Hidden symmetries in the theory of complex multiplication

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To Yuri Manin on the occasion of his 70th birthday, with admiration

0. Introduction

(0.1) Let F be a totally real number field of degree d. It is well known that one can associate to any cuspidal Hilbert eigenform f over F of parallel weight 2 a compatible system of two-dimensional l-adic Galois representations $V_l(f)$ of $\Gamma_F = \mathrm{Gal}(\overline{\mathbf{Q}}/F)$ over $\overline{\mathbf{Q}}_l$ (having fixed embeddings $\overline{\mathbf{Q}} \hookrightarrow \mathbf{C}$ and $\overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_l$).

(0.2) On the other hand, the Shimura variety X associated to $R_{F/\mathbf{Q}}GL(2)_F$ has reflex field $\mathbf{Q}$, which means that its étale cohomology groups give rise to l-adic representations of $\Gamma_{\mathbf{Q}} = \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$. The action of $\Gamma_{\mathbf{Q}}$ on the intersection cohomology of the Baily-Borel compactification X^* of X was determined, up to semi-simplification, by Brylinski and Labesse [Br-La]: non-primitive cohomology (into which we include IH^0) occurs in even degrees and decomposes as

$$IH^{2j}_{et}(X^* \otimes_{\mathbf{Q}} \overline{\mathbf{Q}}, \overline{\mathbf{Q}}_l)_{\mathrm{non-prim}} \xrightarrow{\sim} \bigoplus_{\chi} \chi(-j),$$

where each χ is a finite-order character of $\Gamma_{\mathbf{Q}}$. Primitive cohomology occurs only in degree d; it decomposes as

$$IH^{d}_{et}(X^* \otimes_{\mathbf{Q}} \overline{\mathbf{Q}}, \overline{\mathbf{Q}}_l)_{\mathrm{prim}} \xrightarrow{\sim} \bigoplus_{f} \pi(f) \otimes W_l(f),$$

where f is as above, $\pi(f)$ is the (non-archimedean part of the) automorphic representation of $GL(2, \mathbf{A}_F)$ associated to f, and $W_l(f)$ is a 2^d-dimensional l-adic representation of $\Gamma_{\mathbf{Q}}$ whose semi-simplification $W_l(f)^{\mathrm{ss}}$ is isomorphic to the tensor induction of $V_l(f)$,

$$\bigotimes \mathrm{Ind}_{F/\mathbf{Q}} V_l(f),$$

which is defined as follows. A choice of coset representatives

$$\Gamma_{\mathbf{Q}} = \coprod_{i=1}^{d} g_i \Gamma_F \tag{0.2.1}$$

defines an injective group homomorphism (see §1.1 below)

$$\Gamma_{\mathbf{Q}} \hookrightarrow S_d \ltimes \Gamma_F^d, \qquad g \mapsto (\sigma, (h_1, \ldots, h_d)), \qquad g g_i = g_{\sigma(i)} h_i, \tag{0.2.2}$$

and $\bigotimes \mathrm{Ind}_{F/\mathbf{Q}} V_l(f)$ is obtained from the $(S_d \ltimes \Gamma_F^d)$-module $V_l(f)^{\otimes d}$ by pull-back via the map (0.2.2).

(0.3) In particular, the action of $\Gamma_{\mathbf{Q}}$ on $IH^d_{et}(X^* \otimes_{\mathbf{Q}} \overline{\mathbf{Q}}, \overline{\mathbf{Q}}_l)^{\mathrm{ss}}_{\mathrm{prim}}$ extends to an action of $S_d \ltimes \Gamma_F^d$. The same should be true for the action on $IH^d_{et}(X^* \otimes_{\mathbf{Q}} \overline{\mathbf{Q}}, \overline{\mathbf{Q}}_l)_{\mathrm{prim}}$, since general conjectures predict that $\Gamma_{\mathbf{Q}}$ should act semi-simply on $IH^*_{et}(Y \otimes_{\mathbf{Q}} \overline{\mathbf{Q}}, \overline{\mathbf{Q}}_l)$, for any proper scheme Y over $\mathrm{Spec}(\mathbf{Q})$.

The representations $\chi(-j)$ of $\Gamma_{\mathbf{Q}}$ occurring in the non-primitive cohomology of X^* do not extend to representations of $S_d \ltimes \Gamma_F^d$, but they extend to representations of the group $(S_d \ltimes \Gamma_F^d)_0$, which is defined as the fibre product

$$\begin{array}{ccc} (S_d \ltimes \Gamma_F^d)_0 & \longrightarrow & S_d \ltimes \Gamma_F^d \\ \downarrow & & \downarrow \\ \Gamma_{\mathbf{Q}}^{ab} & \xrightarrow{V_{F/\mathbf{Q}}} & \Gamma_F^{ab}, \end{array} \tag{0.3.1}$$

where the right vertical arrow is trivial on S_d and is given by the product map on Γ_F^d. As the field F is totally real, the transfer map $V_{F/\mathbf{Q}}$ is injective (see 1.2.5 below), which means that we can (and will) consider $(S_d \ltimes \Gamma_F^d)_0$ as a subgroup of $S_d \ltimes \Gamma_F^d$. The inclusion (0.2.2) factors through an inclusion $\Gamma_{\mathbf{Q}} \hookrightarrow (S_d \ltimes \Gamma_F^d)_0$.

(0.4) Question. *To sum up: the results of [Br-La] combined with the semi-simplicity conjecture imply that the action of* $\Gamma_{\mathbf{Q}}$ *on* $IH^*_{et}(X^* \otimes_{\mathbf{Q}} \overline{\mathbf{Q}}, \overline{\mathbf{Q}}_l)$ *should extend to an action of* $(S_d \ltimes \Gamma_F^d)_0$*. Is there a geometric explanation of this hidden symmetry of* $IH^*_{et}(X^* \otimes_{\mathbf{Q}} \overline{\mathbf{Q}}, \overline{\mathbf{Q}}_l)$*?*

(0.5) This question admits a more invariant formulation. Recall that the inclusion (0.2.2) depends on the choice of coset representatives (0.2.1). The same choice defines an isomorphism of F-algebras

$$F \otimes_{\mathbf{Q}} \overline{\mathbf{Q}} \xrightarrow{\sim} \overline{F}^d, \qquad a \otimes b \mapsto (a \otimes g_i^{-1}(b))_i,$$

hence a group isomorphism

$$S_d \ltimes \Gamma_F^d \xrightarrow{\sim} \mathrm{Aut}_{F-alg}(F \otimes_{\mathbf{Q}} \overline{\mathbf{Q}}), \tag{0.5.1}$$

the composition of which with (0.2.2) coincides with the canonical injective map

$$\Gamma_{\mathbf{Q}} = \mathrm{Aut}_{\mathbf{Q}-alg}(\overline{\mathbf{Q}}) \hookrightarrow \mathrm{Aut}_{F-alg}(F \otimes_{\mathbf{Q}} \overline{\mathbf{Q}}), \qquad g \mapsto \mathrm{id}_F \otimes g. \tag{0.5.2}$$

The subgroup $\mathrm{Aut}_{F-alg}(F \otimes_{\mathbf{Q}} \overline{\mathbf{Q}})_0$ of $\mathrm{Aut}_{F-alg}(F \otimes_{\mathbf{Q}} \overline{\mathbf{Q}})$ corresponding to $(S_d \ltimes \Gamma_F^d)_0$ under the isomorphism (0.5.1) is independent of any choices, which means that we should restate Question 0.4 as follows.

(0.6) Question. *Is there a geometric explanation of the fact that the action of* $\Gamma_{\mathbf{Q}}$ *on* $IH^*_{et}(X^* \otimes_{\mathbf{Q}} \overline{\mathbf{Q}}, \overline{\mathbf{Q}}_l)$ *extends to an action of* $\mathrm{Aut}_{F-alg}(F \otimes_{\mathbf{Q}} \overline{\mathbf{Q}})_0$*? For example, does* $X^* \otimes_{\mathbf{Q}} \overline{\mathbf{Q}}$ *(or a related space) admit an action of* $\mathrm{Aut}_{F-alg}(F \otimes_{\mathbf{Q}} \overline{\mathbf{Q}})_0$*?*

(0.7) Idle speculation. The recipe (0.2.2) defines an inclusion

$$G \hookrightarrow S_d \ltimes H^d \tag{0.7.1}$$

(depending on chosen coset representatives of H in G) whenever H is a subgroup of index d of a group G.

If $p : Y \longrightarrow Z$ is an unramified covering of degree d between "nice" connected topological spaces and $H = \pi_1(Y, y)$, $G = \pi_1(Z, p(y))$, then there are at least two geometric incarnations of (0.7.1).

Firstly, if $\widetilde{Z}$ is the universal covering of Z, then

$$G \xrightarrow{\sim} \mathrm{Aut}(\widetilde{Z}/Z), \qquad S_d \ltimes H^d \xrightarrow{\sim} \mathrm{Aut}(Y \times_Z \widetilde{Z}/Y)$$

and (0.7.1) comes from the canonical map

$$\mathrm{Aut}(\widetilde{Z}/Z) \longrightarrow \mathrm{Aut}(Y \times_Z \widetilde{Z}/Y), \qquad g \mapsto \mathrm{id}_Y \times g. \tag{0.7.2}$$

In our situation, the rôle of p (resp., by $\widetilde{Z}$) is played by the structure map $\mathrm{Spec}(F) \longrightarrow \mathrm{Spec}(\mathbf{Q})$ (resp., by $\mathrm{Spec}(\overline{\mathbf{Q}})$), and (0.7.2) is nothing but (0.5.2).

Secondly, $S_d \ltimes H^d$ is closely related to $\pi_1(Y^d/S_d, p^{-1}(p(y)))$, and there is a canonical map

$$Z \longrightarrow Y^d/S_d, \qquad z \mapsto p^{-1}(z). \tag{0.7.3}$$

In other words, the map induced by (0.7.3),

$$\pi_1(Z, z) \longrightarrow \pi_1(Y^d/S_d, p^{-1}(z)),$$

is an approximate version of (0.7.1).

In our situation, in which the rôle of Y (resp., of Z) is played by $\mathrm{Spec}(F)$ (resp., by $\mathrm{Spec}(\mathbf{Q})$), we are confronted with the fact that the analogue of Y^d/S_d should be the d-th symmetric power of $\mathrm{Spec}(F)$ over the elusive absolute point $\mathrm{Spec}(\mathbf{F}_1)$. A Grothendieckean approach to Question 0.6 would then involve

- making sense of the d-th symmetric power $\mathrm{Sym}^d(F/\mathbf{F}_1)$ of $\mathrm{Spec}(F)$ over $\mathrm{Spec}(\mathbf{F}_1)$;
- extending X^* to an object $\widetilde{X}^*$ defined over (a desingularisation of) $\mathrm{Sym}^d(F/\mathbf{F}_1)$;

- relating l-adic intersection cohomology groups[1] of X^* and $\widetilde{X}^*$.

At present, this seems beyond reach, but as A. Genestier pointed out to us, everything makes sense for Drinfeld modular varieties over global fields of positive characteristic.

(0.8) Leaving speculations aside, in the present article we test Question 0.6 by studying the action of $\Gamma_{\mathbf{Q}}$ on the set of CM points. It is convenient to replace $R_{F/\mathbf{Q}}GL(2)_F$ by the group G defined as the fibre product

$$\begin{array}{ccc} G & \longrightarrow & R_{F/\mathbf{Q}}(GL(2)_F) \\ \big\downarrow & & \big\downarrow \mathrm{det} \\ \mathbf{G}_{m,\mathbf{Q}} & \longrightarrow & R_{F/\mathbf{Q}}(\mathbf{G}_{m,F}), \end{array}$$

since the corresponding Shimura variety is a moduli space for polarised Hilbert-Blumenthal abelian varieties (HBAV) equipped with adelic level structures.

The first main result of the present article (see 2.2.5 below) is the following.

(0.9) Theorem. *The group $\mathrm{Aut}_{F-alg}(F \otimes_{\mathbf{Q}} \overline{\mathbf{Q}})_0$ acts naturally on the set of CM points of the Shimura variety $Sh(G, \mathscr{X})$ associated to G. This action extends the natural action of $\Gamma_{\mathbf{Q}}$ and commutes with the action of $G(\mathbf{A}_f) = G(\widehat{\mathbf{Q}})$ on $Sh(G, \mathscr{X})$.*

The key point in the proof is to show that the (reverse) 1-cocycle $f_\Phi : \Gamma_{\mathbf{Q}} \longrightarrow \widehat{K}^*/K^*$ ("the Taniyama element"), which describes the Galois action on the set of CM points by K, naturally extends to a 1-cocycle $\widetilde{f}_\Phi : \mathrm{Aut}_{F-alg}(F \otimes_{\mathbf{Q}} \overline{\mathbf{Q}})_0 \longrightarrow \widehat{K}^*/K^*$ (above, K is a totally imaginary quadratic extension of F, $\widehat{K}$ is the ring of finite adèles of K and Φ is a CM type of K). In fact, f_Φ extends to a 1-cocycle $\widetilde{f}_\Phi$ defined on a slightly bigger subgroup $\mathrm{Aut}_{F-alg}(F \otimes_{\mathbf{Q}} \overline{\mathbf{Q}})_1$ of $\mathrm{Aut}_{F-alg}(F \otimes_{\mathbf{Q}} \overline{\mathbf{Q}})$, which corresponds to the fibre product

$$\begin{array}{ccc} (S_d \ltimes \Gamma_F^d)_1 & \longrightarrow & S_d \ltimes \Gamma_F^d \\ \big\downarrow & & \big\downarrow (1,\mathrm{prod}) \\ \Gamma_{\mathbf{Q}}^{ab}/\langle c\rangle & \overset{V_{F/\mathbf{Q}}}{\hookrightarrow} & \Gamma_F^{ab}/\langle c_1, \ldots, c_d\rangle, \end{array}$$

where $c \in \Gamma_{\mathbf{Q}}^{ab}$ (resp., $c_1, \ldots, c_d \in \Gamma_F^{ab}$) is the complex conjugation (resp., are the complex conjugations at the infinite primes of F). We have

$$\mathrm{Aut}_{F-alg}(F \otimes_{\mathbf{Q}} \overline{\mathbf{Q}})_1/\mathrm{Aut}_{F-alg}(F \otimes_{\mathbf{Q}} \overline{\mathbf{Q}})_0 \xrightarrow{\sim} (\mathbf{Z}/2\mathbf{Z})^{d-1},$$

but only the elements of $\mathrm{Aut}_{F-alg}(F \otimes_{\mathbf{Q}} \overline{\mathbf{Q}})_0$ preserve the positivity of the polarisations.

(0.10) A more abstract formulation of this result (§2.4) involves a generalisation of the Taniyama group $\mathscr{T}$ and its finite-level quotients ${}_K\mathscr{T}$. More precisely, in the special case when K is a Galois extension of $\mathbf{Q}$, the maps $\widetilde{f}_\Phi$ factor through $\mathrm{Aut}_{F-alg}(F \otimes_{\mathbf{Q}} K^{ab})_1 = \mathrm{Im}\left(\mathrm{Aut}_{F-alg}(F \otimes_{\mathbf{Q}} \overline{\mathbf{Q}})_1 \longrightarrow \mathrm{Aut}_{F-alg}(F \otimes_{\mathbf{Q}} K^{ab})\right)$ and can be put together, yielding a 1-cocycle

$$\widetilde{f} : \mathrm{Aut}_{F-alg}(F \otimes_{\mathbf{Q}} K^{ab})_1 \longrightarrow {}_K\mathscr{S}(\widehat{K})/{}_K\mathscr{S}(K), \tag{0.10.1}$$

where ${}_K\mathscr{S}$ is the Serre torus associated to K (see §1.5).

Our second main result (see 2.4.2-3 below) states that the coboundary of $\widetilde{f}$ gives rise to an exact sequence of affine group schemes over $\mathbf{Q}$

$$1 \longrightarrow {}_K\mathscr{S} \xrightarrow{\widetilde{\imath}} {}_K\widetilde{\mathscr{T}} \xrightarrow{\widetilde{\pi}} \mathrm{Aut}_{F-\mathrm{alg}}(F \otimes_{\mathbf{Q}} K^{ab})'_1 \longrightarrow 1, \tag{0.10.2}$$

where $\mathrm{Aut}_{F-\mathrm{alg}}(F \otimes_{\mathbf{Q}} K^{ab})'_1$ is a certain $F/\mathbf{Q}$-form of the constant group scheme $\mathrm{Aut}_{F-\mathrm{alg}}(F \otimes_{\mathbf{Q}} K^{ab})_1$. Moreover, there is a group homomorphism $\widetilde{sp} : \mathrm{Aut}_{F-\mathrm{alg}}(F \otimes_{\mathbf{Q}} K^{ab})_1 \longrightarrow {}_K\widetilde{\mathscr{T}}(\widehat{F})$ satisfying $\widetilde{\pi} \circ \widetilde{sp} = \mathrm{id}$. The pull-back of (0.10.2) to $\mathrm{Aut}_{\mathbf{Q}-\mathrm{alg}}(K^{ab}) = \mathrm{Gal}(K^{ab}/\mathbf{Q})$ is the level-K Taniyama extension

[1] Establishing a relation between de Rham cohomology of X^* and $\widetilde{X}^*$ would also be of interest, in view of potential applications to period relations for Hilbert modular forms.

$$1 \longrightarrow {}_K\mathscr{S} \xrightarrow{i} {}_K\mathscr{T} \xrightarrow{\pi} \mathrm{Gal}(K^{ab}/\mathbf{Q}) \longrightarrow 1.$$

For varying K, the 1-cocycles $\widetilde{f}$ are compatible. When put together, they give rise to an exact sequence of affine group schemes over $\mathbf{Q}$

$$1 \longrightarrow \mathscr{S} \longrightarrow \widetilde{\mathscr{T}} \longrightarrow \varinjlim_F \mathrm{Aut}_{F-alg}(F \otimes_{\mathbf{Q}} \overline{\mathbf{Q}})'_1 \longrightarrow 1 \tag{0.10.3}$$

(where $\mathscr{S}$ is the inverse limit of the tori ${}_K\mathscr{S}$ with respect to the norm maps, and the direct limit is taken with respect to the transition maps $\mathrm{id}_{F'} \otimes_F -$, for $F \subseteq F'$), whose pull-back to $\Gamma_{\mathbf{Q}}$ coincides with the Taniyama extension

$$1 \longrightarrow \mathscr{S} \longrightarrow \mathscr{T} \longrightarrow \Gamma_{\mathbf{Q}} \longrightarrow 1.$$

(0.11) Question. *As shown in [De], the Taniyama group $\mathscr{T}$ has a natural Tannakian interpretation. Does $\widetilde{\mathscr{T}}$, or its subgroup scheme $\widetilde{\mathscr{T}}_0 \subset \widetilde{\mathscr{T}}$ sitting in the exact sequence*

$$1 \longrightarrow \mathscr{S} \longrightarrow \widetilde{\mathscr{T}}_0 \longrightarrow \varinjlim_F \mathrm{Aut}_{F-\mathrm{alg}}(F \otimes \overline{\mathbf{Q}})'_0 \longrightarrow 1,$$

have a similar interpretation?

(0.12) If A is a polarised HBAV over $\overline{\mathbf{Q}}$, then $H^1_{dR}(A/\overline{\mathbf{Q}})$ is a free $F \otimes_{\mathbf{Q}} \overline{\mathbf{Q}}$-module of rank 2, and for each prime p the $F \otimes_{\mathbf{Q}} \overline{\mathbf{Q}} \otimes_{\mathbf{Q}} \mathbf{Q}_p$-module $H^1_{dR}(A/\overline{\mathbf{Q}}) \otimes_{\mathbf{Q}} \mathbf{Q}_p$ has an additional crystalline structure. The comparison theorems between étale and crystalline cohomology together with Faltings's isogeny theorem imply that the F-linear isogeny class of A is determined by $H^1_{dR}(A/\overline{\mathbf{Q}})$ with this additional structure. It is very likely (even though we have not checked this) that the action (0.9) of $\mathrm{Aut}_{F-alg}(F \otimes_{\mathbf{Q}} \overline{\mathbf{Q}})_0$ on the set of CM points of $Sh(G, \mathscr{X})$ is compatible, via the functor $A \mapsto H^1_{dR}(A/\overline{\mathbf{Q}})$, with the natural action of $\mathrm{Aut}_{F-alg}(F \otimes_{\mathbf{Q}} \overline{\mathbf{Q}})$ on the category of $F \otimes_{\mathbf{Q}} \overline{\mathbf{Q}}$-modules.

(0.13) Question. *What happens for non-CM points? In other words, for what $g \in \mathrm{Aut}_{F-alg}(F \otimes_{\mathbf{Q}} \overline{\mathbf{Q}})_0$ is there a polarised HBAV A' over $\overline{\mathbf{Q}}$ such that*

$$H^1_{dR}(A'/\overline{\mathbf{Q}}) = g^* \, H^1_{dR}(A/\overline{\mathbf{Q}}),$$

with all the additional structure?

1. Background material

In §1.4-1.7 of this chapter we recall the main results of the theory of complex multiplication. In §1.1-1.3 we collect some elementary background material.

Notation and conventions: An action of a group on a set always means a left action. We write $A \otimes B$ instead of $A \otimes_{\mathbf{Z}} B$ and denote by $\overline{\mathbf{Q}}$ the algebraic closure of $\mathbf{Q}$ in $\mathbf{C}$. By a number field we always understand a subfield of $\overline{\mathbf{Q}}$ of finite degree over $\mathbf{Q}$. The ring of integers of a number field k will be denoted by O_k. For each subfield L of $\overline{\mathbf{Q}}$ we put $\Gamma_L = \mathrm{Gal}(\overline{\mathbf{Q}}/L)$ and $X(L) = \mathrm{Hom}_{\mathbf{Q}-\mathrm{alg}}(L, \overline{\mathbf{Q}})$. The restriction map $g \mapsto g|_L$ defines an isomorphism of left $\Gamma_{\mathbf{Q}}$-sets $\Gamma_{\mathbf{Q}}/\Gamma_L \xrightarrow{\sim} X(L)$. Denote by $c \in \Gamma_{\mathbf{Q}}$ the complex conjugation. For any abelian group A, put $\widehat{A} = A \otimes \widehat{\mathbf{Z}}$. If A is a ring, so is $\widehat{A}$ (if k is a number field, then $\widehat{k}$ is the ring of finite adèles of k).

1.1 Wreath products and Galois theory

(1.1.1) Notation. If X and Y are sets, we denote by $Y^X = \{f : X \longrightarrow Y\}$ the set of maps from X to Y. If Y is a group, so is Y^X. The group of permutations of the set X, denoted by $S_X = \{$bijective maps $\sigma :$

$X \longrightarrow X\}$, acts on Y^X by ${}^\sigma f = f \circ \sigma^{-1}$. For any group H, the semi-direct product of H^X and S_X (with respect to this action of S_X on H^X) is equal to

$$S_X \ltimes H^X = \{(\sigma, h) \mid \sigma \in S_X,\, h : X \longrightarrow H\}, \qquad (\sigma, h)(\sigma', h') = (\sigma\sigma', (h \circ \sigma')h').$$

If Y is a left H-set, then Y^X is a left $(S_X \ltimes H^X)$-set via the action

$$(\sigma, h)(y) = (hy) \circ \sigma^{-1}, \qquad h \in H^X,\, y \in Y^X,\, (hy)(x) = (h(x))(y(x)). \tag{1.1.1.1}$$

(1.1.2) Basic construction. Let H be a subgroup of a group G. Fix a section $s : X = G/H \longrightarrow G$ of the natural projection $G \longrightarrow G/H$. Left multiplication by $g \in G$ defines a permutation $\sigma = (x \mapsto gx) \in S_X$. For each $x \in X$,

$$gs(x) = s(gx)h(x), \qquad h(x) \in H,$$

and the map

$$g \mapsto (\sigma, h) = ((x \mapsto gx),\, (x \mapsto s(gx)^{-1}gs(x))) \in S_X \ltimes H^X$$

is an injective group homomorphism

$$\rho_s : G \hookrightarrow S_X \ltimes H^X \qquad\qquad (X = G/H). \tag{1.1.2.1}$$

If $s' : X = G/H \longrightarrow G$ is another section, then $s' = st$, $t \in H^X$, and

$$\forall g \in G \qquad \rho_{s'}(g) = (1, t)^{-1}\rho_s(g)(1, t). \tag{1.1.2.2}$$

If $(G : H) < \infty$, then the diagram

$$\begin{array}{ccc} G & \xrightarrow{\rho_s} & S_X \ltimes H^X \\ \big\downarrow & & \big\downarrow{\scriptstyle (1,\mathrm{prod})} \\ G^{ab} & \xrightarrow{V} & H^{ab} \end{array} \tag{1.1.2.3}$$

is commutative, where prod is the product map $h \mapsto \prod_{x \in X} h(x) \pmod{[H, H]}$ and V is the transfer. The map ρ_s factors through an injective group homomorphism

$$G \hookrightarrow (S_X \ltimes H^X)_0,$$

where $(S_X \ltimes H^X)_0$ is the group defined as the fibre product

$$\begin{array}{ccc} (S_X \ltimes H^X)_0 & \longrightarrow & S_X \ltimes H^X \\ \big\downarrow & & \big\downarrow{\scriptstyle (1,\mathrm{prod})} \\ G^{ab} & \xrightarrow{V} & H^{ab}. \end{array} \tag{1.1.2.4}$$

If V is injective, we can (and will) identify $(S_X \ltimes H^X)_0$ with its image in $S_X \ltimes H^X$.

(1.1.3) Proposition. *Let k'/k be a Galois extension (not necessarily finite) and X a finite set. The action of $\Gamma_{k'/k} = \mathrm{Gal}(k'/k) = \mathrm{Aut}_{k-\mathrm{alg}}(k')$ on k' gives rise, as in (1.1.1.1), to an action of $S_X \ltimes \Gamma_{k'/k}^X$ on $(k')^X$ by k-algebra automorphisms, and each k-algebra automorphism of $(k')^X$ arises in this way:*

$$S_X \ltimes \Gamma_{k'/k}^X = \mathrm{Aut}_{k-\mathrm{alg}}((k')^X), \qquad (\sigma, h) \mapsto (a \mapsto (ha) \circ \sigma^{-1}).$$

Proof. Any k-algebra automorphism f of $(k')^X$ must permute the set of irreducible idempotents $\{1_x \mid x \in X\}$ of $(k')^X$: $f(1_x) = 1_{\sigma(x)}$, $\sigma \in S_X$. This implies that $(\sigma, 1) \circ f$ preserves the decomposition $(k')^X = \prod_{x \in X} k' \cdot 1_x$, hence $(\sigma, 1) \circ f \in \mathrm{Aut}_{k-\mathrm{alg}}(k')^X = \Gamma_{k'/k}^X$, which implies that $f \in S_X \ltimes \Gamma_{k'/k}^X$.

(1.1.4) Proposition. *Let k'/k be as in Proposition 1.1.3. Let F/k be a finite subextension of k'/k; put $X = \mathrm{Hom}_{k-\mathrm{alg}}(F, k')$. Fix a section $s : X \longrightarrow \Gamma_{k'/k}$ of the restriction map $\Gamma_{k'/k} \longrightarrow \Gamma_{k'/k}/\Gamma_{k'/F} = X$, $g \mapsto g|_F$. The chosen section induces an isomorphism of k-algebras*

$$s : (k')^X \longrightarrow (k')^X, \qquad u \mapsto (x \mapsto s(x)(u(x))).$$

(i) *The map*

$$\alpha : F \otimes_k k' \longrightarrow (k')^X, \qquad a \otimes b \mapsto (x \mapsto x(a)b)$$

is an isomorphism of k-algebras.
(ii) *The map*

$$\beta_s : F \otimes_k k' \xrightarrow{\alpha} (k')^X \xleftarrow{s} (k')^X, \qquad a \otimes b \mapsto (x \mapsto as(x)^{-1}(b))$$

is an isomorphism of F-algebras.
(iii) *The map β_s satisfies*

$$\forall g \in \mathrm{Aut}_{k-\mathrm{alg}}(k') = \Gamma_{k'/k} \qquad \beta_s \circ (\mathrm{id}_F \otimes g) = \rho_s(g)\beta_s,$$

hence induces a group isomorphism

$$\beta_{s*} : \mathrm{Aut}_{F-\mathrm{alg}}(F \otimes_k k') \xrightarrow{\sim} \mathrm{Aut}_{F-\mathrm{alg}}((k')^X) = S_X \ltimes \Gamma_{k'/F}^X, \qquad f \mapsto \beta_s \circ f \circ \beta_s^{-1}$$

satisfying $\beta_{s*}(\mathrm{id}_F \otimes g) = \rho_s(g)$, for all $g \in \Gamma_{k'/k}$.
(iv) *If $s' = st : X \longrightarrow \Gamma_{k'/k}$ is another section of the restriction map $g \mapsto g|_F$ ($t : X \longrightarrow \Gamma_{k'/F}$), then*

$$\forall g \in \mathrm{Aut}_{F-\mathrm{alg}}(F \otimes_k k') \qquad \beta_{s'*}(g) = (1,t)^{-1}\, \beta_{s*}(g)\, (1,t),$$

i.e., $\beta_{s*} = \mathrm{Ad}(1,t) \circ \beta_{s'*}$.

Proof. (i) This is a well-known fact from Galois theory.
(ii) The map β_s is an isomorphism of k-algebras, by (i). For each $a \in F$, we have $\beta_s(a) : x \mapsto a$, which means that β_s is a morphism of F-algebras.
(iii) Let $a \in F$, $b \in k'$, $g \in \Gamma_{k'/k} = G$, $H = \Gamma_{k'/F}$; put $\rho_s(g) = (\sigma, h)$. For each $x \in X$ we have

$$\sigma(x) = gx, \quad h(x) = s(gx)^{-1}gs(x) = s(\sigma(x))^{-1}gs(x) \in H, \quad \beta_s(a \otimes b)(x) = as(x)^{-1}(b),$$

hence

$$\beta_s \circ (\mathrm{id}_F \otimes g)(a \otimes b) = \beta_s(a \otimes g(b)) : x \mapsto as(x)^{-1}(g(b)).$$

On the other hand,

$$(\sigma, h) \circ \beta_s(a \otimes b) : x \mapsto h(\sigma^{-1}(x))\left(a\, s(\sigma^{-1}(x))^{-1}(b)\right) = a\left(s(x)^{-1}g\right)(b),$$

which proves that $\beta_s \circ (\mathrm{id}_F \otimes g) = \rho_s(g) \circ \beta_s$, as claimed.
(iv) We have $\beta_{s'} = t^{-1}\beta_s$, since

$$\forall x \in X \qquad \beta_{s'}(a \otimes b)(x) = at(x)^{-1} \circ s(x)^{-1}(x) = t(x)^{-1}\left(as(x)^{-1}(b)\right) = t(x)^{-1}\left(\beta_s(a \otimes b)(x)\right),$$

in the notation of the proof of (iii). It follows that

$$\beta_{s'*}(g) = \beta_{s'} \circ g \circ \beta_{s'}^{-1} = t^{-1}\beta_s \circ g \circ \beta_s^{-1}t = t^{-1}\beta_{s*}(g)t,$$

as claimed.
(1.1.5) To sum up the discussion from 1.1.3-4, the natural map

$$(\mathrm{id}_F \otimes -) : \Gamma_{k'/k} = \mathrm{Aut}_{k-\mathrm{alg}}(k') \longrightarrow \mathrm{Aut}_{F-\mathrm{alg}}(F \otimes_k k'), \qquad g \mapsto \mathrm{id}_F \otimes g$$

is a canonical incarnation of the morphism $\rho_s : \Gamma_{k'/k} \hookrightarrow S_X \ltimes \Gamma_{k'/F}^X$, since $\beta_{s*} \circ (\mathrm{id}_F \otimes -) = \rho_s$.

(1.1.6) Proposition. *Let $k \subset F \subset k'$ and $s : X \longrightarrow \Gamma_{k'/k}$ be as in Proposition 1.1.4. Given $\widetilde{u} \in \Gamma_{k'/k}$, put $u = \widetilde{u}|_F$, $F' = u(F)$ and $X' = \mathrm{Hom}_{k-\mathrm{alg}}(F', k')$. The bijection $X \xrightarrow{\sim} X'$ ($x \mapsto x' = xu^{-1}$) gives rise to a section $s' : X' \longrightarrow \Gamma_{k'/k}$ of the restriction map $g \mapsto g|_{F'}$, given by $s'(x') = s(x)\widetilde{u}^{-1}$.*
(i) *The map*

$$\widetilde{u}_* : S_X \ltimes \Gamma^X_{k'/F} \longrightarrow S_{X'} \ltimes \Gamma^{X'}_{k'/F'}, \qquad (\sigma, h) \mapsto (\sigma', h')$$
$$\sigma'(x') = \sigma(x)' \ (\iff \sigma'(xu^{-1}) = \sigma(x)u^{-1}), \qquad h'(x') = \widetilde{u}h(x)\widetilde{u}^{-1} \ (\iff h'(xu^{-1}) = \widetilde{u}h(x)\widetilde{u}^{-1})$$

is a group isomorphism satisfying $\widetilde{u}_* \circ \rho_s = \rho_{s'}$.
(ii) $\forall \widetilde{u}, \widetilde{u}' \in \Gamma_{k'/k} \qquad \widetilde{u}'_* \widetilde{u}_* = (\widetilde{u}'\widetilde{u})_*$.

Proof. Easy calculation.

(1.1.7) Proposition. *In the situation of Proposition 1.1.6,*
(i) *the map*

$$\begin{array}{rcl} [u] : \mathrm{Aut}_{F-\mathrm{alg}}(F \otimes_k k') & \longrightarrow & \mathrm{Aut}_{F'-\mathrm{alg}}(F' \otimes_k k') \\ g & \mapsto & (u \otimes \mathrm{id}_{k'}) \circ g \circ (u^{-1} \otimes \mathrm{id}_{k'}) \end{array}$$

is a group isomorphism satisfying $[u'u] = [u'] \circ [u]$ *and*

$$\forall g \in \Gamma_{k'/k} \qquad [u](\mathrm{id}_F \otimes g) = \mathrm{id}_{F'} \otimes g.$$

(ii) *The following diagram is commutative.*

$$\begin{array}{ccc} \mathrm{Aut}_{F-\mathrm{alg}}(F \otimes_k k') & \xrightarrow{\beta_{s*}} & S_X \ltimes \Gamma^X_{k'/F} \\ \Big\downarrow [u] & & \Big\downarrow \widetilde{u}_* \\ \mathrm{Aut}_{F'-\mathrm{alg}}(F' \otimes_k k') & \xrightarrow{\beta_{s'*}} & S_{X'} \ltimes \Gamma^{X'}_{k'/F'} \end{array}$$

(iii) *If $F' = F$, then the group automorphism*

$$\beta_{s*} \circ [u] \circ \beta_{s*}^{-1} : S_X \ltimes \Gamma^X_{k'/F} \longrightarrow S_X \ltimes \Gamma^X_{k'/F}$$

is given by the formula $(\sigma, h) \mapsto (\sigma_u, h_u)$, *where*

$$\forall x \in X \qquad \sigma_u(x) = \sigma(xu)u^{-1}, \qquad h_u(x) = s\left(\sigma_u(x)\right)^{-1} s\left(\sigma_u(x)u\right) h(xu) s(xu)^{-1} s(x).$$

(iv) *If F is a Galois extension of k, then the maps $[u]$ define an action of $\Gamma_{F/k}$ on $\mathrm{Aut}_{F-\mathrm{alg}}(F \otimes_k k')$, whose set of fixed points is equal to* $\mathrm{id}_F \otimes \Gamma_{k'/k}$.

Proof. (i) Straightforward. (ii) Let $g \in \mathrm{Aut}_{F-\mathrm{alg}}(F \otimes_k k')$; put $(\sigma, h) = \beta_{s*}(g)$ and $(\sigma', h') = \widetilde{u}_*(\sigma, h)$. For $a \otimes b \in F \otimes_k k'$, write $g(1 \otimes b) = \sum a_i \otimes b_i$; then $g(a \otimes b) = \sum aa_i \otimes b_i$. As $\beta_s(a \otimes b)(x) = as(x)^{-1}(b)$, the equalities

$$\beta_s(g(a \otimes b))(x) = ((\sigma, h)\beta_s(a \otimes b))(x) \qquad\qquad (x \in X)$$

read as

$$\sum aa_i s(x)^{-1}(b_i) = ah(\sigma^{-1}(x))s(\sigma^{-1}(x))^{-1}(b) \qquad\qquad (x \in X). \qquad\qquad (1.1.7.1)$$

As $([u](g))(1 \otimes b) = \sum u(a_i) \otimes b_i$, the statement to be proved, namely

$$\forall x' \in X' \ \forall a' \in F' \ \forall b \in k' \qquad \beta_{s'}(([u](g))(a' \otimes b))(x') \stackrel{?}{=} ((\sigma', h')\beta_{s'}(a' \otimes b))(x'),$$

reads as

$$\sum a'u(a_i)s'(x')^{-1}(b_i) \stackrel{?}{=} a'h'(\sigma'^{-1}(x'))s'(\sigma'^{-1}(x'))^{-1}(b),$$

which is obtained from (1.1.7.1) (with $x = x'u$) by applying u, since

$$s'(x')^{-1} = \widetilde{u}s(x)^{-1}, \qquad s'(\sigma'^{-1}(x'))^{-1} = \widetilde{u}s(\sigma^{-1}(x))^{-1}, \qquad h'(\sigma'^{-1}(x')) = \widetilde{u}h(\sigma^{-1}(x))\widetilde{u}^{-1}.$$

(iii) The assumption $F' = F$ implies that $s' = st$, where $t : X \longrightarrow \Gamma_{k'/F}$ is given by $t(x) = s(x)^{-1}s(xu)\widetilde{u}^{-1}$. It follows from (ii) and Proposition 1.1.4(iv) that

$$\beta_{s*} \circ [u] \circ \beta_{s*}^{-1} = \beta_{s*} \circ \beta_{s'*}^{-1} \circ \widetilde{u}_* = \mathrm{Ad}(1,t) \circ \widetilde{u}_*,$$

hence

$$(\sigma_u, h_u) = (1,t)(\sigma', h')(1,t)^{-1} = (\sigma', (t\circ\sigma')h't^{-1}), \qquad \sigma_u(x) = \sigma'(x) = \sigma(xu)u^{-1},$$
$$h_u(x) = t(\sigma_u(x))h'(x)t(x)^{-1} = s\left(\sigma_u(x)\right)^{-1} s\left(\sigma_u(x)u\right) h(xu)s(xu)^{-1}s(x).$$

(1.1.8) Proposition. *In the situation of Proposition 1.1.4, let F'/F be a subextension of k'/F; put $X' = \mathrm{Hom}_{k-\mathrm{alg}}(F', k')$ and fix a section $s' : X' \longrightarrow \Gamma_{k'/k}$ of the restriction map $g \mapsto g|_{F'}$. For each $x' \in X'$, define $t(x') \in \Gamma_{k'/F}$ by the relation $s'(x') = s(x'|_F)t(x')$.*
(i) *The map*

$$\rho_{s,s'} : S_X \ltimes \Gamma_{k'/F}^X \longrightarrow S_{X'} \ltimes \Gamma_{k'/F'}^{X'}, \qquad (\sigma, h) \mapsto (\sigma', h'),$$
$$\sigma'(x') = s(\sigma(x))h(x)s(x)^{-1}x', \qquad h'(x') = t(\sigma'(x'))^{-1}h(x)t(x'), \qquad x = x'|_F$$

is a group homomorphism satisfying

$$\sigma'(x')|_F = \sigma(x), \qquad s'(\sigma'(x'))h'(x')s'(x')^{-1} = s(\sigma(x))h(x)s(x)^{-1}.$$

(ii) *The following diagram is commutative:*

$$\begin{array}{ccc} \mathrm{Aut}_{F-\mathrm{alg}}(F\otimes_k k') & \xrightarrow{\beta_{s*}} & S_X \ltimes \Gamma_{k'/F}^X \\ \big\downarrow{\scriptstyle (\mathrm{id}_{F'}\otimes_F -)} & & \big\downarrow{\scriptstyle \rho_{s,s'}} \\ \mathrm{Aut}_{F'-\mathrm{alg}}(F'\otimes_k k') & \xrightarrow{\beta_{s'*}} & S_{X'} \ltimes \Gamma_{k'/F'}^{X'} \end{array}$$

Proof. (i) Easy calculation. (ii) As in the proof of Proposition 1.1.7, fix $a\otimes b \in F\otimes_k k'$, $g \in \mathrm{Aut}_{F-\mathrm{alg}}(F\otimes_k k')$ and put $(\sigma, h) = \beta_{s*}(g)$. Writing $g(1\otimes b) = \sum a_i \otimes b_i$, then (1.1.7.1) (for $\sigma(x)$ instead of x) reads as

$$\sum a_i s(\sigma(x))^{-1}(b_i) = h(x)s(x)^{-1}(b) \qquad (x \in X). \tag{1.1.8.1}$$

Define $(\sigma', h') := \rho_{s,s'}(\sigma, h)$; we must show that

$$\forall x' \in X' \ \forall a' \in F' \ \forall b \in k' \qquad \beta_{s'}\left((\mathrm{id}_{F'}\otimes_F g)(a'\otimes b)\right)(x') \stackrel{?}{=} ((\sigma', h')\beta_{s'}(a'\otimes b))(x'),$$

which can be rewritten (again using (1.1.7.1) and replacing x' by $\sigma'(x')$) as follows:

$$\sum a'a_i s'(\sigma'(x'))^{-1}(b_i) \stackrel{?}{=} a'h'(x')s'(x')^{-1}(b) \qquad (x' \in X'). \tag{1.1.8.2}$$

As $\sigma'(x')|_F = \sigma(x'|_F)$, the equality (1.1.8.2) is obtained by multiplying (1.1.8.1) (for $x = x'|_F$) by $t(\sigma'(x'))^{-1}$ on the left.

1.2 Class Field Theory

(1.2.1) Let k be a number field. Denote by

$$k_+^* = \mathrm{Ker}\,(k^* \longrightarrow \pi_0((k \otimes \mathbf{R})^*))\,, \qquad O_{k,+}^* = O_k^* \cap k_+^*$$

the set of totally positive elements and the set of totally positive units of k, respectively. Let $\mathbf{A}_k$ be the adèle ring of k and $C_k = \mathbf{A}_k^*/k^*$ the idèle class group of k. The reciprocity map

$$\mathrm{rec}_k : C_k \longrightarrow \Gamma_k^{ab}$$

will be normalised by letting local uniformisers correspond to **geometric** Frobenius elements. As rec_k induces an isomorphism $\pi_0(C_k) \xrightarrow{\sim} \Gamma_k^{ab}$, its restriction to the group of finite idèles gives rise to a surjective continuous morphism

$$r_k : \widehat{k}^*/k_+^* \longrightarrow \Gamma_k^{ab}.$$

(1.2.2) It follows from the structure of the connected component of C_k ([Ar-Ta], ch. 9, Thm. 3) that the kernel of r_k is isomorphic, as an $\mathrm{Aut}(k/\mathbf{Q})$-module, to $O_{k,+}^* \otimes (\widehat{\mathbf{Z}}/\mathbf{Z}) = O_{k,+}^* \otimes (\widehat{\mathbf{Q}}/\mathbf{Q})$.

(1.2.3) For $k = \mathbf{Q}$, the map $r_{\mathbf{Q}}$ is an isomorphism, and its composition with the canonical isomorphism $\widehat{\mathbf{Z}}^* \xrightarrow{\sim} \widehat{\mathbf{Q}}^*/\mathbf{Q}_+^*$ (induced by the inclusion of $\widehat{\mathbf{Z}}$ into $\widehat{\mathbf{Q}}$) is inverse to the cyclotomic character

$$\chi : \Gamma_{\mathbf{Q}}^{ab} \xrightarrow{\sim} \widehat{\mathbf{Z}}^*, \qquad g(\zeta) = \zeta^{\chi(g)} \qquad (\forall \zeta \text{ a root of unity in } \overline{\mathbf{Q}}).$$

(1.2.4) If k'/k is a finite extension of number fields, then the inclusion $k \hookrightarrow k'$ and the norm $N_{k'/k} : k'^* \longrightarrow k^*$ induce commutative diagrams

$$\begin{array}{ccc} \widehat{k}^*/k_+^* & \xrightarrow{i_{k'/k}} & \widehat{k}'^*/k_+'^* \\ \downarrow r_k & & \downarrow r_k' \\ \Gamma_k^{ab} & \xrightarrow{V_{k'/k}} & \Gamma_{k'}^{ab} \end{array} \qquad\qquad \begin{array}{ccc} \widehat{k}'^*/k_+'^* & \xrightarrow{N_{k'/k}} & \widehat{k}^*/k_+^* \\ \downarrow r_k' & & \downarrow r_k \\ \Gamma_{k'}^{ab} & \xrightarrow{j_{k'/k}} & \Gamma_k^{ab}, \end{array} \tag{1.2.4.1}$$

where $V_{k'/k}$ is the transfer map and $j_{k'/k}$ is given by the restriction map $g \mapsto g|_{k^{ab}}$.

(1.2.5) Proposition. *For any number field L,*

$$\mathrm{Ker}\,\big(V_{L/\mathbf{Q}} : \Gamma_{\mathbf{Q}}^{ab} \longrightarrow \Gamma_L^{ab}\big) = \begin{cases} \{1, c\}, & \textit{if } L \textit{ is totally complex,} \\ \{1\}, & \textit{otherwise.} \end{cases}$$

Proof. Let L' be the Galois closure of L over $\mathbf{Q}$. As

$$\mathrm{Im}\,\big(i_{L/\mathbf{Q}}\big) \cap \mathrm{Ker}(r_L) \subseteq \Big(O_{L',+}^* \otimes \widehat{\mathbf{Q}}/\mathbf{Q}\Big)^{\mathrm{Gal}(L'/\mathbf{Q})} = \mathbf{Z}_+^* \otimes \widehat{\mathbf{Q}}/\mathbf{Q} = \{1\},$$

the first commutative diagram (1.2.4.1) for $L/\mathbf{Q}$ implies that $i_{L/\mathbf{Q}}^{-1}\,(\mathrm{Ker}(r_L)) = \mathrm{Ker}(r_L \circ i_{L/\mathbf{Q}})$ is equal to

$$\mathrm{Ker}\,\big(i_{L/\mathbf{Q}}\big) = \big(\mathbf{Q}^* \cap L_+^*\big)\,/\mathbf{Q}_+^* = \begin{cases} \mathbf{Q}^*/\mathbf{Q}_+^* = \{\pm 1\}, & \text{if } L \text{ is totally complex,} \\ \{1\}, & \text{otherwise.} \end{cases}$$

As $r_{\mathbf{Q}}$ is an isomorphism and $r_{\mathbf{Q}}(-1) = c$, the statement follows.

1.3 CM fields

Let K be a CM number field; let F be its maximal totally real subfield (in other words, $c(K) = K$, $\tau c = c\tau \neq \tau$ for all $\tau \in X(K)$, and $F = K^{c=1}$). Put $X = X(F)$.

(1.3.1) Complex conjugations. Fix a section $s : X \longrightarrow \Gamma_{\mathbf{Q}}$ of the restriction map $g \mapsto g|_F$. For each $x \in X$, the image of the element $s(x)^{-1} c s(x) \in \Gamma_F$ in Γ_F^{ab} is independent of the chosen section; denote it by

$c_x \in \Gamma_F^{ab}$ (this is the complex conjugation defined by the real place x of F). Denote by $\langle c_X \rangle$ the subgroup of Γ_F^{ab} generated by all c_x ($x \in X$). The signs at the real places induce an isomorphism

$$(\mathrm{sgn} \circ x)_{x \in X} : F^*/F_+^* \xrightarrow{\sim} \{\pm 1\}^X.$$

Compatibility of the local and global reciprocity maps implies that

$$\forall a \in F^* \qquad r_F(aF_+^*) = \prod_{x \in X} c_x^{a_x}, \qquad (-1)^{a_x} = \mathrm{sgn}(x(a)).$$

As $\mathrm{Ker}(r_F)$ is a $\mathbf{Q}$-vector space, we have $\mathrm{Ker}(r_F) \cap F^*/F_+^* = \{1\}$, which means that r_F induces an isomorphism $F^*/F_+^* \xrightarrow{\sim} \langle c_X \rangle$.

(1.3.2) Transfer maps. If we denote by

$$R : \Gamma_F \longrightarrow \Gamma_K, \qquad g, cg \mapsto g \qquad (g \in \Gamma_K)$$

the "retraction map" from Γ_F to Γ_K, then

$$\forall h \in \Gamma_F \qquad V_{K/F}\left(h|_{F^{ab}}\right) = V_{K/F}\left(ch|_{F^{ab}}\right) = hchc|_{K^{ab}} = {}^{1+c}\left(R(h)|_{K^{ab}}\right). \tag{1.3.2.1}$$

As noted in 1.2.5,

$$\mathrm{Ker}\left(V_{K/\mathbf{Q}} : \Gamma_{\mathbf{Q}}^{ab} \longrightarrow \Gamma_K^{ab}\right) = r_{\mathbf{Q}}\left(\mathrm{Ker}(i_{K/\mathbf{Q}})\right) = r_{\mathbf{Q}}\left(\mathbf{Q}^*/\mathbf{Q}_+^*\right) = \{1, c\} = \langle c \rangle. \tag{1.3.2.2}$$

The equality $\mathrm{Ker}(r_F) = O_{F,+}^* \otimes \widehat{\mathbf{Q}}/\mathbf{Q} = O_K^* \otimes \widehat{\mathbf{Q}}/\mathbf{Q} = \mathrm{Ker}(r_K)$ implies, thanks to (1.2.4.1), that

$$\mathrm{Ker}\left(V_{K/F} : \Gamma_F^{ab} \longrightarrow \Gamma_K^{ab}\right) = r_F\left(\mathrm{Ker}(i_{K/F})\right) = r_F\left(F^*/F_+^*\right) = \langle c_X \rangle. \tag{1.3.2.3}$$

As a result, the map

$$\overline{V}_{F/\mathbf{Q}} : \Gamma_{\mathbf{Q}}^{ab}/\langle c \rangle \hookrightarrow \Gamma_F^{ab}/\langle c_X \rangle \tag{1.3.2.4}$$

induced by $V_{F/\mathbf{Q}}$ is injective and

$$\{h \in \Gamma_F^{ab} \mid V_{K/F}(h) \in V_{K/\mathbf{Q}}(\Gamma_{\mathbf{Q}}^{ab})\} = \langle c_X \rangle\, V_{F/\mathbf{Q}}(\Gamma_{\mathbf{Q}}^{ab}). \tag{1.3.2.5}$$

It also follows that

$$V_{F/\mathbf{Q}}(\Gamma_{\mathbf{Q}}^{ab}) \cap \langle c_X \rangle = \langle V_{F/\mathbf{Q}}(c) \rangle \tag{1.3.2.6}$$

is the cyclic group of order 2 generated by $V_{F/\mathbf{Q}}(c) = \prod_{x \in X} c_x$.

(1.3.3) As observed in [Ta, Lemma 1], the finiteness of $O_K^*/O_{F,+}^*$ implies that c (resp., $1+c$) acts trivially (resp., invertibly) on the $\mathbf{Q}$-vector space $\mathrm{Ker}(r_K)$.

(1.3.4) Proposition. (i) *The continuous homomorphism (induced by r_K)*

$$\{a \in \widehat{K}^* \mid {}^{1+c}a \in \widehat{\mathbf{Z}}^* K^*\}/K^* \longrightarrow \{g \in \Gamma_K^{ab} \mid g|_{F^{ab}} \in \langle c_X \rangle\, V_{F/\mathbf{Q}}(\Gamma_{\mathbf{Q}}^{ab})\}$$

is bijective. Denote by ℓ_K its inverse; then ${}^{1+c}\ell_K(g) = \chi(u(g))K^*$*, where $u(g) \in \Gamma_{\mathbf{Q}}^{ab}/\langle c \rangle$ is the (unique) element satisfying $\overline{V}_{F/\mathbf{Q}}\left(u(g)\right) = \langle c_X \rangle g|_{F^{ab}}$ (equivalently, $V_{K/\mathbf{Q}}\left(u(g)\right) = {}^{1+c}g$).*

(ii) *More precisely, if $g \in \Gamma_K^{ab}$ satisfies*

$$g|_{F^{ab}} = V_{F/\mathbf{Q}}(u(g)) \prod_{x \in X} c_x^{a_x} \qquad (u(g) \in \Gamma_{\mathbf{Q}}^{ab},\ a_x \in \mathbf{Z}/2\mathbf{Z}),$$

then $N_{K/F}(\ell_K(g)) = \chi(u(g))\alpha F_+^* \in \widehat{F}^*/F_+^*$*, where $\alpha \in F^*$ and*

$$\forall x \in X \qquad \mathrm{sgn}(x(\alpha)) = (-1)^{a_x}.$$

(iii) *The canonical morphism (induced by the inclusion* $\widehat{O}_K \hookrightarrow \widehat{K}$*)*

$$\{x \in \widehat{O}_K^* \mid {}^{1+c}x \in \widehat{\mathbf{Z}}^*\} \longrightarrow \{a \in \widehat{K}^* \mid {}^{1+c}a \in \widehat{\mathbf{Z}}^* K^*\}/K^*$$

has finite kernel and cokernel.
(iv) *The morphism* ℓ_K *defined in (i) admits a lift*

$$\widetilde{\ell}_K : \{g \in \Gamma_K^{ab} \mid g|_{F^{ab}} \in \langle c_X \rangle\, V_{F/\mathbf{Q}}(\Gamma_{\mathbf{Q}}^{ab})\} \longrightarrow \{a \in \widehat{K}^* \mid {}^{1+c}a \in \widehat{\mathbf{Z}}^* K^*\},$$

which is a homomorphism when restricted to a suitable open subgroup.

Proof. (i) In the following commutative diagram the right column is exact and $r_{\mathbf{Q}}$ is an isomorphism:

$$\begin{array}{ccc} & & 0 \\ & & \downarrow \\ & & \mathrm{Ker}(r_K) \\ & & \downarrow \\ \widehat{\mathbf{Z}}^* & \longrightarrow & \widehat{K}^*/K^* \\ \downarrow r_{\mathbf{Q}} & & \downarrow r_K \\ \Gamma_{\mathbf{Q}}^{ab} & \xrightarrow{V_{K/\mathbf{Q}}} & \Gamma_K^{ab} \\ & & \downarrow \\ & & 0 \end{array}$$

As $1+c$ acts invertibly on $\mathrm{Ker}(r_K)$, the snake lemma implies that r_K induces an isomorphism

$$\mathrm{Ker}\left(\widehat{K}^*/K^* \xrightarrow{1+c} \widehat{K}^*/K^*\widehat{\mathbf{Z}}^*\right) \xrightarrow{\sim} \mathrm{Ker}\left(1+c = V_{K/F} \circ j_{K/F} : \Gamma_K^{ab} \longrightarrow \Gamma_K^{ab}/V_{K/\mathbf{Q}}(\Gamma_{\mathbf{Q}}^{ab})\right);$$

by (1.3.2.5), the second group is equal to $\{g \in \Gamma_K^{ab} \mid g|_{F^{ab}} \in \langle c_X \rangle\, V_{F/\mathbf{Q}}(\Gamma_{\mathbf{Q}}^{ab})\}$. The remaining statement follows from the fact that

$$\begin{aligned} r_K\left({}^{1+c}\ell_K(g)\right) &= {}^{1+c}g = V_{K/F} \circ j_{K/F}(g) = V_{K/F}\left(g|_{F^{ab}}\right) = V_{K/F} \circ V_{F/\mathbf{Q}}\left(u(g)\right) = \\ &= V_{K/\mathbf{Q}}\left(u(g)\right) = r_K \circ i_{K/\mathbf{Q}} \circ r_{\mathbf{Q}}^{-1}\left(u(g)\right) = r_K\left(\chi(u(g))\right). \end{aligned}$$

(ii) Let $a \in \widehat{K}^*$ be a lift of $\ell_K(g)$ such that ${}^{1+c}a = b\alpha'$, where $b \in \widehat{\mathbf{Z}}^*$, $\alpha' \in K^*$; then $\alpha' \in (K^*)^{c=1} = F^*$. As

$$g|_{F^{ab}} = r_F(N_{K/F}(a)) = r_F(b) r_F(\alpha') = V_{F/\mathbf{Q}}(r_{\mathbf{Q}}(b)) \prod_{x \in X} c_x^{a'_x}, \qquad (-1)^{a'_x} = \mathrm{sgn}(x(\alpha')),$$

it follows from (1.3.2.6) that there is $t \in \mathbf{Z}/2\mathbf{Z}$ such that

$$u(g) = r_{\mathbf{Q}}(b) c^t, \qquad \forall x \in X \quad a'_x = a_x + t.$$

This implies that $\chi(u(g)) = b(-1)^t$ and

$$N_{K/F}\left(\ell_K(g)\right) = {}^{1+c}aF_+^* = \chi(u(g))\alpha F_+^*$$

with $\alpha = \alpha'(-1)^t$, hence

$$\forall x \in X \quad \mathrm{sgn}(x(\alpha)) = \mathrm{sgn}(x(\alpha'))(-1)^t = (-1)^{a'_x + t} = (-1)^{a_x}.$$

(iii) This follows from the finiteness of the groups $\mathrm{Ker}, \mathrm{Coker}(1 + c : O_K^* \longrightarrow O_K^*)$ and $Cl_K = \widehat{K}^*/\widehat{O}_K^* K^*$, combined with the snake lemma applied to the diagrams

$$\begin{array}{ccccccccc}
0 & \longrightarrow & O_K^* & \longrightarrow & \widehat{O}_K^* & \longrightarrow & \widehat{O}_K^*/O_K^* & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle 1+c} & & \Big\downarrow{\scriptstyle 1+c} & & \Big\downarrow{\scriptstyle 1+c} & & \\
0 & \longrightarrow & O_K^*/\mathbf{Z}^* & \longrightarrow & \widehat{O}_K^*/\widehat{\mathbf{Z}}^* & \longrightarrow & \widehat{O}_K^*/\widehat{\mathbf{Z}}^* O_K^* & \longrightarrow & 0
\end{array}$$

and

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \widehat{O}_K^*/O_K^* & \longrightarrow & \widehat{K}^*/K^* & \longrightarrow & Cl_K & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle 1+c} & & \Big\downarrow{\scriptstyle 1+c} & & \Big\downarrow{\scriptstyle 1+c} & & \\
0 & \longrightarrow & \widehat{O}_K^*/\widehat{\mathbf{Z}}^* O_K^* & \longrightarrow & \widehat{K}^*/\widehat{\mathbf{Z}}^* K^* & \longrightarrow & Cl_K & \longrightarrow & 0.
\end{array}$$

Above, $\widehat{O}_K^*$ is a shorthand for $(\widehat{O_K})^*$. Note also that $\widehat{\mathbf{Z}}^* \cap O_K^* = \mathbf{Z}^*$ inside $\widehat{O}_K^*$.
(iv) By (i) and (iii), r_K induces a continuous homomorphism of pro-finite abelian groups

$$f : A = \{x \in \widehat{O}_K^* \mid {}^{1+c}x \in \widehat{\mathbf{Z}}^*\} \longrightarrow B = \{g \in \Gamma_K^{ab} \mid g|_{F^{ab}} \in \langle c_X \rangle \, V_{F/\mathbf{Q}}(\Gamma_{\mathbf{Q}}^{ab})\}$$

with finite kernel and cokernel. This implies that there exists an open subgroup (= a compact subgroup of finite index) $A' \subset A$ such that $A' \cap \mathrm{Ker}(f) = \{1\}$. Then $B' = f(A')$ is a compact subgroup of finite index (= an open subgroup) of B, and f induces a topological isomorphism $f' : A' \xrightarrow{\sim} B'$. Fix coset representatives $B = \bigcup_i b_i B'$ (disjoint union) and lifts $\widetilde{a}_i \in \widehat{K}^*$ of $\ell_K(b_i) \in \widehat{K}^*/K^*$ such that $b_{i_0} = 1$ and $\widetilde{a}_{i_0} = 1$; the map

$$\widetilde{\ell}_K : B \longrightarrow \widehat{K}^*, \qquad b_i f'(a') \mapsto \widetilde{a}_i a' \qquad (a' \in A')$$

has the required properties.

1.4 Tate's construction

Let Φ be a CM type of K, i.e., a subset $\Phi \subset X(K)$ such that $X(K) = \Phi \cup c\Phi$ (disjoint union).
(1.4.1) Tate's half-transfer [Ta] is the continuous map $F_\Phi : \Gamma_{\mathbf{Q}} \longrightarrow \Gamma_K^{ab}$ defined by the formula

$$F_\Phi(g) = \prod_{\varphi \in \Phi} w(g\varphi)^{-1} g w(\varphi) \ (\mathrm{mod}\ \Gamma_{K^{ab}}), \tag{1.4.1.1}$$

where $w : X(K) \longrightarrow X(\overline{\mathbf{Q}}) = \Gamma_{\mathbf{Q}}$ is any section of the restriction map $g \mapsto g|_K$ satisfying $w(cy) = cw(y)$, for all $y \in X(K)$.
The restriction map $g \mapsto g|_F$ defines a bijection $\Phi \xrightarrow{\sim} X(F)$. Composing its inverse with w, we obtain a section $t : X(F) \longrightarrow X(\overline{\mathbf{Q}}) = \Gamma_{\mathbf{Q}}$ of the restriction map to F, which implies that

$$F_\Phi(g)|_{F^{ab}} = \prod_{x \in X(F)} t(gx)^{-1} c^{a(g,x)} g t(x) \ (\mathrm{mod}\ \Gamma_{F^{ab}}) \in \langle c_X \rangle V_{F/\mathbf{Q}}(g) \qquad (a(g,x) \in \mathbf{Z}/2\mathbf{Z}). \tag{1.4.1.2}$$

The maps F_Φ satisfy

$$F_\Phi(gg') = F_{g'\Phi}(g) F_\Phi(g') \qquad (g, g' \in \Gamma_{\mathbf{Q}}) \tag{1.4.1.3}$$

and

$$u \circ F_\Phi(g) \circ u^{-1} = F_{\Phi u^{-1}}(g) \qquad (g \in \Gamma_{\mathbf{Q}}), \tag{1.4.1.4}$$

for any isomorphism of CM number fields $u : K \xrightarrow{\sim} K'$. In addition, if K' is a CM number field containing K and $\Phi' = \{y \in X(K') \mid y|_K \in \Phi\}$ is the CM type of K' induced from Φ, then

$$F_{\Phi'}(g) = V_{K'/K}\left(F_\Phi(g)\right) \qquad (g \in \Gamma_{\mathbf{Q}}). \tag{1.4.1.5}$$

(1.4.2) The Taniyama element is the map $f_\Phi : \Gamma_{\mathbf{Q}} \longrightarrow \widehat{K}^*/K^*$ defined as

$$f_\Phi(g) = \ell_K\left(F_\Phi(g)\right), \tag{1.4.2.1}$$

where

$$\ell_K : \{g \in \Gamma_K^{ab} \mid g|_{F^{ab}} \in \langle c_X\rangle\, V_{F/\mathbf{Q}}(\Gamma_{\mathbf{Q}}^{ab})\} \xrightarrow{\sim} \{a \in \widehat{K}^* \mid {}^{1+c}a \in \widehat{\mathbf{Z}}^* K^*\}/K^* \subset \widehat{K}^*/K^*$$

is the morphism from 1.3.4(i). It follows that

$$\begin{aligned} {}^{1+c}F_\Phi(g) &= V_{K/F}\left(F_\Phi(g)|_{F^{ab}}\right) = V_{K/F}\circ V_{F/\mathbf{Q}}(g) = V_{K/\mathbf{Q}}(g) = \\ &= r_K \circ i_{K/\mathbf{Q}} \circ r_{\mathbf{Q}}^{-1}\left(g|_{\mathbf{Q}^{ab}}\right) = r_K(\chi(g)). \end{aligned}$$

As in the proof of 1.3.4(i), this implies that

$${}^{1+c}f_\Phi(g) = \chi(g)K^*, \qquad r_K\left(f_\Phi(g)\right) = F_\Phi(g). \tag{1.4.2.2}$$

In Tate's original definition, the properties (1.4.2.2) were used to characterise $f_\Phi(g)$.

The identities (1.4.1.3-5) imply that

$$f_\Phi(gg') = f_{g'\Phi}(g) f_\Phi(g') \qquad\qquad (g, g' \in \Gamma_{\mathbf{Q}}), \tag{1.4.2.3}$$

$${}^u f_\Phi(g) = f_{\Phi u^{-1}}(g) \qquad\qquad (g \in \Gamma_{\mathbf{Q}},\ u : K \xrightarrow{\sim} K') \tag{1.4.2.4}$$

and

$$f_{\Phi'}(g) = i_{K'/K}\left(f_\Phi(g)\right) \qquad\qquad (K \subset K',\ \Phi' \text{ induced from } \Phi). \tag{1.4.2.5}$$

(1.4.3) Tate [Ta] conjectured that the idèle class $f_\Phi(g)$ determines the action of $g \in \Gamma_{\mathbf{Q}}$ on abelian varieties with complex multiplication and on their torsion points (this was, essentially, the zero-dimensional case of an earlier conjecture of Langlands [L] about conjugation of Shimura varieties). Building upon earlier results of Shimura and Taniyama, Tate proved his conjecture up to an element of $\widehat{F}^*$ of square 1. The full conjecture was subsequently proved by Deligne [La, ch. 7, §4].

More precisely, if A is a CM abelian variety of type (K, Φ, a, t) in the sense of [La, ch. 7, §3] (see 2.2.5 below), then ${}^gA$ is of type $(K, g\Phi, af, t\chi(g)/{}^{1+c}f)$, where $f \in \widehat{K}^*$ is any lift of $f_\Phi(g)$. Furthermore, for each complex uniformisation

$$\theta : \mathbf{C}^\Phi/a \xrightarrow{\sim} A(\mathbf{C}),$$

there is a unique uniformisation

$$\theta' : \mathbf{C}^{g\Phi}/af \xrightarrow{\sim} {}^gA(\mathbf{C})$$

such that the action of g on $A(\overline{\mathbf{Q}})_{\text{tors}} = A(\mathbf{C})_{\text{tors}}$ is given by

$$g : A(\overline{\mathbf{Q}})_{\text{tors}} \xrightarrow{\theta^{-1}} K/a \xrightarrow{[\times f]} K/af \xrightarrow{\theta'} {}^gA(\overline{\mathbf{Q}})_{\text{tors}}.$$

This implies that, for each full level structure $\eta : (F/O_F)^2 \xrightarrow{\sim} A(\overline{\mathbf{Q}})_{\text{tors}}$, the level structure ${}^g\eta$ is equal to

$${}^g\eta : (F/O_F)^2 \xrightarrow{\eta} A(\overline{\mathbf{Q}})_{\text{tors}} \xrightarrow{\theta^{-1}} K/a \xrightarrow{[\times f]} K/af \xrightarrow{\theta'} {}^gA(\overline{\mathbf{Q}})_{\text{tors}}. \tag{1.4.3.1}$$

1.5 The Serre torus

Let K be as in 1.3.

(1.5.1) The torus ${}_KT = R_{K/\mathbf{Q}}(\mathbf{G}_m)$ represents the functor $A \mapsto {}_KT(A) = (K \otimes_{\mathbf{Q}} A)^*$ on $\mathbf{Q}$-algebras A. The $\Gamma_{\mathbf{Q}}$-equivariant bijections

$$\begin{array}{ccccccc} {}_KT(\overline{\mathbf{Q}}) & = & (K \otimes_{\mathbf{Q}} \overline{\mathbf{Q}})^* & \xrightarrow{\sim} & (\overline{\mathbf{Q}}^*)^{X(K)} = \mathrm{Hom}_{Sets}(X(K), \overline{\mathbf{Q}}^*) & \xrightarrow{\sim} & \mathrm{Hom}_{\mathbf{Z}}(\mathbf{Z}[X(K)], \overline{\mathbf{Q}}^*) \\ & & a \otimes b & \mapsto & (y \mapsto y(a)b) & & (y \in X(K)) \end{array}$$

imply that the character group of ${}_KT$ is equal to

$$X^*({}_KT) = \mathbf{Z}[X(K)] = \{ \sum_{y \in X(K)} n_y[y] \mid n_y \in \mathbf{Z}\},$$

with $g \in \Gamma_{\mathbf{Q}}$ acting on $X^*({}_KT)$ by

$$\lambda = \sum n_y[y] \mapsto {}^g\lambda = \sum n_y[gy] = \sum n_{g^{-1}y}[y]. \tag{1.5.1.1}$$

(1.5.2) The **Serre torus** of K is the quotient ${}_K\mathscr{S}$ of ${}_KT$ (defined over $\mathbf{Q}$) whose character group is equal to

$$X^*({}_K\mathscr{S}) = \{\lambda \in X^*({}_KT) \mid {}^{1+c}\lambda \in \mathbf{Z} \cdot N_{K/\mathbf{Q}}\} \qquad (N_{K/\mathbf{Q}} = \sum_{y \in X(K)} [y]).$$

Each CM type Φ of K defines a character $\lambda_\Phi \in X^*({}_K\mathscr{S})$: $\lambda_\Phi(y) = 1$ (resp., $= 0$) if $y \in \Phi$ (resp., if $y \in c\Phi$). Moreover, the abelian group $X^*({}_K\mathscr{S})$ is generated by the characters λ_Φ ([Sch, 1.3.2]), and

$$\forall g \in \Gamma_{\mathbf{Q}} \qquad {}^g\lambda_\Phi = \lambda_{g\Phi}.$$

(1.5.3) Tate's half-transfer satisfies the following identity: if n is a function

$$n : \{\text{CM types of } K\} \longrightarrow \mathbf{Z}, \qquad \Phi \mapsto n_\Phi,$$

such that $\sum_\Phi n_\Phi \lambda_\Phi = 0$, then

$$\forall g \in \Gamma_{\mathbf{Q}} \qquad \prod_\Phi F_\Phi(g)^{n_\Phi} = 1 \in \Gamma_K^{ab}. \tag{1.5.3.1}$$

Applying ℓ_K, we deduce from (1.5.3.1) that

$$\forall g \in \Gamma_{\mathbf{Q}} \qquad \prod_\Phi f_\Phi(g)^{n_\Phi} = 1 \in \widehat{K}^*/K^*. \tag{1.5.3.2}$$

(1.5.4) In the special case when K is a Galois extension of $\mathbf{Q}$, the action (1.5.1.1) of $\Gamma_{\mathbf{Q}}$ factors through $\mathrm{Gal}(K/\mathbf{Q})$, which implies that the tori ${}_KT$ and ${}_K\mathscr{S}$ are split over K.
In addition, the action of $\mathrm{Gal}(K/\mathbf{Q})$ on K induces an action of $\mathrm{Gal}(K/\mathbf{Q})$ on the $\mathbf{Q}$-group scheme ${}_KT$, which will be denoted by $t \mapsto g * t$ ($g \in \mathrm{Gal}(K/\mathbf{Q})$). The corresponding action on the character group

$$(h * \lambda)(t) = \lambda(h^{-1} * t) \qquad (\lambda \in X^*({}_KT)) \tag{1.5.4.1}$$

is given by

$$\lambda = \sum n_y[y] \mapsto h * \lambda = \sum n_y[yh^{-1}] = \sum n_{yh}[y].$$

The two actions are related by

$$\iota({}^h\lambda) = h * \iota(\lambda) \qquad (h \in \mathrm{Gal}(K/\mathbf{Q}),\ \lambda \in X^*({}_KT)), \tag{1.5.4.2}$$

where

$$\iota : X^*({}_KT) \longrightarrow X^*({}_KT), \qquad \sum n_y[y] \mapsto \sum n_y[y^{-1}] = \sum n_{y^{-1}}[y] \tag{1.5.4.3}$$

is the involution induced by the inverse map $g \mapsto g^{-1}$ on $\mathrm{Gal}(K/\mathbf{Q}) = X(K)$. As $\iota(\lambda_\Phi) = \lambda_{\Phi^{-1}}$, the involution ι and the action (1.5.4.1) preserve $X^*({}_K\mathscr{S})$, and we have

$$h * \lambda_\Phi = \lambda_{\Phi h^{-1}}. \tag{1.5.4.4}$$

We denote by

$$\iota : {}_K\mathscr{S}_K = {}_K\mathscr{S} \otimes_{\mathbf{Q}} K \longrightarrow {}_K\mathscr{S}_K$$

the morphism corresponding to ι.

1.6 Universal Taniyama elements [Mi], [Sch]

In this section, we assume that K is a CM number field which is a Galois extension of $\mathbf{Q}$.

(1.6.1) The two actions of $\mathrm{Gal}(K/\mathbf{Q})$ on $X^*({}_K\mathscr{S})$ correspond to two actions of $\mathrm{Gal}(K/\mathbf{Q})$ on ${}_K\mathscr{S}(\widehat{K})$: the Galois action $t \mapsto {}^g t$ and the algebraic action $t \mapsto h * t$, which commute with each other and satisfy

$$({}^g\lambda)({}^g t) = {}^g(\lambda(t)), \qquad (h*\lambda)(h*t) = \lambda(t) \qquad\qquad (\lambda \in X^*({}_K\mathscr{S}),\ t \in {}_K\mathscr{S}(\widehat{K})),$$

respectively.

(1.6.2) Proposition. (i) *There exists a unique map $f' : \Gamma_{\mathbf{Q}} \longrightarrow {}_K\mathscr{S}(\widehat{K})/{}_K\mathscr{S}(K)$ such that $\lambda_\Phi \circ f' = f_\Phi$, for all CM types Φ of K. The map f' factors through* $\mathrm{Gal}(K^{ab}/\mathbf{Q})$.
(ii) *For each $\lambda \in X^*({}_K\mathscr{S})$, put $f'_\lambda = \lambda \circ f' : \Gamma_{\mathbf{Q}} \longrightarrow \widehat{K}^*/K^*$; then $f'_{\lambda+\mu}(g) = f'_\lambda(g) f'_\mu(g)$.*
(iii) $\forall \lambda \in X^*({}_K\mathscr{S})\ \forall g, g' \in \Gamma_{\mathbf{Q}} \qquad f'_\lambda(gg') = f'_{g'\lambda}(g) f'_\lambda(g')$.
(iv) $\forall h \in \mathrm{Gal}(K/\mathbf{Q}) \qquad {}^h(f'_\lambda(g)) = f'_{h*\lambda}(g)$.

Proof. (i) As the torus ${}_K\mathscr{S}$ is split over K and $X^*({}_K\mathscr{S})$ is a free abelian group generated by the CM characters λ_Φ, we have

$$\begin{aligned} {}_K\mathscr{S}(\widehat{K})/{}_K\mathscr{S}(K) &= \mathrm{Hom}_{\mathbf{Z}}(X^*({}_K\mathscr{S}), \widehat{K}^*)/\mathrm{Hom}_{\mathbf{Z}}(X^*({}_K\mathscr{S}), K^*) = \mathrm{Hom}_{\mathbf{Z}}(X^*({}_K\mathscr{S}), \widehat{K}^*/K^*) = \\ &= \{\alpha : \{\text{CM types of } K\} \longrightarrow \widehat{K}^*/K^* \mid \prod \alpha(\Phi)^{n_\Phi} = 1 \text{ whenever } \sum n_\Phi \lambda_\Phi = 0\}. \end{aligned}$$

The existence and uniqueness of f' then follow from (1.5.3.2). As K is a Galois extension of $\mathbf{Q}$, the maps F_Φ (hence f_Φ, too) factor through $\mathrm{Gal}(K^{ab}/\mathbf{Q})$.
(ii) This is a consequence of (the proof of) (i).
(iii), (iv) If $\lambda = \lambda_\Phi$, the statement of (iii) (resp., of (iv)) is just (1.4.2.3) (resp., (1.4.2.4)). The general case then follows from (ii).

(1.6.3) Proposition. (i) *Define the map $f : \Gamma_{\mathbf{Q}} \longrightarrow {}_K\mathscr{S}(\widehat{K})/{}_K\mathscr{S}(K)$ by the formula $f(g) = (\iota\,(f'(g)))^{-1}$. The map f factors through* $\mathrm{Gal}(K^{ab}/\mathbf{Q})$ *and has the following properties.*
(ii) *The maps $f_\lambda = \lambda \circ f : \Gamma_{\mathbf{Q}} \longrightarrow \widehat{K}^*/K^*$ $(\lambda \in X^*({}_K\mathscr{S}))$ satisfy*

$$f_{\lambda+\mu}(g) = f_\lambda(g) f_\mu(g), \qquad f_\lambda(g) = f'_{\iota(\lambda)}(g)^{-1}, \qquad f_\lambda(gg') = f_{g'*\lambda}(g) f_\lambda(g').$$

(iii) $\forall h \in \mathrm{Gal}(K/\mathbf{Q})\ \forall g \in \Gamma_{\mathbf{Q}} \qquad {}^h(f_\lambda(g)) = f_{{}^h\lambda}(g), \qquad {}^h(f(g)) = f(g)$.
(iv) $\forall g, g' \in \Gamma_{\mathbf{Q}} \quad f(gg') = \left(g'^{-1} * f(g)\right) f(g')$.

Proof. The statements of (i), (ii) and the first part of (iii) are immediate consequences of 1.6.2, thanks to (1.5.4.2). The second part of (iii) follows from

$$\left({}^h\lambda\right)\left({}^h(f(g))\right) = {}^h(\lambda\,(f(g))) \overset{\text{(iii)}}{=} \left({}^h\lambda\right)(f(g)) \qquad\qquad (\lambda \in X^*({}_K\mathscr{S})),$$

while (iv) is a consequence of the last formula from (ii) and

$$\lambda\left(g'^{-1} * f(g)\right) = (g' * \lambda)\left(f(g)\right).$$

(1.6.4) For each CM type Φ of K, the map f_{λ_Φ} is given by

$$f_{\lambda_\Phi}(g) = f_{\Phi^{-1}}(g)^{-1},$$

which implies that

$$r_K \circ f_{\lambda_\Phi}(g) = F_{\Phi^{-1}}(g)^{-1}.$$

In the notation of ([Sch], 4.2), we have $f_\lambda(g) = f_K(g, \lambda)$. The map f is the "universal Taniyama element" of ([Mi], I.5.7).

(1.6.5) Proposition. *If K' is a CM number field, which is a Galois extension of $\mathbf{Q}$ and contains K, then the universal Taniyama elements $f_K : \Gamma_{\mathbf{Q}} \longrightarrow {}_K\mathscr{S}(\widehat{K})/{}_K\mathscr{S}(K)$ and $f_{K'} : \Gamma_{\mathbf{Q}} \longrightarrow {}_{K'}\mathscr{S}(\widehat{K}')/{}_{K'}\mathscr{S}(K')$ over K and K', respectively, satisfy $f_K = N_{K'/K} \circ f_{K'}$.*

Proof. As the map $i_{K'/K} : \widehat{K}^*/K^* \longrightarrow \widehat{K}'^*/K'^*$ is injective, it is enough to check that, for any CM type Φ of K and $g \in \Gamma_{\mathbf{Q}}$,

$$i_{K'/K} \circ \lambda_\Phi \circ f_K(g) \stackrel{?}{=} i_{K'/K} \circ \lambda_\Phi \circ N_{K'/K} \circ f_{K'}(g) \in \widehat{K}'^*/K'^*,$$

which follows from (1.4.2.5), since

$$i_{K'/K} \circ \lambda_\Phi \circ f_K(g) = i_{K'/K}\left(f_{\Phi^{-1}}(g)^{-1}\right) \stackrel{(1.4.2.5)}{=} f_{\Phi'^{-1}}(g)^{-1} = \lambda_{\Phi'} \circ f_{K'}(g) = i_{K'/K} \circ \lambda_\Phi \circ N_{K'/K} \circ f_{K'}(g),$$

where Φ' is the CM type of K' induced from Φ.

1.7 The Taniyama group ([Mi], [Mi-Sh], [Sch])

Let K be as in §1.6.

(1.7.1) The **Taniyama group** of level K sits in an exact sequence of affine group schemes over $\mathbf{Q}$,

$$1 \longrightarrow {}_K\mathscr{S} \stackrel{i}{\longrightarrow} {}_K\mathscr{T} \stackrel{\pi}{\longrightarrow} \mathrm{Gal}(K^{ab}/\mathbf{Q}) \longrightarrow 1,$$

such that the action of (the constant group scheme) $\mathrm{Gal}(K^{ab}/\mathbf{Q})$ on ${}_K\mathscr{S}$ defined by this exact sequence is given by the algebraic action $(g, t) \mapsto g * t$. In addition, there exists a continuous group homomorphism

$$sp : \mathrm{Gal}(K^{ab}/\mathbf{Q}) \longrightarrow {}_K\mathscr{T}(\widehat{\mathbf{Q}})$$

satisfying $\pi \circ sp = \mathrm{id}$.

(1.7.2) Choose a section

$$\alpha : \mathrm{Gal}(K^{ab}/\mathbf{Q}) \longrightarrow {}_K\mathscr{T}(K)$$

of the map ${}_K\mathscr{T}(K) \longrightarrow \mathrm{Gal}(K^{ab}/\mathbf{Q})$ (which is surjective, since the torus ${}_K\mathscr{S}$ is split over K and $H^1(K, \mathbf{G}_m) = 0$); the map

$$b : \mathrm{Gal}(K^{ab}/\mathbf{Q}) \longrightarrow {}_K\mathscr{S}(\widehat{K}), \qquad b(g) = sp(g)\alpha(g)^{-1},$$

has the following properties:

(1.7.2.1) The induced map $\bar{b} : \mathrm{Gal}(K^{ab}/\mathbf{Q}) \longrightarrow {}_K\mathscr{S}(\widehat{K})/{}_K\mathscr{S}(K)$ does not depend on the choice of α.
(1.7.2.2) $\forall g, g' \in \mathrm{Gal}(K^{ab}/\mathbf{Q}) \qquad \bar{b}(gg') = \left(g'^{-1} * \bar{b}(g)\right)\bar{b}(g')$.
(1.7.2.3) $\forall h \in \mathrm{Gal}(K/\mathbf{Q})\ \forall g \in \mathrm{Gal}(K^{ab}/\mathbf{Q}) \qquad {}^h\left(\bar{b}(g)\right) = \bar{b}(g)$.

(1.7.2.4) The "coboundary" $d_{g,g'} = \big(g'^{-1} * b(g)\big)\, b(g')\, b(gg')^{-1}$ is a locally constant function on $\mathrm{Gal}(K^{ab}/\mathbf{Q})^2$.

(1.7.3) Conversely, any map b satisfying (1.7.2.1-4) gives rise to an object from 1.7.1 ([Mi-Sh], Prop. 2.7): firstly, the reverse 2-cocycle $d_{g,g'}$ with values in ${}_K\mathscr{S}(K)$ defines an exact sequence of affine group schemes over K,

$$1 \longrightarrow {}_K\mathscr{S}_K \xrightarrow{\ i\ } G' \xrightarrow{\ \pi\ } \mathrm{Gal}(K^{ab}/\mathbf{Q}) \longrightarrow 1, \tag{1.7.3.1}$$

equipped with a section $\alpha : \mathrm{Gal}(K^{ab}/\mathbf{Q}) \longrightarrow G'(K)$ such that

$$\forall g, g' \in \mathrm{Gal}(K^{ab}/\mathbf{Q}) \qquad \alpha(gg') = \alpha(g)\alpha(g')d_{g,g'}.$$

Secondly, the map

$$sp : \mathrm{Gal}(K^{ab}/\mathbf{Q}) \longrightarrow G'(\widehat{K}), \qquad sp(g) = b(g)\alpha(g),$$

is a group homomorphism satisfying $\pi \circ sp = \mathrm{id}$. Thirdly, each element $h \in \Gamma_K$ acts on $G'(\overline{\mathbf{Q}})$ by

$${}^{h}(s\,\alpha(g)) = {}^{h}s\,\alpha(g) \qquad\qquad (s \in {}_K\mathscr{S}(\overline{\mathbf{Q}})). \tag{1.7.3.2}$$

In order to descend the sequence (1.7.3.1) to an exact sequence of group schemes over $\mathbf{Q}$

$$1 \longrightarrow {}_K\mathscr{S} \xrightarrow{\ i\ } G \xrightarrow{\ \pi\ } \mathrm{Gal}(K^{ab}/\mathbf{Q}) \longrightarrow 1,$$

it is enough to extend the action of Γ_K from (1.7.3.2) to an action of $\Gamma_{\mathbf{Q}}$ compatible with i and π. This is done by putting

$${}^{h}(s\,\alpha(g)) = c_h(g)\,{}^{h}s\,\alpha(g), \qquad c_h(g) = b(g)\,{}^{h}(b(g))^{-1} \in {}_K\mathscr{S}(K) \qquad\qquad (h \in \Gamma_{\mathbf{Q}},\ g \in \mathrm{Gal}(K^{ab}/\mathbf{Q})).$$

As ${}^{h}(sp(g)) = sp(g)$ for all $h \in \Gamma_{\mathbf{Q}}$ and $g \in \mathrm{Gal}(K^{ab}/\mathbf{Q})$, the map sp has values in $G(\widehat{\mathbf{Q}})$. Up to isomorphism, the quadruple (G, i, π, sp) obtained by this method depends only on $\overline{b}$, not on its lift b.

(1.7.4) The Taniyama group ${}_K\mathscr{T}$ of level K is defined by applying the construction from 1.7.3 to the universal Taniyama element f, which satisfies (1.7.2.2-3), by Proposition 1.6.3. The existence of a lift b of f satisfying (1.7.2.4) is established in the following proposition.

(1.7.5) Proposition. *There exists a lift* $b : \mathrm{Gal}(K^{ab}/\mathbf{Q}) \longrightarrow {}_K\mathscr{S}(\widehat{K})$ *of* f *whose "coboundary"* $d_{g,g'} = \big(g'^{-1} * b(g)\big)\, b(g')\, b(gg')^{-1}$ *is a locally constant function on* $\mathrm{Gal}(K^{ab}/\mathbf{Q})^2$.

Proof. Let $\widetilde{\ell}_K$ be as in 1.3.4(iv). As the maps F_Φ (which factor through $\mathrm{Gal}(K^{ab}/\mathbf{Q})$) are continuous, there exists an open subgroup $U \subset \Gamma_K^{ab}$ such that $\widetilde{\ell}_K$, when restricted to $\bigcup_\Phi F_\Phi(U)$, is a homomorphism. If $n_\Phi \in \mathbf{Z}$ satisfy $\sum_\Phi n_\Phi \lambda_\Phi = 0$, then the relation (1.5.3.1) implies that

$$\forall u \in U \qquad \prod_\Phi \widetilde{\ell}_K\,(F_\Phi(u))^{n_\Phi} = 1 \in \widehat{K}^*.$$

As in the proof of 1.6.2(i), we conclude that, for each $u \in U$, there exists a unique element $b'(u) \in {}_K\mathscr{S}(\widehat{K})$ satisfying $\lambda_\Phi(b'(u)) = \widetilde{\ell}_K\,(F_\Phi(u))$. Fix coset representatives $\mathrm{Gal}(K^{ab}/\mathbf{Q}) = \bigcup_j g_j U$ (disjoint union) and lifts $\widetilde{s}_j \in {}_K\mathscr{S}(\widehat{K})$ of $f'(g_j) \in {}_K\mathscr{S}(\widehat{K})/{}_K\mathscr{S}(K)$ such that $g_{j_0} = 1$ and $\widetilde{s}_{j_0} = 1$; define a map $b' : \mathrm{Gal}(K^{ab}/\mathbf{Q}) \longrightarrow {}_K\mathscr{S}(\widehat{K})$ by

$$b'(g_j u) = \widetilde{s}_j b'(u) \qquad\qquad (u \in U).$$

The map $b(g) := (\iota(b'(g)))^{-1}$ then has the required property.

(1.7.6) Proposition 1.6.5 implies that the pull-backs of the group schemes ${}_K\mathscr{T}$ via $\Gamma_{\mathbf{Q}} \longrightarrow \mathrm{Gal}(K^{ab}/\mathbf{Q})$ form, for varying K, a projective system compatible with the norm maps $N_{K'/K} : {}_{K'}\mathscr{S} \longrightarrow {}_K\mathscr{S}$. In the limit, they give rise to an exact sequence

$$1 \longrightarrow \mathscr{S} \xrightarrow{i} \mathscr{T} \xrightarrow{\pi} \Gamma_{\mathbf{Q}} \longrightarrow 1 \qquad (1.7.6.1)$$

equipped with a splitting $sp : \Gamma_{\mathbf{Q}} \longrightarrow \mathscr{T}(\widehat{\mathbf{Q}})$. The main result of [De] states that the affine group scheme $\mathscr{T}$ (= the Taniyama group) is the Tannaka dual of the category $CM_{\mathbf{Q}}$ of CM motives (for absolute Hodge cycles) defined over $\mathbf{Q}$. The group scheme ${}_K\mathscr{T}$ corresponds to the full Tannakian subcategory of $CM_{\mathbf{Q}}$ consisting of objects with reflex field in K.

2. Hidden symmetries in the CM theory

Throughout this chapter, K and F are as in 1.3. Put $X = X(F)$. In §2.1 (resp., §2.2) we extend Tate's half-transfer F_Φ (resp., the Taniyama element f_Φ) from $\Gamma_{\mathbf{Q}}$ to $\mathrm{Aut}_{F-\mathrm{alg}}(F \otimes \overline{\mathbf{Q}})$ (resp., to $\mathrm{Aut}_{F-\mathrm{alg}}(F \otimes \overline{\mathbf{Q}})_1$). In §2.3-2.4 we use our generalisation of the Taniyama element to construct a generalised Taniyama group.

2.1 Generalised half-transfer

(2.1.1) Fix a section $s : X \longrightarrow \Gamma_{\mathbf{Q}}$ of the restriction map $g \mapsto g|_F$. As in 1.1.2-4, the choice of s determines the following objects:

(2.1.1.1) An injection $\rho_s : \Gamma_{\mathbf{Q}} \hookrightarrow S_X \ltimes \Gamma_F^X$.

(2.1.1.2) An isomorphism $\beta_{s*} : \mathrm{Aut}_{F-\mathrm{alg}}(F \otimes \overline{\mathbf{Q}}) \xrightarrow{\sim} \mathrm{Aut}_{F-\mathrm{alg}}(\overline{\mathbf{Q}}^X) = S_X \ltimes \Gamma_F^X$ satisfying $\beta_{s*}(\mathrm{id}_F \otimes g) = \rho_s(g)$.

In addition, we obtain

(2.1.1.3) A bijection between $(\mathbf{Z}/2\mathbf{Z})^X$ and the set of CM types of K: a function $\alpha : X \longrightarrow \mathbf{Z}/2\mathbf{Z}$ corresponds to the CM type $\{c^{\alpha(x)} s(x)|_K = s(x) c^{\alpha(x)}|_K\}_{x \in X}$.

(2.1.1.4) A section $w_s : X(K) \longrightarrow \Gamma_{\mathbf{Q}}$ of the restriction map $g \mapsto g|_K$ satisfying $w_s(cy) = c w_s(y)$, namely $w_s(c^a s(x)|_K) = c^a s(x)$ $(x \in X,\ a \in \mathbf{Z}/2\mathbf{Z})$.

For $h \in \Gamma_F^X$, we denote by $\overline{h} : X \longrightarrow \mathbf{Z}/2\mathbf{Z}$ the image of h in $\mathrm{Gal}(K/F)^X \xrightarrow{\sim} (\mathbf{Z}/2\mathbf{Z})^X$. In other words,

$$\forall x \in X \qquad h(x)|_K = c^{\overline{h}(x)}, \qquad R(h(x)) = c^{\overline{h}(x)} h(x),$$

where $R : \Gamma_F \longrightarrow \Gamma_K$ is the retraction map from 1.3.2. We let $S_X \ltimes \Gamma_F^X$ act on $(\mathbf{Z}/2\mathbf{Z})^X$ via (1.1.1.1) and the natural projection $(\sigma, h) \mapsto (\sigma, \overline{h})$:

$$(\sigma, h)\alpha = (\alpha + \overline{h}) \circ \sigma^{-1}. \qquad (2.1.1.5)$$

(2.1.2) Rewriting Tate's half-transfer in terms of ρ_s. Let Φ be a CM type of K. If $g \in \Gamma_{\mathbf{Q}}$, then

$$\rho_s(g) = (\sigma, h) \in S_X \ltimes \Gamma_F^X, \qquad \forall x \in X \quad \sigma(x) = gx, \qquad h(x) = s(gx)^{-1} g s(x) = s(\sigma(x))^{-1} g s(x) \in \Gamma_F.$$

Let $\alpha \in (\mathbf{Z}/2\mathbf{Z})^X$ correspond to Φ, as in (2.1.1.3). For each $x \in X$, the element

$$\varphi_x = c^{\alpha(x)} s(x)|_K = s(x) c^{\alpha(x)}|_K \in \Phi$$

satisfies $w_s(\varphi_x) = c^{\alpha(x)} s(x)$ and

$$g\varphi_x = g s(x) c^{\alpha(x)}|_K = s(\sigma(x)) h(x) c^{\alpha(x)}|_K = c^{\alpha(x) + \overline{h}(x)} s(\sigma(x))|_K,$$

which implies that $w_s(g\varphi_x) = c^{\alpha(x) + \overline{h}(x)} s(\sigma(x))$ and

$$\begin{aligned} w_s(g\varphi_x)^{-1} g w_s(\varphi_x) &= s(\sigma(x))^{-1} c^{\alpha(x)+\overline{h}(x)} g c^{\alpha(x)} s(x) = s(\sigma(x))^{-1} c^{\alpha(x)+\overline{h}(x)} s(\sigma(x)) h(x) s(x)^{-1} c^{\alpha(x)} s(x) = \\ &= \left[s(\sigma(x))^{-1} c^{\alpha(x)+\overline{h}(x)} s(\sigma(x)) c^{\alpha(x)+\overline{h}(x)} \right] \left[c^{\alpha(x)+\overline{h}(x)} h(x) c^{\alpha(x)} \right] \left[c^{\alpha(x)} s(x)^{-1} c^{\alpha(x)} s(x) \right]. \end{aligned}$$

Denote by $\gamma_{x,s}$ the image of $s(x)^{-1}cs(x)c \in \Gamma_K$ in Γ_K^{ab}. As each of the three elements in square brackets lies in Γ_K, we have

$$F_\Phi(g) = \prod_{x\in X} w_s(g\varphi_x)^{-1} g w_s(\varphi_x)|_{K^{ab}} = \prod_{x\in|(\sigma,h)\alpha|} \gamma_{x,s} \prod_{x\in|\alpha|} \gamma_{x,s}^{-1} \prod_{x\in X} c^{\alpha(x)} R(h(x)) c^{\alpha(x)}|_{K^{ab}},$$

where we have denoted by $|\alpha| = \{x \in X \mid \alpha(x) \neq 0\}$ the support of α. This calculation justifies the following:

(2.1.3) Proposition-Definition. *For each $\alpha \in (\mathbf{Z}/2\mathbf{Z})^X$, the formula*

$$\begin{aligned} {}_s\widetilde{F}_\alpha(\sigma,h) &= \prod_{x\in X} s(\sigma(x))^{-1} c^{\alpha(x)+\overline{h}(x)} s(\sigma(x)) h(x) s(x)^{-1} c^{\alpha(x)} s(x)|_{K^{ab}} = \\ &= \prod_{x\in|(\sigma,h)\alpha|} \gamma_{x,s} \prod_{x\in|\alpha|} \gamma_{x,s}^{-1} \prod_{x\in X} c^{\alpha(x)} R(h(x)) c^{\alpha(x)}|_{K^{ab}} \end{aligned}$$

defines a map

$${}_s\widetilde{F}_\alpha : S_X \ltimes \Gamma_F^X \longrightarrow \Gamma_K^{ab}$$

(depending on s and α) satisfying ${}_s\widetilde{F}_\alpha \circ \rho_s = F_\Phi$, where Φ is the CM type corresponding to α, as in (2.1.1.3).

(2.1.4) Proposition. *The maps ${}_s\widetilde{F}_\alpha$ have the following properties:*
(i) $\forall g, g' \in S_X \ltimes \Gamma_F^X \qquad {}_s\widetilde{F}_\alpha(gg') = {}_s\widetilde{F}_{g'\alpha}(g)\, {}_s\widetilde{F}_\alpha(g')$.
(ii) *For each $(\sigma,h) \in S_X \ltimes \Gamma_F^X$,*

$${}_s\widetilde{F}_\alpha(\sigma,h)|_{F^{ab}} = \prod_{x\in|(\sigma,h)\alpha|} c_x \prod_{x\in|\alpha|} c_x \prod_{x\in X} h(x)|_{F^{ab}}, \qquad {}^{1+c}\Big({}_s\widetilde{F}_\alpha(\sigma,h)\Big) = \widetilde{V}_{K/F}(\sigma,h) = \prod_{x\in X} {}^{1+c}R(h(x))|_{K^{ab}},$$

where $\widetilde{V}_{K/F}(\sigma,h) = \prod_{x\in X} V_{K/F}(h(x)|_{F^{ab}})$.
(iii) *Each map ${}_s\widetilde{F}_\alpha$ factors through $S_X \ltimes \mathrm{Gal}(K^{ab}/F)^X$.*
(iv) *If $g = (\sigma,h) \in S_X \ltimes \Gamma_F^X$ satisfies $g\alpha = \alpha$, then*

$${}_s\widetilde{F}_\alpha(g) = \prod_{x\in X} c^{\alpha(x)} R(h(x)) c^{\alpha(x)}|_{K^{ab}}.$$

(v) $\forall(\sigma,h) \in S_X \ltimes \Gamma_K^X \qquad {}_s\widetilde{F}_0(\sigma,h) = \prod_{x\in X} h(x)|_{K^{ab}}$.
(vi) $\forall\alpha \in (\mathbf{Z}/2\mathbf{Z})^X \qquad {}_s\widetilde{F}_0(1,c^\alpha) = \prod_{x\in|\alpha|} \gamma_{x,s}$.

Proof. (i) If $g = (\sigma,h)$ and $g' = (\sigma',h')$, then $gg' = (\sigma\sigma',(h\circ\sigma')h')$ and $\alpha' := g'\alpha = (\alpha + \overline{h}') \circ \sigma'^{-1}$, which implies that ${}_s\widetilde{F}_\alpha(gg')\, {}_s\widetilde{F}_\alpha(g')^{-1}\, {}_s\widetilde{F}_{g'\alpha}(g)^{-1}$ is equal to

$$\begin{aligned} &\prod_{x\in X} \Big(c^{\alpha(x)+\overline{h}(\sigma'(x))+\overline{h}'(x)} h(\sigma'(x)) h'(x) c^{\alpha(x)}\Big) \Big(c^{\alpha(x)+\overline{h}'(x)} h'(x) c^{\alpha(x)}\Big)^{-1} \Big(c^{\alpha'(x)+\overline{h}(x)} h(x) c^{\alpha'(x)}\Big)^{-1} = \\ &= \prod_{x\in X} \Big(c^{\alpha'(\sigma'(x))+\overline{h}(\sigma'(x))} h(\sigma'(x)) c^{\alpha'(\sigma'(x))}\Big) \Big(c^{\alpha'(x)+\overline{h}(x)} h(x) c^{\alpha'(x)}\Big)^{-1} = 1. \end{aligned}$$

(ii) The first formula is a consequence of the fact that

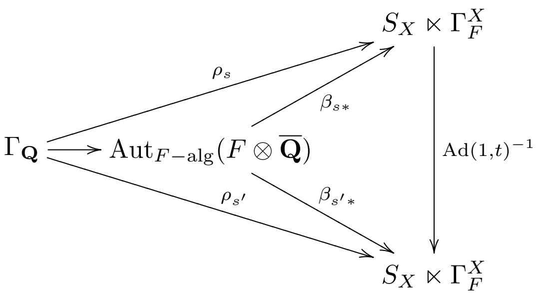
$$\forall x \in X \qquad \gamma_{x,s}|_{F^{ab}} = c_x c, \qquad c^{\alpha(x)} R(h(x)) c^{\alpha(x)}|_{F^{ab}} = c^{\overline{h}(x)} h(x)|_{F^{ab}};$$

applying (1.3.2.1), we obtain the second formula.
The statements (iii)-(vi) follow directly from the definitions.

(2.1.5) Change of s. Let $s, s' \longrightarrow \Gamma_{\mathbf{Q}}$ be two sections of the restriction map $g \mapsto g|_F$. We have $s' = st$, where $t : X \longrightarrow \Gamma_F$. As in 2.1.1, we write, for each $x \in X$, $t(x)|_K = c^{\bar{t}(x)}$ ($\bar{t}(x) \in \mathbf{Z}/2\mathbf{Z}$); then $R(t(x)) = c^{\bar{t}(x)} t(x) \in \Gamma_K$. The recipe (2.1.1.3), applied to s and s', respectively, associates to each CM type Φ of K two functions $\alpha = \alpha_{\Phi,s}, \alpha' = \alpha_{\Phi,s'} : X \longrightarrow \mathbf{Z}/2\mathbf{Z}$ such that

$$\Phi = \{c^{\alpha(x)} s(x)|_K\} = \{c^{\alpha'(x)} s'(x)|_K\} \qquad (\Longrightarrow \alpha' = \alpha + \bar{t}).$$

According to Proposition 1.1.4, the following diagram is commutative:



(2.1.5.1)

For $(\sigma, h) \in S_X \ltimes \Gamma_F^X$, put

$$(\sigma', h') := \mathrm{Ad}(1,t)^{-1}(\sigma, h) = (1,t)^{-1}(\sigma,h)(1,t) = (\sigma, (t \circ \sigma)^{-1} h t) \in S_X \ltimes \Gamma_F^X. \tag{2.1.5.2}$$

The map $\widetilde{V}_{K/F}$ from Proposition 2.1.4(ii) satisfies $\widetilde{V}_{K/F}(\sigma, h) = \widetilde{V}_{K/F}(\sigma', h')$, which means that the map

$$\widetilde{V}_{K/F} \circ \beta_{s*} : \mathrm{Aut}_{F-\mathrm{alg}}(F \otimes \overline{\mathbf{Q}}) \longrightarrow \Gamma_K^{ab} \tag{2.1.5.3}$$

does not depend on s; we denote it again by $\widetilde{V}_{K/F}$. The equalities

$$(\sigma, h)\alpha = (\alpha + \bar{h}) \circ \sigma^{-1}, \qquad (\sigma', h')\alpha' = (\alpha' + \bar{h}') \circ \sigma^{-1} = (\alpha + \bar{h}) \circ \sigma^{-1} + \bar{t} \in (\mathbf{Z}/2\mathbf{Z})^X$$

imply that the action of $S_X \ltimes \Gamma_F^X$ on $(\mathbf{Z}/2\mathbf{Z})^X$ defined in (2.1.1.5) gives rise to an action of the group $\mathrm{Aut}_{F-\mathrm{alg}}(F \otimes \overline{\mathbf{Q}})$ on the set of CM types of K, which is characterised by

$$\forall g \in \mathrm{Aut}_{F-\mathrm{alg}}(F \otimes \overline{\mathbf{Q}}) \qquad \alpha_{g\Phi,s} = \beta_{s*}(g)\alpha_{\Phi,s}, \tag{2.1.5.4}$$

but which does not depend on s.

(2.1.6) Proposition. *In the notation of (2.1.5.2), we have* ${}_s\widetilde{F}_\alpha(\sigma, h) = {}_{s'}\widetilde{F}_{\alpha'}(\sigma', h') \in \Gamma_K^{ab}$.

Proof. The relations $s' = st$, $(\sigma', h') = (\sigma, (t \circ \sigma)^{-1} h t)$, $\bar{h}' = \bar{h} + \bar{t} + \bar{t} \circ \sigma$, $\alpha' = \alpha + \bar{t}$, $(\sigma, h)\alpha = (\alpha + \bar{h}) \circ \sigma^{-1}$ and $(\sigma', h')\alpha' = (\sigma, h)\alpha + \bar{t}$ imply that

$$\begin{aligned}{}_{s'}\widetilde{F}_{\alpha'}(\sigma', h') &= \prod_{x \in X} t(\sigma(x))^{-1} s(\sigma(x))^{-1} c^{(\alpha+\bar{h})(x) + \bar{t}(\sigma(x))} s(\sigma(x)) h(x) s(x)^{-1} c^{(\alpha+\bar{t})(x)} s(x) t(x)|_{K^{ab}} = \\ &= \prod_{x \in X} A'((\sigma,h)\alpha, x)^{-1} B(\alpha, x) A'(\alpha, x),\end{aligned}$$

where

$$A'(\alpha, x) = c^{\alpha(x)} s(x)^{-1} c^{(\alpha+\bar{t})(x)} s(x) t(x)|_{K^{ab}}, \qquad B(\alpha, x) = c^{\alpha(x)} R(h(x)) c^{\alpha(x)}|_{K^{ab}}.$$

As

$${}_s\widetilde{F}_\alpha(\sigma, h) = \prod_{x \in X} A((\sigma,h)\alpha, x)^{-1} B(\alpha, x) A(\alpha, x),$$

where

$$A(\alpha, x) = c^{\alpha(x)} s(x)^{-1} c^{\alpha(x)} s(x)|_{K^{ab}},$$

the equality ${}_s\widetilde{F}_\alpha(\sigma, h) = {}_{s'}\widetilde{F}_{\alpha'}(\sigma', h')$ follows from the fact that

$$\forall x \in X \qquad A(\alpha, x)^{-1} A'(\alpha, x) = s(x)^{-1} c^{\bar{t}(x)} s(x) t(x)|_{K^{ab}}$$

does not depend on α.

(2.1.7) Proposition-Definition. *In the notation of 2.1.5, the map*

$$\widetilde{F}_\Phi = {}_s\widetilde{F}_\alpha(\sigma, h) \circ \beta_{s*} : \mathrm{Aut}_{F-\mathrm{alg}}(F \otimes \overline{\mathbf{Q}}) \longrightarrow \Gamma_K^{ab}$$

depends on Φ, but not on s; it has the following properties:
(i) $\forall g \in \Gamma_{\mathbf{Q}} \qquad \widetilde{F}_\Phi(\mathrm{id}_F \otimes g) = F_\Phi(g)$.
(ii) $\forall g, g' \in \mathrm{Aut}_{F-\mathrm{alg}}(F \otimes \overline{\mathbf{Q}}) \qquad \widetilde{F}_\Phi(gg') = \widetilde{F}_{g'\Phi}(g)\widetilde{F}_\Phi(g')$.
(iii) $\forall g \in \mathrm{Aut}_{F-\mathrm{alg}}(F \otimes \overline{\mathbf{Q}}) \qquad {}^{1+c}\widetilde{F}_\Phi(g) = \widetilde{V}_{K/F}(g)$ *(in the notation of (2.1.5.3)).*

Proof. The independence of $\widetilde{F}_\Phi$ of the choice of s follows from Proposition 2.1.6 and the commutative diagram (2.1.5.1). The remaining statements are consequences of Proposition 2.1.4.

(2.1.8) Galois functoriality of $\widetilde{F}_\Phi$. Given an element $\widetilde{u} \in \Gamma_{\mathbf{Q}}$, define $u := \widetilde{u}|_K$, $u_F := u|_F$, $K' := u(K)$, $F' = u_F(F)$ and $X' = X(F')$. As in Proposition 1.1.6 (for $k = \mathbf{Q}$ and $k' = \overline{\mathbf{Q}}$), a fixed section $s : X \longrightarrow \Gamma_{\mathbf{Q}}$ of the restriction map $g \mapsto g|_F$ defines a section $s' : X' \longrightarrow \Gamma_{\mathbf{Q}}$ of the restriction map $g \mapsto g|_{F'}$, given by

$$s'(x') = s'(xu_F^{-1}) = s(x) \circ \widetilde{u}^{-1} \qquad\qquad (x \in X).$$

(2.1.9) Proposition. *For each $\alpha : X \longrightarrow \mathbf{Z}/2\mathbf{Z}$, the diagram*

$$\begin{array}{ccc} S_X \ltimes \Gamma_F^X & \xrightarrow{{}_s\widetilde{F}_\alpha} & \Gamma_K^{ab} \\ \Big\downarrow{\scriptstyle \widetilde{u}_*} & & \Big\downarrow{\scriptstyle u} \\ S_{X'} \ltimes \Gamma_{F'}^{X'} & \xrightarrow{{}_{s'}\widetilde{F}_{\alpha'}} & \Gamma_{K'}^{ab} \end{array}$$

is commutative, where $\widetilde{u}_*$ is the map defined in Proposition 1.1.6, $\alpha' : X' \longrightarrow \mathbf{Z}/2\mathbf{Z}$ is given by $\alpha'(x') = \alpha(x)$ $(x = x'u_F)$ and the right vertical map (which depends only on u) is given by $g \mapsto \widetilde{u}g\widetilde{u}^{-1}$.

Proof. For $(\sigma, h) \in S_X \ltimes \Gamma_F^X$, we have $\widetilde{u}_*(\sigma, h) = (\sigma', h')$, where $\sigma'(x') = \sigma(x)u_F^{-1}$, $h'(x') = \widetilde{u}h(x)\widetilde{u}^{-1}$ $(x' = xu_F^{-1})$. The relations $s'(\sigma'(x')) = s(\sigma(x))\widetilde{u}^{-1}$, $s'(x') = s(x)\widetilde{u}^{-1}$, $\overline{h}'(x') = \overline{h}(x)$ and $\alpha'(x') = \alpha(x)$ imply

$$\begin{aligned} {}_{s'}\widetilde{F}_{\alpha'}(\sigma', h') &= \prod_{x' \in X'} s'(\sigma'(x'))^{-1} c^{\alpha'(x') + \overline{h}'(x')} s'(\sigma'(x')) h'(x') s'(x')^{-1} c^{\alpha'(x')} s'(x')|_{K'^{ab}} = \\ &= \widetilde{u} \prod_{x \in X} s(\sigma(x))^{-1} c^{\alpha(x) + \overline{h}(x)} s(\sigma(x)) h(x) s(x)^{-1} c^{\alpha(x)} s(x)|_{K^{ab}}\, \widetilde{u}^{-1} = u\left({}_s\widetilde{F}_\alpha(\sigma, h)\right). \end{aligned}$$

(2.1.10) Corollary. *For each CM type Φ of K, the diagram*

$$\begin{array}{ccc} \mathrm{Aut}_{F-\mathrm{alg}}(F \otimes \overline{\mathbf{Q}}) & \xrightarrow{\widetilde{F}_\Phi} & \Gamma_K^{ab} \\ \Big\downarrow{\scriptstyle [u_F]} & & \Big\downarrow{\scriptstyle u} \\ \mathrm{Aut}_{F'-\mathrm{alg}}(F' \otimes \overline{\mathbf{Q}}) & \xrightarrow{\widetilde{F}_{\Phi u^{-1}}} & \Gamma_{K'}^{ab} \end{array}$$

is commutative, where $[u_F]$ *is the map defined in Proposition 1.1.7(i).*

Proof. This follows from Proposition 2.1.9 combined with Proposition 1.1.7(ii) (for $k = \mathbf{Q}$ and $k' = \overline{\mathbf{Q}}$), if we take into account the fact that

$$\{c^{\alpha'(x')} s'(x')|_{K'}\}_{x' \in X'} = \{c^{\alpha(x)} s(x)|_K \, u^{-1}\}_{x \in X}.$$

2.2 Generalised Taniyama elements

(2.2.1) Let $(S_X \ltimes \Gamma_F^X)_1$ be the group defined as the fibre product

$$\begin{array}{ccc} (S_X \ltimes \Gamma_F^X)_1 & \longrightarrow & S_X \ltimes \Gamma_F^X \\ \big\downarrow & & \big\downarrow {\scriptstyle (1,\mathrm{prod})} \\ \Gamma_{\mathbf{Q}}^{ab}/\langle c \rangle & \overset{\overline{V}_{F/\mathbf{Q}}}{\hookrightarrow} & \Gamma_F^{ab}/\langle c_X \rangle. \end{array}$$

As the morphism $\overline{V}_{F/\mathbf{Q}}$ is injective (1.3.2.4), we can (and will) identify $(S_X \ltimes \Gamma_F^X)_1$ with its image in $S_X \ltimes \Gamma_F^X$. The group $(S_X \ltimes \Gamma_F^X)_0$, defined in (1.1.2.4), sits in an exact sequence

$$1 \longrightarrow (S_X \ltimes \Gamma_F^X)_0 \longrightarrow (S_X \ltimes \Gamma_F^X)_1 \longrightarrow \langle c_X \rangle / V_{F/\mathbf{Q}}(\langle c \rangle) \longrightarrow 1.$$

For $i = 0, 1$, the subgroups $\beta_{s*}^{-1}\left((S_X \ltimes \Gamma_F^X)_i\right)$ of $\mathrm{Aut}_{F-\mathrm{alg}}(F \otimes \overline{\mathbf{Q}})$ are independent of the choice of a section $s : X \longrightarrow \Gamma_{\mathbf{Q}}$; we denote them by

$$\mathrm{Aut}_{F-\mathrm{alg}}(F \otimes \overline{\mathbf{Q}})_0 \subset \mathrm{Aut}_{F-\mathrm{alg}}(F \otimes \overline{\mathbf{Q}})_1 \subset \mathrm{Aut}_{F-\mathrm{alg}}(F \otimes \overline{\mathbf{Q}}).$$

(2.2.2) Definition. *For each CM type* Φ *of* K*, define a map*

$$\widetilde{f}_\Phi : \mathrm{Aut}_{F-\mathrm{alg}}(F \otimes \overline{\mathbf{Q}})_1 \longrightarrow \widehat{K}^*/K^*$$

by

$$\widetilde{f}_\Phi(g) = \ell_K\left(\widetilde{F}_\Phi(g)\right),$$

where ℓ_K *is the morphism from Proposition 1.3.4(i). [This definition makes sense, by Proposition 2.1.4(ii).]*

(2.2.3) Proposition. *The maps* $\widetilde{f}_\Phi$ *have the following properties:*
(i) $r_K \circ \widetilde{f}_\Phi = \widetilde{F}_\Phi$.
(ii) $\forall g \in \Gamma_{\mathbf{Q}} \qquad \widetilde{f}_\Phi(\mathrm{id}_F \otimes g) = f_\Phi(g)$.
(iii) *Each map* $\widetilde{f}_\Phi$ *factors through*

$$\mathrm{Aut}_{F-\mathrm{alg}}(F \otimes K^{ab})_1 := \mathrm{Im}\left(\mathrm{Aut}_{F-\mathrm{alg}}(F \otimes \overline{\mathbf{Q}})_1 \longrightarrow \mathrm{Aut}_{F-\mathrm{alg}}(F \otimes K^{ab})\right).$$

(iv) $\forall g, g' \in \mathrm{Aut}_{F-\mathrm{alg}}(F \otimes \overline{\mathbf{Q}})_1 \qquad \widetilde{f}_\Phi(gg') = \widetilde{f}_{g'\Phi}(g)\widetilde{f}_\Phi(g')$.
(v) *If* $u : K \xrightarrow{\sim} K'$ *is an isomorphism of CM number fields, then*

$$\widetilde{f}_{\Phi u^{-1}} \circ [u|_F] = u \circ \widetilde{f}_\Phi.$$

(vi) *For* $g \in \mathrm{Aut}_{F-\mathrm{alg}}(F \otimes \overline{\mathbf{Q}})_1$*, denote by* $u(g) \in \Gamma_{\mathbf{Q}}^{ab}/\langle c \rangle$ *the unique element satisfying* $V_{K/\mathbf{Q}}(u(g)) = {}^{1+c}\widetilde{F}_\Phi(g)$*; then* ${}^{1+c}\widetilde{f}_\Phi(g) = \chi(u(g))K^*$.
(vii) *For* $g \in \mathrm{Aut}_{F-\mathrm{alg}}(F \otimes \overline{\mathbf{Q}})_0$*, denote by* $u(g) \in \Gamma_{\mathbf{Q}}^{ab}$ *the unique element satisfying* $V_{F/\mathbf{Q}}(u(g)) = \widetilde{F}_\Phi(g)|_{F^{ab}}$*; then* $N_{K/F}(\widetilde{f}_\Phi(g)) = \chi(u(g))\alpha F_+^* \in \widehat{F}^*/F_+^*$*, where* $\alpha \in F^*$ *satisfies*

$$\forall x \in X \quad \mathrm{sgn}(x(a)) = \begin{cases} 1, & \text{if } \Phi \text{ and } g\Phi \text{ agree at } x \\ -1, & \text{if } \Phi \text{ and } g\Phi \text{ do not agree at } x \end{cases}$$

(we say that two CM types Φ *and* Φ' *of* K *agree at* $x \in X$ *if the unique element of* Φ *whose restriction to* F *is* x *is equal to the unique element of* Φ' *whose restriction to* F *is* x*).*

Proof. statement (i) holds by definition, while (ii)-(v) follow from the corresponding assertions for $\widetilde{F}_\Phi$, proved in Proposition 2.1.7 and Corollary 2.1.10. Property (vi) (resp., (vii)) is a consequence of Proposition 1.3.4(i) (resp., 1.3.4(ii)) combined with the second (resp., the first) formula in Proposition 2.1.4(ii).

(2.2.4) Proposition. *Let K' be a CM number field containing K; put $X' = X(F')$, where F' is the maximal totally real subfield of K'. If Φ is a CM type of K and Φ' is the induced CM type of K', then:*

(i) $\forall g \in \mathrm{Aut}_{F-\mathrm{alg}}(F \otimes \overline{\mathbf{Q}}) \qquad \widetilde{F}_{\Phi'}(\mathrm{id}_{F'} \otimes_F g) = V_{K'/K}\left(\widetilde{F}_{\Phi}(g)\right) \in \Gamma_{K'}^{ab}.$

(ii) $\forall i = 0, 1\ \forall g \in \mathrm{Aut}_{F-\mathrm{alg}}(F \otimes \overline{\mathbf{Q}})_i \qquad \mathrm{id}_{F'} \otimes_F g \in \mathrm{Aut}_{F'-\mathrm{alg}}(F' \otimes \overline{\mathbf{Q}})_i.$

(iii) $\forall g \in \mathrm{Aut}_{F-\mathrm{alg}}(F \otimes \overline{\mathbf{Q}})_1 \qquad \widetilde{f}_{\Phi'}(\mathrm{id}_{F'} \otimes_F g) = i_{K'/K}\left(\widetilde{f}_{\Phi}(g)\right) \in \widehat{K}'^*/K'^*.$

Proof. (i) Fix a section $s : X \longrightarrow \Gamma_{\mathbf{Q}}$; let $\alpha : X \longrightarrow \mathbf{Z}/2\mathbf{Z}$ correspond to Φ, as in (2.1.1.3). The sets $\Gamma_K/\Gamma_{K'}$ and $\Gamma_F/\Gamma_{F'}$ are canonically identified. Fix a section $u : \Gamma_K/\Gamma_{K'} = \mathrm{Hom}_{K-\mathrm{alg}}(K', \overline{\mathbf{Q}}) \longrightarrow \Gamma_K$ of the restriction map $g \mapsto g|_K$ and define a section $s' : X' \longrightarrow \Gamma_{\mathbf{Q}}$ by

$$s'(s(x)y|_{F'}) = s(x)u(y) \qquad\qquad (x \in X,\ y \in \Gamma_F/\Gamma_{F'});$$

then Φ' corresponds to $\alpha' = \alpha \circ p : X' \longrightarrow \mathbf{Z}/2\mathbf{Z}$, where we have denoted by $p : X' \longrightarrow X$ the restriction map $g \mapsto g|_F$. Proposition 1.1.8 implies that the elements

$$(\sigma, h) = \beta_{s*}(g) \in S_X \ltimes \Gamma_F^X, \qquad (\sigma', h') = \beta_{s'*}(\mathrm{id}_{F'} \otimes_F g) \in S_{X'} \ltimes \Gamma_{F'}^{X'}$$

are related by

$$\sigma'(s(x)y|_{F'}) = s(\sigma(x))h(x)y|_{F'}, \qquad s'(\sigma'(s(x)y|_{F'})) = s(\sigma(x))u(h(x)y),$$
$$h'(s(x)y|_{F'}) = u(h(x)y)^{-1}h(x)u(y) \qquad\qquad (x \in X,\ y \in \Gamma_F/\Gamma_{F'}),$$

hence $\overline{h}' = \overline{h} \circ p$. For $x \in X$ and $x' \in X'$, put

$$k(x) = s(\sigma(x))^{-1}c^{\alpha(x)+\overline{h}(x)}s(\sigma(x))h(x)s(x)^{-1}c^{\alpha(x)}s(x) \in \Gamma_K$$
$$k'(x') = s'(\sigma'(x'))^{-1}c^{\alpha'(x')+\overline{h}'(x')}s'(\sigma'(x'))h'(x')s'(x')^{-1}c^{\alpha'(x')}s'(x') \in \Gamma_{K'}.$$

By definition,

$$\widetilde{F}_{\Phi}(g) = {}_s\widetilde{F}_{\alpha}(\sigma, h) = \prod_{x \in X} k(x)|_{K^{ab}} \in \Gamma_K^{ab}, \qquad \widetilde{F}_{\Phi'}(\mathrm{id}_{F'} \otimes_F g) = {}_{s'}\widetilde{F}_{\alpha'}(\sigma', h') = \prod_{x' \in X'} k'(x')|_{K'^{ab}} \in \Gamma_{K'}^{ab}.$$

For each $x \in X$ and $y \in \Gamma_F/\Gamma_{F'}$,

$$k'(s(x)y|_{F'}) = u(h(x)y)^{-1}k(x)u(y) \in \Gamma_{K'},$$

which implies that $k(x)y = k(x)u(y)|_{K'} = u(h(x)y)|_{K'} = h(x)y$, hence $u(h(x)y) = u(k(x)y)$ and

$$\prod_{x' \in p^{-1}(x)} k'(x')|_{K'^{ab}} = \prod_{y \in \Gamma_K/\Gamma_{K'}} u(k(x)y)^{-1}k(x)u(y)|_{K'^{ab}} = V_{K'/K}\left(k(x)|_{K^{ab}}\right).$$

Taking the product over all $x \in X$ yields (i). The statement (ii) follows from the fact that, in the notation used in the proof of (i),

$$\prod_{x' \in p^{-1}(x)} h'(x')|_{F'^{ab}} = \prod_{y \in \Gamma_F/\Gamma_{F'}} u(h(x)y)^{-1}h(x)u(y)|_{F'^{ab}} = V_{F'/F}\left(h(x)|_{F^{ab}}\right).$$

Finally, (iii) follows by applying $\ell_{K'}$ to the statement of (i) (which makes sense, by (ii) for $i = 1$).

(2.2.5) Action of $\mathrm{Aut}_{F-\mathrm{alg}}(F \otimes \overline{\mathbf{Q}})_0$ on CM points of Hilbert modular varieties. In this section we prove Theorem 0.9. Given a polarised HBAV (Hilbert-Blumenthal abelian variety) A relative to F with CM, then A is defined over $\overline{\mathbf{Q}}$, and there exist

- a CM field K of degree 2 over F;
- a CM type Φ of K (which defines an embedding $K \hookrightarrow \mathbf{C}^{\Phi}$, $\alpha \mapsto (\varphi \mapsto \varphi(\alpha))_{\varphi\in\Phi}$);
- a fractional ideal a of K;
- an element $t \in K^*$ such that $t \notin F^*$, $t^2 \in F^*$ and $\forall \varphi \in \Phi \quad \mathrm{Im}(\varphi(t)) < 0$;
- an O_K-linear isomorphism $\theta : \mathbf{C}^{\Phi}/a \xrightarrow{\sim} A(\mathbf{C})$ such that the Riemann form of the pull-back of the polarisation of A by θ is induced by the form $E_t(x,y) = \mathrm{Tr}_{K/\mathbf{Q}}(tx\,{}^c y)$ on K.

One says that A is a CM abelian variety of type (K, Φ, a, t) (via θ). The type is determined up to transformations $(K, \Phi, a, t) \mapsto (K, \Phi, a\alpha, t/{}^{1+c}\alpha)$ $(\alpha \in K^*)$, and it determines A with its polarisation up to isomorphism.

Given $g \in \mathrm{Aut}_{F-\mathrm{alg}}(F \otimes \overline{\mathbf{Q}})_0$, let $u(g) \in \Gamma^{ab}_{\mathbf{Q}}$ be as in Proposition 2.2.3(vii). Fix a lift $\widetilde{f} \in \widehat{K}^*$ of $\widetilde{f}_{\Phi}(g) \in \widehat{K}^*/K^*$ and define $A' = \mathbf{C}^{g\Phi}/a\widetilde{f}$, with polarisation given by $E_{t'}$, where

$$t' = t\,\chi(u(g))/{}^{1+c}\widetilde{f} \in K^*$$

(t' satisfies $t' \notin F^*$, $t'^2 \in F^*$ and $\forall \varphi' \in g\Phi \quad \mathrm{Im}(\varphi'(t')) < 0$, the last condition by Proposition 2.2.3(vii)).

Given, in addition, a full level structure $\eta : (F/O_F)^2 \xrightarrow{\sim} A(\overline{\mathbf{Q}})_{\mathrm{tors}}$ of A under which the Weil pairing associated to the given polarisation is a $\widehat{\mathbf{Q}}^*$-multiple of the standard form $\mathrm{Tr}_{\widehat{F}/\widehat{\mathbf{Q}}} \circ \det_{\widehat{F}}$ on $\widehat{F}^2$, let η' be the following level structure of A':

$$\eta' : (F/O_F)^2 \xrightarrow{\eta} A(\mathbf{C})_{\mathrm{tors}} \xrightarrow{\theta^{-1}} K/a \xrightarrow{[\times\widetilde{f}]} K/a\widetilde{f} = A'(\mathbf{C})_{\mathrm{tors}}.$$

The isomorphism class of the triple $(A', E_{t'}, \eta')$ depends only on g and on the isomorphism class $[(A, E_t, \eta)]$ of (A, E_t, η). Proposition 2.2.3 implies that the assignment

$${}^g[(A, E_t, \eta)] = [(A', E_{t'}, \eta')]$$

defines an action of $\mathrm{Aut}_{F-\mathrm{alg}}(F \otimes \overline{\mathbf{Q}})_0$ on the isomorphism classes of polarised HBAV (relative to F) with CM, equipped with a full level structure. Moreover, this action commutes with the action of $G(\widehat{F})$ on η (by $\gamma : \eta \mapsto \eta \circ \gamma$), where G is the fibre product

$$\begin{array}{ccc} G & \longrightarrow & R_{F/\mathbf{Q}}(GL(2)_F) \\ \downarrow & & \downarrow \scriptstyle{\det} \\ \mathbf{G}_{m,\mathbf{Q}} & \longrightarrow & R_{F/\mathbf{Q}}(\mathbf{G}_{m,F}). \end{array}$$

In view of the results of Tate and Deligne that were recalled in 1.4.3, it follows from Proposition 2.2.3 that the action of $\mathrm{Aut}_{F-\mathrm{alg}}(F \otimes \overline{\mathbf{Q}})_0$ we have just defined extends the usual Galois action of $\Gamma_{\mathbf{Q}}$.

Recall that $\widetilde{f}_{\Phi}(g)$ is defined even for $g \in \mathrm{Aut}_{F-\mathrm{alg}}(F \otimes \overline{\mathbf{Q}})_1$. However, the positivity of polarisations implies that the above recipe makes sense only if the conlusion of Proposition 2.2.3(vii) is satisfied, namely if $g \in \mathrm{Aut}_{F-\mathrm{alg}}(F \otimes \overline{\mathbf{Q}})_0$.

(2.2.6) Proposition-Definition. *Fix $s : X \longrightarrow \Gamma_{\mathbf{Q}}$ as in 2.1.1; then $X(K) = \{s(x)c^a|_K \mid x \in X, a \in \mathbf{Z}/2\mathbf{Z}\}$.*

(i) *Let $g \in \mathrm{Aut}_{F-\mathrm{alg}}(F \otimes \overline{\mathbf{Q}})$; put $(\sigma, h) = \beta_{s*}(g) \in S_X \ltimes \Gamma_F^X$. The formula*

$${}^g\left(s(x)c^a|_K\right) := s(\sigma(x))c^{\overline{h}(x)+a}|_K = s(\sigma(x))h(x)c^a|_K$$

defines an action of $\mathrm{Aut}_{F-\mathrm{alg}}(F \otimes \overline{\mathbf{Q}})$ on $X(K)$. The action of g on $X(K)$ depends only on the image of (σ, h) in $S_X \ltimes \mathrm{Gal}(K/F)^X$.

(ii) *This action does not depend on the choice of s.*

(iii) *For each CM type $\Phi \subset X(K)$ of K, the set ${}^g\Phi = \{{}^g y \mid y \in \Phi\}$ coincides with $g\Phi$, defined in (2.1.5.4).*

(iv) *If $g = \mathrm{id}_F \otimes u$, $u \in \Gamma_{\mathbf{Q}}$, then ${}^g y = u \circ y = uy$, for each $y \in X(K)$.*

Proof. Easy calculation.

(2.2.7) Corollary-Definition. (i) *The induced action of* $\mathrm{Aut}_{F-\mathrm{alg}}(F \otimes \overline{\mathbf{Q}})$ *on* $X^*({}_KT) = \mathbf{Z}[X(K)]$

$$\lambda = \sum n_y[y] \mapsto {}^g\lambda = \sum n_y[{}^gy] \qquad\qquad (g \in \mathrm{Aut}_{F-\mathrm{alg}}(F \otimes \overline{\mathbf{Q}}))$$

extends the action (1.5.1.1) of $\Gamma_{\mathbf{Q}}$ *and leaves stable the subgroup* $X^*({}_K\mathscr{S})$ *of* $X^*({}_KT)$ *spanned by the CM characters* λ_Φ.
(ii) *In the special case when* K *is a Galois extension of* $\mathbf{Q}$, *the involution* ι *from (1.5.4.3) gives rise to another action of* $\mathrm{Aut}_{F-\mathrm{alg}}(F \otimes \overline{\mathbf{Q}})$ *on* $X^*({}_KT)$, *namely*

$$g * \iota(\lambda) = \iota({}^g\lambda) \qquad\qquad (\lambda \in X^*({}_KT)).$$

This action extends the action (1.5.4.1) of $\Gamma_{\mathbf{Q}}$ *and leaves stable* $X^*({}_K\mathscr{S})$.

(2.2.8) Proposition. *Let* $n : \{\text{CM types of } K\} \longrightarrow \mathbf{Z}$ *be a function satisfying*

$$\sum_\Phi n_\Phi \lambda_\Phi = w \cdot N_{K/\mathbf{Q}} = w \sum_{y \in X(K)} [y] \in X^*({}_K\mathscr{S}) \qquad\qquad (w \in \mathbf{Z}). \quad \text{Then :}$$

(i) $\forall g \in \mathrm{Aut}_{F-\mathrm{alg}}(F \otimes \overline{\mathbf{Q}}) \qquad \prod_\Phi \widetilde{F}_\Phi(g)^{n_\Phi} = \widetilde{V}_{K/F}(g)^w$.
(ii) *If* $w = 0$, *then* $\forall g \in \mathrm{Aut}_{F-\mathrm{alg}}(F \otimes \overline{\mathbf{Q}})_1 \qquad \prod_\Phi \widetilde{f}_\Phi(g)^{n_\Phi} = 1 \in \widehat{K}^*/K^*$.

Proof. (i) Fix $s : X \longrightarrow \Gamma_{\mathbf{Q}}$ as in 2.1.1, and parameterise the CM types by functions $\alpha : X \longrightarrow \mathbf{Z}/2\mathbf{Z}$, as in (2.1.1.3): we write $\Phi_\alpha = \{s(x)c^{\alpha(x)}|_K\}_{x \in X}$, $n_\alpha = n_{\Phi_\alpha}$ and $\lambda_\alpha = \lambda_{\Phi_\alpha}$. The condition $\sum_\Phi n_\Phi \lambda_\Phi = w \cdot N_{K/\mathbf{Q}}$ is equivalent to

$$\forall x \in X \qquad \sum_\alpha n_\alpha \lambda_\alpha(x) = \sum_\alpha n_\alpha (1 - \lambda_\alpha(x)) = w.$$

Statement (i) follows from the fact that, for each $g = (\sigma, h) \in S_X \ltimes \Gamma_F^X$,

$$\prod_\alpha {}_s\widetilde{F}_\alpha(g)^{n_\alpha} = \prod_{x \in X} \gamma_{x,s}^{\sum_\alpha (n_\alpha \lambda_{g\alpha}(x) - n_\alpha \lambda_\alpha(x))} R(h(x))^{\sum_\alpha n_\alpha (1 - \lambda_\alpha(x))} ({}^cR(h(x)))^{\sum_\alpha n_\alpha \lambda_\alpha(x)} =$$

$$= \prod_{x \in X} {}^{1+c}R(h(c))^w = \widetilde{V}_{K/F}(g)^w.$$

If $w = 0$, statement (ii) follows by applying ℓ_K to (i).

2.3 Generalised universal Taniyama elements

As in §1.6, we assume that K is a CM number field which is a Galois extension of $\mathbf{Q}$.

(2.3.1) Proposition. (i) *There exists a unique map* $\widetilde{f}' : \mathrm{Aut}_{F-\mathrm{alg}}(F \otimes \overline{\mathbf{Q}})_1 \longrightarrow {}_K\mathscr{S}(\widehat{K})/{}_K\mathscr{S}(K)$ *such that* $\lambda_\Phi \circ \widetilde{f}' = \widetilde{f}_\Phi$, *for all CM types* Φ *of* K. *The map* $\widetilde{f}'$ *factors through* $\mathrm{Aut}_{F-\mathrm{alg}}(F \otimes K^{ab})_1$.
(ii) *For each* $\lambda \in X^*({}_K\mathscr{S})$, *put* $\widetilde{f}'_\lambda = \lambda \circ \widetilde{f}' : \mathrm{Aut}_{F-\mathrm{alg}}(F \otimes \overline{\mathbf{Q}})_1 \longrightarrow \widehat{K}^*/K^*$; *then* $\widetilde{f}'_{\lambda+\mu}(g) = \widetilde{f}'_\lambda(g)\widetilde{f}'_\mu(g)$.
(iii) $\forall \lambda \in X^*({}_K\mathscr{S})\ \forall g, g' \in \mathrm{Aut}_{F-\mathrm{alg}}(F \otimes \overline{\mathbf{Q}})_1 \qquad \widetilde{f}'_\lambda(gg') = \widetilde{f}'_{g'\lambda}(g)\widetilde{f}'_\lambda(g')$.
(iv) $\forall u \in \mathrm{Gal}(K/\mathbf{Q}) \qquad {}^u(\widetilde{f}'_\lambda(g)) = \widetilde{f}'_{u*\lambda}([u|_F]g)$.
(v) $\forall g \in \Gamma_{\mathbf{Q}} \qquad \widetilde{f}'(\mathrm{id}_F \otimes g) = f'(g)$.

Proof. Statements (i) and (ii) follow from Proposition 2.2.8(ii) by the same argument as in the proof of Proposition 1.6.2. If $\lambda = \lambda_\Phi$, then (iii) (resp., (iv)) is just the statement of Proposition 2.2.3 (iv) (resp., (v)); the general case follows from (ii). Finally, (v) is a consequence of the uniqueness of f', since

$$\forall \Phi \qquad \lambda_\Phi(\widetilde{f}'(\mathrm{id}_F \otimes g)) = \widetilde{f}_\Phi(\mathrm{id}_F \otimes g) = f_\Phi(g) = \lambda_\Phi(f'(g)),$$

by Proposition 2.2.3(ii).

(2.3.2) Proposition. (i) *Define the map* $\widetilde{f} : \mathrm{Aut}_{F-\mathrm{alg}}(F \otimes \overline{\mathbf{Q}})_1 \longrightarrow {}_K\mathscr{S}(\widehat{K})/{}_K\mathscr{S}(K)$ *by the formula* $\widetilde{f}(g) = (\iota(\widetilde{f}'(g)))^{-1}$. *This map factors through* $\mathrm{Aut}_{F-\mathrm{alg}}(F \otimes K^{ab})_1$ *and has the following properties:*
(ii) *The maps* $\widetilde{f}_\lambda = \lambda \circ \widetilde{f} : \mathrm{Aut}_{F-\mathrm{alg}}(F \otimes \overline{\mathbf{Q}})_1 \longrightarrow \widehat{K}^*/K^*$ $(\lambda \in X^*({}_K\mathscr{S}))$ *satisfy*

$$\widetilde{f}_{\lambda+\mu}(g) = \widetilde{f}_\lambda(g)\widetilde{f}_\mu(g), \qquad \widetilde{f}_\lambda(g) = \widetilde{f}'_{\iota(\lambda)}(g)^{-1}, \qquad \widetilde{f}_\lambda(gg') = \widetilde{f}_{g'*\lambda}(g)\widetilde{f}_\lambda(g').$$

(iii) $\forall u \in \mathrm{Gal}(K/\mathbf{Q})\ \forall g \in \mathrm{Aut}_{F-\mathrm{alg}}(F \otimes \overline{\mathbf{Q}})_1 \qquad {}^u(\widetilde{f}_\lambda(g)) = \widetilde{f}_{^u\lambda}([u|_F]g), \qquad {}^u(\widetilde{f}(g)) = \widetilde{f}([u|_F]g).$
(iv) $\forall g, g' \in \mathrm{Aut}_{F-\mathrm{alg}}(F \otimes \overline{\mathbf{Q}})_1 \qquad \widetilde{f}(gg') = (g'^{-1} * \widetilde{f}(g))\, \widetilde{f}(g').$
(v) $\forall g \in \Gamma_{\mathbf{Q}} \qquad \widetilde{f}(\mathrm{id}_F \otimes g) = f(g).$

Proof. As in the proof of Proposition 1.6.3, everything follows from Proposition 2.3.1.

(2.3.3) Proposition. *There exists a lift* $\widetilde{b} : \mathrm{Aut}_{F-\mathrm{alg}}(F \otimes K^{ab})_1 \longrightarrow {}_K\mathscr{S}(\widehat{K})$ *of* $\widetilde{f}$ *whose "coboundary"* $\widetilde{d}_{g,g'} = (g'^{-1} * \widetilde{b}(g))\, \widetilde{b}(g')\, \widetilde{b}(gg')^{-1}$ *is a locally constant function on* $(\mathrm{Aut}_{F-\mathrm{alg}}(F \otimes K^{ab})_1)^2$.

Proof. The argument from the proof of Proposition 1.7.5 applies.

(2.3.4) Proposition. *If* K' *is a CM number field, which is a Galois extension of* $\mathbf{Q}$ *and contains* K*, then the generalised universal Taniyama elements* $\widetilde{f}_K : \mathrm{Aut}_{F-\mathrm{alg}}(F \otimes \overline{\mathbf{Q}})_1 \longrightarrow {}_K\mathscr{S}(\widehat{K})/{}_K\mathscr{S}(K)$ *and* $\widetilde{f}_{K'} : \mathrm{Aut}_{F'-\mathrm{alg}}(F' \otimes \overline{\mathbf{Q}})_1 \longrightarrow {}_{K'}\mathscr{S}(\widehat{K}')/{}_{K'}\mathscr{S}(K')$ *over* K *and* K'*, respectively, satisfy*

$$\forall g \in \mathrm{Aut}_{F-\mathrm{alg}}(F \otimes \overline{\mathbf{Q}})_1 \qquad \widetilde{f}_K(g) = N_{K'/K}\left(\widetilde{f}_{K'}(\mathrm{id}_{F'} \otimes_F g)\right).$$

Proof. This follows from Proposition 2.2.4(iii), as in the proof of Proposition 1.6.5.

2.4 Generalised Taniyama group

Let K be as in §2.3.

(2.4.1) Let us try to apply the method of [Mi-Sh, Prop. 2.7] (see 1.7.3 above) to the generalised universal Taniyama element $\widetilde{f}$ and its lift $\widetilde{b}$. The reverse 2-cocycle $\widetilde{d}_{g,g'}$ with values in ${}_K\mathscr{S}(K)$ gives rise to an exact sequence of affine group schemes over K

$$1 \longrightarrow {}_K\mathscr{S}_K \xrightarrow{\widetilde{\imath}} \widetilde{G}' \xrightarrow{\widetilde{\pi}} \mathrm{Aut}_{F-\mathrm{alg}}(F \otimes K^{ab})_1 \longrightarrow 1 \tag{2.4.1.1}$$

(where the term on the right is considered as a constant group scheme), equipped with a section $\widetilde{\alpha} : \mathrm{Aut}_{F-\mathrm{alg}}(F \otimes K^{ab})_1 \longrightarrow \widetilde{G}'(K)$ such that

$$\forall g, g' \in \mathrm{Aut}_{F-\mathrm{alg}}(F \otimes K^{ab})_1 \qquad \widetilde{\alpha}(gg') = \widetilde{\alpha}(g)\widetilde{\alpha}(g')\widetilde{d}_{g,g'}.$$

The map

$$\widetilde{sp} : \mathrm{Aut}_{F-\mathrm{alg}}(F \otimes K^{ab})_1 \longrightarrow \widetilde{G}'(\widehat{K}), \qquad \widetilde{sp}(g) = \widetilde{b}(g)\widetilde{\alpha}(g),$$

is a group homomorphism satisfying $\widetilde{\pi} \circ \widetilde{sp} = \mathrm{id}$.

(2.4.2) Each element $u \in \Gamma_K$ acts on $\widetilde{G}'(\overline{\mathbf{Q}})$ by

$${}^u(s\,\widetilde{\alpha}(g)) = {}^us\,\widetilde{\alpha}(g) \qquad\qquad (s \in {}_K\mathscr{S}(\overline{\mathbf{Q}})). \tag{2.4.2.2}$$

We extend this action to an action of $\Gamma_{\mathbf{Q}}$: for $u \in \Gamma_{\mathbf{Q}}$ and $g \in \mathrm{Aut}_{F-\mathrm{alg}}(F \otimes K^{ab})_1$, put

$$\widetilde{c}_u(g) = \widetilde{b}([u|_F]g)\, {}^u(\widetilde{b}(g))^{-1} \in {}_K\mathscr{S}(K).$$

As

$$\forall u, u' \in \Gamma_{\mathbf{Q}}\ \forall g \in \mathrm{Aut}_{F-\mathrm{alg}}(F \otimes K^{ab})_1 \qquad \widetilde{c}_{uu'}(g) = \widetilde{c}_u([u'|_F]g)\, {}^{u}(\widetilde{c}_{u'}(g)),$$

the formula

$$^{u}(s\,\widetilde{\alpha}(g)) = \widetilde{c}_u(g)\, {}^{u}s\,\widetilde{\alpha}(g) \qquad\qquad (s \in {}_K\mathscr{S}(\overline{\mathbf{Q}}),\ g \in \mathrm{Aut}_{F-\mathrm{alg}}(F \otimes K^{ab})_1) \tag{2.4.2.3}$$

defines an action of $\Gamma_{\mathbf{Q}}$ on $\widetilde{G}'(\overline{\mathbf{Q}})$ which extends the action (2.4.2.2) of Γ_K.

We define ${}_K\widetilde{\mathscr{T}}$ to be the affine group scheme over $\mathbf{Q}$ such that ${}_K\widetilde{\mathscr{T}}(\overline{\mathbf{Q}}) = \widetilde{G}'(\overline{\mathbf{Q}})$, with the $\Gamma_{\mathbf{Q}}$-action given by (2.4.2.3). The exact sequence (2.4.1.1) descends to an exact sequence

$$1 \longrightarrow {}_K\mathscr{S} \xrightarrow{\ \widetilde{\imath}\ } {}_K\widetilde{\mathscr{T}} \xrightarrow{\ \widetilde{\pi}\ } \mathrm{Aut}_{F-\mathrm{alg}}(F \otimes K^{ab})'_1 \longrightarrow 1, \tag{2.4.2.4}$$

where we have denoted by $\mathrm{Aut}_{F-\mathrm{alg}}(F \otimes K^{ab})'_1$ a twisted form of the constant group scheme $\mathrm{Aut}_{F-\mathrm{alg}}(F \otimes K^{ab})_1$, for which $u \in \Gamma_{\mathbf{Q}}$ acts on

$$\mathrm{Aut}_{F-\mathrm{alg}}(F \otimes K^{ab})'_1(\overline{\mathbf{Q}}) = \mathrm{Aut}_{F-\mathrm{alg}}(F \otimes K^{ab})_1$$

by $[u|_F]$. Note that

$$\mathrm{Aut}_{F-\mathrm{alg}}(F \otimes K^{ab})'_1(\mathbf{Q}) = \mathrm{id}_F \otimes \mathrm{Gal}(K^{ab}/\mathbf{Q}), \tag{2.4.2.5}$$

by Proposition 1.1.6(iv).

(2.4.3) As $\widetilde{f}$ extends f (and the restriction of $\widetilde{b}$ to $\mathrm{Gal}(K^{ab}/\mathbf{Q})^2$ satisfies 1.7.5), there is a commutative diagram of affine group schemes over $\mathbf{Q}$ with exact rows:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & {}_K\mathscr{S} & \xrightarrow{i} & {}_K\mathscr{T} & \xrightarrow{\pi} & \mathrm{Gal}(K^{ab}/\mathbf{Q}) & \longrightarrow & 1 \\
 & & \| & & \downarrow & & \downarrow{\scriptstyle (\mathrm{id}_F \otimes -)} & & \\
1 & \longrightarrow & {}_K\mathscr{S} & \xrightarrow{\widetilde{\imath}} & {}_K\widetilde{\mathscr{T}} & \xrightarrow{\widetilde{\pi}} & \mathrm{Aut}_{F-\mathrm{alg}}(F \otimes K^{ab})'_1 & \longrightarrow & 1.
\end{array}$$

Moreover, there is a commutative diagram of groups

$$\begin{array}{ccc}
{}_K\mathscr{T}(\widehat{\mathbf{Q}}) & \xleftarrow{sp} & \mathrm{Gal}(K^{ab}/\mathbf{Q}) \\
\downarrow & & \downarrow{\scriptstyle (\mathrm{id}_F \otimes -)} \\
{}_K\widetilde{\mathscr{T}}(\widehat{K}) & \xleftarrow{\widetilde{sp}} & \mathrm{Aut}_{F-\mathrm{alg}}(F \otimes K^{ab})_1
\end{array}$$

such that $\pi \circ sp = \mathrm{id}$, $\widetilde{\pi} \circ \widetilde{sp} = \mathrm{id}$. As

$$\begin{aligned}
\forall u \in \Gamma_{\mathbf{Q}}\ \forall g \in \mathrm{Aut}_{F-\mathrm{alg}}(F \otimes K^{ab})_1 \qquad {}^{u}\widetilde{sp}(g) &= {}^{u}(\widetilde{b}(g)\widetilde{\alpha}(g)) = \widetilde{c}_u(g)\, {}^{u}\widetilde{b}(g)\, \widetilde{\alpha}([u|_F]g) = \\
&= \widetilde{b}([u|_F]g)\, \widetilde{\alpha}([u|_F]g) = \widetilde{sp}([u|_F]g),
\end{aligned}$$

the map $\widetilde{sp}$ is $\Gamma_{\mathbf{Q}}$-equivariant. As $[u|_F]$ depends only on the image of u in $\mathrm{Gal}(F/\mathbf{Q})$, it follows that the image of $\widetilde{sp}$ is contained in ${}_K\widetilde{\mathscr{T}}(\widehat{F})$, and that $\widetilde{sp}$ is $\mathrm{Gal}(F/\mathbf{Q})$-equivariant.

(2.4.4) Proposition 2.3.4 implies that the pull-backs of ${}_{K'}\widetilde{\mathscr{T}}$ to $\mathrm{Aut}_{F-\mathrm{alg}}(F \otimes \overline{\mathbf{Q}})'_1$ (for varying $K' \supset K$) give rise to an extension of $\mathrm{Aut}_{F-\mathrm{alg}}(F \otimes \overline{\mathbf{Q}})'_1$ by $\mathscr{S}$. These extensions for varying F are again compatible; they give rise to an extension of affine group schemes over $\mathbf{Q}$,

$$1 \longrightarrow \mathscr{S} \longrightarrow \widetilde{\mathscr{T}} \longrightarrow \varinjlim_{F} \mathrm{Aut}_{F-\mathrm{alg}}(F \otimes \overline{\mathbf{Q}})'_1 \longrightarrow 1,$$

whose pull-back to $\Gamma_{\mathbf{Q}}$ coincides with (1.7.6.1). The direct limit is taken with respect to the transition maps $\mathrm{id}_{F'} \otimes_F -$ (for $F \subseteq F'$).

(2.4.5) It would be of interest to give an "abstract" definition of $\widetilde{\mathscr{T}}$ along the lines of [De]. As observed in 2.2.5, it is the group $\mathrm{Aut}_{F-\mathrm{alg}}(F \otimes \overline{\mathbf{Q}})_0$ rather than $\mathrm{Aut}_{F-\mathrm{alg}}(F \otimes \overline{\mathbf{Q}})_1$ which has a geometric significance, which means that one should rather consider the subgroup scheme $\widetilde{\mathscr{T}_0} \subset \widetilde{\mathscr{T}}$ sitting in the exact sequence

$$1 \longrightarrow \mathscr{S} \longrightarrow \widetilde{\mathscr{T}_0} \longrightarrow \varinjlim_{F} \mathrm{Aut}_{F-\mathrm{alg}}(F \otimes \overline{\mathbf{Q}})'_0 \longrightarrow 1.$$

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