p-adic Abel-Jacobi maps and p-adic heights

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This is a slightly expanded version of the author's lecture at the 1998 Banff Conference "The Arithmetic and Geometry of Algebraic Cycles". Its aim is to give a survey of foundational results on p-adic Abel-Jacobi maps and p-adic height pairings for algebraic cycles on varieties over number fields. Some of the results are new; their proofs will be given elsewhere.

1. p-adic cycle class maps

(1.0) Let X be a proper and smooth variety over a field K. Fix a separable closure K^{sep} of K and put $\overline{X} = X \times_K K^{\mathrm{sep}}$, $G_K = \mathrm{Gal}(K^{\mathrm{sep}}/K)$. We also fix a prime number $p \neq \mathrm{char}(K)$. Throughout this article, all cohomology groups with $\mathbb{Q}_p$-coefficients refer to continuous cohomology in the sense of [Ja 1].

(1.1) The fundamental object of our interest is the p-adic cycle class map

$$cl_X : CH^n(X) \longrightarrow H^{2n}(X_{\mathrm{et}}, \mathbb{Q}_p(n))$$

with values in arithmetic étale cohomology of X. The link to geometric étale cohomology $V^j = H^j(\overline{X}_{\mathrm{et}}, \mathbb{Q}_p)$ is provided by the Hochschild-Serre spectral sequence

$$^{\mathrm{et}}E_2^{i,j} = H^i(G_K, V^j(n)) \Longrightarrow H^{i+j}(X_{\mathrm{et}}, \mathbb{Q}_p(n)), \tag{1.1.1}$$

which can also be viewed as a Leray spectral sequence for the structure morphism $X \longrightarrow \mathrm{Spec}(K)$.

(1.2) It is a non-trivial fact that (1.1.1) degenerates at E_2, i.e. that $^{\mathrm{et}}E_2^{i,j} = {}^{\mathrm{et}}E_\infty^{i,j}$. For X smooth and projective over K, this follows from Deligne's Thesis [De 1] and his proof of the Hard Lefschetz Theorem [De 2] (together with the existence of an appropriate formalism of derived categories for continuous cohomology [Ek]). The general case reduces to the previous one by using de Jong's theorem on alterations [deJ].

As a consequence of this degeneration, the filtration $F^*H^*(X_{\mathrm{et}}, \mathbb{Q}_p(n))$ defined by (1.1.1) satisfies

$$gr_F^j H^m(X_{\mathrm{et}}, \mathbb{Q}_p(n)) \xrightarrow{\sim} H^j(G_K, V^{m-j}(n)).$$

One obtains, therefore, a sequence of induced maps

$$\begin{aligned} cl_{0,X} &: CH^n(X) \longrightarrow H^0(G_K, V^{2n}(n)) \\ cl_{1,X} &: \mathrm{Ker}(cl_{0,X}) \longrightarrow H^1(G_K, V^{2n-1}(n)) \\ cl_{j,X} &: \mathrm{Ker}(cl_{j-1,X}) \longrightarrow H^j(G_K, V^{2n-j}(n)) \\ cl_{j,X} \otimes \mathbb{Q}_p &: \mathrm{Ker}(cl_{j-1,X}) \otimes \mathbb{Q}_p \longrightarrow H^j(G_K, V^{2n-j}(n)) \end{aligned}$$

(1.3) The subgroup

$$CH^n(X)_{\mathrm{hom}\sim 0} := \mathrm{Ker}(cl_{0,X}) \subseteq CH^n(X)$$

is the subgroup of *homologically trivial cycles* (modulo torsion) and $cl_{1,X}$ is the p-adic Abel-Jacobi map.

If $\mathrm{char}(K) = 0$, then one can assume without loss of generality that there is an embedding $\sigma : K^{\mathrm{sep}} \hookrightarrow \mathbb{C}$. Putting $X_{\mathbb{C}} = \overline{X} \times_{K^{\mathrm{sep}},\sigma} \mathbb{C}$ and using standard comparison isomorphisms between étale and singular cohomology over $\mathbb{C}$

$$\begin{array}{ccccc}
CH^n(\overline{X}) & \longrightarrow & H^{2n}(\overline{X}_{\mathrm{et}}, \mathbb{Q}_p(n)) & \xrightarrow[\sigma_*]{\sim} & H^{2n}((X_{\mathbb{C}})_{\mathrm{et}}, \mathbb{Q}_p(n)) \\
\downarrow{\scriptstyle \sigma_*} & & & & \downarrow{\scriptstyle \simeq} \\
CH^n(X_{\mathbb{C}}) & & \longrightarrow & & H^{2n}(X_{\mathbb{C}}^{\mathrm{an}}, \mathbb{Q}(n)) \otimes_{\mathbb{Q}} \mathbb{Q}_p
\end{array}$$

we see that $\mathrm{Ker}(cl_{0,X})$ is independent on the choice of p.

If $\mathrm{char}(K) > 0$, then the independence of $\mathrm{Ker}(cl_{0,X})$ on p is unknown in general.

2. Varieties over number fields

(2.0) From now on, K will be a finite extension of $\mathbb{Q}$ (hence $H^i(G_K, V^j(n)) = 0$ for $i > 2$). In this case the cycle class maps are expected to behave reasonably well.

(2.1) Conjecture. (1) $\mathrm{Im}\,(cl_{0,X} \otimes \mathbb{Q}_p) \stackrel{?}{=} H^0(G_K, V^{2n}(n))$
(a special case of Tate's conjectures [Ta]).
(2) $\mathrm{Ker}(cl_{1,X} \otimes \mathbb{Q}_p) \stackrel{?}{=} 0$ and $\mathrm{Im}\,(cl_{1,X} \otimes \mathbb{Q}_p) \stackrel{?}{=} H^1_f(G_K, V^{2n-1}(n))$
(a special case of conjectures of Bloch-Kato [BK]).

(2.2) Let us recall the definition of Bloch-Kato style Selmer groups [BK] appearing in Conjecture 2.1.2. Let V be a $\mathbb{Q}_p$-representation of G_K, i.e. a finite dimensional vector space over $\mathbb{Q}_p$ with a continuous $\mathbb{Q}_p$-linear action of G_K (e.g. $V = V^j(n)$). The subspace $H^1_f(G_K, V) \subset H^1(G_K, V)$ is defined by the cartesian square

$$\begin{array}{ccc}
H^1_f(G_K, V) & \hookrightarrow & H^1(G_K, V) \\
\downarrow & \blacksquare & \downarrow \\
\prod_v H^1_f(G_v, V) & \hookrightarrow & \prod_v H^1(G_v, V),
\end{array}$$

in which v runs through all non-archimedean primes of K, $G_v = \mathrm{Gal}(\overline{K}_v/K_v)$ and

$$H^1_f(G_v, V) = \begin{cases} H^1_{\mathrm{ur}}(G_v, V) = H^1(G_v/I_v, V^{I_v}), & \text{if } v \not| p \\ \mathrm{Ker}(H^1(G_v, V) \longrightarrow H^1(G_v, V \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}})), & \text{if } v | p \end{cases}$$

Replacing B_{cris} by B_{st} (resp. by B_{dR}), one obtains – for $v|p$ – groups $H^1_{\mathrm{st}}(G_v, V)$ (resp. $H^1_g(G_v, V)$) satisfying

$$H^1_f(G_v, V) \subseteq H^1_{\mathrm{st}}(G_v, V) \subseteq H^1_g(G_v, V) \subseteq H^1(G_v, V).$$

We refer to [Fo 1] and [Fo 2] for more details on Fontaine's rings $B_{\mathrm{cris}} \subset B_{\mathrm{st}} \subset B_{\mathrm{dR}}$.

(2.3) Examples. (1) $n = 1$. In this case $V^1(1) = V_p(A) = T_p(A) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$, where $A = \mathrm{Pic}^0_{X/K}$ is the connected Picard variety of X. Descent theory for the abelian variety A gives maps into classical Selmer groups

$$A(K) \otimes \mathbb{Z}/p^n\mathbb{Z} \hookrightarrow \mathrm{Sel}(A/K, p^n) \subset H^1(G_K, A_{p^n}).$$

Passing to the projective limit and tensoring with $\mathbb{Q}_p$ one obtains the cycle class map for homologically trivial divisors

$$cl_{1,X} \otimes \mathbb{Q}_p : A(K) \otimes \mathbb{Q}_p \hookrightarrow H^1_f(G_K, V_p(A)) \subset H^1(G_K, V_p(A))$$

(as $A(K)$ is finitely generated by the Mordell-Weil theorem, and

$$\left(\varprojlim_n \mathrm{Sel}(A/K, p^n)\right) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = H^1_f(G_K, V_p(A))$$

by [BK, 3.11]). The conjectural equality 2.1.2

$$\mathrm{Im}\,(cl_{1,X} \otimes \mathbb{Q}_p) \stackrel{?}{=} H^1_f(G_K, V_p(A))$$

is equivalent to the finiteness of $\text{Ш}(A/K)_{p^\infty}$, the p-primary part of the Tate-Šafarevič group of A over K.
(2) $n = d = \dim(X)$ (assuming that X is equidimensional). The Albanese variety $\mathrm{Alb}_{X/K}$ can be identified with the dual $\widehat{A}$ of $A = \mathrm{Pic}^0_{X/K}$ and $V^{2d-1}(d)$ with $V_p(\widehat{A})$. The cycle class map $cl_{d,X} \otimes \mathbb{Q}_p$ is the composition of the (surjective) Albanese map

$$CH_0(X) \otimes \mathbb{Q}_p \twoheadrightarrow \widehat{A}(K) \otimes \mathbb{Q}_p$$

and the (injective) descent map $\widehat{A}(K) \otimes \mathbb{Q}_p \hookrightarrow H^1_f(G_K, V_p(\widehat{A}))$. The injectivity of the Albanese map is a very difficult open problem. The surjectivity of the descent map is, as before, equivalent to the finiteness of $\text{Ш}(A/K)_{p^\infty}$.

(2.4) As a basic check of consistency of the Conjecture 2.1.2 one would like to show that

$$\mathrm{Im}\,(cl_{1,X} \otimes \mathbb{Q}_p) \stackrel{?}{\subseteq} H^1_f(G_K, V^{2n-1}(n)) \tag{2.4.1}$$

For $n = 1$ or $n = \dim(X)$ this follows from the discussion in Examples 2.3. In general, (2.4.1) boils down to a local question at each non-archimedean prime v of K. Changing perspective, we fix v and consider a proper and smooth variety X_v over K_v. The question is whether

$$\mathrm{Im}\,(cl_{1,X_v} \otimes \mathbb{Q}_p) \stackrel{?}{\subseteq} H^1_f(G_v, V^{2n-1}(n)) \tag{2.4.2}$$

(now with $V^j = H^j((\overline{X}_v)_{\mathrm{et}}, \mathbb{Q}_p)$). At primes not dividing p the situation is more or less under control.

(2.5) Proposition. *Let $v \nmid p$. If X_v has potentially good reduction or, more generally, if the purity conjecture for the monodromy filtration ([Ja 2], [RZ]) on V^{2n-1} holds, then $H^1(G_v, V^{2n-1}(n)) = 0$.*

Proof. Tate's local Euler characteristic formula [Mi, Thm. I.2.8], [Se, II.5.7 Thm. 5] gives (for $V = V^{2n-1}(n)$)

$$\dim_{\mathbb{Q}_p} H^1(G_v, V) = \dim_{\mathbb{Q}_p} H^0(G_v, V) + \dim_{\mathbb{Q}_p} H^0(G_v, V^*(1)). \tag{2.5.1}$$

Our assumptions, together with Weil conjectures proved by Deligne [De 2], imply that the geometric Frobenius $Fr(v) \in G_v/I_v$ acts on both V^{I_v} and $(V^*(1))^{I_v}$ with weights < 0. It follows that the R.H.S. of (2.5.1) is equal to zero.

3. Varieties over p-adic fields

(3.0) In this section, $v|p$ (i.e. K_v is a finite extension of $\mathbb{Q}_p$) and X_v is proper and smooth over K_v. Our goal is to verify (2.4.2).

(3.1) Theorem. *(i) If X_v has potentially good reduction, then (2.4.2) holds.*
(ii) In general,

$$\mathrm{Im}\,(cl_{1,X_v} \otimes \mathbb{Q}_p) \subseteq H^1_{\mathrm{st}}(G_v, V^{2n-1}(n)) = H^1_g(G_v, V^{2n-1}(n)).$$

If the p-adic version of the purity conjecture for the monodromy filtration for V^{2n-1} holds [Mo, Conj. 3.27], then $H^1_{\mathrm{st}}(G_v, V^{2n-1}(n)) = H^1_f(G_v, V^{2n-1}(n))$ and (2.4.2) again holds.

(3.2) The purity conjecture for the monodromy filtration is known to hold (for all V^j and all v) for abelian varieties and for compactified Kuga-Sato varieties over modular curves [Sa]. The validity of (2.4.1) in the latter case implies that the main result of [Ne 1] holds for all p.

(3.3) We now explain an approach to the proof of Theorem 3.1 based on [Ne 3]. The part (i) was proved independently by Nizioł[Ni] (she assumed X_v to be smooth and projective, but the general case follows by the same argument as in 3.9 below). A special case of (ii) is due to A. Langer [La]. Let us distinguish two cases.

(3.4) Good reduction case. Assume that X_v has a model $\mathfrak{X}_v$, proper and smooth over $\mathcal{O}_v$ (the ring of integers of K_v). According to Fontaine's conjecture C_{cris}, which is now a theorem ([FM], [Fa], [KaM], [Ts]), all V^j are *crystalline* representations of G_v ([Fo 1], [Fo 2]). One defines

$$H^i_f(G_v, -) = \mathrm{Ext}^i_{\mathrm{Rep}_{\mathrm{cris}}(G_v)}(\mathbb{Q}_p, -),$$

where $\mathrm{Rep}_{\mathrm{cris}}(G_v)$ – a full subcategory of $\mathrm{Rep}(G_v)$ – is the category of crystalline representations of G_v (for $i = 1$ this is compatible with the definition given in 2.2). For an object V of $\mathrm{Rep}_{\mathrm{cris}}(G_v)$ one has

$$H^0_f(G_v, V) = H^0(G_v, V), \quad H^1_f(G_v, V) \subseteq H^1(G_v, V), \quad H^i_f(G_v, V) = 0 \quad (i > 1).$$

The key ingredient of the proof of Theorem 3.1.in the good reduction case is the *syntomic cohomology* $H^i((\mathfrak{X}_v)_{\mathrm{syn}}, s_{\mathbb{Q}_p}(n))$ of Fontaine-Messing [FM], which is a p-adic analogue of Deligne cohomology (more precisely, of Beilinson's "absolute Hodge cohomology" [Be]). Recall that there are two kinds of cohomology theories, geometric and arithmetic (= absolute), that come in pairs [Ja 3, 10.5]. For example, we have:

Geometric theory	Arithmetic theory
étale cohomology of $\overline{X}$	étale cohomology of X
Betti cohomology (over $\mathbb{C}$)	absolute Hodge cohomology
crystalline cohomology of $\mathfrak{X}_v$	syntomic cohomology

Given the limited scope of this article, we have to treat syntomic cohomology as a black box. One of its important features is the existence of a commutative diagram of cycle class maps

$$\begin{array}{rccc} cl^{\mathrm{syn}}_{\mathfrak{X}_v}: & CH^n(\mathfrak{X}_v) & \longrightarrow & H^{2n}((\mathfrak{X}_v)_{\mathrm{syn}}, s_{\mathbb{Q}_p}(n)) \\ & \Big\downarrow{\scriptstyle res} & & \Big\downarrow{\scriptstyle \alpha_{FM}} \\ cl^{\mathrm{et}}_{X_v}: & CH^n(X_v) & \longrightarrow & H^{2n}((X_v)_{\mathrm{et}}, \mathbb{Q}_p(n)), \end{array} \tag{3.4.1}$$

where α_{FM} is the (highly non-trivial) Fontaine-Messing map (for $p > 2$ it was constructed in [FM]; the case of $p = 2$ is treated in [Ts]).

One possible definition of the syntomic cycle class $cl^{\mathrm{syn}}_{\mathfrak{X}_v}$ uses the fact that $CH^n(\mathfrak{X}_v) \otimes \mathbb{Q}$ is canonically isomorphic to the weight n eigenspace of $K_0(\mathfrak{X}_v) \otimes \mathbb{Q}$ with respect to Adams operations ψ^k; the usual formula $cl^{\mathrm{syn}}_{\mathfrak{X}_v} = (-1)^{n-1} c^{\mathrm{syn}}_n/(n-1)!$ then makes sense. Similarly, one has $cl^{\mathrm{et}}_{X_v} = (-1)^{n-1} c^{\mathrm{et}}_n/(n-1)!$. The compatibility between the syntomic c^{syn}_n and étale c^{et}_n Chern classes of vector bundles with respect to the Fontaine-Messing map reduces, by the splitting principle, to the compatibility of the two c_1's of a line bundle, which can be verified explicitly. See [Ts] for more details, which are also valid in the logarithmic case discussed in 3.5 below.

(3.5) Semistable reduction case. Assume that X_v has a proper semistable model $\mathfrak{X}_v$ over $\mathcal{O}_v$ (i.e., étale locally, $\mathfrak{X}_v$ is given by $\mathrm{Spec}(\mathcal{O}_v[T_1, \ldots, T_n]/(T_1 \ldots T_m - \pi_v))$, where $m \leq n$ and π_v is a uniformizer of $\mathcal{O}_v$). In this case $\mathfrak{X}_v$ has a canonical log-structure $\mathfrak{X}^\times_v$ induced from X_v (see [HK] for a crash course on log-structures). According to Fontaine-Jannsen's conjecture C_{st}, proved in ([Ka], [Ts]), all V^j are *semistable* representations of G_v ([Fo 1], [Fo 2]). As in 3.4, one defines

$$H^i_{\mathrm{st}}(G_v, -) = \mathrm{Ext}^i_{\mathrm{Rep}_{\mathrm{st}}(G_v)}(\mathbb{Q}_p, -),$$

where $\mathrm{Rep}_{\mathrm{st}}(G_v)$ – a full subcategory of $\mathrm{Rep}(G_v)$ – is the category of semistable representations of G_v (again, for $i = 1$ this is compatible with the definition given in 2.2). For an object V of $\mathrm{Rep}_{\mathrm{st}}(G_v)$ one has [FPR, Chap. I, Prop. 3.3.8]

$$H^0_{\rm st}(G_v, V) = H^0(G_v, V), \quad H^1_{\rm st}(G_v, V) \subseteq H^1(G_v, V),$$
$$H^2_{\rm st}(G_v, V) \twoheadrightarrow H^2(G_v, V), \quad H^i_{\rm st}(G_v, V) = 0 \quad (i > 2).$$

More precisely, [FPR] gives an explicit complex computing cohomology $H^i_{\rm st}(G_v, V)$. The map

$$H^2_{\rm st}(G_v, V) \twoheadrightarrow H^2(G_v, V)$$

is surjective, being dual to the inclusion

$$H^0(G_v, V^*(1)) = D_{\rm st}(V^*(1))^{N=0,\varphi=1} \cap F^0 D_{\rm dR}(V^*(1)) \hookrightarrow D_{\rm st}(V^*(1))^{N=0,\varphi=1}.$$

Here $D_{\rm st}(V^*(1)) = H^0(G_v, V^*(1) \otimes_{\mathbb{Q}_p} B_{\rm st})$ denotes the filtered (φ, N)-module associated to $V^*(1)$ and $F^i D_{\rm dR}(V^*(1)) = H^0(G_v, V^*(1) \otimes_{\mathbb{Q}_p} F^i B_{\rm dR})$.

There is a logarithmic version of syntomic cohomology and of the diagram (3.4.1):

$$\begin{array}{ccc} CH^n(\mathfrak{X}_v) & \longrightarrow & H^{2n}((\mathfrak{X}_v^{\times})_{\rm syn}, s^{\log}_{\mathbb{Q}_p}(n)) \\ \Big\downarrow{\scriptstyle res} & & \Big\downarrow{\scriptstyle \alpha_{FM}} \\ CH^n(X_v) & \longrightarrow & H^{2n}((X_v)_{\rm et}, \mathbb{Q}_p(n)), \end{array} \tag{3.5.1}$$

Again, the log-syntomic cycle class map can be defined using Chern classes with values in log-syntomic cohomology [Ts].

(3.6) Theorem. *If X_v has good (resp. semistable) reduction and if $\mathfrak{X}_v$ is a proper and smooth (resp. proper and semistable) model of X_v over $\mathcal{O}_v$, then there is a canonical spectral sequence*

$${}^{\rm syn}E_\infty^{i,j} = {}^{\rm syn}E_2^{i,j} = H^i_f(G_v, V^j(n)) \Longrightarrow H^{i+j}((\mathfrak{X}_v)_{\rm syn}, s_{\mathbb{Q}_p}(n))$$

(resp.

$${}^{\log-{\rm syn}}E_2^{i,j} = H^i_{\rm st}(G_v, V^j(n)) \Longrightarrow H^{i+j}((\mathfrak{X}_v^{\times})_{\rm syn}, s^{\log}_{\mathbb{Q}_p}(n)))$$

compatible (via α_{FM}) with the Hochschild-Serre spectral sequence

$${}^{\rm et}E_2^{i,j} = H^i(G_v, V^j(n)) \Longrightarrow H^{i+j}((X_v)_{\rm et}, \mathbb{Q}_p(n)).$$

(3.7) See [Ne 3] for the proof in the good reduction case. The semistable reduction case will be treated in an updated version of [Ne 3]; a special case was proved by A. Langer [La]. In the good reduction case there is also a version with $\mathbb{Z}_p$-coefficients, valid for $K_v/\mathbb{Q}_p$ unramified and $p-1 > \max(i+j, n)$. These results have interesting applications to the study of the image of the cycle class map:

(3.8) Corollary. *If X_v has potentially good reduction, then*

$${\rm Im}\left(cl_{X_v} : CH^n(X_v) \longrightarrow H^{2n}((X_v)_{\rm et}, \mathbb{Q}_p(n))\right) \cap F^2 H^{2n}((X_v)_{\rm et}, \mathbb{Q}_p(n)) = 0.$$

(3.9) *Proof of Theorem 3.1.* (assuming Theorem 3.6). For every finite Galois extension K'_v/K_v and $* \in \{\emptyset, f, g, {\rm st}\}$ one has

$$H^1_*(\overline{K}_v/K_v, V) = H^1_*(\overline{K}_v/K'_v, V)^{{\rm Gal}(K'_v/K_v)}.$$

One is free, therefore, to replace K_v by K'_v and X_v by its base change to K'_v. This reduces (i) to the case of good reduction, in which case everything follows from the commutative diagram (3.4.1) and Theorem 3.6 (as the restriction map $CH^n(\mathfrak{X}_v) \longrightarrow CH^n(X_v)$ is surjective).

(ii) According to de Jong's theorem on alterations [deJ] there is – possibly after a finite extension of scalars K'_v/K_v – an alteration $\pi : X'_v \longrightarrow X_v$ such that X'_v has semistable reduction. The existence of trace maps in étale cohomology implies that the induced maps $\pi^* : V^j \longrightarrow V'^j$ are split injective. It follows that the statement

$$\mathrm{Im}\,(cl_{1,X_v} \otimes \mathbb{Q}_p) \overset{?}{\subseteq} H^1_{\mathrm{st}}(G_v, V^{2n-1}(n))$$

reduces to a similar statement for X'_v, in which case we conclude as before by the commutative diagram (3.5.1) and Theorem 3.6.
The quotient $H^1_{\mathrm{st}}(G_v, V)/H^1_f(G_v, V)$ (for $V = V^{2n-1}(n)$) is dual to $D_{\mathrm{st}}(V^*(1))^{N=0,\varphi=1}$ ([Ne 2, Cor. 1.18]). The p-adic version of the purity conjecture for the monodromy filtration for V^{2n-1} ([Mo, Conj. 3.27]; cf. [Il, Quest. 3.13.2], [Fo 3, Conj. 2.4.3-5]) implies that φ acts on $D_{\mathrm{st}}(V^*(1))^{N=0}$ with weights < 0; thus $D_{\mathrm{st}}(V^*(1))^{N=0,\varphi=1}$ vanishes. The equality $H^1_{\mathrm{st}} = H^1_g$ for potentially semistable representations is proved in [Hy] (see also [Ne 2, Prop. 1.24.3]). This concludes the proof of Theorem 3.1.

(3.10) Perhaps it is worth pointing out that in the good reduction case the weights of the action of φ on $D_{\mathrm{cris}}(V^j) = H^0(G_v, V^j \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}})$ are equal to j ("crystalline Weil conjectures"). For $\mathfrak{X}_v$ smooth and projective over $\mathrm{Spec}(\mathcal{O}_v)$ this was proved by Katz and Messing [KM], the general case by Chiarellotto and Le Stum [CLeS].

(3.11) If $Z \subset X_v$ is a homologically trivial cycle of codimension n, the statement

$$cl_{1,X_v}(Z) \overset{?}{\in} H^1_{\mathrm{st}}(G_v, V^{2n-1}(n))$$

would follow if one could prove directly that $H^{2n-1}((\overline{X_v - |Z|})_{\mathrm{et}}, \mathbb{Q}_p)$ is a semistable representation of $\mathrm{Gal}(\overline{K}_v/K'_v)$ for some finite extension K'_v/K_v (i.e. it is potentially semistable). This is because of an alternative description of $cl_{1,X_v}(Z)$ in terms of the extension class of Galois representations arising from the exact cohomology sequence

$$0 \longrightarrow H^{2n-1}(\overline{X}_v, \mathbb{Q}_p)(n) \longrightarrow H^{2n-1}((\overline{X_v - |Z|})_{\mathrm{et}}, \mathbb{Q}_p)(n) \longrightarrow H^{2n}_{|\overline{Z}|}(\overline{X}_v, \mathbb{Q}_p)(n) \longrightarrow \cdots$$

(see [Ja 3, Lemma 9.4]).

4. p-adic height pairings

(4.0) As in Sect. 2, let X be a proper and smooth variety over a number field K, equidimensional of dimension $\dim(X) = d$. Fix integers $i, j \geq 1$ satisfying $i+j = d+1$. In the notation of 1.1, put $V = V^{2i-1}(i)$. Poincaré duality

$$V^{2i-1}(i) \times V^{2j-1}(j) \longrightarrow V^{2d}(d+1) \xrightarrow{Tr} \mathbb{Q}_p(1)$$

then identifies $V^{2j-1}(j)$ with $\mathrm{Hom}_{\mathbb{Q}_p}(V, \mathbb{Q}_p(1)) = V^*(1)$.

(4.1) We assume that the following assumption is satisfied:
(A) At each non-archimedean prime v of K where X has bad reduction, the purity conjecture for the monodromy filtration for V^{2i-1} (or, equivalently, for V^{2j-1}) holds.

It follows from Theorem 3.1 that under the assumption (A) there are p-adic Abel-Jacobi maps

$$CH^i(X)_{\mathrm{hom}\sim 0} \longrightarrow H^1_f(G_K, V)$$
$$CH^j(X)_{\mathrm{hom}\sim 0} \longrightarrow H^1_f(G_K, V^*(1))$$

(4.2) Theorem. *([Ne 2]) Under the assumption (A) there are height pairings*

$$\langle\ ,\ \rangle : H^1_f(G_K, V) \times H^1_f(G_K, V^*(1)) \longrightarrow \mathbb{Q}_p$$

depending on the following additional data:
(a) A continuous homomorphism $\ell : \mathbb{A}_K^*/K^* \longrightarrow \mathbb{Q}_p$.
(b) $\mathbb{Q}_p$*-linear splittings of Hodge filtrations*

$$F^i H^{2i-1}_{\mathrm{dR}}(X_v/K_v) \overset{\longleftarrow}{\longrightarrow} H^{2i-1}_{\mathrm{dR}}(X_v/K_v)$$

for all primes $v|p$ *(where* $X_v = X \times_K K_v$*).*

(4.3) Corollary. *There are p-adic height pairings*

$$CH^i(X)_{\mathrm{hom}\sim 0} \times CH^j(X)_{\mathrm{hom}\sim 0} \longrightarrow \mathbb{Q}_p$$

depending on the data (a), (b) and factoring through p-adic Abel-Jacobi maps.

(4.4) Remarks. (1) The assumptions of [Ne 2, 2.1.2, 7.2], under which the height pairings are constructed, follow from (A), possibly after a finite extension of the field K (cf. the discussion in 3.9-10 above).
(2) It is still an open problem whether the real-valued height pairing factors through the complex-analytic Abel-Jacobi map (if $\min(i,j) > 1$).
(3) Sometimes the assumption (A) can be verified only for a suitable direct factor $W \subset V$ coming from a "submotive" of $h^{2i-1}(X)(i)$. In that case one gets only pairings

$$H^1_f(G_K, W) \times H^1_f(G_K, W^*(1)) \longrightarrow \mathbb{Q}_p$$

depending on a similar set of data.
(4) By class field theory, ℓ can be viewed as a $\mathbb{Z}_p$-linear map $\mathrm{Gal}(K_\infty/K) \longrightarrow \mathbb{Q}_p$, where K_∞ is the compositum of all $\mathbb{Z}_p$-extensions of K inside $\overline{K}$. The dependence of $\langle\ ,\ \rangle$ on ℓ is linear, which means that once the choices of splittings in (b) are fixed one obtains a "universal" height pairing with values in the finite-dimensional $\mathbb{Q}_p$-vector space $\mathrm{Gal}(K_\infty/K) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$.
(5) The dependence on ℓ can be seen already in the following simple example: let $X = \mathbb{P}^1_K$, $x \in K^* \setminus \{1\}$, $Z = (0) - (\infty)$, $Z' = (1) - (x)$. Consider first the real valued height: as $CH^1(X)_{\mathrm{hom}\sim 0} = 0$, the global height $\langle Z, Z'\rangle = 0 \in \mathbb{R}$. However, the local heights $\langle Z, Z'\rangle_v = \log |x|_v$ are non-zero (unles x is a root of unity) and

$$\langle Z, Z'\rangle = \sum_v \langle Z, Z'\rangle_v = \log\left(\prod_v |x|_v\right) = 0$$

vanishes by the product formula. In the p-adic situation, ℓ defines continuous homomorphisms $\ell_v : K_v^* \longrightarrow \mathbb{Q}_p$ for all primes v, and the local p-adic heights are equal to $\langle Z, Z'\rangle_v = \ell_v(x)$. The global height

$$\langle Z, Z'\rangle = \sum_v \langle Z, Z'\rangle_v = \sum_v \ell_v(x) = 0$$

vanishes, as $\ell|_{K^*} = 0$.
(6) In certain cases there are canonical splittings of the Hodge filtrations in (b), which then define "canonical p-adic height pairings". This happens whenever there is, for each prime $v|p$, an exact sequence of representations of G_v

$$0 \longrightarrow V_v^+ \longrightarrow V \overset{\pi}{\longrightarrow} V_v^- \longrightarrow 0$$

with the property that π induces an isomorphism $F^0 D_{\mathrm{dR}}(V) \overset{\sim}{\longrightarrow} D_{\mathrm{dR}}(V_v^-)$. For example, for $i = 1$ this happens iff $A = \mathrm{Pic}^0_{X/K}$ has ordinary reduction at all $v|p$ (see [Ne 2] for more details).

5. p-adic height pairings in families

(5.0) The p-adic heights discussed in Sect. 4 appear in the p-adic version of the conjecture of Birch and Swinnerton-Dyer (and its higher-dimensional analogue). In the formulation of [PR 1], [PR 2] *both sides of the* *conjecture* depend on the data (a), (b) from Theorem 4.2. Traditionally, p-adic L-functions have characters of $\mathrm{Gal}(K_\infty/K)$ as arguments. The homomorphism ℓ then defines a direction in which derivatives of the p-adic L-function are taken. Perrin-Riou's p-adic L-function depends on an extra parameter, of which the splittings (b) are a special case.

(5.1) Apart from Perrin-Riou's new parameter, p-adic L-functions tend to depend on additional variables. For example, Hida's theory of ordinary families of p-adic modular forms [Hi] produces two-variable p-adic L-functions, with weight as the second variable [GS], [Ki].

On the algebraic side, the theory gives p-adic families of representations of G_K of the following type (for $K = \mathbb{Q}$):

(1) R is a complete local ring, finite and flat over $\Lambda = \mathcal{O}_F[[X]]$, where $\mathcal{O}_F$ is the ring of integers in a finite extension F of $\mathbb{Q}_p$.

(2) T is an R-module of finite type, flat over Λ, with an R-linear continuous action of G_K. The action is unramified outside a finite set S of primes of K. One can assume that S contains all primes dividing p and all archimedean primes of K.

(3) For every prime $v|p$ in K there is an exact sequence of $R[G_v]$-modules

$$0 \longrightarrow T_v^+ \longrightarrow T \longrightarrow T_v^- \longrightarrow 0,$$

with $T_v^\pm$ flat over Λ.

(5.2) In concrete terms, T constructed by Hida interpolates Galois representations associated to modular forms as follows: for every even integer $k \geq 2$ in a fixed residue class modulo $2(p-1)$ and a prime ideal $\mathfrak{p}_k \subset R$ above $(X + 1 - (1+p)^{k-2}) \subset \Lambda$, $V_k = (T/\mathfrak{p}_k T) \otimes_{\mathcal{O}_F} F$ is a two-dimensional representation of $G_\mathbb{Q}$ over F, associated to - up to a Tate twist - a Hecke eigenform of weight k on $\Gamma_1(Np)$ (for $p \nmid 6N$). Our normalization of the Tate twist (cf. [Gr]) is such that V_k is a direct factor of $H^{k-1}(\overline{X}_{\mathrm{et}}, \mathbb{Q}_p(k/2))$, where X is a compactified Kuga-Sato variety of dimension $k-1$, fibered over the modular curve $X_1(Np)$. In this case

$$\wedge_F^2 V_k \xrightarrow{\sim} F(1), \qquad \mathrm{Hom}_F(V_k, F(1)) \xrightarrow{\sim} V_k. \tag{5.2.1}$$

The ring R is a local component of the ordinary part of the projective limit of Hecke algebras (with $\mathbb{Z}_p$-coefficients) acting on the Jacobians $J_1(Np^r)$ of modular curves $X_1(Np^r)$. Up to a twist, T is the Pontrjagin dual of the R-part of

$$\varinjlim_r J_1(Np^r)_{p^\infty}(\overline{\mathbb{Q}})^{\mathrm{ord}},$$

where the superscript "ord" refers to the subgroup on which the Hecke operator $T(p)$ is not topologically nilpotent [Hi].

(5.3) In the abstract setup of 5.1.1-3, put $\omega_R = \mathrm{Hom}_\Lambda(R, \Lambda)$ and

$$T^* = \mathrm{Hom}_\Lambda(T, \Lambda) = \mathrm{Hom}_R(T, \omega_R), \qquad (T^*)_v^\pm = \mathrm{Hom}_\Lambda(T_v^\mp, \Lambda) = \mathrm{Hom}_R(T_v^\mp, \omega_R).$$

In the situation of 5.2 there is a canonical skew-symmetric isomorphism $T^*(1) \xrightarrow{\sim} T$, which interpolates the isomorphisms (5.2.1).

Plater [Pl] defined Selmer groups

$$H_f^1(G_K, T)^0 \subset H_f^1(G_K, T) \subset H^1(G_K, T)$$

and - under certain assumptions - height pairings

$$H_f^1(G_K, T)^0 \times H_f^1(G_K, T^*(1))^0 \longrightarrow R \tag{5.3.1}$$

that interpolate the height pairings

$$H_f^1(G_K, V_k) \times H_f^1(G_K, V_k^*(1)) \longrightarrow F$$

of Theorem 4.2.

(5.4) Selmer complexes It seems that the correct setup for duality theorems and p-adic height pairings involves *Selmer complexes* [Ne 4] rather than Selmer groups. The idea is that one should consider local conditions for complexes rather than for cohomology groups, as in 2.2. In the situation of 5.1, let K_S be the maximal extension of K (inside $\overline{K}$) unramified outside S and $G_{K,S} = \mathrm{Gal}(K_S/K)$. Denote by $\mathcal{C}^\bullet_{\mathrm{cont}}(G_{K,S}, T)$ the complex of non-homogeneous continuous cochains (with respect to the pro-finite topologies) on $G_{K,S}$ with values in T and put, for every non-archimedean prime $v \in S$,

$$U_v^+(T) = \begin{cases} \mathcal{C}^\bullet_{\mathrm{cont}}(G_v/I_v, T^{I_v}), & \text{if } v \not| p \\ \mathcal{C}^\bullet_{\mathrm{cont}}(G_v, T_v^+), & \text{if } v | p \end{cases}$$

The *Selmer complex* $\mathbb{R}\widetilde{\Gamma}_f(G_{K,S}, T)$ is defined as the simple complex associated to

$$\begin{pmatrix} \mathcal{C}^\bullet_{\mathrm{cont}}(G_{K,S}, T) & \longrightarrow & \bigoplus_{v \in S_f} \mathcal{C}^\bullet_{\mathrm{cont}}(G_v, T) \\ & & \uparrow \\ & & \bigoplus_{v \in S_f} U_v^+(T) \end{pmatrix}$$

(where $S_f \subset S$ is the subset of non-archimedean primes of S). Its cohomology sits in an exact sequence

$$\cdots \longrightarrow \widetilde{H}^i_f(G_{K,S}, T) \longrightarrow H^i(G_{K,S}, T) \oplus \bigoplus_{v \in S_f} H^i(U_v^+(T)) \longrightarrow$$
$$\longrightarrow \bigoplus_{v \in S_f} H^i(G_v, T) \longrightarrow \widetilde{H}^{i+1}_f(G_{K,S}, T) \longrightarrow \cdots$$

The appropriate generalization of (5.3.1) is given by the following

(5.5) Theorem. *[Ne 4] In the situation of 5.1.1-3, assume that either $p > 2$ or K is totally imaginary. Then, for every continuous homomorphism $\ell : \mathrm{Gal}(K_\infty/K) \longrightarrow R$, there is a "height pairing"*

$$\mathbb{R}\widetilde{\Gamma}_f(G_{K,S}, T) \overset{\mathbb{L}}{\otimes}_R \mathbb{R}\widetilde{\Gamma}_f(G_{K,S}, T^*(1)) \longrightarrow \omega_R[-2],$$

which depends R-linearly on ℓ.

(5.6) Selmer complexes $\mathbb{R}\widetilde{\Gamma}_f(G_{K,S}, T)$ are better than classical Selmer groups even for elliptic curves (when R and Λ are replaced by $\mathbb{Z}_p$). For example, if $K = \mathbb{Q}$ and $T = T_p(E)$ is the Tate module of an elliptic curve E over $\mathbb{Q}$ with *split multiplicative* reduction at p, then $T_p^+ = \mathbb{Z}_p(1)$ and $T_p^- = \mathbb{Z}_p$. The group $H^0(G_p, T_p^-) = \mathbb{Z}_p$ then contributes to the kernel of $\widetilde{H}^1_f(G_{K,S}, T) \longrightarrow H^1(G_{K,S}, T)$; there is an exact sequence

$$0 \longrightarrow \mathbb{Q}_p \longrightarrow \widetilde{H}^1_f(G_{K,S}, T_p(E)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \longrightarrow H^1(G_K, V_p(E)) \longrightarrow 0.$$

This is consistent with the phenomenon discovered by Mazur, Tate and Teitelbaum [MTT]: the term $H^0(G_p, T_p^-)$ is responsible for a trivial zero of the p-adic L-function and

$$\widetilde{H}^1_f(G_{K,S}, V_p(E)) = \widetilde{H}^1_f(G_{K,S}, T_p(E)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$$

is the corresponding "extended Selmer group". A variant of Theorem 5.5 then defines a height pairing

$$\widetilde{H}^1_f(G_{K,S}, T_p(E)) \times \widetilde{H}^1_f(G_{K,S}, T_p(E)) \longrightarrow \mathbb{Z}_p.$$

(5.7) Coleman [Co] constructed p-adic analytic families of non-ordinary modular forms. It would be interesting to extend previous constructions to this case.

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