Values of L-functions and p-adic cohomology

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In this article we give a survey of two relatively recent developments in number theory: (1) the method of "Euler systems"; (2) new ideas and techniques coming from p-adic cohomology (or, rather, p-adic Hodge theory).

The common thread underlying these two themes is a relationship (still largely conjectural) between

(L) Values of L-functions
(A) Arithmetic invariants

Well-known prototypes of such relationships are the Iwasawa Main Conjecture or the conjecture of Birch and Swinnerton-Dyer. In this context, Euler systems have been used to prove theorems about (A) $\longleftrightarrow$ (L) relations, while results about p-adic cohomology suggested formulation of precise conjectures about (A) $\longleftrightarrow$ (L) relations in a general 'motivic' setting.

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1. A brief history

What we now call "Euler systems" is a new descent method developed in pioneering works of F. Thaine, K. Rubin and V.A. Kolyvagin. Their most spectacular results are summed up in the following table:

1986	Thaine	– new bounds for ideal class groups of cyclotomic fields [77]
	Rubin	– finiteness of Ш for some elliptic curves with complex multiplication [57]
1987	Kolyvagin	– finiteness of Ш for some modular elliptic curves [35],[36]
1989	Kolyvagin	– determines the structure of Ш in some cases [38]
		– gives a general treatment of "Euler systems" used by Rubin and himself [37]
	Rubin	– new proof of the Iwasawa Main Conjecture over $\mathbf{Q}$ [58] (much simpler than the proof given by Mazur and Wiles [43])
1990	Rubin	– proof of the Iwasawa Main Conjecture over imaginary quadratic fields [59]
	Kolyvagin	– determines the structure of Selmer groups in some cases [39]

Since then, the method has been applied in other situations ([6],[9],[16], [34],[40],[41],[42],[45],[61],[69]) and has gradually acquired more cohomological flavour.

As for p-adic Hodge theory, it owes its origin to the works of Tate [74] and Grothendieck [26]. Today we view it as a theory of p-adic periods, comparing étale and de Rham comology of algebraic varieties over p-adic numbers. The current state of the art is described in [27] and is a result of a collective effort by many people.

In their seminal paper [7], Bloch and Kato used p-adic Hodge theory to formulate a conjecture about precise values of L-functions of (pure) motives, improving thus on previous conjectures, which left an undetermined rational factor:

1978	Deligne	– conjecture about critical values of L-functions, up to a rational factor [11]
1983	Beilinson	– conjectures about all special values of motivic L-functions, up to a rational factor [1],[2],[3],[4]
1988	Bloch, Kato	– precise conjecture about special values [7]
1991	Kato	– formulation of the Iwasawa Main Conjecture for motivic L-functions [33]
	Fontaine, Perrin-Riou	– generalization and simplification of the conjecture of Bloch and Kato [18],[19]

One of the central objects of the theory of Bloch–Kato are Selmer groups associated to general p-adic Galois representations. As we shall see below in Sec. 9, that is where cohomological Euler systems lie.

In general, an Euler system is formed by elements "explaining" values of a given L-function in a tower of abelian extensions of a base number field. These elements are not independent, but satisfy several compatibility relations. These relations can be plugged into a descent machinery due to Kolyvagin and Rubin, giving a non-trivial information about relevant arithmetic invariants (A). In all known cases the Euler system can be constructed with no apriori reference to L-functions. The relation to L-values must then be established separately.

Symbolically, an Euler system (E) is an "incarnation" of L-values (L), and provides a sought-for link between (A) and (L):

$$(L) \longleftarrow (E) \longrightarrow (A)$$

For a fixed prime number p, the "p-part" of (A) is controlled by the Euler system restricted to (some) abelian extensions of degrees divisible by a sufficiently high power of p, but ramified at various primes $l \neq p$ – this is a basic difference from Iwasawa theory (cf., however, [69]).

2. Euler systems – classical examples

(E1) E : "cyclotomic units" (or, rather, cyclotomic numbers) in real
  abelian extensions of $\mathbf{Q}$ ([37],[58])

  L : $L'(0,\chi)$ for even Dirichlet characters χ over $\mathbf{Q}$

  A : ideal class groups of real abelian extensions of $\mathbf{Q}$

Fix, for every integer $n > 1$, a primitive n-th root of unity $\zeta_n \in \mu_n$ in such a way that $\zeta_{mn}^m = \zeta_n$ (e.g. $\zeta_n = \exp(2\pi i/n)$). Then, for $n > 1$,

$$u_n := (1 - \zeta_n)(1 - \zeta_n^{-1})$$

lies in $\mathbf{Q}(\mu_n)^+$ (the maximal real subfield of $\mathbf{Q}(\mu_n)$) and is a unit if n is divisible by two different primes, or a p-unit if $n = p^r$.

Basic relations: let l be a prime number not dividing $n > 1$. Then

$$\begin{aligned} N_l\, u_{nl} &= (1 - Fr_l)u_n \\ u_{nl} &\equiv Fr_l(u_n) \pmod{\lambda} \end{aligned} \tag{2.1}$$

Here N_l denotes the norm map in the extension $\mathbf{Q}(\mu_{nl})^+/\mathbf{Q}(\mu_n)^+$, Fr_l the geometric Frobenius at l (the inverse of which raises the roots of unity in μ_n to the l-th power) and λ is any prime in $\mathbf{Q}(\mu_{nl})^+$ above l.

(E2) E : "elliptic units" in abelian extensions of an imaginary quadratic
  field $K = \mathbf{Q}(\sqrt{-D})$ ([59])

  L : $L'(0,\chi)$ for abelian Artin characters χ over K

  A : ideal class groups of abelian extensions of K

There exist various versions of elliptic units, which satisfy relations similar to (2.1) – see [10, II.2],[21],[59].

(E3) E : "Stark units" in abelian extensions of a number field K with at
  most one complex place

  L : $L'(0,\chi)$ for abelian extensions of K

  A : ideal class groups of (some) abelian extensions of K

Stark units ([61],[72],[76]) conjecturally appear whenever we are given an abelian extension F/K and a character $\chi : G(F/K) \longrightarrow \mathbf{C}^*$ such that the L-function $L(s,\chi)$ has zero of order 1 at $s = 0$. This can happen only if all infinite places of K except for one, say v_0, are real and v_0 splits completely in F/K. Stark [72] conjectures that in this case there is a number $\epsilon \in F^*$ (in fact a unit, unless $K = \mathbf{Q}$ or $\mathbf{Q}(\sqrt{-D})$ and the conductor of F/K is a prime power) such that for each character χ of $G(F/K)$ we have

$$L'_S(0,\chi) = -\frac{1}{m} \sum_{\sigma \in G(F/K)} \chi(\sigma) \log |\epsilon^\sigma|_{w_0} \tag{2.2}$$

for a suitable integer m. Here w_0 is a fixed place of F above v_0 and L_S is the incomplete L-function with Euler factors at primes dividing the conductor of F/K deleted.

If χ is faithful, then $L_S(s, \chi) = L(s, \chi)$ is the complete L-function. In this case, $L(s, \chi)$ has a simple zero at $s = 0$ iff all infinite places $v \neq v_0$ ramify in F/K.

(E4) E :   "Gauss sums" ([37],[60],[66])
     L : $L(0, \chi)$ for odd Dirichlet characters over $\mathbf{Q}$
     A :   odd part of ideal class groups of abelian extensions of $\mathbf{Q}$

A typical Gauss sum is obtained as follows: let $K = \mathbf{Q}(\mu_m)$ and let $\wp$ be a prime of K above $p \nmid m$. Write $\omega_\wp$ for the character $(\mathcal{O}_K/\wp\mathcal{O}_K)^* \longrightarrow \mu_m$ such that

$$\omega_\wp(a) \equiv a^{(N\wp-1)/m} \pmod{\wp} \qquad a \in \mathcal{O}_K$$

and let $\varepsilon : \mathcal{O}_K/\wp\mathcal{O}_K \longrightarrow \mu_p$ be a non-trivial additive character factoring through the trace to $\mathbf{Z}/p\mathbf{Z}$. The Gauss sum

$$S = \sum_{a \in (\mathcal{O}_K/\wp\mathcal{O}_K)^*} \omega_\wp(a)^{-1}\varepsilon(a)$$

lies in $K(\mu_p)^*$ and is a unit outside of primes above p. If we denote by σ_b (for an integer b prime to m) the automorphism of $K(\mu_p)$ which leaves μ_p fixed and raises elements of μ_m to the b-th power, then $S^{b-\sigma_b}$ lies in K^* (in fact in $\mathcal{O}_K[1/p]^*$).

Varying m and p, Kolyvagin [37] and Rubin [60] define an Euler system formed by certain products of Gauss sums; their definitions are notationally quite involved, one has to keep track of a lot of different indices, so they will not be reproduced here.

(E5) E :   "Heegner points" on Jacobians of modular (or Shimura) curves ([6],[9],[23],[24],[25],[35],[36],[37],[38],[39],[40],[41]) defined over ring class fields of an imaginary quadratic field $K = \mathbf{Q}(\sqrt{-D})$
     L : $L'(f \otimes K, 1)$ for modular forms of weight 2
     A :   rational points and Tate-Šafarevič groups of the Jacobians of modular (Shimura) curves over ring class fields

For an integer $N \geq 1$, the modular curve $Y_0(N)$ classifies, in a suitable sense, isogenies $\lambda : E \longrightarrow E'$ between elliptic curves E, E' with $\text{Ker}(\lambda) \xrightarrow{\sim} \mathbf{Z}/N\mathbf{Z}$. For example, its complex points correspond bijectively to isogenies between elliptic curves defined over complex numbers. $Y_0(N)$ has a natural compactification $X_0(N)$ and both curves are defined over $\mathbf{Q}$. For a fixed order $\mathcal{O}$ in an imaginary quadratic field K, a *Heegner point* corresponding to $\mathcal{O}$ is any point on $X_0(N)$ represented by an isogeny λ between two

curves with complex multiplication by $\mathcal{O}$, i.e. $\mathrm{End}(E) = \mathrm{End}(E') = \mathcal{O}$. The conductor of the Heegner point is, by definition, the conductor of $\mathcal{O}$, equal to $[\mathcal{O}_K : \mathcal{O}]$. Suppose that N is prime to the discriminant of K. Then a Heegner point of conductor n on $X_0(N)$ exists iff the following two conditions are satisfied:

$$\text{Each prime } q|N \text{ splits completely in } K \qquad (2.3a)$$
$$N = N_0 d^2, \quad n = n_0 d, \quad (N_0, n_0) = 1 \qquad (2.3b)$$

In fact, such a point is defined over K_n – the ring class field of K of conductor n. Suppose, for simplicity, that n is prime to N. Choosing an ideal $\mathcal{A} \subset \mathcal{O}_K$ with $\mathcal{O}_K/\mathcal{A} \simeq \mathbf{Z}/N\mathbf{Z}$ (such ideals exist, if (2.3a) is satisfied) and writing $\mathcal{O}_n = \mathbf{Z} + n\mathcal{O}_K$ for the order of conductor n, then the isogeny

$$\mathbf{C}/\mathcal{O}_n \longrightarrow \mathbf{C}/(\mathcal{O}_n \cap \mathcal{A})^{-1}$$

represents a Heegner point P_n on $X_0(N)$, defined over K_n. The divisor $(P_n) - (\infty)$ then defines a point x_n on the Jacobian $J_0(N)$ of $X_0(N)$, also defined over K_n.

Recall that K_1 is nothing else than the Hilbert class field of K and that $G(K_n/K)$ is canonically isomorphic to $\mathrm{Pic}(\mathcal{O}_n)$. From the exact sequence

$$\frac{(\mathbf{Z}/n\mathbf{Z})^*}{\mathbf{Z}^*} \longrightarrow \frac{(\mathcal{O}_K/n\mathcal{O}_K)^*}{\mathcal{O}_K^*} \longrightarrow \mathrm{Pic}(\mathcal{O}_n) \longrightarrow \mathrm{Pic}(\mathcal{O}_K) \longrightarrow 0$$

we see that

$$G(K_n/K_1) \overset{\sim}{\longrightarrow} \frac{(\mathcal{O}_K/n\mathcal{O}_K)^*}{\mathcal{O}_K^*(\mathbf{Z}/n\mathbf{Z})^*}$$

Basic relations: let l be a prime not dividing n and $(nl, ND) = 1$. Then

$$u_l\, N_l x_{nl} = \begin{cases} T_l x_n & \text{if } l \text{ is inert in } K \\ T_l x_n - \sigma(x_n) - \sigma^{-1}(x_n) & \text{if } l \text{ splits in } K \end{cases} \qquad (2.4)$$
$$x_{nl} \equiv Fr_l(x_n) \pmod{\lambda}$$

Here T_l is the l-th Hecke operator, u_l the order of the image of $\mathcal{O}_K^*/\mathbf{Z}^*$ in $(\mathcal{O}_K/l\mathcal{O}_K)^*/(\mathbf{Z}/l\mathbf{Z})^*$, $\sigma \in G(K_n/K)$ corresponds by class field theory to one of the factors of l in K and λ is any prime above l in K_{nl}.

3. Relations   (E) ⟷ (L)

Let us make explicit how various Euler systems from the previous section "compute" values of L-functions.

(E1)   Let $\chi : (\mathbf{Z}/N\mathbf{Z})^* \longrightarrow \mathbf{C}^*$ be an even ($\chi(-1) = 1$) Dirichlet character. Then the (incomplete) L-function

$$L(s, \chi) = \sum_{\substack{n=1 \\ (n,N)=1}}^{\infty} \chi(n)\, n^{-s}$$

has a first order zero at $s = 0$ and its derivative is given by

$$L'(0, \chi) = -\frac{1}{2} \sum_{\sigma} \chi(\sigma) \log |\epsilon^{\sigma}| \tag{3.1}$$

where $\epsilon = (1 - \zeta_N)(1 - \zeta_N^{-1})$, $\zeta_N = \exp(2\pi i/N)$ and σ runs through $G(\mathbf{Q}(\mu_N)^+/\mathbf{Q}) \xrightarrow{\sim} (\mathbf{Z}/N\mathbf{Z})^*/\{\pm 1\}$. This follows from the following elementary fact:

$$\frac{d}{ds} \sum_{\substack{n \in \mathbf{Z} \\ n \equiv a \pmod{N}}} |n|^{-s} \Big|_{s=0} = -\log|1 - \zeta_N^a|.$$

(E2)   If K is an imaginary quadratic field and χ any ray class character over K, say, of conductor f, then there is a similar formula

$$L'(0, \chi) = -\frac{1}{m} \sum \chi(\sigma) \log |\epsilon^{\sigma}| \tag{3.2}$$

for a suitable integer m (see [10, II.5.1]) with σ running through $G(K(f)/K)$ ($K(f)$ denoting the ray class field of conductor f over K) and ϵ a suitable "elliptic unit" in $K(f)$.

(E3)   The formula (2.2), which generalizes (3.2), is one of the defining properties of the Stark units.

(E4)   The classical factorization formula for the Gauss sums ([78, ch. 6]) relates them to the values of Dirichlet L-functions at $s = 0$. In the notation of the previous section, fix a non-trivial character χ of $G = G(K/\mathbf{Q})$ with values in a suitable ring R in which $[K : \mathbf{Q}]$ is invertible. Then the factorization of the χ-component of $S^{b-\sigma_b}$ is given by the formula

$$(S^{b-\sigma_b})^\chi = (\chi(b) - b)L(0,\overline{\chi})[\wp]^\chi$$

Here the equality takes place in the free R-module generated by the primes of K above p and

$$a^\chi = \left(\frac{1}{\sharp(G)} \sum_{\sigma \in G} \chi(\sigma)^{-1} \sigma \right) a$$

for each element a of an arbitrary $R[G]$-module.

(E5) In this case, the result is due to Gross–Zagier [25]. Let, as before, K be an imaginary quadratic field of discriminant $-D$, let f be a newform of weight 2 on $\Gamma_0(N)$. Assume that all primes $q|N$ split completely in K. If n is an integer prime to N, then there is a Heegner point $x_n \in J_0(N)(K_n)$ on the Jacobian of $X_0(N)$, defined over the ring class field of conductor n. Associated to f, there is a quotient $J_0(N) \longrightarrow A$ of $J_0(N)$ (determined up to an isogeny), also defined over $\mathbf{Q}$. Let y_n be the image of x_n in $A(K_n)$. For each ring class character $\chi : G(K_n/K) \longrightarrow \mathbf{C}^*$ we can take the χ-component of y_n:

$$y_n^\chi \in (A(K_n) \otimes \mathbf{C})^\chi$$

The main identity expresses the derivative of the L-series of A over K, twisted by the character χ, in terms of y_n^χ:

$$\frac{L'(A \otimes_{\mathbf{Q}} K, \chi, 1)}{\Omega_{A,K}} \doteq \langle y_n^\chi, y_n^\chi \rangle \tag{3.3}$$

Here $\Omega_{A,K}$ denotes certain period associated to A and $\langle \, , \, \rangle$ is (the hermitian extension of) the Néron-Tate height pairing on $A(K_n) \otimes \mathbf{C}$ (the sign $\doteq$ means equality up to an elementary factor which can be made explicit). The formula (3.3) has been proved so far only for $n = 1$ ([25]).

In a special case when the form f has rational coefficients, $A = E$ is an elliptic curve. If we take $n = 1$, $\chi = 1$, (3.3) reads as

$$\frac{L'(E/K, 1)}{\Omega_{E/K}} \doteq \langle y, y \rangle \tag{3.4}$$

for $y = Tr_{K_1/K} y_1 \in E(K)$.

4. Classical relations   (A) ⟷ (L)

Dedekind's class number formula relates the behaviour of the zeta function of a number field F at $s = 1$ (or, equivalently, at $s = 0$, thank to the functional equation) to the basic arithmetic invariants of F. If, in the standard notation, r_1 and r_2 denote the number of real and complex places of F respectively, then

$$r := \operatorname{ord}_{s=0} \zeta_F(s) = r_1 + r_2 - 1$$

$$\zeta_F^*(0) := \lim_{s \to 0} \zeta_F(s)\, s^{-r} = -\frac{h_F \cdot R_F}{w_F},$$

where $w_F = \sharp(\mathcal{O}_F^*)_{\mathrm{tors}}$ is the number of roots of unity in F, $h_F = \sharp \operatorname{Pic}(\mathcal{O}_F)$ the class number and R_F the regulator of units – the absolute value of the determinant of the $(r \times r)$ matrix $(\log |\sigma_i(\epsilon_j)|)$ (here ϵ_j is a basis of $\mathcal{O}_F^*$ modulo torsion and $\sigma_i : F \hookrightarrow \mathbf{C}$ embeddings given by (some) r distinct archimedean places of F).

Combined with relations (3.1-2) of the previous section, one gets purely algebraic statements, involving only (A) and (E).

(E1)   For $F = \mathbf{Q}(\mu_N)^+$, let C_F be the intersection of the group of units $\mathcal{O}_F^*$ with the group generated by ± 1, $\epsilon_m = (1 - \zeta_m)(1 - \zeta_m^{-1})$ and their conjugates for all divisors m of N (the group of "cyclotomic units"). Write A_F for the ideal class group of $\mathcal{O}_F$. Then

$$\sharp A_F \,(= h_F) = [\mathcal{O}_F^* : C_F], \tag{4.1}$$

possibly up to a power of 2 ([68],[78, ch.8]).

This result admits a stronger version, known as a conjecture of Gras [20] (proved in [43] as a corollary of the Iwasawa Main Conjecture). For simplicity, we consider only the special case when $F = \mathbf{Q}(\mu_N)^+$ with $p \nmid [F : \mathbf{Q}]$. For a fixed irreducible $\mathbf{Z}_p$-representation χ of $G = G(F/\mathbf{Q})$ and a $\mathbf{Z}_p[G]$-module M write

$$M^\chi = \left(\frac{1}{\sharp(G)} \sum_{\sigma \in G} \operatorname{Tr}(\chi(\sigma^{-1}))\, \sigma \right) M$$

for the χ-component of M. The conjecture of Gras says that

$$\sharp(A_F \otimes \mathbf{Z}_p)^\chi = \sharp((\mathcal{O}_F^*/C_F) \otimes \mathbf{Z}_p)^\chi \tag{4.2}$$

(E2–3)   If F is a ray class field of $K = \mathbf{Q}(\sqrt{-D})$, the formula (4.1) holds true with C_F replaced by the group of "elliptic units" [59]. The formula

(4.2) is also true whenever $p \nmid [F : K]$ [59]. Both are also expected to hold in the context of Stark units (see [61]).

(E4) A special case of the Iwasawa Main Conjecture for the odd part of the class groups of cyclotomic fields is the following ([58],[60],[78, 13.6]): let $F = \mathbf{Q}(\mu_p)$ and let $\chi : G(F/\mathbf{Q}) \longrightarrow \mathbf{Z}_p^*$ be an odd ($\chi(-1) = -1$) character different from the Teichmüller character ω. Then

$$\sharp(A_F \otimes \mathbf{Z}_p)^\chi = [\mathbf{Z}_p : L(0, \overline{\chi})\mathbf{Z}_p] \tag{4.3}$$

(E5) Let us consider an elliptic curve E, which is a factor of $J_0(N)$ corresponding to a newform f with rational coefficients. In this context, the conjecture of Birch and Swinnerton Dyer predicts the following ([44, I.7]):

Conjecture (BSD). *For any number field F, $\mathrm{ord}_{s=1} L(E/F, s) = 1$ iff $E(F)/E(F)_{\mathrm{tors}}$ is isomorphic to $\mathbf{Z}$. If $P \in E(F)$ is any element of infinite order, then*

$$\frac{L'(E/F, 1)}{\Omega_{E/F}} \doteq \frac{\sharp \text{III}(E/F)\langle P, P\rangle}{[E(F) : \mathbf{Z}P]^2}$$

In particular, if $F = K = \mathbf{Q}(\sqrt{-D})$ and $y \in E(K)$ are as in Sec. 2, then (BSD) together with (3.4) predict that

$$y \in E(K) \text{ is of infinite order} \implies \begin{cases} [E(K) : \mathbf{Z}y] \text{ is finite} \\ \sharp\text{III}(E/K) \doteq [E(K) : \mathbf{Z}y]^2 \end{cases}$$

5. Euler systems and descent

Let us first describe some of the results. The descent machine of Thaine–Rubin–Kolyvagin operates as follows: the given Euler system serves as an imput; and the output is a bound on (A) in terms of (E), or, if we are lucky, a precise structure of (A):

(E1) In the situation of Sec. 4, one gets ([37],[58])

$$\sharp(A_F \otimes \mathbf{Z}_p)^\chi \ \Big|\ \sharp((\mathcal{O}_F^*/C_F) \otimes \mathbf{Z}_p)^\chi \tag{5.1}$$

It follows from (4.1) that (5.1) is in fact an equality. The method even describes elementary divisors of $(A_F \otimes \mathbf{Z}_p)^\chi$. Using further tools from Iwasawa theory, Rubin [58] is able to deduce the whole Iwasawa Main Conjecture.

(E2-3) These results are extended over $\mathbf{Q}(\sqrt{-D})$ in [59] and also into the (still conjectural) context of Stark units [61].

(E4) In the notation of Sec. 4, it is proved in [37],[60] that

$$\sharp(A_F \otimes \mathbf{Z}_p)^\chi \,\Big|\, [\mathbf{Z}_p : L(0, \overline{\chi})\mathbf{Z}_p] \tag{5.2}$$

for all odd characters of $G(F/\mathbf{Q}) = G(\mathbf{Q}(\mu_p)/\mathbf{Q})$ different from ω. The class number formula for the odd part of A_F ([78, 4.17]) then implies that there must be an equality in (5.2), proving thus (4.3). In his forthcoming work [66], Schoof will give an interpretation of the descent in terms of the Fitting ideals of $A_F \otimes \mathbf{Z}_p$.

(E5) In [37], improving upon previous works [35],[36], Kolyvagin proves that, if $y \in E(K)$ is of infinite order, then

$$[E(K) : \mathbf{Z}y] \text{ is finite} \tag{5.3a}$$

$$\sharp\text{Ш}(E/K) \,\Big|\, [E(K) : \mathbf{Z}y]^2 \cdot \text{ controlled factor} \tag{5.3b}$$

A version of (5.3a) over K_n with $(n, N) = 1$ is proved in [6]. In [38], a precise description of the structure of $\text{Ш}(E/K)_{p^\infty}$ in terms of the Euler system $\{y_n\}$ is given (for almost all p), again assuming that y is not torsion. This result is further generalized in [39].

An interested reader can visualize the descent machine on the picture at the end of this article. The Euler system gets in, elementary divisors of (A) are falling out. The procedure, however, requires some effort on the part of the descenter or descentress. I am greatly indebted to my sister Tereza Nekovářová for drawing the picture.

We now describe, briefly, the main ideas involved (cf. [49]). In the context of ideal class groups, the classical way how to find upper bounds for A_F is to produce sufficiently many elements in F^* with an explicit factorization into prime ideals (using Gauss sums, this leads to the Stickelberger theorem). The descent method of Thaine–Rubin–Kolyvagin does the same, but inside $F^* \otimes \mathbf{Z}/p^M\mathbf{Z}$, where p^M is a fixed power of prime p. The starting point is an observation that we have an isomorphism

$$F^* \otimes \mathbf{Z}/p^M\mathbf{Z} \overset{\sim}{\longrightarrow} (F'^* \otimes \mathbf{Z}/p^M\mathbf{Z})^{G(F'/F)},$$

whenever F' is a Galois extension of F not containing μ_p.

The following notation will be useful: let $I = \bigoplus_v \mathbf{Z} \cdot [v]$ be the group of (fractional) ideals of F, i.e. the free abelian group on the set of nonarchimedean primes v. For $x \in F^* \otimes \mathbf{Z}/p^M\mathbf{Z}$ and a rational prime l write $[x]_l \in \bigoplus_{v|l} \mathbf{Z}/p^M\mathbf{Z} \cdot [v]$ for the factorization of x at primes dividing l.

Suppose we want to prove (5.1). Take $F' = F_n = \mathbf{Q}(\mu_{Nn})^+$ for suitable integers n prime to N. The cyclotomic units

$$x_n = (1 - \zeta_{Nn})(1 - \zeta_{Nn}^{-1}) \in \mathcal{O}_{F_n}^*$$

satisfy the relations (2.1). If $n = \prod_{l|n} l$ is square-free, then the Galois group $G_n = G(F_n/F)$ is the product of $G_l \xrightarrow{\sim} \mathbf{Z}/(l-1)\mathbf{Z}$. Fixing a generator σ_l of G_l, define

$$N_l = \sum_{i=0}^{l-2} \sigma_l^i, \quad D_l = \sum_{i=1}^{l-2} i\sigma_l^i \in \mathbf{Z}[G_l], \quad D_n = \prod_{l|n} D_l \in \mathbf{Z}[G_n]$$

Then

$$(\sigma_l - 1)D_l = (l-1) - N_l, \tag{5.4}$$

i.e., modulo $l-1$, N_l is "divisible" by $\sigma_l - 1$.

Suppose that n is square-free and such that each of its prime factors l satisfies $l \equiv 1 \pmod{p^M}$, $l \equiv \pm 1 \pmod N$. It then follows from the relations (2.1) and (5.4) that

(1) The class of $D_n x_n$ in $F_n^* \otimes \mathbf{Z}/p^M\mathbf{Z}$ is G_n-invariant, hence comes from an element $c_M(n) \in F^* \otimes \mathbf{Z}/p^M\mathbf{Z}$.
(2) As F_n/F is unramified outside n, $c_M(n)$ lies, in fact, in $\mathcal{O}_F[1/n]^* \otimes \mathbf{Z}/p^M\mathbf{Z}$, i.e. $[c_M(n)]_l = 0$ for $l \nmid n$.
(3) If $l|n$, then $[c_M(n)]_l$ is *explicitly* determined by the image of $c_M(n/l)$ in $(\mathcal{O}_F/l\mathcal{O}_F)^* \otimes \mathbf{Z}/p^M\mathbf{Z}$ (this makes sense by the previous remark). More precisely, we have

$$[c_M(n)]_l = \varphi_l(c_M(n/l))$$

for a natural $G(F/\mathbf{Q})$-equivariant surjective homomorphism (see [58])

$$\varphi_l : (\mathcal{O}_F/l\mathcal{O}_F)^* \otimes \mathbf{Z}/p^M\mathbf{Z} \longrightarrow \bigoplus_{v|l} \mathbf{Z}/p^M\mathbf{Z} \cdot [v]$$

We now sketch the proof of (5.1) in the simplest case $F = \mathbf{Q}(\mu_p)^+$ (see [58] for more details). Note that for $\chi \neq 1$ (for trivial χ both sides of (5.1) are equal to one) the statement of (5.1) is equivalent to

$$\sharp(A_F \otimes \mathbf{Z}/p^M\mathbf{Z})^\chi \mid \sharp((\mathcal{O}_F^* \otimes \mathbf{Z}/p^M\mathbf{Z})^\chi/\langle c_M(1)\rangle^\chi)$$

for sufficiently large M, where $\langle c_M(1) \rangle^\chi$ denotes the subgroup of $\mathcal{O}_F^* \otimes \mathbf{Z}/p^M\mathbf{Z}$ generated by $c_M(1)^\chi$. This is true because $c_M(1)$ is the image of $(1-\zeta_p)(1-\zeta_p^{-1})$ in $F^* \otimes \mathbf{Z}/p^M\mathbf{Z}$ (and its χ-component lies in $\mathcal{O}_F^* \otimes \mathbf{Z}/p^M\mathbf{Z}$ for non-trivial χ).

Fix sufficiently large M and ideal classes $\mathcal{A}_1, \ldots, \mathcal{A}_k$ generating $(A_F \otimes \mathbf{Z}_p)^\chi$. We shall construct by induction square-free integers $n_r = l_1 \ldots l_r$ with $l_i \equiv 1 \pmod{p^M}$ (for $r \leq k$) and interpret the explicit factorization of $c_M(n_r)^\chi \in (F^* \otimes \mathbf{Z}/p^M\mathbf{Z})^\chi$ as a non-trivial relation in $(A_F \otimes \mathbf{Z}/p^M\mathbf{Z})^\chi$. Put $n_0 = 1$ and assume that n_r has been defined in such a way that, for each $i \leq r$, the class of $[\lambda_i]^\chi$ (for a fixed prime λ_i in F over l_i) is equal to $\mathcal{A}_i$. Let t_r be the largest integer $t \leq M$ such that $c_M(n_r)^\chi$ is divisible by p^t in $(F^* \otimes \mathbf{Z}/p^M\mathbf{Z})^\chi$. We claim that there exists a prime $l \equiv 1 \pmod{p^M}$ not dividing n_r with the following two properties:

(A) The class of $[\lambda]^\chi$ is equal to $\mathcal{A}_{r+1}$, where λ is one of the primes of F above l.

(B) $\varphi_l(c_M(n_r)^\chi)$ is divisible by p^{t_r}, but not by p^{t_r+1} in $\oplus_{v|l}\mathbf{Z}/p^M\mathbf{Z} \cdot [v]$. This means, in fact, that

$$\varphi_l(c_M(n_r)^\chi) = p^{t_r} \cdot u \cdot [\lambda]^\chi, \qquad u \in (\mathbf{Z}/p^M\mathbf{Z})^*$$

To show this, it is necessary to reformulate both conditions (A),(B) in terms of the splitting of l in a suitable Galois extension of $\mathbf{Q}$ and then apply the Čebotarev density theorem (see [58, 3.1]). We put $l_{r+1} = l$ and continue until n_k has been defined. According to (3) and (B) above, the divisor of $c_M(n_{r+1})^\chi$ is equal, modulo a linear combination of $[\lambda_1]^\chi, \ldots, [\lambda_r]^\chi$, to $u \cdot p^{t_r} \cdot [\lambda_{r+1}]^\chi$. As it is divisible by $p^{t_{r+1}}$, we must have $t_{r+1} \leq t_r$ (hence $t_{r+1} \leq t_0$). Dividing by $p^{t_{r+1}}$, we see that the class of

$$p^{t_r-t_{r+1}}[\lambda_{r+1}]^\chi$$

in $(A_F \otimes \mathbf{Z}/p^{M-t_0}\mathbf{Z})^\chi$ $(= (A_F \otimes \mathbf{Z}_p)^\chi$ if M was chosen large enough) lies in the group generated by $\mathcal{A}_1, \ldots, \mathcal{A}_r$. This shows that

$$\sharp(A_F \otimes \mathbf{Z}_p)^\chi \,\Big|\, p^{t_0-t_k} \,\Big|\, p^{t_0} = \sharp((\mathcal{O}_F^* \otimes \mathbf{Z}/p^M\mathbf{Z})^\chi/\langle c_M(1)\rangle^\chi),$$

proving thus (5.1).

The descent on modular elliptic curves using Heegner points goes essentially along the same lines, but in the framework of Galois cohomology (the above construction could be interpreted cohomologically as well, as $F^* \otimes \mathbf{Z}/p^M\mathbf{Z} \xrightarrow{\sim} H^1(F, \mu_{p^M})$ by Kummer theory). For an elliptic curve E over a field F (say, of characteristic 0) there is the standard exact sequence (cf. [44, I.6])

$$0 \longrightarrow E(F) \otimes \mathbf{Z}/p^M\mathbf{Z} \xrightarrow{\delta} H^1(F, E_{p^M}) \longrightarrow H^1(F, E)_{p^M} \longrightarrow 0 \qquad (5.5)$$

coming from the short exact sequence of $G(\overline{F}/F)$-modules

$$0 \longrightarrow E_{p^M} \longrightarrow E(\overline{F}) \xrightarrow{p^M} E(\overline{F}) \longrightarrow 0$$

If F is a number field, the Selmer group relative to the multiplication by p^M on E, defined as

$$S(E/F, p^M) = \mathrm{Ker}\left(H^1(F, E_{p^M}) \longrightarrow \bigoplus_v H^1(F_v, E)_{p^M} \right),$$

sits in an exact sequence

$$0 \longrightarrow E(F) \otimes \mathbf{Z}/p^M\mathbf{Z} \longrightarrow S(E/F, p^M) \longrightarrow \text{Ш}(E/F)_{p^M} \longrightarrow 0,$$

where the third term is the p^M-torsion of the Tate-Šafarevič group of E over F

$$\text{Ш}(E/F) = \mathrm{Ker}\left(H^1(F, E) \longrightarrow \bigoplus_v H^1(F_v, E) \right)$$

Passing to the projective limit gives

$$0 \longrightarrow E(F) \otimes \mathbf{Z}_p \longrightarrow S_p(E/F) \longrightarrow T_p\text{Ш}(E/F) \longrightarrow 0$$

Here, for any abelian group A, $T_pA = \varprojlim A_{p^M}$ denotes its (p-)Tate module. The p-primary part of Ш has finite corank and is conjectured to be finite. If this is the case, then $S_p(E/F)$ is isomorphic to $E(F) \otimes \mathbf{Z}_p$.

If F'/F is a Galois extension, there are maps

$$(E(F') \otimes \mathbf{Z}/p^M\mathbf{Z})^{G(F'/F)} \xhookrightarrow{\delta} H^1(F', E_{p^M})^{G(F'/F)} \xleftarrow{res} H^1(F, E_{p^M}) \quad (5.6)$$

In the situation of (3.4), there are Heegner points $y_n \in E(K_n)$ (for n prime to N) satisfying (if $-D \neq -3, -4$)

$$\begin{aligned} N_l y_{nl} &= a_l y_n \\ y_{nl} &\equiv Fr_l(y_n) \pmod{\lambda} \end{aligned} \qquad (5.7)$$

for each prime $l \nmid n$ which is inert in K (here a_l is the l-th coefficient in the L-series $L(E/\mathbf{Q}, s)$ and λ any prime above l in K_{nl}).

Kolyvagin takes n square-free, divisible only by primes l inert in K and such that $a_l, l+1$ are both divisible by p^M (this is a condition on Fr_l in the extension $\mathbf{Q}(E_{p^M})/\mathbf{Q}$), so there is a sufficient supply of such primes by the Čebotarev density theorem). Writing $G_n = G(K_n/K_1)$, then $G_n = \prod_{l|n} G_l$. Fixing a generator σ_l of $G_l \xrightarrow{\sim} \mathbf{Z}/(l+1)\mathbf{Z}$, we have operators $D_n \in \mathbf{Z}[G_n]$ as above (with $l-1$ replaced by $l+1$). Thank to the relations (5.4) and (5.7), the image of $D_n y_n$ in $E(K_n) \otimes \mathbf{Z}/p^M\mathbf{Z}$ is G_n-invariant. The restriction map in (5.6) is for all practical purposes an isomorphism (if $F'/F = K_n/K_1$); its "inverse" carries $\delta([D_n y_n])$ into $H^1(K_1, E_{p^M})$. Finally, taking the trace $\mathrm{cor}_{K_1/K}$ we get a cohomology class

$$c_M(n) \in H^1(K, E_{p^M})$$

Denote by $d_M(n)$ its image in $H^1(K, E)_{p^M}$. These classes enjoy the following properties:

(1) $c_M(1) = \delta(y) \in S(E/K, p^M)$
(2) For each prime $v \nmid n$ of K, the localization $d_M(n)_v$ vanishes in $H^1(K_v, E)_{p^M}$ (in reality, one has to be careful with primes of bad reduction; multiplying $c_M(n)$ by a small power of p takes care of that).
(3) If $v = l$ for $l|n$, then the localization $d_M(n)_l$ can be explicitly computed in terms of the localization $c_M(n/l)_l$.

These classes give a plenty of annihilators of the Selmer group $S(E/K, p^M)$: The Weil pairing $E_{p^M} \times E_{p^M} \longrightarrow \mu_{p^M}$ induces local pairings

$$\langle \ , \ \rangle_v : H^1(K_v, E_{p^M}) \times H^1(K_v, E_{p^M}) \longrightarrow H^2(K_v, \mu_{p^M}) \xrightarrow{\sim} \mathbf{Z}/p^M\mathbf{Z}$$

According to the Tate duality theorem ([44, I.3]), these pairings are non-degenerate and $\delta(E(K_v) \otimes \mathbf{Z}/p^M\mathbf{Z})$ is its own orthogonal complement. This implies, in view of (2) above and (5.5) over K_v, that for each element $s \in S(E/K, p^M)$ of the Selmer group, $\langle s, c_M(n) \rangle_v$ vanishes for $v \nmid n$. The reciprocity law says that the sum of the local pairings between two global elements $x, y \in H^1(K, E_{p^M})$ vanishes. In particular,

$$\sum_v \langle s, c_M(n) \rangle_v = \sum_{l|n} \langle s, c_M(n) \rangle_l = 0 \in \mathbf{Z}/p^M\mathbf{Z} \qquad (5.8)$$

for each $s \in S(E/K, p^M)$. Each of the local terms in the last formula involves only $s_l, c_M(n/l)_l$ by (3).

Assuming that y is of infinite order, an inductive argument relying on (5.8) and Čebotarev shows that (see [37])

- $S(E/K, p^M)/\langle c_M(1)\rangle$ is killed by a fixed power of p; consequently, $\text{Ш}(E/K)_{p^\infty}$ is finite and $S_p(E/K) = E(K) \otimes \mathbf{Z}_p$
- The order of $\text{Ш}(E/K)_{p^\infty}$ divides (up to a controlled factor) $[S_p(E/K) : \mathbf{Z}_p c_\infty(1)]^2 = [E(K) \otimes \mathbf{Z}_p : \mathbf{Z}_p y]^2$.

There are variations of this method, which treat factors of $J_0(N)$ of higher dimensions ([40]), factors of Jacobians of Shimura curves ([41]), precise structure of Ш ([38]) or $S(E/K, p^\infty)$ ([39]) (even in the case when y is not a torsion point).

The assertions (5.1-2) involve L-functions of the base field twisted by characters of finite order. Once the Iwasawa Main Conjecture has been established, it is possible to say something about L-functions twisted by more general Hecke characters. Over $\mathbf{Q}$, this amounts to a Tate twist, which merely shifts the variable s (cf. (6.1)); one obtains a relation between the values $\zeta(1 - 2r)$ and orders of étale cohomology groups $H^i((\text{Spec } \mathbf{Z}[1/p])_{et}, \mathbf{Z}_p(2r))$ ([43]). Over $\mathbf{Q}(\sqrt{-D})$, there are more algebraic Hecke characters, some of them are attached to elliptic curves with complex multiplication. In this way, certain non-trivial versions of the BSD conjecture can be deduced (see [59]).

6. Motives and L-functions

In both of the examples treated in the previous section we could have started from the beginning with cohomology classes, not with units or points on the elliptic curve: no p-adic information is lost if we pass from F_n^* to $F_n^* \widehat{\otimes} \mathbf{Z}_p = H^1(F_n, \mathbf{Z}_p(1))$ or from $E(K_n)$ to $H^1(K_n, T_p E)$ (here we take *continuous* Galois cohomology in the sense of [29],[75]).

In order to understand which Galois cohomology groups are related to more general L-functions, we recall the conjectures of Beilinson, which interpret special values of L-functions in terms of motivic extensions (see [1],[2],[3],[4],[5],[12],[14],[30],[47],[54],[65]). The general philosophy of mixed motives (due to Beilinson and Deligne), which underlies these conjectures, is still largely a dream (cf. [31]). Nevertheless, one can produce a lot of examples which fit into the picture predicted by the conjectures ([5],[14],[54],[55]).

Fix a number field F and assume, for the moment, that all dreams have come true and that we know what (mixed) motives over F are. For our purposes, a motive M over F is the most general arithmetic object which possesses an L-function. The latter is defined using various *realizations* of M:

- the étale realization M_p (for each prime number p) — a finite dimensional p-adic (continuous) representation of $G(\overline{F}/F)$.

- the Betti realization $M_{B,v}$ (for each infinite place v of F, where we fix an isomorphism between $\overline{F_v}$ and $\mathbf{C}$) — a mixed $\mathbf{Q}$-Hodge structure over F_v (see [19],[31],[47]). Recall that a mixed $\mathbf{Q}$-Hodge structure H over $\mathbf{C}$ is a $\mathbf{Q}$-vector space with an increasing filtration $W_n H$ of H and a decreasing

filtration $F^p H_{\mathbf{C}}$ of its complexification, satisfying

$$(W_n H / W_{n-1} H)_{\mathbf{C}} = F^p((W_n H / W_{n-1} H)_{\mathbf{C}}) \oplus \overline{F}^{n+1-p}((W_n H / W_{n-1} H)_{\mathbf{C}})$$

for all integers n, p (here $\overline{F}^p$ denotes the filtration obtained from F^p by complex conjugation). A mixed $\mathbf{Q}$-Hodge structure over $\mathbf{R}$ has, in addition, an involution Φ_∞ preserving the filtration $W_n H$ and such that its anti-linear extension to $H_{\mathbf{C}}$ preserves the filtration $F^p H_{\mathbf{C}}$.

- the de Rham realization M_{dR} — a finite dimensional F-vector space with a decreasing filtration.

There are comparison isomorphisms between different realizations:
- $\quad I_v : M_{B,v} \otimes_{\mathbf{Q}} \mathbf{C} \xrightarrow{\sim} M_{dR} \otimes_{F,v} \mathbf{C}$
- $\quad I_p : M_{B,v} \otimes_{\mathbf{Q}} \mathbf{Q}_p \xrightarrow{\sim} M_p$

(I_p depending on the choice of an embedding $\overline{F} \hookrightarrow \overline{F_v}$).

At each place v of F, the motive M has a local L-factor ([67],[11]): for non-archimedean v,

$$L_v(M, s) = \det(1 - Fr_v (Nv)^{-s} \mid M_p^{I_v})^{-1} \qquad (p \nmid Nv)$$

(where I_v is the inertia group of v), conjecturally independent of p.

For an archimedean place v, $L_v(M, s)$ is a product of gamma factors depending only on the real Hodge structure $M_{(v)} = M_{B,v} \otimes_{\mathbf{Q}} \mathbf{R}$. See [13] for a unified treatment of the local L-factors at all places.

For any finite set of (non-archimedean) places S, write

$$L_S(M, s) = \prod_{v \notin S} L_v(M, s),$$

where the product is taken over non-archimedean places outside S. If S is empty, we write $L(M, s)$ instead of $L_\emptyset(M, s)$.

The total L-factor at infinity will be denoted by

$$L_\infty(M, s) = \prod_{v \mid \infty} L_v(M, s).$$

A typical motive should be of the form $M = h^m(X)(n)$ for a variety X over F (not necessarily smooth or irreducible). Its realizations are

$$M_p = H^m((X \times_F \overline{F})_{et}, \mathbf{Q}_p)(n)$$
$$M_{B,v} = H^m((X \times_{F,v} \mathbf{C})(\mathbf{C}), \mathbf{Q})(n)$$
$$M_{dR} = H_{dR}^m(X_{/F})(n)$$

Various constructions of linear algebra should make sense for motives. Above $M(n)$ refers to the Tate twist, given by the tensor product with $\mathbf{Q}(1)^{\otimes n}$, where $\mathbf{Q}(1)$ is the Tate motive, dual to $h^2(\mathbf{P}^1)$, with realizations

$$\mathbf{Q}(1)_p = \mathbf{Q}_p(1) = \mathbf{Z}_p(1) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p = \varprojlim(\mu_{p^M}(\overline{F})) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$$
$$\mathbf{Q}(1)_{B,v} = (2\pi i)\mathbf{Q}, \quad \text{Hodge type } (-1,-1)$$
$$\mathbf{Q}(1)_{dR} = F = F^{-1}\mathbf{Q}(1)_{dR} \supset F^0\mathbf{Q}(1)_{dR} = 0$$

We have

$$L_v(M(n), s) = L_v(M, s+n) \tag{6.1}$$

for each place v.

Let us mention two standard examples:
(1) The unit motive $\mathbf{Q}(0) = h^0(\mathrm{Spec}(F))$. Then $L(M, s) = \zeta_F(s)$ is the Dedekind zeta function of F.
(2) Let X be smooth and projective over F and $M = h^1(X)(1)$. Let $A = \mathrm{Pic}^o(X)$, $\hat{A} = \mathrm{Alb}(X)$ be the Picard and Albanese variety of X respectively. Then $M_p = T_p(A) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$ and $L(M, s) = L(\hat{A}, s) = L(A, s)$.

If X is smooth and projective over F, then $M = h^m(X)(n)$ is a pure motive of weight $w = m - 2n$. In realizations, all $M_{B,v}$ are pure Hodge structures of weight w; there is a finite set S_{bad} of places at which X has bad reduction. If $v \notin S_{bad}$ is nonarchimedean and prime to p, then M_p is unramified at v and all eigenvalues of the geometric Frobenius Fr_v on M_p are algebraic numbers with absolute value $(Nv)^{w/2}$. According to the monodromy conjecture, a more refined version of the purity should hold also at bad places, see [28]. In general, every motive M should have a canonical (increasing) weight filtration $W_{\cdot}M$ with $W_iM/W_{i-1}M$ pure of weight i.

For our $M = h^m(X)(n)$, the series $L_{S_{bad}}(M, s)$ is absolutely convergent and non-zero for $\mathrm{Re}(s) > 1 + \frac{w}{2}$; if the monodromy conjecture holds, the same is true for $L(M, s)$. The gamma factors $L_\infty(M, s)$ have no poles or zeroes in this region either.

Conjecturally, the L-function has a meromorphic continuation to $\mathbf{C}$, with only possible pole at $1 + \frac{w}{2}$ if w is even, and satisfies the functional equation

$$L_\infty \cdot L(M, s) = \varepsilon(M, s)L_\infty \cdot L(M^*(1), -s) \tag{6.2}$$

with an ε-factor of the form $\varepsilon(M, s) = a \cdot b^s$. For our $M = h^m(X)(n)$, we have $M^*(1) = M(w+1)$, so (6.2) becomes

$$L_\infty \cdot L(M, s) = \varepsilon(M, s)L_\infty \cdot L(M, w+1-s) \tag{6.3}$$

Suppose that we are interested in the behaviour of $L(M, s)$ at an integer $s = j$. Applying a Tate twist, we can assume (by (6.1)) that $j = 0$. Assuming that the functional equation (6.3) holds, it suffices to treat only the case when $s = 0$ lies to the right of the central point $(w + 1)/2$ (or coincides with it), which happens iff $w \leq -1$. In fact,

$$w = -1 \Longleftrightarrow s = 0 \text{ is the central value } (w + 1)/2$$
$$w = -2 \Longleftrightarrow s = 0 \text{ is the near central value } w/2 + 1$$
$$w \leq -3 \Longleftrightarrow s = 0 \text{ is in the convergence region}$$

The collection of Hodge structures $M_\infty = (M_{(v)})_{v|\infty}$ can be viewed as an object of the category $\mathcal{MH}_{F\otimes\mathbf{R}}$ of mixed Hodge structures over $F \otimes \mathbf{R}$. Writing $M_{(v)}^+$ for the invariants under $G(\overline{F_v}/F_v)$, the comparison maps I_v induce a map ("the Deligne period map")

$$\alpha_M : M_\infty^+ = \bigoplus_{v|\infty} M_{(v)}^+ \longrightarrow \bigoplus_{v|\infty} M_{dR}/F^0 \otimes_F F_v = M_{dR}/F^0 \otimes_{\mathbf{Q}} \mathbf{R}$$

Because of our assumption $w \leq -1$, α_M is injective and we have ([47],[55, ch.1])

$$- \operatorname{ord}_{s=0} L_\infty(M^*(1), s) = \dim_{\mathbf{R}} \operatorname{Coker}(\alpha_M)$$
$$\operatorname{ord}_{s=0} L_\infty(M, s) = 0$$

Note that $\det_{\mathbf{R}} \operatorname{Coker}(\alpha_M)$ has a natural $\mathbf{Q}$-structure, coming from the $\mathbf{Q}$-structures $M_{B,v}^+$ and M_{dR}/F^0. The group $\operatorname{Coker}(\alpha_M)$ has an interpretation in terms of Hodge theory as the group of extensions

$$\operatorname{Ext}^1_{\mathcal{MH}_{F\otimes\mathbf{R}}}(\mathbf{R}(0)_\infty, M_\infty) = H^1(\mathcal{MH}_{F\otimes\mathbf{R}}, M_\infty)$$

of the unit Hodge structure $\mathbf{R}(0)_\infty = (\mathbf{R}(0))_{v|\infty}$ by M_∞ ([3],[31],[47]).

Suppose, for simplicity, that $w \leq -3$ (then $s = 0$ is in the region of absolute convergence), or that $w = -2$ and M contains no copy of $\mathbf{Q}(1)$ (then, according to the Tate conjecture ([73],[31, ch.5],[55, ch.1]), $L(M, s)$ has no pole or zero at $s = 0$ either). The L-function $L(M^*(1), -s) = L(M, w + 1 - s)$ then has a zero of order

$$r = \dim_{\mathbf{R}} H^1(\mathcal{MH}_{F\otimes\mathbf{R}}, M_\infty)$$

at $s = 0$.

The fundamental idea of Beilinson and Deligne is that the value $L(M,0)$ should be given in terms of the group of motivic extensions

$$\mathrm{Ext}^1_{\mathcal{MM}_F}(\mathbf{Q}(0), M) = H^1(\mathcal{MM}_F, M)$$

and the Hodge realization map

$$r_\infty : H^1(\mathcal{MM}_F, M) \longrightarrow H^1(\mathcal{MH}_{F\otimes\mathbf{R}}, M_\infty)$$

More precisely, they conjecture ([3],[12]) that there is a subgroup

$$H^1_f(\mathcal{MM}_F, M) \subseteq H^1(\mathcal{MM}_F, M)$$

formed by extension classes 'with good reduction' everywhere such that
(1) The Hodge realization r_∞ induces an isomorphism

$$r : H^1_f(\mathcal{MM}_F, M) \otimes_{\mathbf{Q}} \mathbf{R} \xrightarrow{\sim} H^1(\mathcal{MH}_{F\otimes\mathbf{R}}, M_\infty) \qquad (6.4)$$

(2) Modulo a non-zero rational factor, the value $L(M,0)$ is equal to the determinant $\det(r) \in \mathbf{R}^*/\mathbf{Q}^*$. Here the determinant is computed with respect to the $\mathbf{Q}$-structures $\det_{\mathbf{Q}} H^1_f(\mathcal{MM}_F, M)$ and the $\mathbf{Q}$-structure of $\det_{\mathbf{R}} \mathrm{Coker}(\alpha_M)$ described above.

In [65], Scholl interprets $H^1_f(\mathcal{MM}_F, M)$ as the group of extensions $H^1(\mathcal{MM}_{\mathcal{O}}, M)$ in a certain subcategory $\mathcal{MM}_{\mathcal{O}}$ (containing all pure motives) of $\mathcal{MM}_F$, consisting of "motives over the ring of integers $\mathcal{O}$".

If $w = -2$ and M contains $\mathbf{Q}(1)$, the conjecture needs some modification: for $M = \mathbf{Q}(1)$, we have $\mathrm{Coker}(\alpha_M) = \mathbf{R}^{r_1+r_2}$ and one expects that $H^1(\mathcal{MM}_F, \mathbf{Q}(1)) = F^* \otimes \mathbf{Q}$, $H^1_f(\mathcal{MM}_F, \mathbf{Q}(1)) = \mathcal{O}^* \otimes \mathbf{Q}$. In this case r is the standard logarithm map and the image of $\mathcal{O}^* \otimes \mathbf{R}$ has codimension 1, which is the order of the pole of $L(\mathbf{Q}(1), s) = \zeta(s+1)$ at $s = 0$.

There is a conjectural description of the motivic H^1, H^1_f in terms of algebraic K-theory (see [1],[2],[4],[14],[47],[55, ch.1]). In particular, $H^1(\mathcal{MM}_F, h^m(X)(n))$ should be isomorphic to the subspace of $K_{2n-m-1}(X) \otimes \mathbf{Q} = K_{-w-1}(X) \otimes \mathbf{Q}$ on which Adams operations act with weight n. For this K-theoretical version of the motivic cohomology, the map r_∞ ("the regulator") was constructed in [1].

If $s = 0$ is the central value, i.e. if $w = -1$, the special value should be given by a generalization of the conjecture of Birch and Swinnerton-Dyer. For $n \geq 1$, let $CH^n(X)_0$ be the group of homologically trivial algebraic cycles of codimension n on X (which is supposed to be smooth and projective as before, but also equidimensional), modulo rational equivalence. Under certain conditions on X, there is a height pairing ([4])

$$\langle\,,\,\rangle : CH^n(X)_0 \times CH_0^{\dim X + 1 - n} \longrightarrow \mathbf{R}$$

The conjecture predicts ([1],[4],[55, ch.1]):

(1) $\langle\,,\,\rangle$ is non-degenerate modulo torsion.

(2) $r := \mathrm{ord}_{s=0} L(h^{2n-1}(X)(n),s)$ is equal to $\dim_{\mathbf{Q}} CH^n(X)_0 \otimes \mathbf{Q} = \dim_{\mathbf{Q}} CH^{\dim X + 1 - n}(X)_0 \otimes \mathbf{Q}$.

(3) $L^*(h^{2n-1}(X)(n),0) := \lim_{s \longrightarrow 0} L(h^{2n-1}(X)(n),s)s^{-r}$ is equal, up to a rational factor, to $\Omega_X \cdot \det\langle\,,\,\rangle$, where Ω_X is a suitable period.

The conjectures predict, in particular, that

$$\mathrm{ord}_{s=0} L(M^*(1),s) = \dim_{\mathbf{Q}} H^1_f(\mathcal{MM}_F, M)$$

in all three cases, i.e. whenever $w \leq -1$. It is also expected that

$$H^1(\mathcal{MM}_F, M) = 0 \qquad \text{if } w \geq 0 \tag{6.5}$$

Indeed, for $w > 0$ every extension of $\mathbf{Q}(0)$ by M would be split by the weight filtration; there should also be no non-trivial extensions of pure motives of the same weight.

7. Selmer groups

In order to determine the missing rational factors in Beilinson's conjectures, Bloch and Kato [7] considered also p-adic realization maps (using the K-theoretic version of the motivic cohomology)

$$r_p : H^1(\mathcal{MM}_F, M) \longrightarrow H^1(F, M_p)$$

into (continuous) Galois cohomology (or, in the notation explained below, to its subgroup $\varinjlim_S H^1(G_S, M_p)$).

One of their main objectives was a definition of $\mathbf{Q}_p$-subspaces (we use a slightly different notation, cf. [19])

$$H^1_f(F, M_p) \subseteq H^1_g(F, M_p) \subseteq H^1(F, M_p)$$

such that, conjecturally, r_p induces isomorphisms

$$\begin{aligned} H^1(\mathcal{MM}_F, M) \otimes_{\mathbf{Q}} \mathbf{Q}_p &\xrightarrow{\sim} H^1_g(F, M_p) \\ H^1_f(\mathcal{MM}_F, M) \otimes_{\mathbf{Q}} \mathbf{Q}_p &\xrightarrow{\sim} H^1_f(F, M_p) \end{aligned} \tag{7.1}$$

for all mixed motives M (although Bloch and Kato considered only pure motives).

Let us first offer a geometric insight into the cohomology groups H_f^1, H_g^1, which is based on a speculation about motivic sheaves. Let S be a finite set of places of F containing all $v|p$. Write $\mathcal{O}_S$ for the localization of $\mathcal{O}$ at S and G_S for the Galois group of the maximal extension of F ramified only at S. Every p-adic representation V of G_S determines a p-adic étale sheaf $V_{S,et}$ on $(\operatorname{Spec} \mathcal{O}_S)_{et}$ and we have the basic isomorphism between Galois and étale cohomology (see [44, ch.2])

$$H^i(G_S, V) = H^i((\operatorname{Spec} \mathcal{O}_S)_{et}, V_{S,et})$$

In the context of motives, the pair

$$\operatorname{Rep}_{\mathbf{Q}_p}(G_S) = \{\, p-\text{adic representations of } G_S \,\}$$
$$\mathcal{S}((\operatorname{Spec} \mathcal{O}_S)_{et}) = \{\, p-\text{adic etale sheaves on } \operatorname{Spec}(\mathcal{O}_S) \,\}$$

should be replaced by

$$\operatorname{Rep}_{S,g,\mathbf{Q}_p}(F) = \{\, \text{``geometric''} \ p-\text{adic representations with good}$$
$$\text{reduction outside } S \,\}$$
$$\mathcal{S}((\operatorname{Spec} \mathcal{O}_S)_{mot}) = \{\, p-\text{adic ``motivic'' sheaves on } \operatorname{Spec}(\mathcal{O}_S) \,\}$$

(in this case, S is not supposed to contain all places dividing p). Every representation V in $\operatorname{Rep}_{S,g,\mathbf{Q}_p}(F)$ shoud determine a motivic sheaf $V_{S,mot}$ in such a way that (for all i)

$$H^i(\operatorname{Rep}_{S,g,\mathbf{Q}_p}(F), V) = H^i((\operatorname{Spec} \mathcal{O}_S)_{mot}, V_{S,mot})$$

This cohomology group will be denoted $H_g^i(\mathcal{O}_S, V)$ and we should have

$$H_g^i(F, V) = \varinjlim_S H_g^i(\mathcal{O}_S, V)$$
$$H_f^i(F, V) = H^i((\operatorname{Spec} \mathcal{O})_{mot}, j_* V_{S,mot}),$$

where $j : \operatorname{Spec}(\mathcal{O}_S) \hookrightarrow \operatorname{Spec}(\mathcal{O})$ is the inclusion map.

There is an algebraic candidate for $\operatorname{Rep}_{S,g,\mathbf{Q}_p}(F)$ proposed in [19]. The definition is local; suppose that, for each non-archimedean place v of F, we are given two subcategories

$$\operatorname{Rep}_{g,\mathbf{Q}_p}(\mathcal{O}_v) \subseteq \operatorname{Rep}_{g,\mathbf{Q}_p}(F_v)$$

of the category $\mathrm{Rep}_{\mathbf{Q}_p}(F_v)$ of all p-adic representations of $G(\overline{F_v}/F_v)$, consisting of "geometric p-adic representations of $G(\overline{F_v}/F_v)$ with good reduction" resp. of "geometric p-adic representations of $G(\overline{F_v}/F_v)$".

A global p-adic representation of $G(\overline{F}/F)$ then belongs to $\mathrm{Rep}_{S,g,\mathbf{Q}_p}(F)$ if its restrictions to $G(\overline{F_v}/F_v)$ lie in $\mathrm{Rep}_{g,\mathbf{Q}_p}(F_v)$ (for $v \in S$) and in $\mathrm{Rep}_{g,\mathbf{Q}_p}(\mathcal{O}_v)$ (for $v \notin S$).

For $v \nmid p$, one defines ([19])

$$\mathrm{Rep}_{g,\mathbf{Q}_p}(F_v) = \mathrm{Rep}_{\mathbf{Q}_p}(F_v)$$
$$\mathrm{Rep}_{g,\mathbf{Q}_p}(\mathcal{O}_v) = \{ \text{ unramified representations of } G(\overline{F_v}/F_v) \}$$

For $v | p$, one has to use Fontaine's theory of Grothendieck's 'mysterious functor' ([26],[27]), which relates p-adic representations of $G(\overline{F_v}/F_v)$ to certain objects of p-linear algebra. The definition given in [19] is the following:

$$\mathrm{Rep}_{g,\mathbf{Q}_p}(F_v) = \{ \text{ potentially semistable representations of } G(\overline{F_v}/F_v) \}$$
$$\mathrm{Rep}_{g,\mathbf{Q}_p}(\mathcal{O}_v) = \{ \text{ crystalline representations of } G(\overline{F_v}/F_v) \}$$

See [19],[27],[46] for more details about p-adic Galois representations.

According to various conjectures in the p-adic Hodge theory, the étale realization M_p of every mixed motive M should be in $\mathrm{Rep}_{S,g,\mathbf{Q}_p}(F)$ for suitable S. This is known to be true for $M = h^m(X)(n)$ with X proper and smooth over F, provided X has good reduction (or a semistable reduction and $p > 2m$) at all $v|p$ ([27]).

We are now ready to define the groups H_f^1, H_g^1. First the local case. Suppose that V is any p-adic representation of $G(\overline{F_v}/F_v)$. Denote by V_g (resp. V_f) its largest subrepresentation which belongs to $\mathrm{Rep}_{g,\mathbf{Q}_p}(F_v)$ (resp. $\mathrm{Rep}_{g,\mathbf{Q}_p}(\mathcal{O}_v)$) and put

$$H_g^i(F_v, V) = H^i(\mathrm{Rep}_{g,\mathbf{Q}_p}(F_v), V_g)$$
$$H_f^i(F_v, V) = H^i(\mathrm{Rep}_{g,\mathbf{Q}_p}(\mathcal{O}_v), V_f)$$

(if $v \nmid p$, then H_g^i coincide with the usual cohomology and H_f^i with the unramified cohomology groups). If T is a submodule (resp. a factor-module) of V, write $H_*^i(F_v, T)$ for the preimage (resp. the image) of $H_*^i(F_v, V)$ in $H^i(F_v, T)$. The most important property of these groups is that under the duality between $H^1(F_v, V)$ and $H^1(F_v, V^*(1))$ given by the cup product ([43, 1.3]), the subgroups $H_f^1(F_v, V)$ and $H_f^1(F_v, V^*(1))$ are orthogonal complements of each other.

For a global p-adic representation V unramified outside $S \cup \{v|p\}$ and its sub- or factor-module T, we define

$$H_g^1(\mathcal{O}_S, T) = \left\{ x \in H^1(G_{S \cup \{v|p\}}, T) \;\middle|\; \begin{array}{ll} x_v \in H_g^1(F_v, T) & v|p, \, v \in S \\ x_v \in H_f^1(F_v, T) & v|p, \, v \notin S \end{array} \right\}$$

$$H_g^1(F, T) = \varprojlim_S H_g^1(\mathcal{O}_S, T)$$
$$H_f^1(F, T) = \{\, x \in H_g^1(\mathcal{O}_S, T) \,|\, x_v \in H_f^1(F_v, T) \quad v|p \text{ or } v \in S \,\}$$

If T is a $\mathbf{Z}_p$-lattice in V, invariant under $G(\overline{F}/F)$, the *cohomological* Tate-Šafarevič group $Ш(F, T)$ is defined as ([15])

$$Ш(F, T) = \mathrm{Coker}(H_f^1(F, V) \longrightarrow H_f^1(F, V/T))$$

It is always finite ([7],[15]).

Example (1): $V = \mathbf{Q}_p(1) \supset T = \mathbf{Z}_p(1)$. Then V is in $\mathrm{Rep}_{g,\mathbf{Q}_p}(\mathcal{O}_v)$ for each finite place v of F and we have

$$H_g^1(F_v, \mathbf{Q}_p(1)) = H^1(F_v, \mathbf{Q}_p(1)) = F_v^* \widehat{\otimes} \mathbf{Q}_p \qquad (\xrightarrow{\sim} \mathbf{Q}_p \text{ if } v \nmid p)$$
$$H_f^1(F_v, \mathbf{Q}_p(1)) = \mathcal{O}_v^* \widehat{\otimes} \mathbf{Q}_p \qquad (= 0 \text{ if } v \nmid p)$$

Here $A \widehat{\otimes} \mathbf{Q}_p = (\varprojlim A/p^M A) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$ denotes the completed tensor product. It follows that (for any finite set S of non-archimedean places)

$$H_g^1(\mathcal{O}_S, \mathbf{Q}_p(1)) = \mathcal{O}_S^* \otimes \mathbf{Q}_p$$
$$H_g^1(F, \mathbf{Q}_p(1)) = F^* \otimes \mathbf{Q}_p$$
$$H_f^1(F, \mathbf{Q}_p(1)) = \mathcal{O}^* \otimes \mathbf{Q}_p$$

The Tate-Šafarevič group

$$Ш(F, \mathbf{Z}_p(1)) = \frac{\mathrm{Ker}(F^* \otimes \mathbf{Q}_p/\mathbf{Z}_p \longrightarrow \oplus_v \mathbf{Q}_p/\mathbf{Z}_p \cdot [v])}{\mathcal{O}^* \otimes \mathbf{Q}_p/\mathbf{Z}_p} = (A_F)_{p^\infty}$$

is equal to the p-part of the class group of F.

Example (2): Let E be an elliptic curve over F and $T = T_p E \subset V = T \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$. In this case, V lies in $\mathrm{Rep}_{g,\mathbf{Q}_p}(F_v)$ (resp. $\mathrm{Rep}_{g,\mathbf{Q}_p}(\mathcal{O}_v)$) for all v (resp. for places v of good reduction of E). The sequence (5.5) over F_v defines, in the limit, an injection $\delta : E(F_v) \widehat{\otimes} \mathbf{Q}_p \hookrightarrow H^1(F_v, V)$. Its image is equal to $H_f^1(F_v, V) = H_g^1(F_v, V)$ for all v ([7]). In fact, $H^1(F_v, V) = 0$ for all primes $v \nmid p$. Consequently,

$$H_f^1(F, V) = H_g^1(F, V) = S_p(E/F) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$$
$$H_f^1(F, T) = H_g^1(F, T) = S_p(E/F)$$

and the Tate-Šafarevič group $\mathrm{III}(F, T)$ is equal to the factor of $\mathrm{III}(E/F)_{p^\infty}$ by its maximal divisible subgroup.

8. Values of L-functions

Let, as before, $M = h^m(X)(n)$ (for a smooth projective variety over F) be a pure motive of weight $w = m - 2n \leq -1$. Let us consider the composite map

$$H_f^1(\mathcal{MM}_F, M) \otimes_{\mathbf{Q}} \mathbf{Q}_p \longrightarrow H_f^1(F, M_p) \longrightarrow H_f^1(F \otimes \mathbf{Q}_p, M_p)$$

Bloch and Kato [7] define a canonical *exponential map*

$$M_{dR}/F^0 \otimes_{\mathbf{Q}} \mathbf{Q}_p \longrightarrow H_f^1(F \otimes \mathbf{Q}_p, M_p) = \bigoplus_{v|p} H_f^1(F_v, M_p)$$

which is conjectured to be an isomorphism for our M (this is known to be true if X has good reduction at all $v|p$).

Tate's global duality ([44, I.4.10]) gives an exact sequence

$$H^2(G_S, M_p^*(1))^* \longrightarrow H_f^1(F, M_p) \longrightarrow \bigoplus_{v \in S} H_f^1(F_v, M_p) \longrightarrow$$
$$\longrightarrow H^1(G_S, M_p^*(1))^* \longrightarrow H_f^1(F, M_p^*(1))^* \longrightarrow 0$$

(for any set of places S containing all places of bad reduction and $v|p$). Suppose now that $w \leq -2$. Then the first term should vanish by a conjecture of Jannsen [28], the last term by a combination of (6.5) and (7.1). The local groups $H_f^1(F_v, V)$ should vanish for $v \nmid p$ by the monodromy conjecture. Assuming all this, we get a conjectural exact sequence [33]

$$0 \longrightarrow H_f^1(M) \otimes_{\mathbf{Q}} \mathbf{Q}_p \longrightarrow M_{dR}/F^0 \otimes_{\mathbf{Q}} \mathbf{Q}_p \longrightarrow H^1(G_S, M_p^*(1))^* \longrightarrow 0$$
$$(8.1)$$

Define a one-dimensional $\mathbf{Q}$-vector space

$$\Theta(M) = \det_{\mathbf{Q}} H_f^1(\mathcal{MM}_F, M) \otimes \det_{\mathbf{Q}}(\oplus_{v|\infty} M_{B,v}^+) \otimes \det_{\mathbf{Q}}(M_{dR}/F^0)^{\otimes -1}$$

Suppose, for simplicity, that M contains no copy of $\mathbf{Q}$ if $w = -2$. Beilinson's conjecture (6.4) and (7.1) then predict that r_∞ and r_p induce isomorphisms

$$s_\infty : \Theta(M) \otimes_{\mathbf{Q}} \mathbf{R} \xrightarrow{\sim} \mathbf{R}$$

$$s_p : \Theta(M) \otimes_{\mathbf{Q}} \mathbf{Q}_p \xrightarrow{\sim} \det_{\mathbf{Q}_p} H^1(G_S, M_p^*(1)) \otimes \det_{\mathbf{Q}_p}(F \otimes \mathbf{R}, M_p)$$

The essential point is that the one-dimensional $\mathbf{Q}_p$-vector space on the right hand side has a *canonical* $\mathbf{Z}_p$-submodule ([18],[19]), say, X_p.

We can now formulate the conjecture specifying the value $L(M, 0)$ ([7],[18],[19],[33]):
There is an element $z_M \in \Theta(M)$ (unique up to a sign) such that $s_p(z_M \otimes 1)$ generates X_p for all p. Then

$$s_\infty(z_M \otimes 1) = \pm L(M, 0) \tag{8.2}$$

This is, in fact, a simplified version of a general conjecture of Fontaine and Perrin-Riou ([17],[18],[19]), which makes sense for all mixed motives. They work consistently with determinants of all cohomology groups, making no appeal to vanishing conjectures à la Jannsen.
Kato [33] considers a fixed abelian extension F'/F with Galois group G and works over F', taking into account the action of G. His 'Iwasawa Main Conjecture for motives' goes essentially along the same lines, one just has to take all determinants over the group ring $\mathbf{Q}[G]$.

In practice, one has to replace $H^1_f(\mathcal{MM}_F, M)$ by a suitable subspace of the K-theory of X, which satisfies (6.4) and (7.1). There are quite a few instances in which one can construct enough elements in K-theory and compute their image under r_∞ (see [1],[2],[14],[54],[55]). In order to verify the conjecture (8.2), one must be able to compute also their image under r_p. This is far from being easy: for example, it is not known, in general, whether K-theory actually maps into $H^1_f(F, V) \subseteq H^1(F, V)$ (see [22],[46],[63] for some partial results; note that [22] contains a fair amount of misprints). The computation of the map r_p can be viewed alternately as a p-adic integration (this is the point of view of [8]), or as an explicit reciprocity law (see [7],[32],[51] for these).

9. Cohomological Euler systems

It is now clear where to look for further Euler systems in Galois cohomology: if we are given a family of extensions F_n of the base number field F together with elements in $H^1_f(\mathcal{MM}_{F_n}, M)$ (or in some more wordly version of this not quite well-defined group, probably given by algebraic K-theory), consider their p-adic realizations in $H^1(F_n, M_p)$. If we can control the denominators, they will in fact lie in (the image of) $H^1(F_n, T)$ for a suitable

Galois invariant $\mathbf{Z}_p$-lattice T in M_p. If we are able to prove that these cohomological elements satisfy relations analogous to (2.1) and if we can control their localizations at various places (most notably those dividing p), we can then apply the descent procedure to obtain various finiteness results for Selmer and Tate-Šafarevič groups involved. We now give a list of cases when this has been done.

(E1–3) The Euler system lies in $H_f^1(F_n, \mathbf{Z}_p(1)) = \mathcal{O}_{F_n}^* \otimes \mathbf{Z}_p$.

(E4) Gauss sums lie in $H_g^1(F_n, \mathbf{Z}_p(1))$, but not in $H_f^1(F_n, \mathbf{Z}_p(1))$ for suitable abelian extensions F_n of $\mathbf{Q}(\mu_p)$.

(E5) Under the boundary map δ in (5.5), the Heegner points define an Euler system in $H_f^1(K_n, T_p E)$.

(E6) E : "Heegner cycles" over ring class fields of an imaginary quadratic field K [45]
    L : $L(f \otimes K, r)$ for modular forms of weight $2r > 2$
    A : Selmer group associated to f over K

Let $f = \sum a_n q^n$ be a normalized newform of weight $2r > 2$ on $\Gamma_0(N)$ with rational coefficients. There is a 'motive' M associated to f such that $L(M, s) = L(f, s) = \sum a_n n^{-s}$. In fact, $M \subset h^{2r-1}(X)$ for a suitable smooth compactification of the $(2r-1)$-dimensional Kuga-Sato variety over (a suitable cover) of $Y_0(N)$ (see [64]). If K is an imaginary quadratic field as in (E5), it is possible to define Heegner cycles $x_n \in CH^r(X \otimes_K K_n)_0 \otimes \mathbf{Q}$. Fix a prime number p not dividing N and write y_n for the image of x_n under the Abel-Jacobi map

$$\phi : CH^r(X \otimes_K K_n)_0 \otimes \mathbf{Q} \longrightarrow H_f^1(K_n, V),$$

where $V = M_p(r)$. They all lie inside $H_f^1(K_n, T)$ for a suitable $\mathbf{Z}_p$-lattice in V and satisfy relations analogous to (2.4). It is possible to construct cohomology classes $c_M(n) \in H^1(K, T/p^M T)$ following the procedure in Sec. 5 and then apply the descent machine.

The result of the descent is the following: if $y = \mathrm{cor}_{K_1/K}(y_1)$ is not torsion in $H_f^1(K, T)$, then ([45])

$$\text{the index } [H_f^1(K, T) : \mathbf{Z}_p y] \text{ is finite} \qquad (9.1)$$

In fact, the methods of [37] extend to this case, showing that the order of $\text{Ш}(T, K)$ is divisible by the square of this index times a controlled power of p.

The field K has a unique extension K_∞ with $G(K_\infty/K) \xrightarrow{\sim} \mathbf{Z}_p^2$. If $p \nmid a_p$, then there is a p-adic L-function L_p defined on continuous characters $\chi : G(K_\infty/K) \longrightarrow \mathbf{C}_p^*$ that interpolates algebraic parts of the values of the complex L-function: if χ is a character of finite order, then

$$L_p(f \otimes K, \chi) \doteq i_p \circ i^{-1} \left(\frac{L(f \otimes K, \chi, r)}{\Omega_{f,K,\chi}} \right) \tag{9.2}$$

Here $\Omega_{f,K,\chi}$ is a suitable period and i (resp. i_p) denotes a fixed embedding of $\overline{\mathbf{Q}}$ into $\mathbf{C}$ (resp. $\overline{\mathbf{Q}_p}$). Restricting to characters of $G(K\mathbf{Q}_\infty/K)$, where $\mathbf{Q}_\infty/\mathbf{Q}$ is the cyclotomic $\mathbf{Z}_p$-extension of $\mathbf{Q}$, we get a function $L_p(f \otimes K, s)$ of a p-adic variable $s \in p\mathbf{Z}_p$, if we use the canonical isomorphisms

$$G(K\mathbf{Q}_\infty/K) \xrightarrow{\sim} G(\mathbf{Q}_\infty/\mathbf{Q}) \xrightarrow{\sim} (1 + p\mathbf{Z}_p) \xrightarrow{\log_p} p\mathbf{Z}_p$$

Under the same conditions ($p \nmid Na_p$), there exists a canonical (symmetric) p-adic height pairing ([46],[50])

$$\langle \, , \, \rangle_p : H^1_f(K, V) \times H^1_f(K, V) \longrightarrow \mathbf{Q}_p$$

It is shown in [48] that (if p splits in K and $p \nmid 2Na_p$)

$$L'_p(f \otimes K, 0) \doteq \langle y, y \rangle_p \tag{9.3}$$

In particular, it follows from (9.1) and (9.3), that

$$\mathrm{ord}_{s=0} L_p(f \otimes K, s) = 1 \Longrightarrow \dim_{\mathbf{Q}_p} H^1_f(K, V) = 1$$

(E7) E : "cyclotomic elements" in $H^1_f(F, \mathbf{Z}_p(r))$ for abelian extensions F of $\mathbf{Q}$ for $r > 1$ odd ([42],[70])

  L : $L'(1 - r, \chi)$ for Dirichlet characters over $\mathbf{Q}$

  A : $H^1_f(\mathbf{Q}, \mathbf{Q}_p/\mathbf{Z}_p(1 - r)) = \text{Ш}(\mathbf{Q}, \mathbf{Z}_p(1 - r))$

The Euler system is formed by cyclotomic elements of Deligne-Soulé ([70]) $y_n \in H^1_f(\mathbf{Q}(\mu_n), \mathbf{Z}_p(r))$ satisfying relations similar to (2.1). Out of these elements one manufactures cohomology classes $c_M(n) \in H^1(\mathbf{Q}, \mathbf{Z}/p^M\mathbf{Z}(r))$ which give a bound on $H^1_f(\mathbf{Q}, \mathbf{Q}_p/\mathbf{Z}_p(1-r))$: if $s \in H^1_f(\mathbf{Q}, \mathbf{Q}_p/\mathbf{Z}_p(1-r))_{p^M}$, the reciprocity law gives a formula analogous to (5.8)

$$\sum_l \langle s, c_M(n) \rangle_l = \sum_{l|n} \langle s, c_M(n) \rangle_l = 0 \in \mathbf{Z}/p^M\mathbf{Z}$$

with

$$\langle \, , \, \rangle_l : H^1(\mathbf{Q}_l, \mathbf{Z}/p^M\mathbf{Z}(1 - r)) \times H^1(\mathbf{Q}_l, \mathbf{Z}/p^M\mathbf{Z}(r)) \longrightarrow \mathbf{Z}/p^M\mathbf{Z}$$

Each term $\langle s, c_M(n)\rangle_l$ can be again computed inductively using $c_M(n/l)_l \in H^1(\mathbf{Q}_l, \mathbf{Z}/p^M\mathbf{Z}(r))$. The descent shows that ([42])

$$\sharp H^1_f(\mathbf{Q}, \mathbf{Q}_p/\mathbf{Z}_p(1-r)) \;\Big|\; [H^1_f(\mathbf{Q}, \mathbf{Z}_p(r)) : \mathbf{Z}_p\, y_1] \qquad (9.3)$$

In [7], Bloch and Kato prove (for $p > 2$) that the Iwasawa Main Conjecture over $\mathbf{Q}$ implies that there is in fact an equality in (9.3), which is a "p-part" of the conjecture (8.2) for $M = \mathbf{Q}(r)$. They also need the fact that Jannsen's vanishing conjecture, formulated in Sec. 8, is true for $M = \mathbf{Q}(r)$ (see [28]).

(E8) E : Beilinson's elements in K_2 of a modular curve [1],[34], [55, ch. 9]
        L : $L'(f,0)$ (or $L(f,2)$ by the functional equation) for modular forms of weight 2

Kato [34] announced a construction of an Euler system $x_n \in H^0(X_1(Np^n)\otimes \mathbf{Q}(\mu_{p^n}), \mathcal{K}_2) \otimes \mathbf{Q}$, using symbols $\{u,v\}$ formed by suitable modular units. If f is a fixed newform of weight 2 on $\Gamma_0(N)$ (for simplicity, with rational coefficients), E the corresponding quotient of $J_0(N)$, which is an elliptic curve, $V = T_p E$, then x_n map to $y_n \in H^1(\mathbf{Q}(\mu_{p^n}), T(1))$ for a suitable lattice $T \subset V$. Applying the 'Tate twisting trick' of Soulé [70, 1.4] backwards, Kato uses relations between y_n to define a new cohomology class $z \in H^1(\mathbf{Q}, T)$, with the property

 • If $L(E/\mathbf{Q}, 1) \neq 0$, then $z_p \notin H^1_f(\mathbf{Q}_p, V)$

Suppose that $L(E/\mathbf{Q}, 1) \neq 0$. If $s \in H^1_f(\mathbf{Q}, V)$, then the localizations s_l vanish for $l \neq p$, as $H^1(\mathbf{Q}_l, V) = 0$. The reciprocity law then implies that

$$\langle s_p, z_p\rangle_p = 0 \in H^2(\mathbf{Q}_p, \mathbf{Q}_p(1)) = \mathbf{Q}_p$$

However, $H^1_f(\mathbf{Q}_p, V)$ is one-dimensional and is equal to its own orthogonal complement, hence $s_p \in H^1_f(\mathbf{Q}_p, V)$ must vanish. This proves that $E(\mathbf{Q})$ is torsion, because the composite map

$$E(\mathbf{Q}) \otimes \mathbf{Q} \longrightarrow H^1_f(\mathbf{Q}, V) \longrightarrow H^1_f(\mathbf{Q}_p, V)$$

is injective.

This proof of the finiteness of $E(\mathbf{Q})$ provided $L(E/\mathbf{Q}, 1) \neq 0$ is purely p-adic; if one could define cohomology classes analogous to $c_M(n)$ in this context as well, that would probably prove the finiteness of $\text{Ш}(E/\mathbf{Q})_{p^\infty}$. Hopefully a written version of [34] will be soon available.

(E9) E : cohomology classes in $H^1(\mathbf{Q}, M_p)$ for $M = \text{Sym}^2(h^1(E)(1))$, where E is a modular elliptic curve over $\mathbf{Q}$ ([16]).
        L : $L(\text{Sym}^2(h^1(E)), 2)$
        A : $H^1_f(\mathbf{Q}, V/T) = \text{Ш}(\mathbf{Q}, T)$ for a suitable $\mathbf{Z}_p$-lattice T in $V = M_p$.

This construction seems to be of a different nature than (E6–8), as it deals with a critical value, when both $L(M, s)$ and $L(M^*(1), -s)$ are non-zero at $s = 0$. Fix a modular parametrization $\phi : X_0(N) \longrightarrow E$ defined over $\mathbf{Q}$. Assuming that E does not have a complex multiplication and p is a fixed prime (outside of a finite set of exceptional primes), Flach constructs, for each prime $l \nmid N$ an element $\epsilon(l) \in H^1(X_0(N) \times X_0(N), \mathcal{K}_2)$, which is then mapped to $c(l) \in H^1(\mathbf{Q}, T)$ (with $T = \mathrm{Sym}^2(H^1((E \otimes_{\mathbf{Q}} \overline{\mathbf{Q}})_{et}, \mathbf{Z}_p(1))))$ via a sequence of maps

$$H^1(X_0(N) \times X_0(N), \mathcal{K}_2) \longrightarrow H^1(E \times E, \mathcal{K}_2) \longrightarrow H^3((E \times E)_{et}, \mathbf{Z}_p(2))$$
$$\longrightarrow H^1(\mathbf{Q}, H^2(((E \times E) \otimes_{\mathbf{Q}} \overline{\mathbf{Q}})_{et}, \mathbf{Z}_p(2))) \longrightarrow H^1(\mathbf{Q}, T)$$

It is shown in [16] that (for $l \nmid pN$) the localizations $c(l)_r$ lie in $H^1_f(\mathbf{Q}_r, T)$ for all primes $r \neq l$, but $c(l)_l \notin H^1_f(\mathbf{Q}_l, T)$. An application of Čebotarev gives the main result of [16]:

$$H^1_f(\mathbf{Q}, V/T) = \mathrm{III}(\mathbf{Q}, T) \quad \text{is killed by } \deg(\phi)$$

10. Concluding remarks

In this survey, we have tried to give examples of Euler systems and their applications. What we are still lacking is a general framework (if only a conjectural one) into which they could be incorporated, although Kato's conjectural 'Iwasawa theory for motives' [33] undoubtedly points in the right direction.

The metaquestion "What is a general motivic Euler system?" seems to be quite important and far from being vacuous; indeed, any answer, however partial, would have to deal with the following topics, which have arithmetic significance in their own right:

(A) p-adic L-functions: Develop a general cohomological theory of p-adic L-functions, or even a finer theory modulo powers of p, which would explain *all* congruences between rational parts of special values arising from congruences between Galois representations. Firsts steps in this direction have been taken in [52],[53],[62].

(B) p-adic regulators: Let X be a smooth projective variety over a finite extension F of $\mathbf{Q}_p$ and $\mathcal{X}$ a regular proper flat model of X over the ring of integers $\mathcal{O}$ of F. For $V = H^m((X \otimes_F \overline{F})_{et}, \mathbf{Q}_p(n))$ with $m - 2n \leq -2$, investigate the Chern character ('p-adic regulator')

$$r_p : K_{2n-m-1}(\mathcal{X}) \otimes \mathbf{Q} \longrightarrow H^1(F, V)$$

In particular, show that the image of r_p is contained in $H_f^1(F, V)$ (this is proved - modulo certain unchecked compatibility - in [22] and [63], provided $p > n$, $\mathcal{X}$ is smooth over $\mathcal{O}$ and $F/\mathbf{Q}_p$ is unramified; the same method works even in the ramified case) and give formulas for r_p.

(C) Explicit reciprocity laws: Formulate and prove general explicit reciprocity laws for p-adic Galois representations (cf. [7],[32]). A conjectural formula for crystalline Galois representations is given in [51],[52].

(D) More examples: Produce new examples; following (E5), (E6), (E8) and (E9) one could try to explore the geometry of higher dimensional Shimura varieties. For example, is it possible to produce Stark units using Hilbert-Blumenthal modular varieties?

These problems, while not unrelated to each other, could be pursued separately. Any progress on them, however, would seem to require a more solid foundational background than is currently available, both in p-adic Hodge theory and in our understanding of L-functions of Shimura varieties. Nevertheless, in view of potential arithmetic consequences, each of them is a worthy cause.

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