcal{X}\right).$$

The special subvarieties of dimension zero are the special points, which correspond to sub Shimura datum $(H, \mathcal{Y})$ for which H is a $\mathbf{Q}$-subtorus of G. The following proposition is well known, see for instance [8, Chapter 13].

Proposition 1.1. *Any special subvariety contains a dense set of special points.*

Conjecture 1.2. *(André-Oort) Any irreducible subvariety Z of M which contains a Zariski dense set of special points is special.*

This conjecture has been proven under GRH [7], as well as unconditionally in many special cases [1, 5, 6, 7]. For various applications [2, 3], one has to list, or at least parametrize in a more explicit way the set of all special subvarieties of a given Shimura variety. We give such a parametrization for M a product of Shimura curves, following Bas Edixhoven's treatment of products of modular curves [5].

2. Statement

Let F be totally real number field, B a quaternion algebra over F and $G = \mathrm{Res}_{F/\mathbf{Q}} GL_1(B)$. Thus G is a reductive group over $\mathbf{Q}$ with center $Z = \mathrm{Res}_{F/\mathbf{Q}} \mathbf{G}_{m,F}$. Let $d = [F : \mathbf{Q}]$ and denote by $\{\tau_1, \cdots, \tau_d\}$ the embeddings $F \hookrightarrow \mathbf{R}$. We assume that $B_j = B \otimes_{F,\tau_j} \mathbf{R}$ splits only for $j = 1$. Put $\mathbf{S} = \mathrm{Res}_{\mathbf{C}/\mathbf{R}} \mathbf{G}_{m,\mathbf{C}}$ and let $\mathcal{X}$ be the $G(\mathbf{R})$-conjugacy class of morphisms $\mathbf{S} \to G_{\mathbf{R}} \simeq \prod_{j=1}^{d} GL_1(B_j)$ whose projection to $GL_1(B_j)$ is trivial for $j \neq 1$, and whose projection $\mathbf{S} \to GL_1(B_1)$ is induced by an embedding of $\mathbf{R}$-algebras $\mathbf{C} \hookrightarrow B_1 \simeq M_2(\mathbf{R})$. Then $(G, \mathcal{X})$ is a Shimura datum with reflex field $\tau_1(F) \subset \mathbf{C}$ and its associated Shimura varieties have dimension one: these are the Shimura curves. The classification of all special subvarieties of a given product of Shimura curves, as sketched without proof in [4], is easily derived from the following proposition.

Proposition 2.1. *Let $I \neq \emptyset$ be a finite set. For any special subvariety Y of $\mathcal{X}^I$, there exists*

1. *a partition $I = J' \coprod J$, $J = \coprod_{\alpha \in A} J(\alpha)$,*
2. *for each $j \in J'$, a special point $y_j \in \mathcal{X}$,*
3. *for each $j \in J$, an element $\sigma_j \in G(\mathbf{Q})$,*
4. *for each $\alpha \in A$, a connected component $\mathcal{X}_\alpha$ of $\mathcal{X}$*

such that

$$Y = \{(y_j)_{j\in J'}\} \times \prod_{\alpha\in A} \Delta_\alpha(\mathcal{X}_\alpha) \quad in \quad \mathcal{X}^I = \mathcal{X}^{J'} \times \prod_{\alpha\in A} \mathcal{X}^{J(\alpha)},$$

where $\Delta_\alpha : \mathcal{X} \to \mathcal{X}^{J(\alpha)}$ *maps* x *to* $(\sigma_j x)_{j\in J(\alpha)}$.

3. Preliminaries

In the sequel, we view F as a subfield of $\mathbf{R} \subset \mathbf{C}$ through $\tau_1 : F \hookrightarrow \mathbf{R}$. For a subfield L of the algebraic closure $\overline{\mathbf{Q}}$ of $\mathbf{Q}$ in $\mathbf{C}$, we put $\Gamma_L = \mathrm{Gal}(\overline{\mathbf{Q}}/L)$. For a torus T over a subfield of $\mathbf{C}$, we denote by $X_\star(T)$ (resp. $X^\star(T)$) the group of cocharacters $\mathbf{G}_{m,\mathbf{C}} \to T_\mathbf{C}$ (resp. characters $T_\mathbf{C} \to \mathbf{G}_{m,\mathbf{C}}$). If T is defined over L, Γ_L acts on these groups and

$$X_\star(\mathrm{Res}_{L/\mathbf{Q}} T) = \mathrm{Ind}_{\Gamma_L}^{\Gamma_\mathbf{Q}} X_\star(T) \quad X^\star(\mathrm{Res}_{L/\mathbf{Q}} T) = \mathrm{Ind}_{\Gamma_L}^{\Gamma_\mathbf{Q}} X^\star(T).$$

Lemma 3.1. *The maximal* $\mathbf{Q}$*-subtori of* G *are in bijection with the quadratic (maximal)* F*-algebras* K *inside* B *via* $K \mapsto T_K = \mathrm{Res}_{F/\mathbf{Q}} GL_1(K)$.

Proof. Let T be a maximal $\mathbf{Q}$-subtorus of G. Then T is the centralizer in G of some regular element $\gamma \in T(\mathbf{Q}) \subset G(\mathbf{Q}) = B^\times$. Let K be the commutant of γ in B. Then K is a maximal (quadratic) commutative F-subalgebra of B and $T = T_K$. □

For any algebraic $\mathbf{Q}$-subgroup H of G, we denote by $\mathcal{X}(H)$ the set of all $x \in \mathcal{X}$ whose corresponding morphism $h_x : \mathbf{S} \to G_\mathbf{R}$ factors through $H_\mathbf{R}$.

Proposition 3.2. *For a connected* $\mathbf{Q}$*-subgroup* H *of* G*,* $\mathcal{X}(H)$ *contains a special point if and only if* $H = G$ *or* $H = T_K$ *for some totally imaginary quadratic extension* K *of* F *inside* B.

Proof. Let $x \in \mathcal{X}(H)$ be a special point, T a maximal $\mathbf{Q}$-subtorus of G such that $h_x : \mathbf{S} \to G_\mathbf{R}$ factors through $T_\mathbf{R}$ and S the connected component of $T \cap H$. Thus S is a maximal $\mathbf{Q}$-subtorus of H and $h_x : \mathbf{S} \to H_\mathbf{R}$ factors through $S_\mathbf{R}$.

Write $T = T_K$ for some quadratic F-subalgebra K of B. Put $K_j = K \otimes_{F,\tau_j} \mathbf{R}$, so that $T_\mathbf{R} \simeq \prod_{j=1}^d GL_1(K_j/\mathbf{R})$. For $j \neq 1$, the projection of $h_x : \mathbf{S} \to T_\mathbf{R}$ to $GL_1(K_j/\mathbf{R})$ is trivial. For $j = 1$, it is induced by an $\mathbf{R}$-algebra isomorphism $K_1 \simeq \mathbf{C}$ (corresponding to an extension of τ_1 to $K \hookrightarrow \mathbf{C}$). Since also $K_j \simeq \mathbf{C}$ for $j \neq 1$ (because B_j does not split), K is a totally imaginary extension of F.

Let $\eta : \mathbf{G}_{m,\mathbf{R}} \to G_\mathbf{R}$ be the restriction of $h_x : \mathbf{S} \to G_\mathbf{R}$ to $\mathbf{G}_{m,\mathbf{R}} \subset \mathbf{S}$. Then η factors through $Z_\mathbf{R}$, and its Galois orbit $\Gamma_\mathbf{Q} \cdot \eta$ spans $X_\star(Z) \simeq \mathrm{Ind}_{\Gamma_F}^{\Gamma_\mathbf{Q}} \mathbf{Z}$. Thus any $\mathbf{Q}$-subtorus of G through which η factors, such as S, contains Z. Put $\overline{S} = S/Z$, $\overline{T} = T/Z$ and $\overline{\mathbf{S}} = \mathbf{S}/\mathbf{G}_{m,\mathbf{R}}$. Then $h_x : \mathbf{S} \to T_\mathbf{R}$ induces a morphism $\overline{h}_x : \overline{\mathbf{S}} \to \overline{T}_\mathbf{R}$. Since $\overline{\mathbf{S}}_\mathbf{C} \simeq \mathbf{G}_{m,\mathbf{C}}$, we may view $\overline{h}_x$ as a cocharacter of $\overline{T}_\mathbf{R}$. One checks again that its Galois orbit $\Gamma_\mathbf{Q} \cdot \overline{h}_x$ spans $X_\star(\overline{T}) \simeq \mathrm{Ind}_{\Gamma_K}^{\Gamma_\mathbf{Q}} \mathbf{Z}/\mathrm{Ind}_{\Gamma_F}^{\Gamma_\mathbf{Q}} \mathbf{Z}$, so that any $\mathbf{Q}$-subtorus of $\overline{T}$ through which $\overline{h}_x$ factors, such as $\overline{S} = S/Z$, actually equals $\overline{T}$. Thus $S = T$.

Now $h_x(i) \in H(\mathbf{R})$ yields a Cartan involution $\mathrm{Ad} h_x(i)$ of $G_\mathbf{R}$, and thus also one of $H_\mathbf{R}$. Therefore H is a reductive group. As such, it is now entirely characterized by the set of roots of T in Lie$H$, which has to be a Galois stable subset of the roots of T in Lie$G$. Since $\Gamma_\mathbf{Q}$ acts transitively on the latter, $H = T$ or $H = G$. This finishes the proof of the non-obvious part of the proposition. □

Proposition 3.3. *Let* $\overline{G} = G/Z$ *be the adjoint group of* G*. Then*

(1) $\overline{G}$ *is* $\mathbf{Q}$*-simple: any normal* $\mathbf{Q}$*-subgroup of* $\overline{G}$ *equals* 1 *or* $\overline{G}$.

(2) *Any* $\mathbf{Q}$*-automorphism of* $\overline{G}$ *is inner.*
(3) *The projection* $G(\mathbf{Q}) \to \overline{G}(\mathbf{Q})$ *is surjective.*

Proof. (1) The Dynkin diagram D of $\overline{G}$ has no edges, and its $\Gamma_{\mathbf{Q}}$-set of vertices is isomorphic to $\Gamma_{\mathbf{Q}}/\Gamma_F$. Since $\Gamma_{\mathbf{Q}}$ acts transitively on D, $\overline{G}$ is almost $\mathbf{Q}$-simple: its connected normal $\mathbf{Q}$-subgroup are 1 and $\overline{G}$. Since $\overline{G}$ is also adjoint, it is $\mathbf{Q}$-simple. (2) Any $\mathbf{Q}$-automorphism σ of $\overline{G}$ induces an automorphism of D which (a) commutes with the transitive action of $\Gamma_{\mathbf{Q}}$ and (b) fixes the node of D which corresponds to the unique non-compact simple component of $\overline{G}_{\mathbf{R}}$. Thus σ acts trivially on D, and therefore is inner. (3) There is an exact sequence

$$1 \to Z(\mathbf{Q}) \to G(\mathbf{Q}) \to \overline{G}(\mathbf{Q}) \to H^1(\mathbf{Q}, Z).$$

By Shapiro's lemma and Hilbert's theorem 90, $H^1(\mathbf{Q}, Z) = H^1(F, \mathbf{G}_m) = 0$. Thus $G(\mathbf{Q}) \to \overline{G}(\mathbf{Q})$ is indeed surjective. □

Proposition 3.4. *Let* $\overline{G}$ *be a* $\mathbf{Q}$*-simple group. Let* I *be a finite set and let* H *be an algebraic* $\mathbf{Q}$*-subgroup of* $\overline{G}^I$ *all of whose projections to* $\overline{G}$ *are surjective. Then there exists a partition* $I = \coprod_{\alpha \in A} I(\alpha)$ *and elements* $\sigma_i \in \mathrm{Aut}_{\mathbf{Q}}\overline{G}$ *(for all* $i \in I$*) such that*

$$H = \prod_{\alpha \in A} \Delta_\alpha(\overline{G}) \quad \textit{in } \overline{G}^I = \prod_{\alpha \in A} \overline{G}^{I(\alpha)},$$

where $\Delta_\alpha : \overline{G} \to \overline{G}^{I(\alpha)}$ *maps* g *to* $(\sigma_i(g))_{i \in I(\alpha)}$*.*

Proof. Call special the subgroups which are of this form. We argue by induction on $|I|$. If $|I| = 1$, there is nothing to prove. If $|I| = 2$, let $p_{1/2} : \overline{G}^2 \to \overline{G}$ be the projections and let $H_{2/1}$ be the kernel of the restriction of $p_{1/2}$ to H, so that $H_1 \times H_2$ is a normal subgroup of H and

$$p_2H/p_2H_2 \simeq H/H_1 \times H_2 \simeq p_1H/p_1H_1.$$

Since $p_1H = p_2H = \overline{G}$, either $p_1H_1 = p_2H_2 = \overline{G}$ (in which case $H = \overline{G}^2$) or $p_1H_1 = p_2H_2 = 1$, in which case $H = \Delta(\overline{G})$ with $\Delta(g) = (g, \theta(g))$ for $\theta = (p_2|_H) \circ (p_1|_H)^{-1} \in \mathrm{Aut}_{\mathbf{Q}}(\overline{G})$. In both cases, H is indeed special.

Suppose now that $|I| \geq 3$. For any subset $J \neq \emptyset$ of I, let $p_J : \overline{G}^I \to \overline{G}^J$ be the projection. By induction, we know already that p_JH is special for every *proper* subset $J \neq \emptyset$ of I. Suppose that there is such a J for which $p_JH \neq \overline{G}^J$. This means that there exists $i_1 \neq i_2 \in I$ and $\sigma_1, \sigma_2 \in \mathrm{Aut}_{\mathbf{Q}}\overline{G}$ such that H is contained in the image of the morphism

$$\begin{array}{ccc} \overline{G}^{I-\{i_1,i_2\}} \times \overline{G} & \to & \overline{G}^I = \overline{G}^{I-\{i_1,i_2\}} \times \overline{G}^{\{i_1,i_2\}} \\ ((g_i), g) & \mapsto & ((g_i), (\sigma_1(g), \sigma_2(g))) \end{array}$$

By induction, the preimage of H under this morphism is special, and so is H.

Finally, suppose that for any proper subset $J \neq \emptyset$ of I, $p_JH = \overline{G}^J$. Then $H = \overline{G}^I$ (which is special). Indeed, it is sufficient to show that $H(\overline{\mathbf{Q}})$ contains every factor $\overline{G}(\overline{\mathbf{Q}})$ of $\overline{G}^I(\overline{\mathbf{Q}})$. Thus, fix $j \in I$, $g \in \overline{G}(\overline{\mathbf{Q}})$, and also pick two other elements $j_\pm \in I$ (recall that $|I| \geq 3$). Since $\overline{G}$ is semi-simple, there exists $g_\pm \in \overline{G}(\overline{\mathbf{Q}})$ such that $g = [g_+, g_-]$. For $\epsilon \in \{\pm 1\}$, we know that $H(\overline{\mathbf{Q}})$ contains an element h_ϵ with $p_j(h_\epsilon) = g_\epsilon$ and $p_i(h_\epsilon) = 1$ for all $i \notin \{j, j_\epsilon\}$. Then $p_i([h_+, h_-]) = g$ if $i = j$ and 1 otherwise. This proves our claim. □

4. Proof of Proposition 2.1

Let $(H, \mathcal{Y})$ be a sub Shimura datum of $(G^I, \mathcal{X}^I)$ such that Y is a connected component of $\mathcal{Y}$. For any subset $J \neq \emptyset$ of I, let $p_J : G^I \to G^J$ and $q_J : \mathcal{X}^I \to \mathcal{X}^J$ be the projections. Define

$$J = \{i \in I \,|\, p_i H = G\} \quad \text{and} \quad J' = \{i \in I \,|\, p_i H \neq G\}.$$

By Propositions 3.4 and 3.3, there exist a partition $J = \coprod_{\alpha \in A} J(\alpha)$ and elements $\sigma_j \in G(\mathbf{Q})$ (for all $j \in J$) such that the two subgroups $p_J H$ and $H_J = \prod_{\alpha \in A} \Delta_\alpha(G)$ of $G^J = \prod_{\alpha \in A} G^{J(\alpha)}$ have the same projection to $(G/Z)^J$, where

$$\Delta_\alpha : G \to G^{J(\alpha)} \quad \text{maps } g \text{ to } (\sigma_j g \sigma_j^{-1})_{j \in J(\alpha)}.$$

According to Proposition 1.1, there exists at least one special point $y \in Y$. For all $i \in I$, put $y_i = q_i(y)$ and for each $\alpha \in A$, denote by $\mathcal{X}_\alpha$ the connected component of $\mathcal{X}$ which contains $x_\alpha = \sigma_{j(\alpha)}^{-1} \cdot y_{j(\alpha)}$ for some arbitrary (but fixed) choice of an element $j(\alpha)$ in $J(\alpha)$ (we will see that x_α does not depend upon that choice).

For any $i \in I$, $y_i = q_i(y)$ is a special point of $q_i(Y) \subset \mathcal{X}(p_i H)$. Thus by Proposition 3.2, for any $j \in J'$, $p_j H$ is a (CM type) maximal subtori of G. Then $\mathcal{X}(p_j H)$ is a finite (two points) set and its connected subset $q_j(Y)$ reduces to $\{y_j\}$. This already shows that

$$Y = \{(y_j)_{j \in J'}\} \times q_J(Y) \quad \text{in} \quad \mathcal{X}^I = \mathcal{X}^{J'} \times \mathcal{X}^J.$$

Put $y_J = q_J(y)$ and $y_\alpha = q_{J(\alpha)}(y)$. Since $Y = H(\mathbf{R})^0 \cdot y$,

$$q_J(Y) = p_J(H(\mathbf{R})^0) \cdot y_J = (p_J H)(\mathbf{R})^0 \cdot y_J = H_J(\mathbf{R})^0 \cdot y_J$$

because $p_J(H(\mathbf{R})^0) = (p_J H)(\mathbf{R})^0$ and $Z^J \cdot p_J H = Z^J \cdot H_J$. Therefore

$$q_J(Y) = \textstyle\prod_{\alpha \in A} q_{J(\alpha)}(Y) \quad \text{with } q_{J(\alpha)}(Y) = \Delta_\alpha(G(\mathbf{R})^0) \cdot y_\alpha.$$

Now $y_\alpha = q_{J(\alpha)}(y) \in \mathcal{X}^{J(\alpha)}$ corresponds to a morphim $h_\alpha : \mathbf{S} \to G_{\mathbf{R}}^{J(\alpha)}$ which factors through $(p_{J(\alpha)} H)_{\mathbf{R}}$ and therefore also through

$$(Z^{J(\alpha)} \cdot \Delta_\alpha(G))_{\mathbf{R}} = (Z^{J(\alpha)} \cdot p_{J(\alpha)} H)_{\mathbf{R}}.$$

Put $Z_\alpha = \{1\} \times Z^{J(\alpha) - \{j(\alpha)\}} \subset Z^{J(\alpha)}$, so that $Z^{J(\alpha)} \cdot \Delta_\alpha(G) = Z_\alpha \times \Delta_\alpha(G)$ and $h_\alpha = h_\alpha^Z \cdot h_\alpha^\Delta$ with $h_\alpha^Z : \mathbf{S} \to (Z_\alpha)_{\mathbf{R}}$ and $h_\alpha^\Delta : \mathbf{S} \to \Delta_\alpha(G)_{\mathbf{R}}$. Write $h_\alpha^\Delta = \Delta_\alpha \circ h$ for some $h : \mathbf{S} \to G_{\mathbf{R}}$. Then

$$\mathrm{Ad}(\sigma_{j(\alpha)}) \circ h = p_{j(\alpha)} \circ h_\alpha^\Delta = p_{j(\alpha)} \circ h_\alpha = h_{y_{j(\alpha)}} = \mathrm{Ad}(\sigma_{j(\alpha)}) \circ h_{x_\alpha}$$

so that $h = h_{x_\alpha}$. For each $j \in J(\alpha)$, choose $g_j \in G(\mathbf{R})$ such that $g_j \cdot y_j = \sigma_j \cdot x_\alpha$ in $\mathcal{X}$. Then for all $s \in \mathbf{S}$,

$$h_\alpha^Z(s) = h_\alpha(s) h_\alpha^\Delta(s)^{-1} = (h_{y_j}(s) \cdot h_{\sigma_j x_\alpha}(s)^{-1})_{j \in J(\alpha)} = (h_{y_j}(s) g_j h_{y_j}(s)^{-1} g_j^{-1})_{j \in J(\alpha)}$$

Thus $h_\alpha^Z : \mathbf{S} \to (Z_\alpha)_{\mathbf{R}} \hookrightarrow Z_{\mathbf{R}}^{J(\alpha)} \hookrightarrow G_{\mathbf{R}}^{J(\alpha)}$ factors through the derived group of $G_{\mathbf{R}}^{J(\alpha)}$. Since the latter has finite intersection with $Z_{\mathbf{R}}^{J(\alpha)}$ while $\mathbf{S}$ is connected, $h_\alpha^Z = 1$ and $h_\alpha = h_\alpha^\Delta = \Delta_\alpha \circ h_{x_\alpha}$. This simply means that $y_\alpha = \Delta_\alpha(x_\alpha)$, so that

$$q_{J(\alpha)}(Y) = \Delta_\alpha(G(\mathbf{R})^0) \cdot y_\alpha = \Delta_\alpha(G(\mathbf{R})^0 \cdot x_\alpha) = \Delta_\alpha(\mathcal{X}_\alpha).$$

Thus $Y = \{(y_j)_{j \in J'}\} \times \prod_{\alpha \in A} \Delta_\alpha(\mathcal{X}_\alpha)$, which proves the Proposition.

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