On the p-adic height of Heegner cycles

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Introduction

(0.1) In [Gr - Za], Gross and Zagier proved a remarkable formula, which relates the first derivative of the L-function of a modular form f of weight 2 on $\Gamma_0(N)$ (over a suitable imaginary quadratic field) and the Néron-Tate height of a "Heegner point" (over the same quadratic field) on the f-part of the Jacobian of the modular curve $X_0(N)$. Later, Perrin-Riou [PR 2] proved a p-adic version of this formula.

Kolyvagin's method of Euler systems [Ko], combined with the formula of Gross-Zagier, proves the conjecture of Birch and Swinnerton-Dyer (up to a controlled rational factor) for all modular elliptic curves over $\mathbf{Q}$ with analytic rank ≤ 1.

In [Ne 1], Kolyvagin's method was applied to certain cohomological objects associated to a modular form f of arbitrary even weight $2r > 2$. In the present work we prove, under suitable hypotheses, a p-adic version of the Gross-Zagier formula in this context; our Theorem A relates the first derivative of a p-adic L-function of f at the "central point" and the p-adic height of a "Heegner cycle" (as mentioned above, this result is due to Perrin-Riou in the weight two case). As a consequence of this formula (and results of [Ne 1]), we obtain a certain weak form (Theorem B) of the conjecture of Beilinson and Bloch [Be] in our situation (this conjecture replaces the conjecture of Birch and Swinnerton-Dyer when one deals with central values of L-functions associated to cohomology groups of degree $2r - 1 > 1$).

(0.2) We now describe the results of the present work in more detail. Let $N \geq 1$ be an integer and $f = \sum_{n\geq 1} a_n q^n$ a normalized ($a_1 = 1$) newform on $\Gamma_0(N)$ of weight $2r > 2$. Let $K = \mathbf{Q}(\sqrt{D})$ be an imaginary quadratic field with an *odd* discriminant $D < 0$ such that all primes dividing N split in K. We further assume that, if $N = 1$, both 2 and 3 split in K. Denote by $\mathbf{N}$ the norm $N_{K/\mathbf{Q}}$ and write τ for the non-trivial element of $G(K/\mathbf{Q})$. Fix an embedding $\overline{\mathbf{Q}} \hookrightarrow \mathbf{C}$; this enables us to view f as a form with Fourier coefficients in $\overline{\mathbf{Q}}$. Let $F(f)$ be the subfield of $\overline{\mathbf{Q}}$ generated by the coefficients a_n; it is a totally real extension of $\mathbf{Q}$ of finite degree. Let p be a prime number not dividing $2N$ that splits in K. We fix an embedding $i_p : \overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_p$ and assume that f is ordinary with respect to i_p, i.e. that $i_p(a_p)$ is a p-adic unit. It is expected that the above conditions are satisfied for infinitely many primes p. Let $\widehat{F(f)}$ be the closure of $i_p(F(f))$. For reasons explained in 0.13 below we also have to assume that $p > 2r - 1$.

(0.3) Let $Y(N)$ be the modular curve over $\mathbf{Q}$ classifying elliptic curves with a full level N structure and $j : Y(N) \hookrightarrow X(N)$ its non-singular compactification. If $N \geq 3$, then there is a universal elliptic curve $f : E \longrightarrow Y(N)$. The $(2r-2)$-fold fibre product of E with itself over $Y(N)$ has a canonical non-singular compactification W ([De 1], [ScA 1]). There are natural maps

$$H^{2r-1}_{\mathrm{et}}(W \otimes_{\mathbf{Q}} \overline{\mathbf{Q}}, \mathbf{Q}_p)(r) \longrightarrow H^1_{\mathrm{et}}(X(N) \otimes_{\mathbf{Q}} \overline{\mathbf{Q}}, j_*\mathrm{Sym}^{2r-2}(R^1 f_* \mathbf{Q}_p))(r) \longrightarrow V_f, \tag{0.3.1}$$

where V_f is (a Tate twist of) the p-adic Galois representation associated to f: V_f is two-dimensional over $\widehat{F(f)}$, unramified outside pN and satisfies

$$\det_{\widehat{F(f)}}(1 - tFr_\ell | V_f) = 1 - a_\ell \ell^{-r} t + \ell^{-1} t^2$$

for all primes $\ell \not| pN$ (here Fr_ℓ denotes a geometric Frobenius at ℓ).

Let H be the Hilbert class field of $K = \mathbf{Q}(\sqrt{D})$. In II.3.6 we define a "Heegner cycle" εY (with rational coefficients) of dimension $r - 1$ supported in the fibre of $W \otimes_{\mathbf{Q}} H$ over a "Heegner point" in $Y_0(N)(H)$.

The cohomology class of εY in $H^{2r}_{\mathrm{et}}(W \otimes_{\mathbf{Q}} \overline{\mathbf{Q}}, \mathbf{Q}_p)(r)$ vanishes, hence εY lies in the domain of the p-adic Abel-Jacobi map

$$\Phi : CH^r(W \otimes_{\mathbf{Q}} H)_0 \otimes \mathbf{Q} \longrightarrow H^1(H, H^{2r-1}_{\mathrm{et}}(W \otimes_{\mathbf{Q}} \overline{\mathbf{Q}}, \mathbf{Q}_p)(r))$$

Let Φ_f be the map induced by Φ and 0.3.1

$$\Phi_f : CH^r(W \otimes_{\mathbf{Q}} H)_0 \otimes \mathbf{Q} \longrightarrow H^1(H, V_f)$$

and put $z_f = \Phi_f(\varepsilon Y)$. In order to define z_f for $N < 3$, one chooses a suitable auxiliary integer $M \geq 3$, performs the above construction for NM and takes an average over $GL_2(\mathbf{Z}/M\mathbf{Z})$; see II.3.8 for more details.

(0.4) Bloch and Kato [Bl - Ka] define a subspace $H^1_f(H, V_f) \subset H^1(H, V_f)$ (their subscript f has nothing to do with our modular form f), which is a cohomological version of classical Selmer groups. Under our assumptions, $\mathrm{Im}(\Phi_f)$ is contained in $H^1_f(H, V_f)$ [1].
Fix a continuous homomorphism $\ell_K : \mathbf{A}^*_K/K^* \longrightarrow \mathbf{Q}_p$ (an "arithmetic logarithm"). The construction of [Ne 2] associates to $\ell_H = \ell_K \circ N_{H/K} : \mathbf{A}^*_H/H^* \longrightarrow \mathbf{Q}_p$ and the canonical splitting of certain Hodge filtration (this comes from the assumption that f is ordinary at p) a symmetric $\widehat{F(f)}$-linear height pairing

$$\langle\ ,\ \rangle : H^1_f(H, V_f) \times H^1_f(H, V_f) \longrightarrow \widehat{F(f)}$$

For a character $\mathcal{C} : G(H/K) \longrightarrow \overline{\mathbf{Q}}^*$, we define

$$z_{f,\mathcal{C}} = \frac{1}{h} \sum_{\sigma \in G(H/K)} \mathcal{C}^{-1}(\sigma) z_f^\sigma \in H^1_f(H, V_f) \otimes_{\widehat{F(f)}} \overline{\mathbf{Q}}_p,$$

where $h = [H : K]$ is the class number of K. Our aim is to compute the height (extended in a $\overline{\mathbf{Q}}_p$-linear way)

$$\langle z_{f,\mathcal{C}}, z_{f,\overline{\mathcal{C}}} \rangle = \frac{1}{h} \sum_{\sigma \in G(H/K)} \langle z_f, z_f^\sigma \rangle$$

(recall that $\langle x^\sigma, y^\sigma \rangle = \langle x, y \rangle$ for all $\sigma \in G(H/K)$).

(0.5) The L-function of f has an Euler product

$$L(f, s) = \sum_{n=1}^{\infty} a_n n^{-s} = \prod_{\ell} [(1 - \alpha_\ell \ell^{-s})(1 - \beta_\ell \ell^{-s})]^{-1}$$

(with $\alpha_\ell \beta_\ell = 0$ for $\ell \mid N$). Let $\mathfrak{f} \subset \mathcal{O}_K$ be an ideal and $K(\mathfrak{f})$ the corresponding ray class field of K. Suppose that $\mathcal{W} : G(K(\mathfrak{f})/K) \longrightarrow \overline{\mathbf{Q}}^*$ is a ray class character with conductor $\mathfrak{f}$. We view $\mathcal{W}$ as a character on ideals prime to $\mathfrak{f}$, via the reciprocity map (normalized in the old-fashioned way, using arithmetic Frobenius elements).
The L-function of f over K, twisted by $\mathcal{W}$, is given by

$$L(f \otimes K, \mathcal{W}, s) = \prod_{\lambda} [(1 - \alpha_{\mathbf{N}(\lambda)} \mathcal{W}(\lambda) \mathbf{N}(\lambda)^{-s})(1 - \beta_{\mathbf{N}(\lambda)} \mathcal{W}(\lambda) \mathbf{N}(\lambda)^{-s})]^{-1},$$

where λ runs through prime ideals of $\mathcal{O}_K$ prime to $\mathfrak{f}$ and $\alpha_{\mathbf{N}(\lambda)} = \alpha_\ell^k$, $\beta_{\mathbf{N}(\lambda)} = \beta_\ell^k$ if $\mathbf{N}(\lambda) = \ell^k$.
Let K_∞/K be the unique $\mathbf{Z}_p^2$-extension of K. In I.5.10 we define a p-adic L-function $L_p(f \otimes K, \mathcal{C})(\lambda)$, defined on continuous characters $\lambda : G(K_\infty/K) \longrightarrow 1 + p\mathbf{Z}_p$, which has the following interpolation property (cf. 5.10, 6.1):
If $\mathcal{W} : G(K_\infty/K) \longrightarrow 1 + p\mathbf{Z}_p$ is a character of finite order with conductor $\mathfrak{f}$, then

[1] See 0.13, however

$$L_p(f\otimes K,\mathcal{C})(\mathcal{W}) = \tau(\mathcal{C}\mathcal{W})\mathcal{W}(N)\overline{\mathcal{C}\mathcal{W}}((\sqrt{D}))V_p(f,\mathcal{C}\mathcal{W})\frac{|D|^{1/2}\mathbf{N}(\mathfrak{f})^{r-1/2}}{\alpha_{\mathbf{N}(\mathfrak{f})}(f)}\frac{2((r-1)!)^2}{(4\pi)^{2r}}\frac{L(f\otimes K,\overline{\mathcal{C}\mathcal{W}},r)}{\langle f,f\rangle_N},$$

where

$$V_p(f,\mathcal{C}\mathcal{W}) = \prod_{\mathfrak{p}|p}\left(1-\frac{\overline{\mathcal{C}\mathcal{W}}(\mathfrak{p})p^{r-1}}{\alpha_p(f)}\right)\left(1-\frac{\mathcal{C}\mathcal{W}(\mathfrak{p})p^{r-1}}{\alpha_p(f)}\right),$$

$\alpha_p(f)$ is the unique root of $t^2-a_pt+p^{2r-1}$ which is a p-adic unit, $\tau(\mathcal{C}\mathcal{W})\in\overline{\mathbf{Q}}^*$ is the root number for the L-function $L(\mathcal{C}\mathcal{W},s)$ (see I.3.3) and

$$\langle f,f\rangle_N = \int_{\Gamma_0(N)\backslash\mathcal{H}} |f(z)|^2 y^{2r-2}\,dxdy$$

In the above formulas, $\overline{\mathcal{W}}=\mathcal{W}^{-1}$, $\overline{\mathcal{C}}=\mathcal{C}^{-1}$.

(0.6) Using the reciprocity map, we can view $\ell_K:\mathbf{A}_K^*/K^*\longrightarrow\mathbf{Q}_p$ as a homomorphism $\ell_K:G(K_\infty/K)\longrightarrow\mathbf{Q}_p$. We have $\ell_K=p^{-n}\log_p\circ\lambda$, for a suitable integer n and a character $\lambda:G(K_\infty/K)\longrightarrow 1+p\mathbf{Z}_p$. Here $\log_p:\mathbf{Q}_p^*\longrightarrow\mathbf{Z}_p$ denotes the Iwasawa logarithm, characterized by $\log_p(p)=0$. We define the derivative of $L_p(f\otimes K,\mathcal{C})$ at the trivial character $\mathbb{1}$ in the direction of ℓ_K as

$$L_p'(f\otimes K,\mathcal{C},\mathbb{1}) = p^{-n}\frac{d}{ds}L_p(f\otimes K,\mathcal{C})(\lambda^s)\Big|_{s=0}$$

(0.7) We are now ready to state our main result.

Theorem A. *For each character* $\mathcal{C}:G(H/K)\longrightarrow\overline{\mathbf{Q}}^*$,

$$L_p'(f\otimes K,\mathcal{C},\mathbb{1}) = -\prod_{\mathfrak{p}|p}\left(1-\frac{\overline{\mathcal{C}}(\mathfrak{p})p^{r-1}}{\alpha_p(f)}\right)\left(1-\frac{\mathcal{C}(\mathfrak{p})p^{r-1}}{\alpha_p(f)}\right)\frac{h\langle z_{f,\mathcal{C}},z_{f,\overline{\mathcal{C}}}\rangle}{u^2(4|D|)^{r-1}},$$

where $u=(\sharp\mathcal{O}_K^*)/2$.

Put $z_{K,f}=\mathrm{cor}_{H/K}(z_f)\in H^1_f(K,V_f)$. As $\mathrm{res}_{H/K}(z_{K,f})=hz_{f,1}$ and $h\langle x,y\rangle_K=\langle\mathrm{res}_{H/K}(x),\mathrm{res}_{H/K}(y)\rangle_H$ for $x,y\in H^1_f(K,V_f)$ (the subscripts K resp. H indicate the field over which the height is computed), we have

$$\langle z_{K,f},z_{K,f}\rangle_K = h\langle z_{f,1},z_{f,1}\rangle_H$$

Applying Theorem A for the trivial character, we obtain (omitting $\mathcal{C}=1$ from the notation)

Corollary.

$$L_p'(f\otimes K,\mathbb{1}) = -\left(1-\frac{p^{r-1}}{\alpha_p(f)}\right)^4\frac{\langle z_{K,f},z_{K,f}\rangle_K}{u^2(4|D|)^{r-1}}$$

The minus sign in the statement of Theorem A depends on the normalizations of the height pairing and the L-function, but not of the reciprocity map. It seems that [PR 2] contains a sign mistake; see II.6.4 for more details.

(0.8) In [Ne 1], we proved the following statement (in a slightly different formulation; see II.6.5 and 0.13 for more comments and precise assumptions).

Theorem. *If $z_{K,f} \neq 0$, then the image of the map*

$$\Phi_{K,f} : CH^r(W \otimes_{\mathbf{Q}} K)_0 \otimes \mathbf{Q} \longrightarrow H^1_f(K, V_f)$$

satisfies

$$\mathbf{Q}_p \cdot \mathrm{Im}(\Phi_{K,f}) = \widehat{F(f)} \cdot z_{K,f} = H^1_f(K, V_f)$$

Combined with the Corollary of Theorem A, this proves

Theorem B. *If $L'_p(f \otimes K, \mathbb{1}) \neq 0$ for some ℓ_K, then*

$$\mathbf{Q}_p \cdot \mathrm{Im}(\Phi_{K,f}) = \widehat{F(f)} \cdot z_{K,f} = H^1_f(K, V_f)$$

(0.9) An arithmetic logarithm $\ell_K : \mathbf{A}^*_K/K^* \longrightarrow \mathbf{Q}_p$ is called cyclotomic (resp. anticyclotomic) if $\ell_K \circ \tau = \ell_K$ (resp. $\ell_K \circ \tau = -\ell_K$). Every ℓ_K can be written uniquely as a sum of a cyclotomic and an anticyclotomic logarithms. For an anticyclotomic ℓ_K, Theorem A is vacuous, as its statement becomes $0 = 0$.

Let $\mathbf{Q}_\infty/\mathbf{Q}$ be the unique $\mathbf{Z}_p$-extension of $\mathbf{Q}$. Write $\mathbf{N} : G(K_\infty/K) \longrightarrow G(\mathbf{Q}_\infty/\mathbf{Q})$ for the restriction map and $\langle\ \rangle$ for the isomorphism $G(\mathbf{Q}_\infty/\mathbf{Q}) \xrightarrow{\sim} 1 + p\mathbf{Z}_p$ induced by the cyclotomic character. Then every cyclotomic ℓ_K is a scalar multiple of $\langle\ \rangle \circ \mathbf{N}$ and we have

$$L_p(f \otimes K, \langle\ \rangle^s \circ \mathbf{N}) = \frac{\langle N\rangle^{2s}(-1)^{r-1}|D|^{1/2}}{2^{2r-1}(2\pi)^2\langle f, f\rangle_N} L_p^{\mathrm{MTT}}(f, x^{r-1}\langle\ \rangle^s) L_p^{\mathrm{MTT}}\left(f \otimes \left(\frac{D}{\cdot}\right), x^{r-1}\langle\ \rangle^s\right), \tag{0.9.1}$$

where $L_p^{\mathrm{MTT}}(f, \lambda)$ is the p-adic L-function of f over $\mathbf{Q}$, in the normalization of [M - T - T] (cf. I.5.13).

The complex functional equation

$$\Lambda(f, s) = N^{s/2}(2\pi)^{-s}\Gamma(s)L(f, s) = w\Lambda(f, 2r - s)$$

with $w = (-1)^{r-1}\lambda_N(f) = \pm 1$ (where $\lambda_N(f)$ is the eigenvalue of f under the Fricke involution) has a p-adic counterpart ([M - T - T, 18.3])

$$\begin{aligned} L_p^{\mathrm{MTT}}(f, x^{r-1}\langle\ \rangle^s) &= w\langle N\rangle^{-s} L_p^{\mathrm{MTT}}(f, x^{r-1}\langle\ \rangle^{-s}) \\ L_p^{\mathrm{MTT}}\left(f \otimes \left(\frac{D}{\cdot}\right), x^{r-1}\langle\ \rangle^s\right) &= -w\langle N|D|\rangle^{-s} L_p^{\mathrm{MTT}}\left(f \otimes \left(\frac{D}{\cdot}\right), x^{r-1}\langle\ \rangle^{-s}\right) \end{aligned} \tag{0.9.2}$$

(0.10) Assume that $w = -1$. In this case

$$z_{K,f} \in H^1_f(K, V_f)^{G(K/\mathbf{Q})} = H^1_f(\mathbf{Q}, V_f)$$

and 0.9.1-2 imply that, for $\ell_K = \langle\ \rangle \circ \mathbf{N}$,

$$L'_p(f \otimes K, \mathbb{1}) = \frac{(-1)^{r-1}|D|^{1/2}}{2^{2r-1}(2\pi)^2\langle f, f\rangle_N} L_p^{\mathrm{MTT}}\left(f \otimes \left(\frac{D}{\cdot}\right), x^{r-1}\right) \frac{d}{ds} L_p^{\mathrm{MTT}}(f, x^{r-1}\langle\ \rangle^s)\Big|_{s=0}$$

with

$$L_p^{\mathrm{MTT}}\left(f \otimes \left(\frac{D}{\cdot}\right), x^{r-1}\right) = \left(1 - \frac{p^{r-1}}{\alpha_p(f)}\right)^2 \frac{(r-1)!}{(-2\pi i)^{r-1}} L\left(f \otimes \left(\frac{D}{\cdot}\right), r\right)$$

According to [Sk - Za], there is a non-zero Jacobi form $\phi = \sum c(n, k)q^n\zeta^k \in J^{\mathrm{cusp}}_{r+1,N}$ (unique up to a scalar multiple) with the same Hecke eigenvalues as f. Multiplying by a constant, we can assume that all coefficients of ϕ are real. By [G - K - Z], we have

$$\frac{L(f\otimes\left(\frac{D}{\cdot}\right),r)}{\langle f,f\rangle_N}=\frac{2^{2r-1}\pi^r N^{r-1}}{(r-1)!|D|^{r-1/2}}\frac{c(n,k)^2}{\langle\phi,\phi\rangle},$$

whenever $D=k^2-4nN$. It follows that

$$L_p'(f\otimes K,\mathbb{1})=\left(1-\frac{p^{r-1}}{\alpha_p(f)}\right)^2\frac{(-i)^{r-1}c(n,k)^2}{2^{r+1}\pi|D|^{r-1}\langle\phi,\phi\rangle}\frac{d}{ds}L_p^{\mathrm{MTT}}(f,x^{r-1}\langle\ \rangle^s)\Big|_{s=0} \tag{0.10.1}$$

Combining Cor. of Thm. A, Thm. 0.8 and the non-vanishing results of [Wa], we obtain the following statement.

Theorem C. *Assume that*

$$\mathrm{ord}_{s=0}L_p^{\mathrm{MTT}}(f,x^{r-1}\langle\ \rangle^s)=1$$

(which implies that $w=-1$). Then
(1) *There exist infinitely many imaginary quadratic fields $\mathbf{Q}(\sqrt{D})$ of odd discriminant, in which all primes dividing pN split and for which $z_{\mathbf{Q}(\sqrt{D}),f}\in H^1_f(\mathbf{Q},V_f)$ is non-zero.*
(2) *For any such $\mathbf{Q}(\sqrt{D})$,*

$$H^1_f(\mathbf{Q},V_f)=\widehat{F(f)}\cdot z_{\mathbf{Q}(\sqrt{D}),f}$$

(3) *If $D_1=k_1^2-4n_1N$, $D_2=k_2^2-4n_2N$, are odd discriminants of two imaginary quadratic fields in which all primes dividing pN split, then*

$$c(n_2,k_2)z_{\mathbf{Q}(\sqrt{D_1}),f}=\pm c(n_1,k_1)z_{\mathbf{Q}(\sqrt{D_2}),f}$$

One would expect that (3) always holds with the plus sign, as in [G - K - Z] in the weight two case (cf. also [ScC] for some calculations involving the complex Abel-Jacobi map).

(0.11) Assume that $w=1$. In this case,

$$z_{K,f}\in H^1_f(K,V_f)^{\tau=-1}$$

and

$$L_p'(f\otimes K,\mathbb{1})=\frac{(-1)^{r-1}|D|^{1/2}}{2^{2r-1}(2\pi)^2\langle f,f\rangle_N}L_p^{\mathrm{MTT}}(f,x^{r-1})\frac{d}{ds}L_p^{\mathrm{MTT}}(f\otimes\left(\frac{D}{\cdot}\right),x^{r-1}\langle\ \rangle^s)\Big|_{s=0}$$

with

$$L_p^{\mathrm{MTT}}(f,x^{r-1})=\left(1-\frac{p^{r-1}}{\alpha_p(f)}\right)^2\frac{(r-1)!}{(-2\pi i)^{r-1}}L(f,r)$$

In the absence of a p-adic analogue of the non-vanishing results of [B - F - H], [Iw], [Mu - Mu], we can prove only a conditional result.

Theorem D. *Assume that $L(f,r)\neq 0$ (which implies that $w=1$). Suppose that there is an imaginary quadratic field $\mathbf{Q}(\sqrt{D})$ with an odd discriminant, in which all primes dividing pN split and such that*

$$\frac{d}{ds}L_p^{\mathrm{MTT}}(f\otimes\left(\frac{D}{\cdot}\right),x^{r-1}\langle\ \rangle^s)\Big|_{s=0}\neq 0$$

Then

$$H^1_f(\mathbf{Q},V_f)=0$$

(0.12) The proof of Theorem A follows rather closely [PR 2]. Essentially, we compute the height $\langle x,T_m x^\sigma\rangle$ for sufficiently many integers $m\geq 1$. Here x is a Heegner cycle, $\sigma\in G(H/K)$ and T_m denotes the m-th

Hecke correspondence on W. More precisely, if x and $T_m x^\sigma$ have disjoint supports, then the height can be expressed in terms of local heights at all (non-archimedean) places v of H:

$$\langle x, T_m x^\sigma \rangle = \sum_v \langle x, T_m x^\sigma \rangle_v$$

We compute the sum over places outside of p

$$\sum_{v \nmid p} \langle x, T_m x^\sigma \rangle_v,$$

which turns out to be almost equal to the m-th coefficient of a certain modular form $G_\sigma \in \overline{M}_{2r}(\Gamma_0(Np^\infty); \mathbf{Q}_p)$ which controls the derivative of the p-adic L-function. To pass from G_σ to the L-function, one has to apply, among other things, Hida's projector e onto the space of ordinary modular forms. A trick from [PR 2, 5.2] then shows that e kills the contribution of local heights at places $v \mid p$ (at least when it comes to the f-component of the height). It is quite essential for the proof that

$$\sum_{m \geq 1} \langle x, T_m x^\sigma \rangle q^m$$

is known to be a modular form in $S_{2r}(\Gamma_0(N); \mathbf{Q}_p)$ (this follows from the fact that Φ commutes with Hecke correspondences and that the p-adic height factors through Φ). This helps to get around the fact that local heights are defined only for cycles with disjoint supports. In [Gr - Za] (and [PR 2]), a deformation argument was used to deal with local heights of divisors on a curve whose supports are not necessarily disjoint. However, we were unable to generalize this argument to our situation.

This is the main reason why an analogue of Theorem A for archimedean heights $\langle\ ,\ \rangle_\infty$ is still lacking. Either we have to know a priori that

$$\sum_{m \geq 1} \langle x, T_m x^\sigma \rangle_\infty q^m$$

is a modular form, or we have to find a way to compute *all* of its coefficients (at least for all m prime to N) and make this conclusion only a posteriori.

We used the ideas of [Br] and worked with sections of the sheaf $\mathrm{Sym}^{2r-2} R^1 f_* \mathbf{Q}_p(r-1)$ over divisors on the modular curve $X_0(N)$, rather than with cycles on W. The point is that the Abel-Jacobi map and local heights for cycles contained in the fibres of the projection $W \longrightarrow X_0(N)$ depend only on the cohomology of the given cycles in those fibres. Local heights at places $v \nmid p$ are then computed by using an intersection formula from [Br] (with a corrected sign) and calculations of [Gr - Za].

It should be clear from this account how much the present work owes to [Gr - Za], [Br] and [PR 1], [PR 2]. When possible, we try to avoid repetition of calculations performed in these papers. The reader is therefore advised to consult them while reading the present work.

(0.13) The present paper and [Ne 1], [Ne 2] all make use of several deep results from p-adic Hodge theory. The "crystalline conjecture" for smooth projective varieties with good reduction is invoked to ensure that V_f is a crystalline representation of $G(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ (so that the height machinery of [Ne 2] can be applied to $H^1_f(H, V_f)$). The "de Rham conjecture" for smooth separated (but not necessarily proper) varieties was used in [Ne 1] to show that the image of Φ_f is contained in $H^1_f(H, V_f)$. However, in a letter to the author dated May 27, 1992, Faltings pointed out a gap in the proof of the "de Rham conjecture" in [Fa] and sketched an idea how to modify the argument. To this day, however, a rigorous proof has not been written down. More recently, a remark of Berthelot [Ber] seems to call into question one step in the proof of the "crystalline conjecture" in [Fa]. Fortunately, at least for sufficiently large primes p, one can rely on proofs of the crystalline conjecture in [Fo - Me] (unramified case) and [KaK - Me] (general case). It will be shown in a future publication that (again for sufficiently large p) the image of Φ_f is indeed contained in $H^1_f(H, V_f)$ without making a reference to [Fa]. For the reader's convenience, we sketch the argument in II.1.4. In our case the restriction is $p > \dim(W) = 2r - 1$; if the results of [Fa] can be corroborated, then this condition can be dropped from the assumptions of Theorems A–D and Theorem 0.8. [1]

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I. Modular forms and p-adic L-functions

In this chapter we define p-adic L-functions referred to in 0.5. Most of the work has been done in the weight two case by Perrin-Riou in [PR 1]; we merely indicate basic changes that have to be made in the higher weight case. This occupies Sec. 1–5 and is carried out in the same generality as [PR 1] is (we could have appealed to the results of [Hi 1], [Hi 2], but the setting of [PR 1] was best suited for our work). In Sec. 6 we specialize to the situation of Thm. A and perform a calculation of Fourier coefficients of certain p-adic integrals after [PR 2].

1. Modular forms

(1.1) Let us recall the usual notation: $\mathcal{H}$ denotes the complex upper half plane with the standard action of $GL_2^+(\mathbf{R})$ by fractional linear transformations. For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(\mathbf{R})$ and any function f on $\mathcal{H}$ we define, for $k \in \mathbf{Z}$,

$$(f|_k\gamma)(z) := (\det(\gamma))^{k/2}(cz+d)^{-k} f(\gamma(z))$$

(for odd k, we take the positive square root of $\det(\gamma)$). If Γ is a subgroup of $SL_2(\mathbf{Z})$ of finite index, we define $\widetilde{M}_k(\Gamma)$ to be the space of complex-valued $\mathcal{C}^\infty$-functions on $\mathcal{H}$ satisfying $f|_k\gamma = f$ for all $\gamma \in \Gamma$ and having at most polynomial growth at the cusps: $|(f|_k\alpha)(x+iy)| = O(y^N)$ as $y \longrightarrow \infty$ for each $\alpha \in SL_2(\mathbf{Z})$. We also write $M_k(\Gamma)$ (resp. $S_k(\Gamma)$) for the space of holomorphic modular forms (resp. cusp forms) of weight k on Γ. If χ is a Dirichlet character modulo $N > 1$, we denote by $M_k(\Gamma_0(N), \chi)$ the set of $f \in M_k(\Gamma_1(N))$ satisfying $f\Big|_k \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \chi(d) f$ for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$ and we put $S_k(\Gamma_0(N), \chi) = S_k(\Gamma_1(N)) \cap M_k(\Gamma_0(N), \chi)$. We also denote by $\widetilde{M}_k(\Gamma_0(N), \chi)$ the corresponding space of $\mathcal{C}^\infty$-functions. If f is a function on $\mathcal{H}$, we define $f^\tau(z) = \overline{f(-\overline{z})}$. For $f \in S_k(\Gamma_0(N), \chi)$, $g \in \widetilde{M}_k(\Gamma_0(N), \chi)$, the Petersson scalar product is defined by

$$\langle f, g \rangle_N := \int\limits_{\Gamma_0(N)\backslash\mathcal{H}} \overline{f(z)} g(z) y^{k-2}\, dx\, dy$$

(1.2) We shall need some facts about Shimura's differential operators (cf. [Sh]), defined on $\mathcal{H}$ by the formulas

$$\begin{aligned} \delta_s &= \frac{1}{2\pi i}\left(\frac{\partial}{\partial z} + \frac{s}{2iy}\right) = q\frac{\partial}{\partial q} - \frac{s}{4\pi y} \\ \delta_s^r &= \underbrace{\delta_{s+2r-2} \circ \cdots \circ \delta_{s+2} \circ \delta_s}_{r \text{ times}} = \sum_{t=0}^{r} \binom{r}{t} \frac{\Gamma(s+r)}{\Gamma(s+t)} (-4\pi y)^{t-r} \left(q\frac{\partial}{\partial q}\right)^t \end{aligned} \tag{1.2.1}$$

[1] *Added in proof.* The assumption $p > 2r - 1$ is indeed unnecessary. An exchange of letters between Berthelot and Faltings (June–August 1994) helped to clarify several obscure points in [Fa]. In particular, the objection raised in [Ber] is not valid. A recent preprint of Tsuji gives a syntomic proof of the crystalline conjecture for all $p > 2$. This implies that the assumptions (a),(b) of Lemma II.1.4 can be replaced by $p > 2$.

(where $z = x + iy$, $q = e^{2\pi i z}$). They satisfy the relations

$$\begin{aligned} \delta_{s+t}(fg) &= g\delta_s(f) + f\delta_t(g) \\ \delta_k^m(f|_k\gamma) &= (\delta_k^m f)|_{k+2m}\gamma \qquad (\forall \gamma \in GL_2^+(\mathbf{R})) \end{aligned} \tag{1.2.2}$$

It follows that for $g \in \widetilde{M}_j(\Gamma_0(N), \chi)$ and $h \in \widetilde{M}_k(\Gamma_0(N), \psi)$,

$$g\delta_k^m(h) \in \widetilde{M}_{j+k+2m}(\Gamma_0(N), \chi\psi) \tag{1.2.3}$$

We shall also need the formulas

$$\begin{aligned} \delta_k^m \left((cz+d)^{-k} \left(\frac{y}{|cz+d|^2} \right)^s \right) &= (-4\pi)^{-m} \frac{\Gamma(k+s+m)}{\Gamma(k+s)} (cz+d)^{-(k+2m)} \left(\frac{y}{|cz+d|^2} \right)^{s-m} \\ \delta_1^{r-1}(q^n) &= \frac{(r-1)!}{(-4\pi y)^{r-1}} q^n \, p_{r,r-1}(4\pi n y) \end{aligned} \tag{1.2.4}$$

where

$$p_{r,r-1}(t) = \sum_{j=0}^{r-1} \binom{r-1}{j} \frac{(-t)^j}{j!}$$

(our notation follows [Gr - Za]).

(1.3) For every integer $k \geq 2$ there is a holomorphic projection defined on a certain subspace $\widetilde{M} \subset \widetilde{M}_k(\Gamma_0(N), \chi)$

$$H : \widetilde{M} \longrightarrow S_k(\Gamma_0(N), \chi)$$

with the properties (see [St], [Gr - Za])

$$\begin{aligned} \langle F, f \rangle_N &= \langle F, H(f) \rangle_N \qquad (\forall F \in S_k(\Gamma_0(N), \chi)) \\ H(f)|_k\gamma &= H(f|_k\gamma) \qquad (\forall \gamma \in GL_2^+(\mathbf{R})) \end{aligned} \tag{1.3.1}$$

In a special situation we are interested in, there is a simple formula for H (cf. [Gr - Za], where the case $\chi = 1$ is discussed):

Suppose that $f \in \widetilde{M}_{2r}(\Gamma_0(N), \chi)$ with $2r > 2$ has a Fourier expansion

$$f(z) = \sum_{m \in \mathbf{Z}} a_m(y) \, e^{2\pi i m z}$$

and satisfies $(f|_{2r}\gamma)(z) = O(y^{-\varepsilon})$ as $y = \mathrm{Im}(z) \longrightarrow \infty$ for some $\varepsilon > 0$ and every $\gamma \in SL_2(\mathbf{Z})$. Then

$$H(f) = \sum_{m=1}^{\infty} a_m \, e^{2\pi i m z}$$

with

$$a_m = \frac{(4\pi m)^{2r-1}}{(2r-2)!} \int_0^\infty a_m(y) e^{-4\pi m y} y^{2r-2} \, dy \tag{1.3.2}$$

For $f = g \, \delta_k^m(h)$ with $g \in M_j(\Gamma_0(N), \chi)$, $h \in M_k(\Gamma_0(N), \psi)$ (hence $f \in \widetilde{M}_{j+k+2m}(\Gamma_0(N), \chi\psi)$), the holomorphic projection of f can be computed as follows (cf. [Hi 1], [Sh]): there is a unique decomposition

$$g \, \delta_k^m(h) = \sum_{t=0}^{m} \delta_{k-2t}^t(g_t)$$

with $g_t \in M_{j+k+2m-2t}(\Gamma_0(N), \chi\psi)$ and $H(f) = g_0$.

(1.4) We now define non-holomorphic Eisenstein series that will be used in the sequel. Fix integers $r \geq 1$, $M \geq 1$. For any function $\phi : \mathbf{Z}/M\mathbf{Z} \longrightarrow \mathbf{C}$, define for $z = x + iy \in \mathcal{H}$, $\mathrm{Re}(s) > 3/2 - r$

$$E_r(z, s, \phi) = \sum_{\substack{m,n\in\mathbf{Z} \\ M|m}}{}' \frac{\phi(n)}{(mz+n)^{2r-1}} \left(\frac{y}{|mz+n|^2}\right)^s$$

$$\widetilde{E}_r(z, s, \phi) = \sum_{m,n\in\mathbf{Z}}{}' \frac{\phi(m)}{(mz+n)^{2r-1}} \left(\frac{y}{|mz+n|^2}\right)^s$$

Both series depend only on $\phi^-(n) = ((\phi(n) - \phi(-n))/2$ and admit a holomorphic continuation to all $s \in \mathbf{C}$ ([Si, Cor. of Thm. I.5.3]). It makes sense, therefore, to define

$$E_r(z, \phi) = E_r(z, 1-r, \phi), \qquad \widetilde{E}_r(z, \phi) = \widetilde{E}_r(z, 1-r, \phi)$$

Some properties of these Eisenstein series are summarized in the following Proposition.

(1.5) Proposition. (1) *For* $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(M)$ *define* $\phi^*(n) = \phi(an)$, $\phi'(m) = \phi(dm)$. *Then*

$$E_r(z, \phi)|_{2r-1}\gamma = E_r(z, \phi^*)$$
$$\widetilde{E}_r(z, \phi)|_{2r-1}\gamma = \widetilde{E}_r(z, \phi')$$

(2) *If* ϕ *is a Dirichlet character modulo* M, *then*

$$E_r(z, \phi) \in \widetilde{M}_{2r-1}(\Gamma_0(M), \overline{\phi})$$
$$\widetilde{E}_r(z, \phi) \in \widetilde{M}_{2r-1}(\Gamma_0(M), \phi)$$

For general ϕ, $E_r, \widetilde{E}_r$ *both lie in* $\widetilde{M}_{2r-1}(\Gamma_1(M))$.
(3)

$$\delta_1^{r-1}(E_1(z, \phi)) = \frac{(r-1)!}{(-4\pi)^{r-1}} E_r(z, \phi)$$
$$\delta_1^{r-1}(\widetilde{E}_1(z, \phi)) = \frac{(r-1)!}{(-4\pi)^{r-1}} \widetilde{E}_r(z, \phi)$$

(4) *The Fourier expansions are*

$$E_r(z, \phi) = \left(-\frac{2\pi i}{M}\right) y^{1-r}[\frac{1}{2}\widetilde{L}(0, \widehat{\phi}) + \sum_{k\cdot m>0} \widehat{\phi}(k)\mathrm{sgn}(k) p_{r,r-1}(4\pi kmy) q^{km}]$$
$$\widetilde{E}_r(z, \phi) = (-2\pi i) y^{1-r}[\frac{1}{2}\widetilde{L}(0, \phi) + \sum_{k\cdot m>0} \phi(k)\mathrm{sgn}(k) p_{r,r-1}(4\pi kmy) q^{km}]$$

where $q = e^{2\pi iz}$ *and*

$$\widetilde{L}(s, \phi) = \sum_{0\neq n\in\mathbf{Z}} \frac{\phi(n)\mathrm{sgn}(n)}{|n|^s}, \qquad \widehat{\phi}(k) = \sum_{n\in\mathbf{Z}/M\mathbf{Z}} \phi(n) e^{2\pi ikn/M}$$

(5) $E_r(z, \phi) = M^{-1}\widetilde{E}(z, \widehat{\phi})$, $\quad E_r(z, \widehat{\phi}) = -\widetilde{E}(z, \phi)$
(6)

$$E_r(z, \phi)\Big|_{2r-1} \begin{pmatrix} 0 & -1 \\ M & 0 \end{pmatrix} = M^{-1/2}\widetilde{E}_r(z, \phi)$$
$$\widetilde{E}_r(z, \phi)\Big|_{2r-1} \begin{pmatrix} 0 & -1 \\ M & 0 \end{pmatrix} = -M^{-1/2} E_r(z, \phi)$$

Proof. (1) is a simple calculation; (2) follows from (1) and (3) from (1.2.4). (4) can be proved in a number of ways – e.g. by an explicit calculation based on the Poisson formula as in [Gr - Za]; or by a reduction to the well-known case $r = 1$ by using (3) and (1.2.4). (5) follows from (4) and (6) is again an easy calculation.

(1.6) The Rankin-Selberg convolution of two modular forms $f \in M_j(\Gamma_0(M), \chi)$, $g \in M_k(\Gamma_0(M), \psi)$ with Fourier expansions

$$f = \sum_{n \geq 0} a(n) q^n, \qquad g = \sum_{n \geq 0} b(n) q^n$$

is defined as

$$D_M(f, g, s) := L_M(2s + 2 - j - k, \chi\psi) \sum_{n \geq 1} \frac{a(n)b(n)}{n^s},$$

where

$$L_M(s, \chi\psi) = \sum_{\substack{n \geq 1 \\ (n,M)=1}} \frac{(\chi\psi)(n)}{n^s} = \prod_{p \nmid M} (1 - (\chi\psi)(p)p^{-s})^{-1}$$

(1.7) If f (resp. g) is a normalized newform of level $N_f | M$ (resp. $N_g | M$), then

$$L(f, s) = \sum_{n \geq 1} a(n) n^{-s} = \prod_p [(1 - \alpha_p p^{-s})(1 - \alpha'_p p^{-s})]^{-1}$$
$$L(g, s) = \sum_{n \geq 1} b(n) n^{-s} = \prod_p [(1 - \beta_p p^{-s})(1 - \beta'_p p^{-s})]^{-1}$$

with $\alpha_p \alpha'_p = \chi(p) p^{2j-1}$, $\beta_p \beta'_p = \psi(p) p^{2k-1}$ for primes $p \nmid M$; $\alpha_p \alpha'_p \beta_p \beta'_p = 0$ for $p | N = \mathrm{lcm}(N_f, N_g)$ and (cf. [Sh, 3.1])

$$D_M(f, g, s) = \prod_p [(1-\alpha_p\beta_p p^{-s})(1-\alpha_p\beta'_p p^{-s})(1-\alpha'_p\beta_p p^{-s})(1-\alpha'_p\beta'_p p^{-s})]^{-1} \prod_{p | M/N} (1-\alpha_p\alpha'_p\beta_p\beta'_p p^{-2s}) \quad (1.7.1)$$

(1.8) Suppose that, in the notation of 1.6, $j = 2r$ is even, $k = 1$ and that f is a cusp form. Write ϕ for the character $\chi\psi$ extended by zero to $\mathbf{Z}/M\mathbf{Z}$. Then

$$E_r(z, s, \phi) = \widetilde{L}_M(2s + 2r - 1, \phi) \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(M)} \phi(d)\; y^s |_{2r-1} \gamma$$

with $\gamma = \begin{pmatrix} * & * \\ c & d \end{pmatrix}$, $\Gamma_\infty = \pm \begin{pmatrix} 1 & \mathbf{Z} \\ 0 & 1 \end{pmatrix}$. The standard unfolding trick gives

$$\langle f^\tau, g E_r(z, s, \phi) \rangle_M = \widetilde{L}_M(2s + 2r - 1, \phi) \int_{\Gamma_\infty \backslash \mathcal{H}} \overline{f^\tau(z)} g(z) y^{s+2r-2}\, dx\, dy$$

with

$$\int_{\Gamma_\infty \backslash \mathcal{H}} \overline{f^\tau(z)} g(z) y^{s+2r-2}\, dx\, dy = \frac{\Gamma(s + 2r - 1)}{(4\pi)^{s+2r-1}} \sum_{n \geq 1} \frac{a(n)b(n)}{n^{s+2r-1}},$$

hence

$$\langle f^\tau, gE_r(z,s,\phi)\rangle_M = (1-\phi(-1))\frac{\Gamma(s+2r-1)}{(4\pi)^{s+2r-1}}D_M(f,g,s+2r-1)$$

$$\langle f^\tau, gE_r(z,\phi)\rangle_M = (1-\phi(-1))\frac{(r-1)!}{(4\pi)^{r}}D_M(f,g,r)$$

Using Prop. 1.5.3, this can be rewritten as

$$\left\langle f^\tau, g\delta_1^{r-1}(E_1(z,\phi))\right\rangle_M = \frac{(1-\phi(-1))(-1)^{r-1}((r-1)!)^2}{(4\pi)^{2r-1}}D_M(f,g,r) \tag{1.8.1}$$

(1.9) Proposition. *Let $f=\sum_{n\geq 1}a(n)q^n$ (resp. $g=\sum_{n\geq 0}b(n)q^n$) be a cusp form (resp. a holomorphic modular form) of weight one on $\Gamma_0(N)$. Then $H(f\delta_1^{r-1}(g))=\sum_{n\geq 1}c(n)q^n$ with*

$$c(n)=\frac{(-1)^{r-1}}{\binom{2r-2}{r-1}}n^{r-1}\sum_{j+k=n}a(j)b(k)P_{r-1}\left(\frac{j-k}{j+k}\right),$$

where

$$P_k(t)=\frac{1}{2^k\cdot k!}\left(\frac{d}{dt}\right)^k[(t^2-1)^k]$$

is the Legendre polynomial of degree k.

Proof. It follows from (1.2.4) that

$$f\delta_1^{r-1}(g)=\frac{(r-1)!}{(-4\pi)^{r-1}}y^{1-r}\sum_{j\geq 1}\sum_{k\geq 0}a(j)b(k)p_{r,r-1}(4\pi ky)q^{j+k}$$

The formula (1.3.2) then gives

$$c(n)=\frac{(r-1)!}{(-4\pi)^{r-1}}\cdot\frac{(4\pi n)^{2r-1}}{(2r-2)!}\sum_{j+k=n}a(j)b(k)\int_0^\infty p_{r,r-1}(4\pi ky)e^{-4\pi(j+k)y}y^{r-1}\,dy$$

and the value of the integral is equal to (cf. [Gr - Za, p.293])

$$\int_0^\infty p_{r,r-1}(4\pi ky)e^{-4\pi(j+k)y}y^{r-1}\,dy=\frac{(r-1)!}{(-4\pi)^{r-1}}P_{r-1}\left(1-\frac{2k}{j+k}\right)$$

(1.10) Let $N>1$ be an integer and Q a divisor of N prime to N/Q. A Dirichlet character ψ modulo N decomposes uniquely as $\psi=\psi_Q\cdot\psi_{N/Q}$, where ψ_Q (resp. $\psi_{N/Q}$) is a character modulo Q (resp. N/Q). The Atkin-Lehner operator (cf. [At - Li, sec. 1])

$$W_Q: M_k(\Gamma_0(N),\psi_Q\cdot\psi_{N/Q})\longrightarrow M_k(\Gamma_0(N),\overline{\psi}_Q\cdot\psi_{N/Q})$$

is defined by

$$f|W_Q=\psi_Q(y)\psi_{N/Q}(x)f\Big|_k\begin{pmatrix}Qx & y\\ Nz & Qw\end{pmatrix},$$

where x,y,z,w are integers satisfying $Qxw-(N/Q)yz=1$.

These operators satisfy, among others, the following relations:

$$f|W_Q|W_Q = \psi_Q(-1)\overline{\psi}_{N/Q}(Q)f$$
$$f|_kT_p|W_Q = \psi_Q(p)f|W_Q|_kT_p, \tag{1.10.1}$$

where T_p (for $p \not| N$) denotes the p-th Hecke operator.
If f is a normalized ($a_1 = 1$) newform, then

$$f|W_Q = \lambda_Q(f)g$$

for a suitable normalized newform g. The complex number $\lambda_Q(f)$ is algebraic and has absolute value equal to 1. If $\psi_Q = 1$ (i.e. $\mathrm{cond}(\psi) \mid N/Q$), then $g = f$ by (1.10.1) and

$$\lambda_Q(f)^2 = \overline{\psi}_{N/Q}(Q) \tag{1.10.2}$$

2. p-adic modular forms

In this section we recall the basic setup of the theory of p-adic modular forms, following [Hi 1].

(2.1) As in the Introduction, we fix embeddings $i_\infty : \overline{\mathbf{Q}} \hookrightarrow \mathbf{C}$, $i_p : \overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_p$ (for a fixed prime number p). For a subring $A \subset \overline{\mathbf{Q}}$ and $\Gamma = \Gamma_0(N)$ or $\Gamma_1(N)$, one defines $M_k(\Gamma; A)$ as the set of all modular forms

$$f = \sum_{n\geq 0} a_n(f)q^n \in M_k(\Gamma)$$

with Fourier coefficients $a_n(f)$ in A (and similarly for $M_k(\Gamma_0(N), \chi; A)$, where χ is any Dirichlet character modulo N with values in A).
There is a natural p-adic norm on these spaces, given by

$$|f|_p = \sup_n |i_p(a_n(F))|_p,$$

where $|\cdot|_p$ is the standard valuation on $\overline{\mathbf{Q}}_p$ (normalized by $|p|_p = p^{-1}$). Denote by $M_k(\Gamma; \widehat{A})$ (resp. $M_k(\Gamma_0(N), \chi; \widehat{A})$) the corresponding completions; here $\widehat{A}$ denotes the closure of $i_p(A)$.
Finally, one defines

$$M_k(\Gamma_i(Np^\infty), \widehat{A}) = \bigcup_{n=0}^{\infty} M_k(\Gamma_i(Np^n), \widehat{A})$$

and writes $\overline{M}_k(\Gamma_i(Np^\infty), \widehat{A})$ for its completion under $|\cdot|_p$.

(2.2) Let $f = \sum a_n q^n$ be a normalized newform in $S_k(\Gamma_0(N), \psi)$ and suppose that $\widehat{A}$ contains all coefficients of f and values of ψ (under the fixed embedding i_p). The form f is called *ordinary* if $|i_p(a_p)|_p = 1$.
If this is the case, then precisely one of the roots of the polynomial $X^2 - a_pX + \psi(p)p^{k-1}$ (call it $\alpha_p(f)$) satisfies $|i_p(\alpha_p)|_p = 1$. Hida [Hi 1, 3.3] defines

$$f_0(z) = f(z) - \frac{\psi(p)p^{k-1}}{\alpha_p(f)} f(pz)$$

(recall that $\psi(p) = 0$ if $p \mid N$). This is a modular form in $M_k(\Gamma_0(Np^\alpha), \psi; \widehat{A})$ for

$$\alpha = \begin{cases} 1 & \text{if } p \not| N \\ 0 & \text{if } p \mid N \end{cases}$$

We have

$$f_0^\tau(z) = f^\tau(z) - \frac{\overline{\psi}(p)p^{k-1}}{\overline{\alpha_p(f)}}$$

(2.3) Hida [Hi 1, p. 172] defines a projection

$$e : \overline{M}_k(\Gamma_0(Np^\infty), \psi; \widehat{A}) \longrightarrow M_k(\Gamma_0(Np^\alpha), \psi; \widehat{A})$$

onto the space of ordinary forms. On $M_k(\Gamma_0(Np^n), \psi; \widehat{A})$, it is equal to the limit

$$\lim_{r\to\infty} U_p^{cp^r}$$

for a suitable integer c (depending on n). Here U_p is given by

$$U_p\left(\sum_{m\geq 1} a(m)q^m\right) = \sum_{m\geq 1} a(mp)q^m$$

Hida [Hi 1, p. 174] also defines a projection (which he denotes by $\mathbb{1}_f$)

$$\mathbb{1}_{f_0} : M_k(\Gamma_0(Np^\alpha), \psi; \widehat{A}) \longrightarrow M_k(\Gamma_0(Np^\alpha), \psi; c(f)^{-1}\widehat{A})$$

onto the f_0-part (here $c(f)$ is a suitable non-zero element of $\widehat{A}$). Following [PR 1, p.7], we define

$$l_{f_0} : M_k(\Gamma_0(Np^\alpha), \psi; \widehat{A}) \longrightarrow c(f)^{-1}\widehat{A}$$

by $l_{f_0}(g) = a_1(g|\mathbb{1}_{f_0})$ and

$$L_{f_0} = l_{f_0} \circ e : \overline{M}_k(\Gamma_0(Np^\infty), \psi; \widehat{A}) \longrightarrow c(f)^{-1}\widehat{A}$$

Note that our L_{f_0} is denoted by l_f in [Hi 1].

(2.4) [Hi 1, 4.5] gives an explicit formula for L_{f_0}: if $g \in M_k(\Gamma_0(Np^\mu), \psi; \overline{\mathbf{Q}})$ with $\mu \geq \alpha$, then

$$L_{f_0}(g) = \left(\frac{p^{k/2-1}}{\alpha_p(f)}\right)^{\mu-\alpha} \frac{\left\langle f_0^\tau\Big|_k \begin{pmatrix} 0 & -1 \\ Np^\mu & 0 \end{pmatrix}, g \right\rangle_{Np^\mu}}{\left\langle f_0^\tau\Big|_k \begin{pmatrix} 0 & -1 \\ Np^\alpha & 0 \end{pmatrix}, f_0 \right\rangle_{Np^\alpha}} \tag{2.4.1}$$

and takes values in $\overline{\mathbf{Q}}$. This implies that

$$L_{f_0}(f) = \left(1 - \frac{\psi(p)p^{2r-1}}{\alpha_p(f)^2}\right)^{-1} \tag{2.4.2}$$

3. Galois groups and basic measures

(3.1) Let $K = \mathbf{Q}(\sqrt{D})$, $D < 0$, be an imaginary quadratic field with discriminant D. Write $\mathcal{O}_K$ for the ring of integers in K and let $h = \sharp(\mathrm{Pic}(\mathcal{O}_K))$, $w = \sharp(\mathcal{O}_K^*)$ be the class number and the number of roots of unity of K respectively. We write $\mathbf{N}$ for the norm $N_{K/\mathbf{Q}}$.

For a non-zero ideal $\mathfrak{f} \subset \mathcal{O}_K$ we denote by $K(\mathfrak{f})$ the ray class field of K (inside the fixed algebraic closure $\overline{\mathbf{Q}}$ of $\mathbf{Q}$). The reciprocity map (= Artin symbol) defines an isomorphism

$$I_\mathfrak{f}/P_\mathfrak{f} \xrightarrow{\sim} G(K(\mathfrak{f})/K),$$

where $I_{\mathfrak{f}}$ is the multiplicative group of fractional ideals of K prime to $\mathfrak{f}$ and $P_{\mathfrak{f}}$ the group of principal ideals (α) with $\alpha \equiv 1 \,(\mathrm{mod}\, \mathfrak{f}\,)^{\times}$.

For an integer $m \geq 1$, let $R_m \subset I_{(m)}$ be the group of principal ideals (β) with $\beta \equiv b \,(\mathrm{mod}\, m\mathcal{O}_K\,)^{\times}$ for some $b \in \mathbf{Z}$ prime to m. The subextension $K \subset H_m \subset K((m))$ satisfying $G(K((m))/H_m) \xrightarrow{\sim} R_m/P_{(m)}$ is the *ring class field* of K of conductor m.

Write $\mathcal{O}_m := \mathbf{Z} + m\mathcal{O}_K$ for the subring of $\mathcal{O}_K$ of index m. An $\mathcal{O}_m$-submodule $\mathcal{A}$ of K is an invertible $\mathcal{O}_m$-module (= projective module of rank 1) iff $\mathcal{O}_m = \{x \in K \mid x\mathcal{A} \subseteq \mathcal{A}\}$. A module with the latter property is called a proper $\mathcal{O}_m$-ideal. The multiplicative group of proper $\mathcal{O}_m$-ideals modulo principal $\mathcal{O}_m$-ideals $\alpha\mathcal{O}_m$ is canonically isomorphic to $\mathrm{Pic}(\mathcal{O}_m)$.

The restriction map $I \mapsto I \cap \mathcal{O}_m$, defined on ideals $I \subset \mathcal{O}_K$ prime to m, extends to a (surjective) homomorphism $I_{(m)} \longrightarrow \mathrm{Pic}(\mathcal{O}_m)$ with kernel R_m, defining thus an isomorphism

$$\mathrm{Pic}(\mathcal{O}_m) \xrightarrow{\sim} I_{(m)}/R_m \xrightarrow{\sim} G(H_m/K)$$

Write H for the Hilbert class field of K, $H = H_1 = K((1))$.

(3.2) For every $m \geq 1$, $K((m))$ is a Galois extension of $\mathbf{Q}$. The Galois group $G(K((m))/\mathbf{Q})$ is a semidirect product of $G(K((m))/K)$ and $G(K/\mathbf{Q})$. The non-trivial element $\tau \in G(K/\mathbf{Q})$ acts on the first group by conjugation: ${}^{\tau}g = \widetilde{\tau}g\widetilde{\tau}^{-1}$, where $\widetilde{\tau}$ is a lift of τ to $G(K((m))/\mathbf{Q})$. We have

$${}^{\tau}g = \begin{cases} g & \text{if } g \in G(K(\mu_m)/K) \\ g^{-1} & \text{if } g \in G(H_m/K) \end{cases}$$

The compositum of all $K((m))$ is the maximal abelian extension K^{ab} of K. Denote by H_∞ the compositum of all H_m.

Fix a prime $p > 2$ and let K_∞ be the compositum of all $\mathbf{Z}_p$-extensions of K. It is contained in $H_\infty(\mu_\infty)$ (as the exponent of $G(K^{\mathrm{ab}}/H_\infty(\mu_\infty))$ is equal to two) and $G(K_\infty/K) \xrightarrow{\sim} \mathbf{Z}_p^2$.

For a non-zero ideal $\mathfrak{f} \subset \mathcal{O}_K$, let $m(\mathfrak{f})$ be the smallest integer $m \geq 1$ such that $\mathfrak{f} \mid (m)$. An easy exercise in class field theory shows that

$$K(\mathfrak{f}) \cap H_\infty(\mu_\infty) = K(\mathfrak{f}) \cap H_n(\mu_n)$$

with

$$n = m(\mathfrak{f}) \times \begin{cases} 1 & \text{if } 8 \nmid m(\mathfrak{f}) \\ 2 & \text{if } 8 \mid m(\mathfrak{f}) \end{cases}$$

This means that every character

$$\Omega : G(K(\mathfrak{f}) \cap H_\infty(\mu_\infty)/K) \longrightarrow \overline{\mathbf{Q}}^*$$

can be decomposed into a product

$$\Omega = \eta \cdot (\alpha \circ \mathbf{N}) \tag{3.2.1}$$

with

$$\eta : G(H_n/K) \longrightarrow \overline{\mathbf{Q}}^*, \qquad \alpha : G(\mathbf{Q}(\mu_n)/\mathbf{Q}) \longrightarrow \overline{\mathbf{Q}}^*,$$

where we use the notation of [PR 1] and write $\mathbf{N}$ for the restriction map $G(\overline{\mathbf{Q}}/K) \longrightarrow G(\overline{\mathbf{Q}}/\mathbf{Q})$. We have $\eta\eta^{\tau} = 1$, $\alpha^{\tau} = \alpha$, where $\mathcal{W}^{\tau}(g) := \mathcal{W}({}^{\tau}g)$. Note that a decomposition in (3.2.1) is not unique in general.

(3.3) Let $\mathcal{W} : G(K(\mathfrak{f})/K) \longrightarrow \mathbf{C}^*$ be a complex-valued ray class character of conductor $\mathfrak{f}$ (in other words, $\mathcal{W}$ does not factor through $G(K(\mathfrak{g})/K)$ for any proper divisor $\mathfrak{g} \mid \mathfrak{f}$). The theta function

$$\Theta(\mathcal{W})(z) = \sum_{(\mathfrak{a},\mathfrak{f})=(1)} \mathcal{W}(\mathfrak{a}) q^{\mathbf{N}(\mathfrak{a})} \quad \left(+\frac{h}{w} \text{ if } \mathfrak{f} = (1)\right)$$

lies in $M_1(\Gamma_0(\Delta(\mathcal{W})), \left(\frac{D}{\cdot}\right)\mathcal{W}|_{\mathbf{Z}})$ (in fact is a newform if $\mathfrak{f} \neq (1)$). Here $\Delta(\mathcal{W}) = |D|\mathbf{N}(\mathfrak{f})$ and $\mathcal{W}|_{\mathbf{Z}}$ denotes the restriction of $\mathcal{W}$ to ideals in $\mathbf{Z}$. If $\mathcal{W} = \rho \cdot (\chi \circ \mathbf{N})$ as in (3.2), then $\mathcal{W}|_{\mathbf{Z}} = \chi^2$. The L-function of $\Theta(\mathcal{W})$ is equal to the L-function of $\mathcal{W}$ over K:

$$L(\Theta(\mathcal{W}), s) = \sum_{(\mathfrak{a},\mathfrak{f})=(1)} \mathcal{W}(\mathfrak{a})\mathbf{N}(\mathfrak{a})^{-s} = L(\mathcal{W}, s)$$

These functions satisfy the following functional equations

$$\Lambda(\mathcal{W}, s) := \Delta(\mathcal{W})^{s/2}(2\pi)^{-s}\Gamma(s)L(\mathcal{W}, s) = \tau(\mathcal{W})\Lambda(\overline{\mathcal{W}}, 1-s)$$
$$\Theta(\mathcal{W})\Big|_1 \begin{pmatrix} 0 & -1 \\ \Delta(\mathcal{W}) & 0 \end{pmatrix} = -i\tau(\mathcal{W})\Theta(\overline{\mathcal{W}})$$

with $\tau(\mathcal{W}) \in \overline{\mathbf{Q}}^*$, $|\tau(\mathcal{W})| = 1$.

(3.4) Suppose $f = \sum_{n\geq 1} a_n q^n \in S_{2r}(\Gamma_0(N), \psi)$ is a normalized ($a_1 = 1$) newform of level N prime to $|D|$. The L-function of f has an Euler product

$$L(f, s) = \sum_{n\geq 1} a_n n^{-s} = \prod_{\ell} (1 - a_\ell \ell^{-s} + \psi(\ell)\ell^{2r-1-s})^{-1} = \prod_{\ell} [(1 - \alpha_\ell \ell^{-s})(1 - \beta_\ell \ell^{-s})]^{-1}$$

(with $\psi(\ell) = 0$ for $\ell \mid N$, of course). The L-function of f over K, twisted by $\mathcal{W}$, is defines by an Euler product

$$L(f \otimes K, \mathcal{W}, s) = \prod_{\lambda} [(1 - \alpha_{\mathbf{N}(\lambda)}\mathcal{W}(\lambda)\mathbf{N}(\lambda)^{-s})(1 - \beta_{\mathbf{N}(\lambda)}\mathcal{W}(\lambda)\mathbf{N}(\lambda)^{-s})]^{-1},$$

where λ runs through prime ideals of $\mathcal{O}_K$, $\alpha_{\ell^k} = \alpha_\ell^k$, $\beta_{\ell^k} = \beta_\ell^k$ and $\mathcal{W}$ is extended by zero to ideals that are not prime to $\mathfrak{f}$. Note that

$$L(f \otimes K, \mathbb{1}, s) = L(f, s)L(f \otimes \left(\frac{D}{\cdot}\right), s)$$

with

$$L(f \otimes \left(\frac{D}{\cdot}\right), s) = \sum_{n\geq 1} a_n \left(\frac{D}{n}\right) q^n$$

(one needs the assumption $(N, |D|) = 1$ at this place) and

$$L(f \otimes K, \mathcal{W}, s) = D_{\mathrm{lcm}(N,\Delta(\mathcal{W}))}(f, \Theta(\mathcal{W}), s)$$

by 1.7.1.

(3.5) Following [PR 1, 2.1], we define basic measures associated to theta functions and Eisenstein series. Let $p > 2$ be a prime number.

Fix an integer $m \geq 1$ and a proper $\mathcal{O}_m$-ideal $\mathcal{A}$. Then $Q_{\mathcal{A}}(x) := \mathbf{N}(x)/\mathbf{N}(\mathcal{A})$ is a quadratic form on $\mathcal{A}$. For a fixed integer c_1 prime to p, define a distribution $\Theta_{\mathcal{A}}$ on $\mathbf{Z}_{p,c_1}^* = \varprojlim_\nu (\mathbf{Z}/c_1 p^\nu \mathbf{Z})^*$ by

$$\Theta_{\mathcal{A}}(a \,(\mathrm{mod}\, c_1 p^\nu))(z) = \sum_{\substack{x\in\mathcal{A} \\ Q_{\mathcal{A}}(x)\equiv a \,(\mathrm{mod}\, c_1 p^\nu)}} q^{Q_{\mathcal{A}}(x)}$$

For $n \geq 1$, denote by $r_{\mathcal{A}}(n)$ the number of proper ideals $I \subseteq \mathcal{O}_m$ with the class $[I] = [\mathcal{A}]$. Note that $r_{\mathcal{A}}(n) = r_{\mathcal{A}^\tau}(n)$, where τ is the non-trivial element of $G(K/\mathbf{Q})$. For $n = 0$, we put $r_{\mathcal{A}}(0) = w_m^{-1}$. If ϕ is a function on $\mathbf{Z}/c_1p^\nu\mathbf{Z}$ with values in A, then

$$\Theta_{\mathcal{A}}(\phi) = \sum_{x\in\mathcal{A}} \phi(Q_{\mathcal{A}}(x))q^{Q_{\mathcal{A}}(x)} = w_m \sum_{n\geq 1} \phi(n)r_{\mathcal{A}}(n)q^n$$

lies in $M_1(\Gamma_1(M); A)$ for $M = \mathrm{lcm}(|D|m^2, c_1p^{2\nu})$ and depends only on the class $[\mathcal{A}]$ of $\mathcal{A}$ in $\mathrm{Pic}(\mathcal{O}_m)$. If ϕ is a character, then $\Theta_{\mathcal{A}}(\phi)$ lies in $M_1(\Gamma_0(M), (\frac{D}{\cdot})\phi^2)$.

(3.6) For $M \geq 1$ and $\alpha \in \mathbf{Z}/M\mathbf{Z}$, define a function $\phi_{\alpha,M} : \mathbf{Z}/M\mathbf{Z} \longrightarrow \mathbf{C}$ by

$$\phi_{\alpha,M}(x) = e^{-2\pi i\alpha x/M}$$

The formulas (for an integer C prime to pM)

$$E_1(\alpha\,(\mathrm{mod}\,Mp^\nu))(z) := (-2\pi i)^{-1}E_1(z, \phi_{\alpha,Mp^\nu}) = \frac{1}{2}\widetilde{L}(0, \delta_{\alpha\,(\mathrm{mod}\,Mp^\nu)}) + \sum_{\substack{k\cdot m>0\\ k\equiv\alpha\,(\mathrm{mod}\,Mp^\nu)}} \mathrm{sgn}(k)q^{km}$$

$$E_1^C(\alpha\,(\mathrm{mod}\,Mp^\nu))(z) := E_1(\alpha\,(\mathrm{mod}\,Mp^\nu))(z) - CE_1(C^{-1}\alpha\,(\mathrm{mod}\,Mp^\nu))(z)$$

then define a distribution E_1 (resp. a measure E_1^C) on $\mathbf{Z}_{p,M}^*$ with values in $\overline{M}_1(\Gamma_1(Mp^\infty))$.

4. The convolution measures

In this section we slightly modify the construction of measures given in [PR 1, sec. 4] in order to obtain p-adic L-functions for modular forms of arbitrary even weight.

(4.1) Fix a prime number $p > 2$. Let $f \in S_{2r}(\Gamma_0(N), \psi)$ be a newform of even weight $2r \geq 2$, ordinary at p. Decompose the level of f into $N = N_1N_2N_3$ with $N_1 = \mathrm{cond}(\psi)$, $N_2 = (N_1^\infty, N/N_1)$. We write $N = N'p^\gamma$ (resp. $N_i = N_i'p^{\gamma_i}$) with N', N_i' prime to p.

Let $K = \mathbf{Q}(\sqrt{D})$ be an imaginary quadratic field with discriminant D prime to N. Let $\Omega : G(H_\infty(\mu_\infty)/K) \longrightarrow \overline{\mathbf{Q}}^*$ be a character of finite order with conductor $\mathfrak{f}(\Omega)$ prime to p. As in (3.2.1), it has a decomposition

$$\Omega = \eta \cdot (\alpha \circ \mathbf{N})$$

Let c_1 be the conductor of the Dirichlet character α over $\mathbf{Q}$ and c_2 the conductor of the ring class field character η. We put, as in [PR 1, 3.1],

$$c = \mathbf{N}(\mathfrak{f}(\Omega)), \qquad c_3 = \mathrm{lcm}(c_1, c_2), \qquad \Delta = \mathrm{lcm}(|D'|c_2^2, c_1^2),$$

where D' is the non-p part of D. We assume that $(N, \Delta) = 1$. Denote by Δ_1 the non-p part of $\mathrm{cond}((\frac{D}{\cdot})\chi^2)$ and by Δ_2 the non-p part of the level of $\Theta(\Omega)$, i.e. of $|D|c$. Put $\Delta_3 = \Delta/\Delta_2$ and $\xi = \psi(\frac{D}{\cdot})$.

The p-adic measures defined in this section will be used to interpolate algebraic parts of the expressions $D(f, \Theta(\mathcal{W}), r)$ for characters $\mathcal{W}$ of $G(H_\infty(\mu_\infty)/K)$ such that Ω is equal to the non-p part of $\mathcal{W}$. We decompose

$$\mathcal{W} = \rho \cdot (\chi \circ \mathbf{N})$$

as in (3.2.1). The level of $\Theta(\mathcal{W})$ is equal to $\Delta(\mathcal{W}) = \Delta_2p^\beta$.

(4.2) Definition. In the notation of 4.1, we choose an integer C prime to $N\Delta pc_3$ and put

$$\Phi^C_{\mathcal{A},c_1}(a \,(\mathrm{mod}\, c_1 p^\nu)) := H\Big[\sum_{\alpha\in(\mathbf{Z}/N_1\Delta_1 p^\nu\mathbf{Z})^*} \xi(\alpha)\Theta_{\mathcal{A}}(\alpha^2 a \,(\mathrm{mod}\, c_1 p^\nu))(N_1N_2z)$$
$$\delta_1^{r-1}(E_1^C(\alpha \,(\mathrm{mod}\, N_1\Delta_1 p^\nu))(N_3'z))\Big]$$

This defines a distribution (in fact a measure – cf. [Hi 1, 5.1]) on $\mathbf{Z}^*_{p,c_1}$ with values in $\overline{M}_{2r}(\Gamma_0(N\Delta p^\infty), \psi; p^{-\delta}\mathbf{Z}_p)$ for some δ depending only on r. Note that the order of brackets matters: by (1.2.2) we have

$$\delta_1^{r-1}(f(Az)) = A^{r-1}(\delta_1^{r-1}f)(Az)$$

(4.3) Definition. We define a measure $\Psi^C_{\mathcal{A},c_1}$ on $\mathbf{Z}^*_{p,c_1}$ with values in $\overline{M}_{2r}(\Gamma_0(Np^\infty), \psi; p^{-\delta}\mathbf{Z}_p)$ by

$$\Psi^C_{\mathcal{A},c_1} := \frac{1}{2w_m}\Phi^C_{\mathcal{A},c_1}\Big|_{2r}\mathcal{T}(\Delta_2)_{N\Delta p^\infty/Np^\infty},$$

where $w_m = \sharp(\mathcal{O}_m^*)$ (the index m was omitted in the Definition 6 in [PR 1]) and, for $Md|M'$, $\mathcal{T}(d)$ denotes the operator

$$M_l(\Gamma_0(M'), \chi) \longrightarrow M_l(\Gamma_0(M), \chi)$$

adjoint to $g \mapsto d^{l/2-1}g\Big|_l\begin{pmatrix} d & 0\\ 0 & 1\end{pmatrix}$, i.e. satisfying

$$d^{l/2-1}\left\langle g\Big|_l\begin{pmatrix} d & 0\\ 0 & 1\end{pmatrix}, h\right\rangle_{M'} = \left\langle g, h\Big|_l\mathcal{T}(d)_{M'/M}\right\rangle_M$$

for all $g \in M_l(\Gamma_0(M), \chi)$, $h \in M_l(\Gamma_0(M'), \chi)$

(4.4) Definition. For a character $\rho : G(H_m/K) \xrightarrow{\sim} \mathrm{Pic}(\mathcal{O}_m) \longrightarrow \overline{\mathbf{Q}}^*$, we define

$$\Phi^C_{\rho,c_1} = \sum_{[\mathcal{A}]\in\mathrm{Pic}(\mathcal{O}_m)} \rho([\mathcal{A}])^{-1}\Phi^C_{\mathcal{A},c_1}$$
$$\Psi^C_{\rho,c_1} = \sum_{[\mathcal{A}]\in\mathrm{Pic}(\mathcal{O}_m)} \rho([\mathcal{A}])^{-1}\Psi^C_{\mathcal{A},c_1}$$

We also define

$$\Psi^C_{f,\rho,c_1} := L_{f_0}(\Psi^C_{\rho,c_1})$$

This is a measure on $\mathbf{Z}^*_{p,c_1}$ with values in $c(f)^{-1}p^{-\delta}\mathcal{O}_{\widehat{F(f)}}$, where $F(f)$ is the field generated by coefficients of f, $\widehat{F(f)}$ the closure of $i_p(F(f))$ and $\mathcal{O}_{\widehat{F(f)}}$ the ring of integers in $\widehat{F(f)}$.

(4.5) Definition. We view the Galois group $G(K(\mu_{c_1p^\infty})/K)$ as a subgroup of $\mathbf{Z}^*_{p,c_1}$ (of index 1 or 2) by the restriction and reciprocity maps

$$G(K(\mu_{c_1p^\infty})/K) \hookrightarrow G(\mathbf{Q}(\mu_{c_1p^\infty})/\mathbf{Q}) \xrightarrow{\sim} \mathbf{Z}^*_{p,c_1}$$

All measures defined in (4.2)–(4.4) have support in $G(K(\mu_{c_1p^\infty})/K)$ (by [PR 1, 5.1b]).
We define a measure Ψ^C_{f,c_1,c_2} on $G(H_{c_2p^\infty}/K) \times G(K(\mu_{c_1p^\infty})/K)$ with values in $c(f)^{-1}p^{-\delta}\mathcal{O}_{\widehat{F(f)}}$.

$$\Psi^C_{f,c_1,c_2}(\sigma \,(\mathrm{mod}\, c_2p^n), \tau \,(\mathrm{mod}\, c_1p^m)) := L_{f_0}\left(\Psi^C_{\mathcal{A},c_1}(a \,(\mathrm{mod}\, c_1p^m))\right)$$

for $\tau\big|_{\mathbf{Q}(\mu_{c_1p^m})}$ corresponding to $a \in (\mathbf{Z}/c_1p^m\mathbf{Z})^*$ under the Artin symbol and $\sigma \pmod{c_2p^n}$ corresponding to $[\mathcal{A}]^{-1} \in \mathrm{Pic}(\mathcal{O}_{c_2p^n})$. The argument of [PR 1, 5.1c] shows that Ψ^C_{f,c_1,c_2} is in fact supported on the subgroup

$$G(H_{c_2p^\infty}(\mu_{c_1p^\infty})/K) \subset G(H_{c_2p^\infty}/K) \times G(K(\mu_{c_1p^\infty})/K)$$

(4.6) Definition. We define modified measures $\widetilde{\Psi}^C_{\mathcal{A},c_1}, \widetilde{\Psi}^C_{\rho,c_1}, \widetilde{\Psi}^C_{f,\rho,c_1}, \widetilde{\Psi}^C_{f,c_1,c_2}$ by replacing the operator $\mathcal{T}(\Delta_2)$ in (4.3) by $\mathcal{T}(1)$ – these measures are used (in a special case) in [PR 2].

5. Integrals of characters

In this section we sketch how the calculations of [PR 1] work in the higher weight case.

(5.1) Let $\chi : (\mathbf{Z}/c_1p^\nu\mathbf{Z})^* \longrightarrow \overline{\mathbf{Q}}^*$ be a character such that its restriction to $(\mathbf{Z}/c_1\mathbf{Z})^*$ is primitive and put $M(\nu) = N_1\Delta_1p^\nu$. It follows from the definitions and Prop. 1.5.4 (cf. [PR 1, Lemma 7]) that

$$\int\limits_{\mathbf{Z}^*_{p,c_1}} \chi \, d\Phi^C_{\mathcal{A},c_1} = (1 - C\xi(C)\overline{\chi}^2(C))H\left[\Theta_{\mathcal{A}}(\chi \circ Q_{\mathcal{A}})(N_1N_2z)\delta_1^{r-1}(E_1(N_3'z,\phi))\right],$$

where $\phi : \mathbf{Z}/M(\nu)\mathbf{Z} \longrightarrow \mathbf{C}$ satisfies $\left(-\frac{2\pi i}{M(\nu)}\right)\widehat{\phi} = \overline{\omega}$ with $\omega = \overline{\psi}(\frac{D}{\cdot})\chi^2$.

(5.2) There is a bijection between pairs $(x \in \mathcal{A} \pmod{\mathcal{O}_m^*}, Q_{\mathcal{A}}(x))$ and $(I \subset \mathcal{O}_m$ proper ideal, $\mathbf{N}(I))$ given by $I = x\mathcal{A}^{-1}$, hence

$$\int\limits_{\mathbf{Z}^*_{p,c_1}} \chi \, d\Phi^C_{\rho,c_1} = w_m(1 - C\xi(C)\overline{\chi}^2(C))H\left[\Theta(\mathcal{W}'')(N_1N_2z)\delta_1^{r-1}(E_1(N_3'z,\phi))\right],$$

where $\mathcal{W}'' = \rho \cdot (\chi \circ \mathbf{N})$, considered as a character modulo $\mathfrak{f} = \mathrm{lcm}(\mathrm{cond}(\rho), \mathrm{cond}(\chi), p)$, and

$$\Theta(\mathcal{W}'')(z) = \sum_{\substack{\mathfrak{a} \subset \mathcal{O} \\ (\mathfrak{a},\mathfrak{f})=1}} \mathcal{W}''(\mathfrak{a})\mathfrak{q}^{\mathbf{N}(\mathcal{A})}$$

(note that [PR 1, Lemma 7'] gives an incorrect factor 2 instead of w_m).

(5.3) Put $h = \Theta(\mathcal{W}'')(N_1N_2z)\delta_1^{r-1}(E_1(N_3'z,\phi))$. Hida's formula (2.4.1) and the basic property (1.3.1) of the holomorphic projection imply that we have, for $\mu >> 0$,

$$\int\limits_{\mathbf{Z}^*_{p,c_1}} \chi \, d\Psi^C_{f,\rho,c_1} = \frac{1}{2}(1 - C\xi(C)\overline{\chi}^2(C))\frac{(\Delta_2p^{\mu-\alpha})^{r-1}}{\alpha_p(f)^{\mu-\alpha}} \frac{\left\langle f_0^\tau\Big|_{2r}\begin{pmatrix} 0 & -1 \\ N\Delta_2p^\mu & 0 \end{pmatrix}, h\right\rangle_{N\Delta p^\mu}}{\left\langle f_0^\tau\Big|_{2r}\begin{pmatrix} 0 & -1 \\ Np^\alpha & 0 \end{pmatrix}, f_0\right\rangle_{Np^\alpha}} \tag{5.3.1}$$

(5.4) Following [PR 1, 4.3] we use the involution

$$\mathcal{F} = \begin{pmatrix} 0 & -1 \\ N\Delta p^\mu & 0 \end{pmatrix}\begin{pmatrix} N_3' & y \\ N\Delta p^\mu t & N_3' \end{pmatrix}$$

with $N_3'xw - \Delta p^\mu tyN/N_3' = 1$ to compute the numerator in (5.3.1). We have (by [PR 1, Lemma 16,17, Cor. 20])

$$f_0\Big|_{2r}\begin{pmatrix}0 & -1\\ N\Delta_2 p^\mu & 0\end{pmatrix}\Big|_{2r}\mathcal{F} = \psi_0(N_3'w)\lambda_{N_3'}(f) f_0\Big|_{2r}\begin{pmatrix}\Delta_3 & 0\\ 0 & 1\end{pmatrix}$$

$$\Theta(\mathcal{W}'')(N_1N_2z)\Big|_1\mathcal{F} = (N_1N_2)^{-1/2}(\left(\frac{D}{\cdot}\right)\overline{\chi}^2)(w)\Theta(\mathcal{W}'')\Big|_1\begin{pmatrix}0 & -1\\ \Delta p^{\mu+\gamma_3} & 0\end{pmatrix}$$

$$E_1(N_3'z,\phi)\Big|_1\mathcal{F} = \frac{\omega_0(N_3'w)}{2\pi i}\left(\frac{N\Delta p^\mu}{N_3'}\right)^{1/2} E_1(z,\overline{\omega}\,(\mathrm{mod}\,N\Delta p^\mu\,))$$

Here ω_0 is the primitive character associated to $\omega = \overline{\psi}(\frac{D}{\cdot})\chi^2 : (\mathbf{Z}/N_1\Delta_1 p^\nu\mathbf{Z})^* \longrightarrow \mathbf{C}^*$. Putting this and (5.3.1) together (and using (1.2.2)), we obtain

$$\int\limits_{\mathbf{Z}_{p,c_1}^*} \chi\, d\Psi_{f,\rho,c_1}^C = (1 - C\xi(C)\overline{\chi}^2(C))\frac{((\frac{D}{\cdot})\chi^2)(N_3')\lambda_{N_3'}(f)(\Delta p^{\gamma_3})^{1/2}}{(4\pi i)\alpha_p(f)^{-\alpha}}\left(\frac{\Delta_2}{p^\alpha}\right)^{r-1}\frac{\Lambda_\mu(\mathcal{W}'')}{\left\langle f_0^\tau\Big|_{2r}\begin{pmatrix}0 & -1\\ Np^\alpha & 0\end{pmatrix}, f_0\right\rangle_{Np^\alpha}}$$

with

$$\Lambda_\mu(\mathcal{W}'') = \frac{p^{\mu(r-1/2)}}{\alpha_p(f)^\mu}\left\langle f_0^\tau\Big|_{2r}\begin{pmatrix}\Delta_3 & 0\\ 0 & 1\end{pmatrix}, g\right\rangle_{N\Delta p^\mu}$$

$$g = \Theta(\mathcal{W}'')\Big|_1\begin{pmatrix}0 & -1\\ \Delta p^{\mu+\gamma_3} & 0\end{pmatrix}\delta_1^{r-1}(E_1(z,\overline{\omega}\,(\mathrm{mod}\,N\Delta p^\mu\,)))$$

(5.5) The theta function $\Theta(\mathcal{W})$ satisfies the functional equation

$$\Theta(\mathcal{W})\Big|_1\begin{pmatrix}0 & -1\\ \Delta_2 p^\beta & 0\end{pmatrix} = -i\tau(\mathcal{W})\Theta(\overline{\mathcal{W}})$$

A calculation similar to [PR 1, 4.4] gives

$$\Lambda_\mu(\mathcal{W}'') = -i\tau(\mathcal{W})\Delta_3^{1/2}\Gamma(\mathcal{W})\sum_{\substack{\mathfrak{a}|p\\ \mathbf{N}(\mathfrak{a})=p^s}}\widetilde{\mu}(\mathfrak{a})\mathcal{W}(\mathfrak{a})\Lambda_{\mu,s}$$

with

$$\Lambda_{\mu,s} = p^{\mu(r-1/2)-s/2}\alpha_p(f)^{-\mu}\left\langle f_0^\tau, \Theta(\overline{\mathcal{W}})\Big|_1\begin{pmatrix}p^x & 0\\ 0 & 1\end{pmatrix}\delta_1^{r-1}(E_1(z,\overline{\omega}\,(\mathrm{mod}\,N\Delta_2 p^\mu\,)))\right\rangle_{N\Delta_2 p^\mu}$$

$x = \mu + \gamma_3 - \beta - s$ and (cf. [PR 1, Lemma 24])

$$\Gamma(\mathcal{W}) = \sum_{\substack{\mathfrak{b}|c_3\\ \mathbf{N}(\mathfrak{b})=m}}\frac{\widetilde{\mu}(\mathfrak{b})\mathcal{W}(\mathfrak{b})a_m(\Theta(\overline{\mathcal{W}}))}{m^r} = \prod_{l|c_3}\left(1 - \frac{|a_l(\Theta(\mathcal{W}))|^2}{l^r}\right) = \prod_{\substack{l|(c_1,c_2)\\ l\nmid c/c_1}}\left(1 - \frac{1}{l^r}\right)$$

(5.6) Finally, a rather tedious calculation along the lines of [PR 1, pp. 23–25] (which will not be reproduced here) yields

$$\Lambda_\mu = V_p(f,\mathcal{W})\left(\frac{p^{r-1/2}}{\alpha_p(f)}\right)^{\beta-\gamma_3}\frac{2((r-1)!)^2(-1)^{r-1}}{(4\pi)^{2r-1}}D_{N\Delta(\mathcal{W})}(f,\Theta(\overline{\mathcal{W}}),r)$$

with

$$V_p(f,\mathcal{W})=\prod_{\mathfrak{p}|p}\left(1-\frac{\psi(\mathbf{N}(\mathfrak{p}))\overline{\mathcal{W}}(\mathfrak{p})}{\alpha_{\mathbf{N}(\mathfrak{p})}(f)}\mathbf{N}(\mathfrak{p})^{r-1}\right)\left(1-\frac{\mathcal{W}(\mathfrak{p})}{\alpha_{\mathbf{N}(\mathfrak{p})}(f)}\mathbf{N}(\mathfrak{p})^{r-1}\right)$$

The constant involving factorials comes from 1.8.1, which replaces Prop. 22 in [PR 1].

(5.7) The denominator in (5.4) is given by (cf. [Hi 2, 9.5])

$$\alpha_p(f)^{-\alpha}\left\langle f_0^\tau\Big|_{2r}\begin{pmatrix}0 & -1\\ Np^\alpha & 0\end{pmatrix}, f_0\right\rangle_{Np^\alpha}=\lambda_N(f)p^{\alpha(1-r)}H_p(f)\langle f,f\rangle_N$$

with

$$H_p(f)=\left(1-\frac{\psi(p)p^{2r-2}}{\alpha_p(f)^2}\right)\left(1-\frac{\psi(p)p^{2r-1}}{\alpha_p(f)^2}\right)$$

(5.8) Putting 5.4-7 together, we obtain a higher weight version of [PR 1, Thm. B]:

Theorem. *In the notation of 4.1, we have*

$$\Delta_3^{-1}(1-C\psi(C)\left(\frac{D}{C}\right)\overline{\mathcal{W}}(C))^{-1}\int\limits_{\mathbf{Z}^*_{p,c_1}}\chi\, d\Psi^C_{f,\rho,c_1}=\frac{\mathcal{L}_p(f,\mathcal{W})\Gamma(\Omega)t_p(f)V_p(f,\mathcal{W})\Delta(\mathcal{W})^{r-1/2}}{\alpha_p(f)^\beta H_p(f)p^{\gamma_3(r-1)}}$$

with $t_p(f)=\lambda_{N_3'}(f)\lambda_N(f)^{-1}\alpha_p(f)^{\gamma_3}$, $\Delta(\mathcal{W})=|D|\mathbf{N}(\mathfrak{f}(\mathcal{W}))=$ *the level of* $\Theta(\mathcal{W})$,

$$\mathcal{L}_p(f,\mathcal{W})=\left(\frac{D}{-N_3'}\right)\mathcal{W}(N_3')\tau(\mathcal{W})\frac{2(-1)^{r-1}((r-1)!)^2}{(4\pi)^{2r}}\frac{D_{N\Delta(\mathcal{W})}(f,\Theta(\overline{\mathcal{W}}),r)}{\langle f,f\rangle_N}$$

$$\Gamma(\Omega)=\prod_{\substack{l|(c_1,c_2)\\ l\nmid c/c_1}}\left(1-\frac{1}{l^r}\right)$$

(5.9) For the modified measures, the same calculation gives

$$\int\limits_{\mathbf{Z}^*_{p,c_1}}\chi\, d\widetilde{\Psi}^C_{f,\rho,c_1}=\frac{1}{2}(1-C\xi(C)\overline{\chi}^2(C))\frac{(p^{\mu-\alpha})^{r-1}}{\alpha_p(f)^{\mu-\alpha}}\frac{\left\langle f_0^\tau\Big|_{2r}\begin{pmatrix}0 & -1\\ Np^\mu & 0\end{pmatrix}, h\right\rangle_{N\Delta p^\mu}}{\left\langle f_0^\tau\Big|_{2r}\begin{pmatrix}0 & -1\\ Np^\alpha & 0\end{pmatrix}, f_0\right\rangle_{Np^\alpha}}$$

$$\int\limits_{\mathbf{Z}^*_{p,c_1}}\chi\, d\widetilde{\Psi}^C_{f,\rho,c_1}=(1-C\xi(C)\overline{\chi}^2(C))\frac{((\frac{D}{\cdot})\chi^2)(N_3')\lambda_{N_3'}(f)(\Delta p^{\gamma_3})^{1/2}}{(4\pi i)\alpha_p(f)^{-\alpha}}\left(\frac{1}{p^\alpha}\right)^{r-1}\frac{\widetilde{\Lambda}_\mu(\mathcal{W}'')}{\left\langle f_0^\tau\Big|_{2r}\begin{pmatrix}0 & -1\\ Np^\alpha & 0\end{pmatrix}, f_0\right\rangle_{Np^\alpha}}$$

with

$$\widetilde{\Lambda}_\mu(\mathcal{W}'')=\frac{p^{\mu(r-1/2)}}{\alpha_p(f)^\mu}\left\langle f_0^\tau\Big|_{2r}\begin{pmatrix}\Delta & 0\\ 0 & 1\end{pmatrix}, g\right\rangle_{N\Delta p^\mu}$$

$$g=\Theta(\mathcal{W}'')\Big|_1\begin{pmatrix}0 & -1\\ \Delta p^{\mu+\gamma_3} & 0\end{pmatrix}\delta_1^{r-1}(E_1(z,\overline{\omega}\ (\mathrm{mod}\ N\Delta p^\mu)))$$

$$\widetilde{\Lambda}_\mu(\mathcal{W}'') = -i\tau(\mathcal{W})\Delta_3^{1/2}\Gamma(\Omega)V_p(f,\mathcal{W})\left(\frac{p^{r-1/2}}{\alpha_p(f)}\right)^{\beta-\gamma_3}\frac{2((r-1)!)^2(-1)^{r-1}}{(4\pi)^{2r-1}}D_{N\Delta(\mathcal{W})}(f\Big|_{2r}\begin{pmatrix}\Delta_2 & 0\\ 0 & 1\end{pmatrix}, \Theta(\overline{\mathcal{W}}), r)$$

As Δ_2 divides the level of $\Theta(\overline{\mathcal{W}})$, we have

$$\frac{D(f\Big|_{2r}\begin{pmatrix}\Delta_2 & 0\\ 0 & 1\end{pmatrix}, \Theta(\overline{\mathcal{W}}), r)}{D(f,\Theta(\overline{\mathcal{W}}), r)} = \Delta_2^r\frac{\sum_{n\geq 1} a(n,f)a(n\Delta_2,\Theta(\overline{\mathcal{W}}))(n\Delta_2)^{-r}}{\sum_{n\geq 1} a(n,f)a(n,\Theta(\overline{\mathcal{W}}))n^{-r}} = a(\Delta_2,\Theta(\overline{\mathcal{W}})) = \overline{\mathcal{W}}(\mathcal{D}'),$$

where $\mathcal{D}'$ is the non-p-part of $\mathcal{D} = (\sqrt{D})$. It follows that

$$\int_{\mathbf{Z}_{p,c_1}^*} \chi\, d\widetilde{\Psi}^C_{f,\rho,c_1} = \Delta_2^{1-r}\overline{\mathcal{W}}(\mathcal{D}')\int_{\mathbf{Z}_{p,c_1}^*}\chi\, d\Psi^C_{f,\rho,c_1}$$

(5.10) In terms of measures on Galois groups, this can be rephrased as the following

Theorem. *If $\mathcal{W} : G(H_{c_2p^\infty}(\mu_{c_1p^\infty})/K) \longrightarrow \overline{\mathbf{Q}}_p^{\,*}$ is a character of finite order with non-p part equal to Ω, then*

$$\Delta_3^{-1}(1-C\psi(C)\left(\frac{D}{C}\right)\overline{\mathcal{W}}(C))^{-1}\int_{G(H_{c_2p^\infty}(\mu_{c_1p^\infty})/K)}\mathcal{W}\, d\widetilde{\Psi}^C_{f,c_1,c_2} = \overline{\mathcal{W}}(\mathcal{D}')\frac{\mathcal{L}_p(f,\mathcal{W})\Gamma(\Omega)t_p(f)V_p(f,\mathcal{W})\Delta(\mathcal{W})^{r-1/2}}{\alpha_p(f)^\beta H_p(f)(\Delta_2p^{\gamma_3})^{r-1}}$$

Definition. For any continuous character $\phi : G(H_{c_2p^\infty}(\mu_{c_1p^\infty})/K) \longrightarrow \overline{\mathbf{Q}}_p^{\,*}$ with $(\phi)^{non-p} = \Omega$, we define (following [PR 2, Def. 2.4])

$$L_p(f\otimes K,\phi) := (-1)^{r-1}H_p(f)\Delta_3^{-1}\left(\frac{D}{-N_3'}\right)(1-C\psi(C)\left(\frac{D}{C}\right)\phi(C)^{-1})^{-1}\int_{G(H_{c_2p^\infty}(\mu_{c_1p^\infty})/K)}\phi\, d\widetilde{\Psi}^C_{f,c_1,c_2}$$

Fix a character $\mathcal{C} : G(H_{c_2p^\infty}(\mu_{c_1p^\infty})/K) \longrightarrow \overline{\mathbf{Q}}_p^{\,*}$ of finite order and consider

$$L_p(f\otimes K,\mathcal{C})(\lambda) := L_p(f\otimes K,\mathcal{C}\lambda)$$

as a function on characters $\lambda : G(H_{c_2p^\infty}(\mu_{c_1p^\infty})/K) \longrightarrow (1+p\mathbf{Z}_p)$ (such characters automatically factor through $G(K_\infty/K)$). It follows from [PR 1, Prop. 29] that $L_p(f\otimes K,\mathcal{C})$ is an Iwasawa function with values in $c(f)^{-1}p^{-\delta}\mathcal{O}_{\widehat{F(f)}}$, provided $p\not|\,\mathrm{cond}(\psi)$.

(5.11) Following [PR 1, 5.2], we show that the p-adic L-function $L_p(f\otimes K,\mathcal{C})$ satisfies a functional equation, provided the following condition is satisfied:

$$(N,\Delta(\Omega)) = 1, \qquad p\not|\,\mathrm{cond}(\psi)$$

If this is the case, then (because f is ordinary) either (a) $p\not|\,N$, $|i_p(a_p)|_p = 1$, or (b) $p\|N$, $p\not|\,\mathrm{cond}(\psi)$. We have $\gamma = \gamma_3 = 0$ (resp. $= 1$) in the case (a) (resp. (b)).

We use the fact that, for a character $\mathcal{W}$ of $G(H_{c_2p^\infty}(\mu_{c_1p^\infty})/K)$ of finite order, we have

$$\frac{a_p(f^\tau)}{a_p(f)} = \frac{\alpha_p(f^\tau)}{\alpha_p(f)} = \overline{\psi}_0(p), \qquad \lambda_N(f^\tau) = \lambda_N(f)^{-1}$$
$$V_p(f^\tau, \overline{\mathcal{W}}) = V_p(f, \mathcal{W}), \quad H_p(f^\tau) = H_p(f), \quad \langle f^\tau, f^\tau \rangle_N = \langle f, f \rangle_N$$
$$\lambda_{N_3'}(f)/\lambda_{N_3'}(f^\tau) = \lambda_{N_3'}(f)^2 = \overline{\psi}_0(N_3'), \quad t_p(f)/t_p(f^\tau) = \overline{\psi}_0(N_3')\psi_0(p)^\gamma \lambda_N(f)^{-2}$$

and the functional equation for the Rankin-Selberg convolution (which can be proved, e.g., by a computation of local constants; see [PR 1, 5.2] for the weight two case)

$$D_{N\Delta(\mathcal{W})}(f, \Theta(\overline{\mathcal{W}}), r) = \left(\frac{D}{-N'}\right)\overline{\mathcal{W}}(N')\lambda_N(f)^2\psi_0(\Delta(\mathcal{W})p^{-\gamma})\tau(\mathcal{W})^{-2}D_{N\Delta(\mathcal{W})}(f^\tau, \Theta(\mathcal{W}), r) \tag{5.11.1}$$

which holds for $\beta = \mathrm{ord}_p\Delta(\mathcal{W})$ big enough.
This implies that

$$\frac{\mathcal{L}_p(f, \mathcal{W})}{\mathcal{L}_p(f^\tau, \overline{\mathcal{W}})} = \mathcal{W}^2(N_3')\left(\frac{D}{-N'}\right)\overline{\mathcal{W}}(N')\psi_0(\Delta(\mathcal{W})p^{-\gamma})\lambda_N(f)^2$$
$$\frac{L_p(f \otimes K, \mathcal{W})}{L_p(f^\tau \otimes K, \overline{\mathcal{W}})} = \overline{\mathcal{W}}^2(\mathcal{D}')\mathcal{W}^2(N_3')\left(\frac{D}{-N'}\right)\overline{\mathcal{W}}(N')\psi_0(\Delta(\Omega))\overline{\psi}_0(N_3')$$

(5.12) In the notation of 5.11, we define

$$\Lambda_p(f \otimes K, \mathcal{C})(\lambda) := (\mathcal{C}\lambda)(\mathcal{D}'N_3'^{-1})\lambda(N')^{1/2}L_p(f \otimes K, \mathcal{C})(\lambda)$$

It follows from (5.11.1) that

Proposition. *Λ_p satisfies the functional equation*

$$\Lambda_p(f \otimes K, \mathcal{C})(\lambda) = \overline{\psi}_0(N_3')\left(\frac{D}{-N'}\right)\psi_0(\Delta(\mathcal{C})^{non-p})\overline{\mathcal{C}}(N')\Lambda_p(f^\tau \otimes K, \overline{\mathcal{C}})(\lambda^{-1})$$

Corollary. *Suppose that $\psi = 1$, $\left(\frac{D}{N'}\right) = 1$ and both $\mathcal{C}$ and λ are anticyclotomic (i.e. $\mathcal{C}\mathcal{C}^\tau = \lambda\lambda^\tau = 1$). Then $L_p(f \otimes K, \mathcal{C}) = 0$.*

Proof. Under our assumptions, we have $f^\tau = f$. Noting that $\Lambda_p(f \otimes K, \mathcal{C})(\lambda) = \Lambda_p(f \otimes K, \mathcal{C}^\tau)(\lambda^\tau)$, the functional equation reads as follows:

$$\Lambda_p(f \otimes K, \mathcal{C})(\lambda) = \left(\frac{D}{-N'}\right)\overline{\mathcal{C}}(N')\Lambda_p(f \otimes K, \overline{\mathcal{C}}^\tau)(\lambda^{-\tau}) = -\Lambda_p(f \otimes K, \mathcal{C})(\lambda)$$

(5.13) We now specialize the p-adic L-function to cyclotomic λ's. Let χ be a Dirichlet character of conductor c_1p^ν and put $\mathcal{W} = \overline{\chi} \circ \mathbf{N}$. Assume that $(pN, |D|c_1) = 1$. In this case we have $c_2 = 1$, $c_3 = c_1$, $c = c_1^2$, $\Delta = \Delta_2 = |D|c_1^2$, $\Delta_3 = 1$, $\Delta(\mathcal{W}) = |D|p^{2\nu}$, $\beta = 2\nu$, $\Gamma(\Omega) = 1$.
Under our assumptions, $\chi\left(\frac{D}{\cdot}\right)$ has conductor $|D|c_1p^\nu$ and

$$D_{N\Delta(\chi\circ\mathbf{N})}(f, \Theta(\chi \circ \mathbf{N}), s) = L(f \otimes \chi, s)L(f \otimes \chi\left(\frac{D}{\cdot}\right), s),$$

where

$$L(f\otimes\chi,s)=\sum_{n\geq 1}a_n\chi(n)n^{-s}$$
$$L(f\otimes\chi\Big(\frac{D}{\cdot}\Big),s)=\sum_{n\geq 1}a_n\chi(n)\Big(\frac{D}{n}\Big)n^{-s}$$

We also have the relations

$$\alpha_p(f\otimes\Big(\frac{D}{\cdot}\Big))=\Big(\frac{D}{p}\Big)\alpha_p(f)$$
$$\tau(\overline{\chi}\circ\mathbf{N})=-iG(\overline{\chi})G(\overline{\chi}\Big(\frac{D}{\cdot}\Big))/(c_1p^\nu)=-i\overline{\chi}(|D|)\Big(\frac{D}{c_1p^\nu}\Big)G(\overline{\chi})^2G(\Big(\frac{D}{\cdot}\Big))/(c_1p^\nu)=$$
$$=\overline{\chi}(|D|)\Big(\frac{D}{c_1p^\nu}\Big)G(\overline{\chi})^2/(c_1p^\nu)=\overline{\chi}(|D|)\Big(\frac{D}{c_1p^\nu}\Big)(c_1p^\nu)G(\chi)^{-2},$$

where

$$G(\chi)=\sum_{x\in(\mathbf{Z}/\mathrm{cond}(\chi)\mathbf{Z})^*}\chi(x)e^{2\pi ix/\mathrm{cond}(\chi)}$$

is the Gauss sum. Theorem 5.10 tells us that

$$L_p(f\otimes K,\overline{\chi}\circ\mathbf{N})=\overline{\chi}(N_3')^2t_p(f)V_p(f,\overline{\chi}\circ\mathbf{N})\frac{|D|^{1/2}p^{2\nu}}{p^{\gamma_3(r-1)}c_1^{2r-1}\alpha_p(f)^{2\nu}}\Big(\frac{D}{c_1p^\nu}\Big)\frac{2((r-1)!)^2}{(4\pi)^{2r}}\frac{D_{N|D|p^{2\nu}}(f,\Theta(\chi\circ\mathbf{N}),r)}{\langle f,f\rangle_N G(\chi)^2} \tag{5.13.1}$$

Over $\mathbf{Q}$, there exist several constructions of p-adic L-functions associated to modular forms. We adopt the normalization of [M - T - T]. Their L-function is defined on continuous characters $\lambda:{\mathbf{Z}_p}^*\longrightarrow\mathbf{C}_p^*$ and satisfies the following interpolation property. For χ as above and $x:\mathbf{Z}_p^*\longrightarrow\mathbf{Z}_p^*$ the identity,

$$L_p^{\mathrm{MTT}}(f,x^{r-1}\overline{\chi})=\frac{p^{r\nu}}{\alpha_p(f)^\nu}\left(1-\frac{\chi(p)\psi(p)}{\alpha_p(f)}p^{r-1}\right)\left(1-\frac{\overline{\chi}(p)p^{r-1}}{\alpha_p(f)}\right)\frac{(r-1)!}{(-2\pi i)^{r-1}}\frac{L(f\otimes\chi,r)}{G(\chi)}$$

We define

$$L_p^{\mathrm{MTT}}(f\otimes K,\lambda)=L_p^{\mathrm{MTT}}(f,\lambda)L_p^{\mathrm{MTT}}(f\otimes\Big(\frac{D}{\cdot}\Big),\lambda)$$

Then

$$L_p^{\mathrm{MTT}}(f\otimes K,x^{r-1}\overline{\chi})=\frac{p^{2r\nu}}{\alpha_p(f)^{2\nu}}\Big(\frac{D}{p^\nu}\Big)V_p(f,\overline{\chi}\circ\mathbf{N})\frac{(-1)^{r-1}((r-1)!)^2}{(2\pi)^{2r-2}}\frac{D_{N|D|p^{2\nu}}(f,\Theta(\chi\circ\mathbf{N}),r)}{G(\chi)^2} \tag{5.13.2}$$

Comparing 5.13.1-2. we obtain

$$L_p(f\otimes K,\overline{\chi}\circ\mathbf{N})=\Big(\frac{D}{c_1}\Big)\frac{\overline{\chi}(N_3')^2(-1)^{r-1}t_p(f)|D|^{1/2}}{(2c_1)^{2r-1}p^{\gamma_3(r-1)}}\frac{L_p^{\mathrm{MTT}}(f\otimes K,x^{r-1}\overline{\chi})}{(2\pi)^2\langle f,f\rangle_N} \tag{5.13.3}$$

Consequently,

$$L_p(f\otimes K,\lambda\circ\mathbf{N})=\Big(\frac{D}{c_1}\Big)\frac{\lambda(N_3')^2(-1)^{r-1}t_p(f)|D|^{1/2}}{(2c_1)^{2r-1}p^{\gamma_3(r-1)}}\frac{L_p^{\mathrm{MTT}}(f\otimes K,x^{r-1}\lambda)}{(2\pi)^2\langle f,f\rangle_N} \tag{5.13.4}$$

for any continuous character $\lambda:G(\mathbf{Q}_\infty/\mathbf{Q})\longrightarrow 1+p\mathbf{Z}_p$.

6. The Fourier expansion

(6.1) In this section we shall investigate more closely the p-adic L-function defined in 5.10. In order to prove Theorem A, it is sufficient to specialize the constants in Sec. 5 as follows: $\Omega = 1$, $\psi = 1$, $p \not| N|D|$, $m = 1$, $2 \not| |D|$. This implies that $(|D|, 2N) = 1$, f is a normalized newform in $S_{2r}(\Gamma_0(N))$, $N_1 = N_2 = 1$, $N_3 = N_3' = N$, $c_1 = c_2 = c = 1$, $\Delta = \Delta_1 = \Delta_2 = |D|$, $\Delta_3 = 1$, $\gamma = \gamma_3 = 0$. Under these assumptions, the interpolation formula from 5.10 (with $\mathcal{W}$ replaced by $\mathcal{CW}$) reduces to that of 0.5.

In particular, for an ideal $\mathcal{A} \subset \mathcal{O}_K$ and an integer C prime to $N|D|p$ we have the following measures on $\mathbf{Z}_p^*$:

$$\Phi_{\mathcal{A}}^C(\beta \,(\mathrm{mod}\, p^\nu)) = H\Big[\sum_{\alpha \in (\mathbf{Z}/|D|p^\nu\mathbf{Z})^*} \left(\frac{D}{\alpha}\right) \Theta_{\mathcal{A}}(\alpha^2\beta \,(\mathrm{mod}\, p^\nu))(z)\delta_1^{r-1}(E_1^C(\alpha \,(\mathrm{mod}\, |D|p^\nu))(Nz))\Big]$$

$$\widetilde{\Psi}_{\mathcal{A}}^C(\beta \,(\mathrm{mod}\, p^\nu)) = \frac{1}{2w}\Phi_{\mathcal{A}}^C(\beta \,(\mathrm{mod}\, p^\nu))\Big|_{2r} \mathcal{T}(1)_{N|D|p^\nu/Np^\nu},$$

where

$$g\Big|_{2r}\mathcal{T}(1)_{N|D|p^\nu/Np^\nu} = \sum_{\gamma \in \Gamma_0(N|D|p^\nu)\backslash\Gamma_0(Np^\nu)} g|_{2r}\gamma$$

is the trace from $M_{2r}(\Gamma_0(N|D|p^\nu))$ to $M_{2r}(\Gamma_0(Np^\nu))$.

(6.2) For a fixed character $\mathcal{C} : G(H/K) \longrightarrow \overline{\mathbf{Q}}_p^{\,*}$ (recall that H is the Hilbert class field of K) we have the p-adic L-function $L_p(f \otimes K, \mathcal{C})$ from 5.10, defined on continuous characters

$$\lambda : G(K_\infty/K) \longrightarrow (1 + p\mathbf{Z}_p)$$

If $\left(\frac{D}{N}\right) = 1$, then, according to Cor. of Prop. 5.12, $L_p(f \otimes K, \mathcal{C})$ vanishes on characters satisfying $\lambda\lambda^\tau = 1$. If $\lambda = \lambda^\tau$, then λ factors through $G(\mathbf{Q}_\infty/\mathbf{Q})$, where $\mathbf{Q}_\infty/\mathbf{Q}$ is the cyclotomic $\mathbf{Z}_p$-extension. Write $\widetilde{\lambda}$ for the composite map

$$\mathbf{Z}_p^* \hookrightarrow \mathbf{Q}_p^* \longrightarrow \mathbf{A}_{\mathbf{Q}}^*/\mathbf{Q}^* \longrightarrow G(\mathbf{Q}^{\mathrm{ab}}/\mathbf{Q}) \longrightarrow G(\mathbf{Q}_\infty/\mathbf{Q}) \xrightarrow{\lambda} 1 + p\mathbf{Z}_p$$

(we have $\lambda = \widetilde{\lambda} \circ \mathbf{N}$ in the notation of 3.2.1). Then

$$L_p(f \otimes K, \mathcal{C})(\lambda) = (-1)^{r-1}H_p(f)\left(\frac{D}{-N}\right)(1 - C\left(\frac{D}{C}\right)\widetilde{\lambda}^{-2}(C))^{-1}L_{f_0}\left[\sum_{[\mathcal{A}] \in \mathrm{Pic}(\mathcal{O}_K)} \mathcal{C}(\mathcal{A})^{-1}\int_{\mathbf{Z}_p^*} \widetilde{\lambda}\, d\widetilde{\Psi}_{\mathcal{A}}^C\right] \quad (6.2.1)$$

(this is true regardless of the value of $\left(\frac{D}{N}\right)$).

(6.3) The rest of this section will be devoted to the calculation of (some) Fourier coefficients of $\int_{\mathbf{Z}_p^*} \lambda\, d\widetilde{\Psi}_{\mathcal{A}}^C$ for an arbitrary continuous function on $\mathbf{Z}_p^*$.

Since D is odd by assumption, $|D|$ is square-free. This means that one can take as representatives of $\Gamma_0(N|D|p^\nu)\backslash\Gamma_0(Np^\nu)$ the set of matrices

$$|D_1|^{-1}W_{D_1}^{(\nu)}\begin{pmatrix} 1 & j \\ 0 & |D_1| \end{pmatrix},$$

where $|D_1|$ runs through positive divisors of D, $j \in \mathbf{Z}/|D_1|\mathbf{Z}$ and

$$W_{D_1}^{(\nu)} = \begin{pmatrix} |D_1|a & b \\ N|D|p^\nu c & |D_1|d \end{pmatrix}, \qquad a, b, c, d \in \mathbf{Z}$$

with $\det(W_{D_1}^{(\nu)}) = |D_1|$ is fixed. We normalize the sign of D_1 by $D_1 \equiv 1 \,(\mathrm{mod}\, 4)$ and put $D_2 = D/D_1$. Let λ be a function on $(\mathbf{Z}/p^\nu\mathbf{Z})^*$ with values in $\mathbf{C}_p$. We define

$$h_{D_1}(\lambda) := \sum_{\beta \in (\mathbf{Z}/p^\nu\mathbf{Z})^*} \sum_{\alpha \in (\mathbf{Z}/|D|p^\nu\mathbf{Z})^*} \left(\frac{D}{\alpha}\right) \lambda(\beta) H\Big[\Theta_{\mathcal{A}}(\alpha^2\beta \,(\mathrm{mod}\, p^\nu\,))(z)\Big|_1 W_{D_1}^{(\nu)} \times \\ \times \delta_1^{r-1}(E_1(\alpha \,(\mathrm{mod}\, |D|p^\nu\,))(Nz)\Big|_1 W_{D_1}^{(\nu)}\Big]$$

Then

$$\int_{\mathbf{Z}_p^*} \lambda \, d\widetilde{\Psi}_{\mathcal{A}} = \frac{1}{2w} \sum_{D = D_1 \cdot D_2} \sum_{j \in \mathbf{Z}/|D_1|\mathbf{Z}} h_{D_1}(\lambda)\Big|_{2r} \begin{pmatrix} 1 & j \\ 0 & |D_1| \end{pmatrix} \tag{6.3.1}$$

Note that we have

$$\int_{\mathbf{Z}_p^*} \lambda \, d\widetilde{\Psi}_{\mathcal{A}}^C = \int_{\mathbf{Z}_p^*} \lambda_C \, d\widetilde{\Psi}_{\mathcal{A}} \tag{6.3.2}$$

with $\lambda_C(\beta) = \lambda(\beta) - C\left(\frac{D}{C}\right)\lambda(C^{-2}\beta)$.

(6.4) Proposition. *We have*

$$\Theta_{\mathcal{A}}(\alpha \,(\mathrm{mod}\, p^\nu\,))(z)\Big|_1 W_{D_1}^{(\nu)} = \left(\frac{D_1}{cp^\nu N}\right)\left(\frac{D_2}{a\mathbf{N}(\mathcal{A})}\right)\kappa(D_1)^{-1}\Theta_{\mathcal{A}\mathcal{D}_1^{-1}}(|D_1|a^2\alpha \,(\mathrm{mod}\, p^\nu\,))(z)$$

$$\sum_{\alpha_1 \,(\mathrm{mod}\, |D_1|\,)} \left(\frac{D_1}{\alpha_1}\right) E_1((\alpha_1, \alpha_2) \,(\mathrm{mod}\, (|D_1|, |D_2|p^\nu)\,))(Nz)\Big|_1 W_{D_1}^{(\nu)} =$$

$$= \left(\frac{D_1}{bN}\right)\kappa(D_1)\sum_{n=1}^{\infty}\left(\sum_{\substack{k|n \\ k \equiv a\alpha_2 \,(\mathrm{mod}\, |D_2|p^\nu\,)}} \mathrm{sgn}(k)\left(\frac{D_1}{n/k}\right)\right) q^{nN} + \begin{cases} 0 & \text{if } D_1 \neq 1 \\ \frac{1}{2}\widetilde{L}(0, \delta_{a\alpha_2 \,(\mathrm{mod}\, |D_2|p^\nu\,)}) & \text{if } D_1 = 1 \end{cases}$$

where: $\kappa(D_1) = 1$ (resp. i) if $D_1 > 0$ (resp. $D_1 < 0$), $\mathcal{D}_1$ is the unique ideal of $\mathcal{O}_K$ with $\mathbf{N}(\mathcal{D}_1) = |D_1|$ and $(\alpha_1, \alpha_2) \,(\mathrm{mod}\, (D_1|, |D_2|p^\nu)\,)$ denotes the unique residue class modulo $|D|p^\nu$ congruent to α_1 modulo $|D_1|$ and to α_2 modulo $|D_2|p^\nu$.

Proof. [PR 2, Prop. 3.8, 3.15]

(6.5) Writing $a_m(g)$ for the m-th Fourier coefficient of a holomorphic modular form g, we obtain from Prop. 6.4 and Prop. 1.9

$$a_m(h_{D_1}(\lambda)) = \frac{(-1)^{r-1}}{\binom{2r-2}{r-1}} w m^{r-1} \left(\frac{D_1}{bcp^\nu}\right)\left(\frac{D_2}{\mathbf{N}(\mathcal{A})}\right) \sum_{\substack{\alpha_2 \in (\mathbf{Z}/|D_2|p^\nu\mathbf{Z})^* \\ \beta \in (\mathbf{Z}/p^\nu\mathbf{Z})^*}} \left(\frac{D_2}{a\alpha_2}\right)\lambda(\beta) \sum_{\substack{j+nN=m \\ j \equiv |D_1|a^2\alpha_2^2\beta \,(\mathrm{mod}\, p^\nu\,)}}$$

$$r_{\mathcal{A}\mathcal{D}_1^{-1}}(j) \sum_{\substack{k|n \\ k \equiv a\alpha_2 \,(\mathrm{mod}\, |D_2|p^\nu\,)}} \mathrm{sgn}(k)\left(\frac{D_1}{n/k}\right) P_{r-1}\left(\frac{j-nN}{j+nN}\right) = \left(\frac{D_1}{-|D_2|N}\right)\left(\frac{D_2}{\mathbf{N}(\mathcal{A})}\right) \times$$

$$\times \sum_{\substack{j+nN=m \\ p\nmid j}} \sum_{\substack{k|n \\ p\nmid k}} r_{\mathcal{A}\mathcal{D}_1^{-1}}(j)\left(\frac{D_2}{k}\right)\left(\frac{D_1}{n/k}\right)\mathrm{sgn}(k)\lambda\left(\frac{j}{|D_1|k^2}\right) P_{r-1}\left(1 - \frac{2nN}{m}\right)$$

plus the contribution of an extra term for $D_1 = 1$ in Prop. 6.4. Applying (6.3.1) and

$$a_m\Big(\sum_{j\,(\mathrm{mod}\,|D_1|\,)} g\Big|_{2r} \begin{pmatrix} 1 & j \\ 0 & |D_1| \end{pmatrix} \Big) = |D_1|^{1-r} a(|D_1|m, g),$$

we see that

$$a_m\left(\int_{\mathbf{Z}_p^*} \lambda\, d\widetilde{\Psi}_{\mathcal{A}} \right) = \frac{1}{2w} \sum_{D = D_1 \cdot D_2} |D_1|^{1-r} a(|D_1|m, h_{D_1}(\lambda)) = \frac{(-1)^{r-1}}{\binom{2r-2}{r-1}} m^{r-1} \left(\frac{D}{-N} \right) \sum_{D=D_1\cdot D_2} \left(\frac{D_2}{\mathbf{N}(\mathcal{A})} \right) \times$$

$$\times \sum_{\substack{j+nN=|D_1|m \\ p\not| j}} \sum_{\substack{p\not| k | n \\ k>0}} r_{\mathcal{A}\mathcal{D}_1^{-1}}(j) \left(\frac{D_2}{-kN} \right) \left(\frac{D_1}{|D_2|n/k} \right) \lambda \left(\frac{m|D_1| - nN}{|D_1|k^2} \right) P_{r-1}\left(1 - \frac{2nN}{m|D_1|} \right) \qquad (6.5.1)$$

plus an extra term for $D_1 = 1$.

(6.6) We define, for integers n, k and an ideal $\mathcal{A} \subset \mathcal{O}_K$,

$$\varepsilon_{\mathcal{A}}(n,k) := \begin{cases} 0 & \text{if } (k, n/k, |D|) \neq 1 \\ \left(\frac{D_1}{k} \right) \left(\frac{D_2}{-nN/k} \right) \left(\frac{D_2}{\mathbf{N}(\mathcal{A})} \right) & \text{if } \left\{ \begin{matrix} (k, n/k, |D|) = 1 \\ (k, |D|) = |D_2| \\ D_1 D_2 = D \end{matrix} \right\} \end{cases}$$

Proposition. *If $p \mid m$, then*

$$a_m\left(\int_{\mathbf{Z}_p^*} \lambda\, d\widetilde{\Psi}_{\mathcal{A}} \right) = \frac{(-1)^{r-1}}{\binom{2r-2}{r-1}} m^{r-1} \left(\frac{D}{-N} \right) \sum_{\substack{1 \le n \le \frac{m|D|}{N} \\ p\not| n}} r_{\mathcal{A}}(m|D| - nN) P_{r-1}\left(1 - \frac{2nN}{m|D|} \right) \times$$

$$\times \sum_{\substack{k|n \\ k>0}} \varepsilon_{\mathcal{A}}(n,k) \lambda \left(\frac{m|D| - nN}{|D|} \cdot \frac{k^2}{n^2} \right)$$

Proof. As $p \mid m$, the extra term with $D_1 = 1$ in (6.5.1) vanishes (cf. [PR 2, p. 488]). We introduce new variables $n_{\text{new}} := n|D_2|$ and $k_{\text{new}} := n|D_2|/k$. Observing that $r_{\mathcal{A}\mathcal{D}_1^{-1}}(j) = r_{\mathcal{A}\mathcal{D}^{-1}}(j|D_2|) = r_{\mathcal{A}}(j|D_2|)$ (the last equality because $\mathcal{D} = (\sqrt{D})$ is a principal ideal), we obtain from (6.5.1) that the L.H.S. is equal to

$$\frac{(-1)^{r-1}}{\binom{2r-2}{r-1}} m^{r-1} \left(\frac{D}{-N} \right) \sum_{D=D_1\cdot D_2} \sum_{\substack{1 \le n \le \frac{m|D|}{N} \\ |D_2|\,|n \\ p\not| m|D| - nN}} r_{\mathcal{A}}(m|D| - nN) \sum_{\substack{|D_2|\,|k|n \\ p\not| n/k \\ k>0}} \left(\frac{D_1}{k} \right) \left(\frac{D_2}{-nN/k} \right) \left(\frac{D_2}{\mathbf{N}(\mathcal{A})} \right) \times$$

$$\times \lambda \left(\frac{m|D| - nN}{|D|} \cdot \frac{k^2}{n^2} \right) P_{r-1}\left(1 - \frac{2nN}{m|D|} \right)$$

The product $\left(\frac{D_1}{k}\right)\left(\frac{D_2}{-nN/k}\right)$ is non-zero (for $|D_2| \mid k \mid n$) iff $(k, \frac{n}{k}, |D|) = 1$ and $(k, |D|) = |D_2|$. Finally, $p \not| \, m|D| - nN$ iff $p \not| \, n$.

Corollary. *If* $\left(\frac{D}{N}\right)=1$ *and* $p \mid m$, *then*

$$a_m\left(\int_{\mathbf{Z}_p^*}\log_p\, d\widetilde{\Psi}_{\mathcal{A}}\right)=\frac{(-1)^r}{\binom{2r-2}{r-1}}m^{r-1}\sum_{\substack{n=1\\ p\nmid n}}^{\frac{m|D|}{N}} r_{\mathcal{A}}(m|D|-nN)\sigma_{\mathcal{A}}(n)P_{r-1}\left(1-\frac{2nN}{m|D|}\right),$$

where

$$\sigma_{\mathcal{A}}(n)=\sum_{\substack{k|n\\ k>0}}\varepsilon_{\mathcal{A}}(n,k)\log_p\left(\frac{n}{k^2}\right)$$

Proof. As in [PR 2, Prop. 3.18].

(6.7) Let $\sigma\in G(H/K)$ correspond to $[\mathcal{A}]^{-1}$ under the reciprocity isomorphism $G(H/K)\xrightarrow{\sim}\mathrm{Pic}(\mathcal{O}_K)$. For every integer $m\geq 1$ we define

$$B_m^\sigma := m^{r-1}\sum_{\substack{n=1\\ p\nmid n}}^{\frac{m|D|}{N}} r_{\mathcal{A}}(m|D|-nN)\sigma_{\mathcal{A}}(n)P_{r-1}\left(1-\frac{2nN}{m|D|}\right)$$

$$C_m^\sigma := m^{r-1}\sum_{n=1}^{\frac{m|D|}{N}} r_{\mathcal{A}}(m|D|-nN)\sigma_{\mathcal{A}}(n)P_{r-1}\left(1-\frac{2nN}{m|D|}\right)$$

Note that, for $\left(\frac{D}{N}\right)=1$ and $p\mid m$, we have

$$a_m\left(\int_{\mathbf{Z}_p^*}\log_p\, d\widetilde{\Psi}_{\mathcal{A}}\right)=\frac{(-1)^r}{\binom{2r-2}{r-1}}B_m^\sigma$$

by the previous corollary.

Proposition. *Suppose that* $p\mathcal{O}_K=\mathfrak{p}\overline{\mathfrak{p}}$ *with* $\mathfrak{p}\neq\overline{\mathfrak{p}}$. *Then*

$$C_{mp^4}^\sigma-2p^{r-1}(C_{mp^3}^{\sigma\sigma_{\mathfrak{p}}}+C_{mp^3}^{\sigma\sigma_{\overline{\mathfrak{p}}}})+p^{2(r-1)}(C_{mp^2}^{\sigma\sigma_{\mathfrak{p}}^2}+4C_{mp^2}^\sigma+C_{mp^2}^{\sigma\sigma_{\overline{\mathfrak{p}}}^2})-2p^{3(r-1)}(C_{mp}^{\sigma\sigma_{\mathfrak{p}}}+C_{mp}^{\sigma\sigma_{\overline{\mathfrak{p}}}})+C_m^\sigma=$$
$$=B_{mp^4}^\sigma-p^{2(r-1)}B_{mp^2}^\sigma,$$

where $\sigma_{\mathfrak{p}}$ *denotes the Frobenius element associated to* $\mathfrak{p}$.

Proof. A calculation along the lines of [PR 2, Prop. 3.20]. The details are omitted.

In more suggestive terms, the above result can be stated as

$$(U_p-p^{r-1}\sigma_{\mathfrak{p}})^2(U_p-p^{r-1}\sigma_{\overline{\mathfrak{p}}})^2\left(\sum_{m\geq 1}C_m^\sigma q^m\right)=(U_p^4-p^{2(r-1)}U_p^2)\left(\sum_{m\geq 1}B_m^\sigma q^m\right)\qquad(6.7.1)$$

II. Heegner cycles and heights

The contents of this chapter is the following. In Sec. 1 we recall basic facts about p-adic heights constructed in [Ne 2]. Sec. 2 deals with the geometry of Kuga-Sato varieties. In Sec. 3 we define Heegner cycles and in Sec. 4–5 we compute their local heights. The final section contains the proof of Thm A.

1. Abel-Jacobi maps and mixed extensions

(1.1) Let X be a proper smooth scheme over a field F of characteristic zero. For an integer $i \geq 0$, let $CH^i(X)$ be the Chow group of algebraic cycles of codimension i on X modulo rational equivalence. For each prime number p, there is a cycle class into p-adic étale cohomology

$$CH^i(X) \longrightarrow H^{2i}_{et}(X \otimes_F \overline{F}, \mathbf{Q}_p(i))^{G(\overline{F}/F)} \tag{1.1.1}$$

and we denote by $CH^i(X)_0$ its kernel (it does not depend on p, by the Lefschetz principle and a comparison between étale and singular cohomology over $\mathbf{C}$). This is a slightly non-standard notation, as we include all torsion into $CH^i(X)_0$.

(1.2) The cycle class map (1.1.1) factors through $H^{2i}_{et}(X, \mathbf{Q}_p(i))$. The Hochschild-Serre spectral sequence

$$H^i(G(\overline{F}/F), H^j_{et}(X \otimes_F \overline{F}, \mathbf{Q}_p(k))) \Longrightarrow H^{i+j}_{et}(X, \mathbf{Q}_p(k))$$

then induces a p-adic analogue of the Abel-Jacobi map (cf. [Ja 1])

$$\Phi : CH^i(X)_0 \longrightarrow H^1(G(\overline{F}/F), H^{2i-1}_{et}(X \otimes_F \overline{F}, \mathbf{Q}_p(k)))$$

Here one must interpret both the Galois and étale cohomology as a continuous cohomology ([Ja 1], [Ta]) – a convention we adopt for the rest of the paper. We shall also omit the subscript "*et*" and use the abbreviation $H^i(F, A)$ for $H^i(G(\overline{F}/F), A)$.
There is a geometric definition of the Abel-Jacobi map Φ ([Ja 2]): let Y be a cycle on X representing an element of $CH^i(X)_0$. Consider the exact cohomology sequence (in which $\overline{X} = X \otimes_F \overline{F}$ etc.)

$$0 \longrightarrow H^{2i-1}(\overline{X}, \mathbf{Q}_p(i)) \longrightarrow H^{2i-1}(\overline{X} - \overline{Y}, \mathbf{Q}_p(i)) \longrightarrow H^{2i}_{\overline{Y}}(\overline{X}, \mathbf{Q}_p(i)) \longrightarrow H^{2i}(\overline{X}, \mathbf{Q}_p(i))$$

of (continuous, linear) p-adic representations of $G(\overline{F}/F)$. Its pull-back by

$$cl(\overline{Y}) : \mathbf{Q}_p \longrightarrow H^{2i}_{\overline{Y}}(\overline{X}, \mathbf{Q}_p(i))$$

gives a short exact sequence

$$0 \longrightarrow V \longrightarrow E \longrightarrow \mathbf{Q}_p \longrightarrow 0 \tag{1.2.1}$$

The class of E in $\mathrm{Ext}^1_{G(\overline{F}/F)}(\mathbf{Q}_p, V) = H^1(F, V)$ is equal to $\Phi(Y)$.

(1.3) Suppose that F is a finite extension of $\mathbf{Q}_\ell$ and V an arbitrary (linear, continuous) p-adic representation of $G(\overline{F}/F)$. Bloch and Kato [Bl - Ka] defined a subspace $H^1_f(F, V) \subseteq H^1(F, V)$. If W is the largest subrepresentation of V "with good reduction", then $H^1_f(F, V)$ is equal to the Ext-group $\mathrm{Ext}^1(\mathbf{Q}_p, W)$ in the category of representations with good reduction, i.e. unramified (resp. crystalline) representations if $\ell \neq p$ (resp. if $\ell = p$) – cf. [Fo - PR].
For a number field F, one defines

$$H^1_f(F, V) = \{x \in H^1(F, V) \mid x_v \in H^1_f(F_v, V)\ \forall v\}$$

(1.4) The main result of [Ne 1], as well as the application of the height machinery of [Ne 2] to algebraic cycles ([Ne 2, Sec. 5]) were both based on the following statement (cf. [Ne 2, 5.4]):

"Would-be Lemma". *Let F be a number field, X a smooth projective scheme over F with good reduction at all places dividing p. Assume that $V := H^{2i-1}(\overline{X}, \mathbf{Q}_p(i))$ satisfies the monodromy conjecture at all places of bad reduction of X. Then:*
(1) *V is a crystalline representation of $G(\overline{F}_v/F_v)$ for each place $v|p$ of F.*
(2) *The image of $\Phi : CH^i(X)_0 \longrightarrow H^1(F, V)$ is contained in $H^1_f(F, V)$.*

Here (1) is the "crystalline conjecture" and (2) was deduced from (1), [Bl - Ka, 3.8.4], the crystalline Weil conjectures [KaN - Me] and the "de Rham conjecture" [Fa]. For reasons explained in 0.13, it would be desirable to have a proof that would not refer to [Fa]. We sketch an argument that works for sufficiently large p, leaving the details for a future publication [Ne 3].

Lemma. *Let F be a number field, X a proper smooth scheme over F with good reduction at all places dividing p. Assume that $V := H^{2i-1}(\overline{X}, \mathbf{Q}_p(i))$ satisfies the monodromy conjecture at all places of bad reduction of X and that either (a) $p > 2\dim(X)+1$; or (b) $p > \dim(X)$ and, for each place $v|p$ of F, $X \otimes_F F_v$ is obtained by base change from a proper smooth scheme with good reduction over an unramified extension of $\mathbf{Q}_p$. Then:*
(1) *V is a crystalline representation of $G(\overline{F}_v/F_v)$ for each place $v|p$ of F.*
(2) *The image of $\Phi : CH^i(X)_0 \longrightarrow H^1(F,V)$ is contained in $H^1_f(F,V)$.*

Proof. (1) Under the assumption (a) (resp. (b)) this is proved in [KaK - Me] (resp. [Fo - Me]).
(2) In fact, we need $H^1(F_v, V) = 0$ for all $v \not| p$, and this follows from our assumptions (cf. [Ne 2, 2.1.3]). Fix $v|p$ and a proper smooth model $\mathcal{X}$ of $X \otimes_F F_v$ over the ring of integers of F_v. Recall that the Abel-Jacobi map over F_v is given by the composition

$$\begin{aligned} CH^i(X\otimes_F F_v)_0 \otimes \mathbf{Q} &\longrightarrow H^{2i}(X\otimes_F F_v, \mathbf{Q}_p(i))_0 := \mathrm{Ker}\left[H^{2i}(X\otimes_F F_v, \mathbf{Q}_p(i)) \xrightarrow{\gamma} \right. \\ &\left. H^0(F_v, H^{2i}(X\otimes_F \overline{F}_v, \mathbf{Q}_p(i)))\right] \xrightarrow{\delta} H^1(F_v, H^{2i-1}(X\otimes_F \overline{F}_v, \mathbf{Q}_p(i))) = H^1(F_v, V), \end{aligned}$$

where the maps γ, δ come from the Hochschild-Serre spectral sequence.
Define syntomic cohomology with $\mathbf{Q}_p$-coefficients of the formal scheme $\widehat{\mathcal{X}} = \varinjlim(\mathcal{X} \otimes \mathbf{Z}/p^n\mathbf{Z})$ as

$$H^*(\widehat{\mathcal{X}}_{\mathrm{syn}}, s_\infty(r)) := \Big(\varprojlim_n H^*(\widehat{\mathcal{X}}_{\mathrm{syn}}, s_n(r))\Big) \otimes \mathbf{Q},$$

where $s_n(r)$ is the sheaf denoted by S^r_n in [Fo - Me]. In our case there is no difference between continuous and naive p-adic étale cohomology, hence the Fontaine-Messing maps [Fo - Me, III.5] define, in the limit, homomorphisms

$$\beta : H^*(\widehat{\mathcal{X}}_{\mathrm{syn}}, s_\infty(r)) \longrightarrow H^*(X \otimes_F F_v, \mathbf{Q}_p(r))$$

(one needs $p > 2$ at this point, but for $\dim(X) = 1$ the statement (2) follows from [Bl - Ka]). Chern classes c_i into syntomic cohomology [Gro] (resp. étale cohomology) - divided by $(-1)^{i-1}(i-1)!$ - define cycle classes, which fit to a commutative diagram (the compatibility of the Chern classes on K_0 follows from the compatibility of c_1, proved in [Fo - Me, p. 205], and the compatibility of the Fontaine-Messing maps β with products; recall also that $i \le \dim(X) < p$)

$$\begin{array}{ccc} CH^i(\mathcal{X}) \otimes \mathbf{Q} & \longrightarrow & CH^i(X \otimes_F F_v) \otimes \mathbf{Q} \\ \big\downarrow \wr & & \big\downarrow \wr \\ (K_0(\mathcal{X}) \otimes \mathbf{Q})^{(i)} & \longrightarrow & (K_0(X \otimes_F F_v) \otimes \mathbf{Q})^{(i)} \\ \big\downarrow & & \big\downarrow \\ H^{2i}(\widehat{\mathcal{X}}_{\mathrm{syn}}, s_\infty(i)) & \stackrel{\beta}{\longrightarrow} & H^{2i}(X \otimes_F F_v, \mathbf{Q}_p(i)) \end{array}$$

(the superscript (i) denotes the weight i eigenspace for the Adams operations). As the upper horizontal arrow is surjective, it follows that

$$\mathrm{Im}(\Phi) \subseteq \mathrm{Im}\left[\mathrm{Ker}(\gamma \circ \beta) \xrightarrow{\delta \circ \beta} H^1(F_v, V)\right]$$

As shown in [Ne 3], under the assumptions (a) or (b), the latter group is equal to $H^1_f(F_v, V)$.

(1.5) In the rest of Sec. 1 we shall assume that X satisfies the assumptions of Lemma 1.4 and is equidimensional of dimension d. Fix integers $i, j \ge 1$ with $i + j = d + 1$. The Poincaré duality

$$H^{2i-1}(\overline{X}, \mathbf{Q}_p(i)) \times H^{2j-1}(\overline{X}, \mathbf{Q}_p(j)) \xrightarrow{\cup} H^{2d}(\overline{X}, \mathbf{Q}_p(d+1)) \xrightarrow{\mathrm{Tr}} \mathbf{Q}_p(1)$$

identifies $W = H^{2j-1}(\overline{X}, \mathbf{Q}_p(j))$ with the twisted dual $V^*(1)$ of $V = H^{2i-1}(\overline{X}, \mathbf{Q}_p(i))$. The Abel-Jacobi maps are

$$\Phi_i : CH^i(X)_0 \longrightarrow H^1_f(F, V)$$
$$\Phi_j : CH^j(X)_0 \longrightarrow H^1_f(F, V^*(1))$$

According to [Ne 2], there are height pairings

$$\langle\ ,\ \rangle : H^1_f(F, V) \times H^1_f(F, V^*(1)) \longrightarrow \mathbf{Q}_p$$

depending on the following data:

(A) A continuous homomorphism $\ell_F : \mathbf{A}_F^*/F^* \longrightarrow \mathbf{Q}_p$, i.e. a collection of continuous homomorphisms $\ell_v : F_v^* \longrightarrow \mathbf{Q}_p$ such that $\sum_v \ell_v(a) = 0$ for all $a \in F^*$. We view each ℓ_v as a map

$$H^1(F_v, \mathbf{Q}_p(1)) = F_v^* \widehat{\otimes} \mathbf{Q}_p := \left(\varprojlim_n F_v^* \widehat{\otimes} \mathbf{Z}/p^n\mathbf{Z} \right) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p \longrightarrow \mathbf{Q}_p$$

(B) For each place $v \mid p$, a $\mathbf{Q}_p$-linear splitting of the Hodge filtration

$$F^i H^{2i-1}_{\mathrm{dR}}(X \otimes_F F_v/F_v) \overset{\longleftarrow}{\longrightarrow} H^{2i-1}_{\mathrm{dR}}(X \otimes_F F_v/F_v) \tag{1.5.1}$$

(1.6) We recall a definition of $\langle\ ,\ \rangle$ in terms of local heights ([Ne 2, Sec. 4]). Given $a_1 \in H^1_f(F, V)$ and $a_2 \in H^1_f(F, V^*(1))$, represent them by extensions of Galois representations

$$e_1 : 0 \longrightarrow V \longrightarrow E_1 \longrightarrow \mathbf{Q}_p \longrightarrow 0$$
$$e_2 : 0 \longrightarrow V^*(1) \longrightarrow E_2^*(1) \longrightarrow \mathbf{Q}_p \longrightarrow 0$$

We claim that the Yoneda product of $[e_1] \in \mathrm{Ext}^1(\mathbf{Q}_p, V)$ and $[e_2^*(1)] \in \mathrm{Ext}^1(V, \mathbf{Q}_p(1))$ in $\mathrm{Ext}^2(\mathbf{Q}_p, \mathbf{Q}_p(1)) = H^2(F, \mathbf{Q}_p(1))$ vanishes. Indeed, it is given (up to a sign) by the cup product $a_1 \cup a_2$; we note that $H^2(F, \mathbf{Q}_p(1))$ injects into $\prod_v H^2(F_v, \mathbf{Q}_p(1))$ and each localization $(a_1)_v \cup (a_2)_v \in H^2(F_v, \mathbf{Q}_p(1))$ vanishes by [Bl - Ka, 3.8]. This implies that there is a mixed extension E of $e_1, e_2^*(1)$, i.e. a Galois representation E fitting into an exact diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbf{Q}_p(1) & \longrightarrow & E_2 & \longrightarrow & V & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbf{Q}_p(1) & \longrightarrow & E & \longrightarrow & E_1 & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathbf{Q}_p & = & \mathbf{Q}_p & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

In fact, the set of all such E's is an $H^1(F, \mathbf{Q}_p(1))$-torsor (cf. [Ne 2, 4.4]). Fix one such E. If v is a place of F not dividing p, then $H^1(F_v, V) = H^1(F_v, V^*(1)) = 0$ by our assumptions, hence E splits, as a representation of $G(\overline{F_v}/F_v)$, into

$$E = V \oplus U_v,$$

where

$$0 \longrightarrow \mathbf{Q}_p(1) \longrightarrow U_v \longrightarrow \mathbf{Q}_p \longrightarrow 0 \tag{1.6.1}$$

is a “Kummer extension” with class $[U_v] \in H^1(F_v, \mathbf{Q}_p(1))$. The local height of a_1, a_2 (depending on E) is defined as

$$\langle a_1, a_2\rangle_{E,v} = -\ell_v([U_v]) \in \mathbf{Q}_p$$

(1.7) For $v \mid p$, there is an exact sequence

$$0 \longrightarrow H^1_f(F_v, \mathbf{Q}_p(1)) \longrightarrow H^1_f(F_v, E_2) \longrightarrow H^1_f(F_v, V) \longrightarrow 0 \tag{1.7.1}$$

with a splitting

$$H^1_f(F_v, \mathbf{Q}_p(1)) \xleftarrow{r_v} H^1_f(F_v, E_2)$$

determined by the splitting (1.5.1) (see [Ne 2]). If E is crystalline as a representation of $G(\overline{F_v}/F_v)$, then its class lies in $H^1_f(F_v, E_2)$ and the local height is

$$\langle a_1, a_2\rangle_{E,v} = -\ell_v(r_v([E])) \tag{1.7.2}$$

If E is not crystalline, one takes instead of (1.7.1) its push-out by $j : H^1_f(F_v, \mathbf{Q}_p(1)) \hookrightarrow H^1(F_v, \mathbf{Q}_p(1))$

$$0 \longrightarrow H^1(F_v, \mathbf{Q}_p(1)) \longrightarrow j_* H^1_f(F_v, E_2) \longrightarrow H^1_f(F_v, V) \longrightarrow 0,$$

again equipped with a splitting

$$H^1(F_v, \mathbf{Q}_p(1)) \xleftarrow{r_v} j_* H^1_f(F_v, E_2)$$

The local height is given by the same formula (1.7.2).
If $a_{1,v}$ vanishes in $H^1(F_v, V)$, then $[E]$ lies in the image of $H^1(F_v, \mathbf{Q}_p(1))$ in $H^1(F_v, E_2)$, which implies that

$$\langle a_1, a_2\rangle_{E,v} = -\ell_v([E]) \tag{1.7.3}$$

does not depend on the choice of the splittings (1.5.1). The difference between two splittings of (1.5.1) is given by a map

$$s : H^1_f(F_v, V) \times H^1_f(F_v, V^*(1)) \longrightarrow \mathbf{Q}_p$$

(using the exponential map of [Bl - Ka], which is in our case an isomorphism). The difference between the corresponding local heights is then equal to

$$^1\langle a_1, a_2\rangle_{E,v} - {}^2\langle a_1, a_2\rangle_{E,v} = s(a_{1,v}, a_{2,v})$$

(1.8) The global height

$$\langle a_1, a_2\rangle = \sum_v \langle a_1, a_2\rangle_{E,v}$$

does not depend on the choice of E – if we replace E by its translation by $x \in H^1(F, \mathbf{Q}_p(1)) = F^* \widehat{\otimes} \mathbf{Q}_p$, then the local pairing at v is changed by $\ell_v(x)$ and the sum $\sum_v \ell_v(x)$ vanishes.

(1.9) Mixed extensions arise in nature as follows: suppose that Y (resp. Z) is a cycle of codimension i (resp. j) on X representing an element of $CH^i(X)_0$ (resp. $CH^j(X)_0$) and assume that Y and Z have disjoint supports. By definition, the Abel-Jacobi image $\Phi_i(Y)$ is represented by the extension

$$0 \longrightarrow V \longrightarrow E_1 \longrightarrow \mathbf{Q}_p \longrightarrow 0$$

obtained from

$$0 \longrightarrow H^{2i-1}(\overline{X}, \mathbf{Q}_p(i)) \longrightarrow H^{2i-1}(\overline{X} - \overline{Y}, \mathbf{Q}_p(i)) \longrightarrow H^{2i}_{\overline{Y}}(\overline{X}, \mathbf{Q}_p(i))$$

by pull-back by $cl(\overline{Y}) : \mathbf{Q}_p \longrightarrow H^{2i}_{\overline{Y}}(\overline{X}, \mathbf{Q}_p(i))$.

Similarly, the push-out of the sequence

$$H^{2i-2}(\overline{Z}, \mathbf{Q}_p(i)) \longrightarrow H^{2i-1}(\overline{X}\ rel\ \overline{Z}, \mathbf{Q}_p(i)) \longrightarrow H^{2i-1}(\overline{X}, \mathbf{Q}_p(i)) \longrightarrow 0$$

by $-\mathrm{Tr}_{\overline{Z}} : H^{2i-2}(\overline{Z}, \mathbf{Q}_p(i)) \longrightarrow \mathbf{Q}_p(1)$ gives an extension

$$0 \longrightarrow \mathbf{Q}_p(1) \longrightarrow E_2 \longrightarrow V \longrightarrow 0,$$

the dual of which represents $\Phi_j(Z)$ ([ScA 2, 7.5]). The mixed extension E is obtained from $H^{2i-1}(\overline{X} - \overline{Y}\ rel\ \overline{Z}, \mathbf{Q}_p(i))$ (sitting in two exact cohomology sequences) by applying both $cl(\overline{Y})^*$ and $(-\mathrm{Tr}_{\overline{Z}})_*$ (see [Ne 2]. [ScA 2] for more details).

We shall write $\langle Y, Z\rangle_v$ for the local height $\langle \Phi_i(Y), \Phi_j(Z)\rangle_{E,v}$ associated to this mixed extension. It has the following compatibility property (cf. [PR 2, 4.3a]): if F' is a finite extension of F, Y a cycle of codimension i on X and Z a cycle of codimension j on $X \otimes_F F'$, then

$$\langle Y, N_{F'/F}(Z)\rangle_v = \langle Y, Z\rangle_w, \tag{1.9.1}$$

where v is a place of F, w a place of F' over v. The logaritm ℓ'_F over F' is equal to $\ell'_F = \ell_F \circ N_{F'/F}$ and the splitting (1.5.1) over F'_w (if $v \mid p$) is induced from the splitting over F_v.

(1.10) We shall need, later on, basic integrality properties of the Abel-Jacobi maps and local heights.

If Y is a cycle representing an element in $CH^i(X)_0$, then its cohomology class in $H^{2i}(\overline{X}, \mathbf{Z}_p(i))$ is torsion. The torsion subgroup $H^{2i}(\overline{X}, \mathbf{Z}_p(i))_{\mathrm{tors}}$ is finite, of order p^a. The construction of the extension class associated to p^aY works with cohomology with $\mathbf{Z}_p$-coefficients. This implies that $p^a\Phi_i$ factors through $H^1(F, H^{2i-1}(\overline{X}, \mathbf{Z}_p(i)))$. In the same way, $p^b\Phi_j$ factors through $H^1(F, H^{2j-1}(\overline{X}, \mathbf{Z}_p(j)))$, where p^b is the order of $H^{2j}(\overline{X}, \mathbf{Z}_p(j))_{\mathrm{tors}}$.

Replacing Y, Z by p^aY and p^bZ, we see that the construction (1.9) works in $\mathbf{Z}_p$-cohomology, giving rise to a mixed extension T_E of extensions

$$0 \longrightarrow T \longrightarrow T_1 \longrightarrow \mathbf{Z}_p \longrightarrow 0$$
$$0 \longrightarrow \mathbf{Z}_p(1) \longrightarrow T_2 \longrightarrow T \longrightarrow 0$$

representing $a_1 = \Phi_i(p^aY) \in H^1_f(F, T)$, $a_2 = \Phi_j(p^bZ) \in H^1_f(F, T^*(1))$. Here

$$T = \mathrm{Im}\left(H^{2i-1}(\overline{X}, \mathbf{Z}_p(i)) \longrightarrow H^{2i-1}(\overline{X}, \mathbf{Q}_p(i))\right)$$

Assume that $v \nmid p$. Then the group $H^1(F_v, T^*(1))$ is finite, of order p^c. As $p^c[T_2] = 0 \in \mathrm{Ext}^1_{G(\overline{F}_v/F_v)}(T, \mathbf{Z}_p(1))$, we have

$$\begin{aligned} p^c\langle a_1, a_2\rangle_{E,v} &\in \ell_v(F_v^*) \\ \langle Y, Z\rangle_v &\in p^{-a-b-c}\ell_v(F_v^*) \end{aligned} \tag{1.10.1}$$

Assume now that $v \mid p$. The torsion subgroup $H^1(F_v, T)_{\mathrm{tors}}$ is finite, of order p^d. If $a_{1,v}$ goes to zero in $H^1_f(F_v, V)$, then $p^d[T_E]$ lies in the image of $H^1(F_v, \mathbf{Z}_p(1)) = F_v^* \widehat{\otimes} \mathbf{Z}_p$ in $H^1(F_v, T_1)$. It then follows from (1.7.3) that

$$\begin{aligned} p^d\langle a_1, a_2\rangle_{E,v} &\in \ell_v(F_v^* \widehat{\otimes} \mathbf{Z}_p) \\ \langle Y, Z\rangle_v &\in p^{-a-b-d}\ell_v(F_v^* \widehat{\otimes} \mathbf{Z}_p) \end{aligned} \tag{1.10.2}$$

(1.11) We need a generalization of (1.10.2). Assume the following conditions are satisfied:
(A) $\ell_v : F_v^* \longrightarrow \mathbf{Q}_p$ is ramified, i.e. ℓ_v does not vanish identically on the group of units $\mathcal{O}_v^*$.
(B) As a representation of $G(\overline{F}_v/F_v)$, V is a direct sum $V = V' \oplus V''$, where V' satisfies the "Pančiškin condition" of [Ne 2], [PR 3] (this happens, for example, when V' is ordinary in the sense of [Gre]). Put $T' = T \cap V'$, $T'' = T \cap V''$. Then $p^{d_0}T \subseteq T' \oplus T'' \subseteq T$ for some integer d_0.
(C) The V'-component of the splitting (1.5.1) is equal to the canonical splitting V' acquires from (B).
(D) The image of $a_{2,v}$ in $H^1_f(F_v, V''^*(1))$ vanishes.

The local reciprocity map $F_v^* \longrightarrow G(F_v^{\mathrm{ab}}/F_v)$ takes $\mathrm{Ker}(\ell_v)$ to $G(F_v^{\mathrm{ab}}/F_{v,\infty})$, where $F_{v,\infty}/F_v$ is a ramified $\mathbf{Z}_p$-extension, by (A). Write $F_{v,n}$ for its n-th layer, $G(F_{v,n}/F_v) \xrightarrow{\sim} \mathbf{Z}/p^n\mathbf{Z}$. Let

$$N_\infty H^1_f(F_v, T') = \bigcap_{n\geq 1} \mathrm{Im}\left(\mathrm{cor}_n : H^1_f(F_{v,n}, T') \longrightarrow H^1_f(F_v, T')\right)$$

be the subgroup of universal norms in $H^1_f(F_v, T')$. According to [Ne 2, 6.9], it is a subgroup of finite index.

Proposition. *Let p^{d_1} be the order of $H^1(F_v, T''^*(1))_{\mathrm{tors}}$ and p^{d_2} the order of $H^1_f(F_v, T')/N_\infty H^1_f(F_v, T')$. Then*

$$p^{d_0+d_1+d_2}\langle a_1, a_2\rangle_{E,v} \in \ell_v(F_v^* \widehat{\otimes} \mathbf{Z}_p)$$

Proof. Replace a_2 by $p^{d_0+d_1}a_2$. Then $a_{2,v} \in H^1_f(F_v, T'^*(1))$ by (B) and (D), hence T_2 (for this new a_2) splits as $T_2 = T_2' \oplus T''$ with

$$0 \longrightarrow \mathbf{Z}_p(1) \longrightarrow T_2' \longrightarrow T' \longrightarrow 0$$

Suppose first that E is crystalline as a representation of $G(\overline{F}_v/F_v)$. Under this assumption, $[T_E] \in H^1_f(F_v, T_2)$. Consider the following exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^1_f(F_v, \mathbf{Z}_p(1)) & \longrightarrow & H^1_f(F_v, T_2') & \xrightarrow{\beta} & H^1_f(F_v, T') & \longrightarrow & 0 \\ 0 & \longrightarrow & H^1_f(F_{v,n}, \mathbf{Z}_p(1)) & \longrightarrow & H^1_f(F_{v,n}, T_2') & \xrightarrow{\beta_n} & H^1_f(F_{v,n}, T') & \longrightarrow & 0 \end{array} \tag{1.11.1}$$

By definition, the local height $\langle a_1, a_2\rangle_{E,v}$ is equal to the image of $[T_E]$ under

$$H^1_f(F_v, T_2) \longrightarrow H^1_f(F_v, E_2) \xrightarrow{r_v} H^1_f(F_v, \mathbf{Q}_p(1)) \xrightarrow{-\ell_v} \mathbf{Q}_p,$$

which is the same as the image of $[T_E]$ under

$$H^1_f(F_v, T_2) \longrightarrow H^1_f(F_v, T_2') \xrightarrow{r_v'} H^1_f(F_v, \mathbf{Q}_p(1)) \xrightarrow{-\ell_v} \mathbf{Q}_p,$$

where r_v' is associated, by our assumption (C), to the canonical splitting. The map $\ell_v \circ r_v'$ can be described in terms of universal norms as follows (see [Ne 2, 6.11.2]). Let $x \in H^1_f(F_v, T_2')$. For each $n \geq 1$, we have $p^{d_2}\beta(x) = \mathrm{cor}_n(y_n)$ for some y_n. Write $y_n = \beta_n(x_n)$ with $x_n \in H^1_f(F_{v,n}, T_2')$. Then $p^{d_2}x - x_n$ lies in the image of $H^1_f(F_v, \mathbf{Z}_p(1))$; put

$$a_n := p^{-d_2}\ell_v(p^{d_2}x - x_n) \in p^{-d_2}\ell_v(F_v^* \widehat{\otimes} \mathbf{Z}_p)$$

If we make another choice of y_n and x_n, then a_n changes by an element of

$$p^{-d_2}\ell_v \circ N_{F_{v,n}/F_v}(F_{v,n}^* \widehat{\otimes} \mathbf{Z}_p) = p^{n-d_2}\ell_v(F_v^* \widehat{\otimes} \mathbf{Z}_p)$$

The sequence a_n is convergent and its limit is equal to $\ell_v \circ r_v'(x)$. It follows that

$$\mathrm{Im}\left(\ell_v \circ r_v' : H^1_f(F_v, T_2') \longrightarrow \mathbf{Q}_p\right) \subseteq p^{-d_2}\ell_v(F_v^* \widehat{\otimes} \mathbf{Z}_p) \tag{1.11.2}$$

If E is not crystalline, we replace the sequences (1.11.1) by their push-outs by j, as in (1.7). The rest of the argument remains unchanged and shows that (1.11.2) holds in this case as well. Recalling that we have multiplied the original a_2 by $p^{d_0+d_1}$, (1.11.2) proves the Proposition.

2. Kuga-Sato varieties

(2.1) For an integer $N \geq 3$, let $Y(N)$ be the affine curve over $\mathbf{Q}$ parametrizing elliptic curves with a full level N structure and let $Y(N) \stackrel{j}{\hookrightarrow} X(N)$ be its smooth projective compactification classifying generalized elliptic curves. We have the universal generalized elliptic curve $\mathcal{E} \longrightarrow X(N)$ which restricts to the universal elliptic curve $f : E \longrightarrow Y(N)$.
For an integer $w \geq 1$, let $\mathcal{E}^w$ be the w-fold fibre product over $X(N)$ of $\mathcal{E}$ with itself and let W be its canonical desingularization described in [De 1], [ScA 1]. The finite group

$$\Gamma = S_w \ltimes \left(\mu_2 \ltimes (\mathbf{Z}/N\mathbf{Z})^2\right)^w$$

acts on $\mathcal{E}^w$ (along the fibres): the symmetric group S_w permutes the factors, $(\mathbf{Z}/N\mathbf{Z})^2$ acts by translations by points of order N and μ_2 by multiplication by ± 1 in the given factor. This action extends canonically to W.

(2.2) Scholl [ScA 1] defines a projector

$$\varepsilon \in \mathbf{Z}\left[\frac{1}{2N \cdot w!}\right][\Gamma]$$

corresponding to the following representation of Γ: trivial on $(\mathbf{Z}/N\mathbf{Z})^2$, the product map on μ_2^w and the sign character on S_w.
The main result of [ScA 1] says that (for any prime number p) there is a canonical isomorphism

$$H^1(X(N) \otimes_{\mathbf{Q}} \overline{\mathbf{Q}}, j_* \mathrm{Sym}^w(R^1 f_* \mathbf{Q}_p)) \xrightarrow{\sim} \varepsilon H^{w+1}(W \otimes_{\mathbf{Q}} \overline{\mathbf{Q}}, \mathbf{Q}_p) = \varepsilon H^*(W \otimes_{\mathbf{Q}} \overline{\mathbf{Q}}, \mathbf{Q}_p) \qquad (2.2.1)$$

The expression (2.2.1) is a p-adic representation of $G(\overline{\mathbf{Q}}/\mathbf{Q})$, which contains representations associated to all cusp forms of weight $w+2$ on $\Gamma(N)$.

(2.3) Suppose, from now on, that $w = 2r-2 \geq 2$ is even. Let F be an extension of $\mathbf{Q}$ and $\widetilde{y}$ a closed point of $Y(N) \otimes_{\mathbf{Q}} F$. The fibre $(W \otimes_{\mathbf{Q}} F)_{\widetilde{y}}$ of $W \otimes_{\mathbf{Q}} F$ over $\widetilde{y}$ is equal to $E_{\widetilde{y}}^{2r-2}$, where $E_{\widetilde{y}}$ is an elliptic curve over $k(\widetilde{y})$, the residue field of $\widetilde{y}$.
If Y is a cycle of dimension $r-1$ supported in $(W \otimes_{\mathbf{Q}} F)_{\widetilde{y}}$, we can form εY (a cycle with rational coefficients), also supported in $(W \otimes_{\mathbf{Q}} F)_{\widetilde{y}}$. It lies in $\varepsilon\,(CH^r(W \otimes_{\mathbf{Q}} F)_0 \otimes \mathbf{Q})$, as

$$\varepsilon H^{2r}(W \otimes_{\mathbf{Q}} \overline{F}, \mathbf{Q}_p(r)) = \varepsilon H^{2r}(W \otimes_{\mathbf{Q}} \overline{\mathbf{Q}}, \mathbf{Q}_p(r)) = 0$$

by (2.2.1).
Denote by $b(Y)$ the cohomology class of $\varepsilon(Y \otimes_F \overline{F})$ in

$$(W \otimes_{\mathbf{Q}} F)_{\widetilde{y}} \otimes_F \overline{F} = \left(W \otimes_{\mathbf{Q}} \overline{F}\right)_{\widetilde{y} \otimes_F \overline{F}}$$

It lies in

$$\varepsilon H^{2r-2}(\left(W \otimes_{\mathbf{Q}} \overline{F}\right)_{\widetilde{y} \otimes_F \overline{F}}, \mathbf{Q}_p(r-1))^{G(\overline{F}/F)} = H^0(\widetilde{y} \otimes_F \overline{F}, \mathcal{B})^{G(\overline{F}/F)},$$

where

$$\mathcal{B} = \mathrm{Sym}^{2r-2}\left(R^1 f_* \mathbf{Q}_p\right)(r-1)$$

is a smooth p-adic sheaf on $Y(N)$, equipped with a non-degenerate symmetric pairing $\mathcal{B} \times \mathcal{B} \longrightarrow \mathbf{Q}_p$ coming from the Poincaré duality.

(2.4) Proposition. (1) *There is a canonical isomorphism*

$$H^1(X(N)\otimes_{\mathbf{Q}}\overline{F}, j_*\mathcal{B})(1) \xrightarrow{\sim} \varepsilon H^{2r-1}(W\otimes_{\mathbf{Q}}\overline{F}, \mathbf{Q}_p(r))$$

(2) *The Abel-Jacobi image of εY in $H^1(F, H^{2r-1}(W\otimes_{\mathbf{Q}}\overline{F}, \mathbf{Q}_p(r)))$ depends only on $b(Y)$. It is represented by the pull-back of the extension (of p-adic representations of $G(\overline{F}/F)$)*

$$0\longrightarrow H^1(X(N)\otimes_{\mathbf{Q}}\overline{F}, j_*\mathcal{B})(1)\longrightarrow H^1(X(N)\otimes_{\mathbf{Q}}\overline{F}-(\widetilde{y}\otimes_F\overline{F}), j_*\mathcal{B})(1)\longrightarrow H^0(\widetilde{y}\otimes_F\overline{F},\mathcal{B})\longrightarrow 0$$

by the map $\mathbf{Q}_p\longrightarrow H^0(\widetilde{y}\otimes_F\overline{F},\mathcal{B})$ sending 1 to $b(Y)$.

Proof. (1) is just a reminder of Scholl's result (2.2.1).
(2) Consider the commutative diagram with exact rows

$$\begin{array}{ccccccc}
0 \longrightarrow \varepsilon H^{2r-1}(W\otimes_{\mathbf{Q}}\overline{F},\mathbf{Q}_p(r)) \longrightarrow & \varepsilon H^{2r-1}(W\otimes_{\mathbf{Q}}\overline{F}-\varepsilon(Y\otimes_F\overline{F}),\mathbf{Q}_p(r)) & \longrightarrow & \varepsilon H^{2r}_{\varepsilon(\widetilde{y}\otimes_F\overline{F})}(W\otimes_{\mathbf{Q}}\overline{F},\mathbf{Q}_p(r)) & \longrightarrow 0\\
\| & \downarrow & & \downarrow & \\
0 \longrightarrow \varepsilon H^{2r-1}(W\otimes_{\mathbf{Q}}\overline{F},\mathbf{Q}_p(r)) \longrightarrow & \varepsilon H^{2r-1}(W\otimes_{\mathbf{Q}}\overline{F}-U,\mathbf{Q}_p(r)) & \longrightarrow & \varepsilon H^{2r}_{U}(W\otimes_{\mathbf{Q}}\overline{F},\mathbf{Q}_p(r)) & \longrightarrow 0
\end{array}$$

where $U=\left(Y\otimes_F\overline{F}\right)_{\widetilde{y}\otimes_F\overline{F}}$. There is a canonical isomorphism

$$\varepsilon H^{2r}_U(W\otimes_{\mathbf{Q}}\overline{F},\mathbf{Q}_p(r))\xrightarrow{\sim}\varepsilon H^{2r-2}(U,\mathbf{Q}_p(r-1))=H^0(\widetilde{y}\otimes_F\overline{F},\mathcal{B})$$

The composite map

$$\mathbf{Q}_p\xrightarrow{cl\left(\varepsilon\left(Y\otimes_F\overline{F}\right)\right)}\varepsilon H^{2r}_{\varepsilon(\widetilde{y}\otimes_F\overline{F})}(W\otimes_{\mathbf{Q}}\overline{F},\mathbf{Q}_p(r))\longrightarrow\varepsilon H^{2r}_U(W\otimes_{\mathbf{Q}}\overline{F},\mathbf{Q}_p(r))=H^0(\widetilde{y}\otimes_F\overline{F},\mathcal{B})$$

takes 1 to $b(Y)$, so it remains to show that the bottom sequence of the diagram is canonically isomorphic to the exact sequence in the statement of the Proposition.
As Γ acts along the fibres, we can apply ε to the Leray spectral sequence for $g: W\otimes_{\mathbf{Q}}\overline{F}\longrightarrow X(N)\otimes_{\mathbf{Q}}\overline{F}$, obtaining

$$H^k(X(N)\otimes_{\mathbf{Q}}\overline{F},\varepsilon R^m g_*\mathbf{Q}_p(r-1))\Longrightarrow\varepsilon H^{k+m}(W\otimes_{\mathbf{Q}}\overline{F},\mathbf{Q}_p(r-1))$$

The sheaves $C_m:=\varepsilon R^m g_*\mathbf{Q}_p(r-1)$ satisfy

$$j^*C_m=\begin{cases}\mathcal{B} & \text{if } m=2r-2\\ 0 & \text{if } m\neq 2r-2\end{cases}$$

The spectral sequence then gives

$$0\longrightarrow H^1(X(N)\otimes_{\mathbf{Q}}\overline{F},C_{q-1})\longrightarrow\varepsilon H^q(W\otimes_{\mathbf{Q}}\overline{F},\mathbf{Q}_p(r-1))\longrightarrow H^0(X(N)\otimes_{\mathbf{Q}}\overline{F},C_q)\longrightarrow 0$$

for all $q<2r$. Combined with (2.2.1), this implies that $H^0(X(N)\otimes_{\mathbf{Q}}\overline{F},C_q)=0$ for $q\leq 2r-2$, hence $C_q=0$ for $q<2r-2$ (as the sheaves C_q are punctual for $q\neq 2r-2$). For $q=2r-1$ we get an exact sequence

$$0\longrightarrow H^1(X(N)\otimes_{\mathbf{Q}}\overline{F},C_{2r-2})\longrightarrow H^1(X(N)\otimes_{\mathbf{Q}}\overline{F},j_*\mathcal{B})\longrightarrow H^0(X(N)\otimes_{\mathbf{Q}}\overline{F},C_{2r-1})\longrightarrow 0$$

As both the kernel and cokernel of the canonical map $C_{2r-2}\longrightarrow j_*\mathcal{B}$ are supported at the cusps, the map

$$H^1(X(N)\otimes_{\mathbf{Q}}\overline{F},C_{2r-2})\longrightarrow H^1(X(N)\otimes_{\mathbf{Q}}\overline{F},j_*\mathcal{B}) \tag{2.4.2}$$

is surjective. This shows that $C_{2r-1}=0$ and that (2.4.2) is in fact an isomorphism.

Restricting g to $W \otimes_{\mathbf{Q}} \overline{F} - U$, we get

$$H^1(X(N) \otimes_{\mathbf{Q}} \overline{F} - (\widetilde{y} \otimes_F \overline{F}), C_{2r-2}) \xrightarrow{\sim} \varepsilon H^{2r-1}(W \otimes_{\mathbf{Q}} \overline{F} - U, \mathbf{Q}_p(r-1))$$

An easy argument based on the relative cohomology sequence for $X(N) \otimes_{\mathbf{Q}} \overline{F} - (\widetilde{y} \otimes_F \overline{F}) \hookrightarrow X(N) \otimes_{\mathbf{Q}} \overline{F}$ then shows that

$$H^1(X(N) \otimes_{\mathbf{Q}} \overline{F} - (\widetilde{y} \otimes_F \overline{F}), C_{2r-2}) \xrightarrow{\sim} \varepsilon H^1(X(N) \otimes_{\mathbf{Q}} \overline{F} - (\widetilde{y} \otimes_F \overline{F}), j_*\mathcal{B}))$$

is an isomorphism as well.

(2.5) The group $GL_2(\mathbf{Z}/N\mathbf{Z})/\{\pm 1\}$ acts on $X(N), Y(N)$ and the sheaves $\mathcal{B}, j_*\mathcal{B}$. The quotients by the Borel subgroup

$$B = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\} / \{\pm 1\}$$

are $X_0(N) = B\backslash X(N)$, $Y_0(N) = B\backslash Y(N)$. The affine curve $Y_0(N)$ is the coarse moduli space (over $\mathbf{Q}$) for pairs (E, C), where E is an elliptic curve and C is a subgroup scheme of E isomorphic to $\mathbf{Z}/N\mathbf{Z}$.

Write j_0 for the inclusion of $Y_0(N)$ to $X_0(N)$ and let $\varepsilon_B = (1/\sharp B) \sum_{g \in B} g$ be the idempotent corresponding to the trivial representation of B. The sheaf $\mathcal{A} = \varepsilon_B \pi_* \mathcal{B} = (\pi_* \mathcal{B})^B$ (where π denotes the projection $Y(N) \longrightarrow Y_0(N)$) is smooth outside of the ramification locus Ram of π and $\pi^* \mathcal{A}$ is isomorphic to $\mathcal{B}$ outside of $\pi^{-1}(\mathrm{Ram})$.

Lemma. *In the notation of Proposition 2.4, the Abel-Jacobi image*

$$\Phi(\varepsilon_B \varepsilon Y) \in H^1(F, \varepsilon_B H^1(X(N) \otimes_{\mathbf{Q}} \overline{F}, j_*\mathcal{B})(1)) = H^1(F, H^1(X_0(N) \otimes_{\mathbf{Q}} \overline{F}, j_{0*}\mathcal{A})(1))$$

is represented by the pull-back of the extension

$$0 \longrightarrow H^1(X_0(N) \otimes_{\mathbf{Q}} \overline{F}, j_{0*}\mathcal{A})(1) \longrightarrow H^1(X_0(N) \otimes_{\mathbf{Q}} \overline{F} - (y \otimes_F \overline{F}), j_{0*}\mathcal{A})(1) \longrightarrow H^0(y \otimes_F \overline{F}, \mathcal{A})^{G(\overline{F}/F)} \longrightarrow 0$$

by the map $\mathbf{Q}_p \longrightarrow H^0(y \otimes_F \overline{F}, \mathcal{A})^{G(\overline{F}/F)}$ sending 1 to $a(Y) := \varepsilon_B b(Y)$. Here $y = \pi(\widetilde{y})$.

Proof. This follows from Prop. 2.4 and the fact that both ε_B and π_* are exact functors.

(2.6) Following (and slightly modifying) [Br], we make the following

Definition. *If F is an extension of $\mathbf{Q}$ and y a closed point of $Y_0(N) \otimes_{\mathbf{Q}} F$, a Tate vector at y is an element of $H^0(y, \mathcal{A}) = H^0(y \otimes_F \overline{F}, \mathcal{A})^{G(\overline{F}/F)}$. A Tate cycle on $Y_0(N)$ over F is a formal finite sum of Tate vectors at several points y.*

Similarly, replacing $Y_0(N)$ by $Y(N)$ and $\mathcal{A}$ by $\mathcal{B}$, we define Tate vectors and cycles on $Y(N)$. We denote by $Z(Y(N), F)$ (resp. $Z(Y_0(N), F)$) the groups of Tate cycles. In the notation of (2.3)and (2.5), we have

$$b(Y) \in Z(Y(N), F), \qquad a(Y) \in Z(Y_0(N), F)$$

There is a map

$$\Phi_T : Z(Y_0(N), F) \longrightarrow H^1(F, H^1(X_0(N) \otimes_{\mathbf{Q}} \overline{F}, j_{0*}\mathcal{A})(1))$$

defined as follows: let

$$a = \sum_{y \in S} a_y \in Z(Y_0(N), F)$$

be a Tate cycle supported at a finite set S of closed points of $Y_0(N) \otimes_{\mathbf{Q}} F$. Then, writing $\overline{S}$ for $S \otimes_F \overline{F}$, $\Phi_T(a)$ is represented by the pull-back of the exact sequence

$$0 \longrightarrow H^1(X_0(N) \otimes_{\mathbf{Q}} \overline{F}, j_{0*}\mathcal{A})(1) \longrightarrow H^1(X_0(N) \otimes_{\mathbf{Q}} \overline{F} - \overline{S}, j_{0*}\mathcal{A})(1) \longrightarrow H^0(\overline{S}, \mathcal{A}) \longrightarrow 0 \qquad (2.6.1)$$

by the map $\mathbf{Q}_p \longrightarrow H^0(\overline{S}, \mathcal{A}) = \oplus_{y \in S} H^0(y \otimes_F \overline{F}, \mathcal{A})$ sending 1 to a.
In this language, Lemma 2.5 says that $\Phi(\varepsilon_B \varepsilon Y) = \Phi_T(a(Y))$.

(2.7) According to [ScA 2, 7.5], the relative cohomology sequence

$$0 \longrightarrow H^0(\overline{S}, \mathcal{A}) \longrightarrow H^1(X_0(N) \otimes_{\mathbf{Q}} \overline{F}\ rel\ \overline{S}, j_{0*}\mathcal{A}) \longrightarrow H^1(X_0(N) \otimes_{\mathbf{Q}} \overline{F}, j_{0*}\mathcal{A}) \longrightarrow 0$$

is dual to the sequence (2.6.1). The pairing between them is given by the Poincaré duality

$$\begin{array}{ccccccc} H^1(X_0(N) \otimes_{\mathbf{Q}} \overline{F}\ rel\ \overline{S}, j_{0*}\mathcal{A}) & \times & H^1(X_0(N) \otimes_{\mathbf{Q}} \overline{F} - \overline{S}, j_{0*}\mathcal{A})(1) & \longrightarrow & H^2(X_0(N) \otimes_{\mathbf{Q}} \overline{F}, \mathbf{Q}_p(1)) & \xrightarrow{\sim} & \mathbf{Q}_p \\ \uparrow & & \downarrow & & \| & & \| \\ H^1(X_0(N) \otimes_{\mathbf{Q}} \overline{F}, j_{0*}\mathcal{A}) & \times & H^1(X_0(N) \otimes_{\mathbf{Q}} \overline{F}, j_{0*}\mathcal{A})(1) & \longrightarrow & H^2(X_0(N) \otimes_{\mathbf{Q}} \overline{F}, \mathbf{Q}_p(1)) & \xrightarrow{\sim} & \mathbf{Q}_p \end{array}$$

and the *opposite* of the pairing $(\ ,\ ) : \mathcal{A} \times \mathcal{A} \longrightarrow \mathbf{Q}_p$ that $\mathcal{A}$ inherits from $\mathcal{B}$.

(2.8) Let $a, b \in Z(Y_0(N), F)$ be Tate cycles supported at disjoint finite sets of closed points S, T. The relative cohomology group

$$H^1(X_0(N) \otimes_{\mathbf{Q}} \overline{F} - \overline{S}\ rel\ \overline{T}, j_{0*}\mathcal{A})(1)$$

is a mixed extension of exact sequences

$$\begin{gathered} 0 \longrightarrow H^1(X_0(N) \otimes_{\mathbf{Q}} \overline{F}, j_{0*}\mathcal{A})(1) \longrightarrow H^1(X_0(N) \otimes_{\mathbf{Q}} \overline{F} - \overline{S}, j_{0*}\mathcal{A})(1) \longrightarrow H^0(\overline{S}, \mathcal{A}) \longrightarrow 0 \\ 0 \longrightarrow H^0(\overline{T}, \mathcal{A}) \longrightarrow H^1(X_0(N) \otimes_{\mathbf{Q}} \overline{F}\ rel\ \overline{T}, j_{0*}\mathcal{A}) \longrightarrow H^1(X_0(N) \otimes_{\mathbf{Q}} \overline{F}, j_{0*}\mathcal{A}) \longrightarrow 0 \end{gathered}$$

Take its pull-back by $\mathbf{Q}_p \longrightarrow H^0(\overline{S}, \mathcal{A})$ $(1 \mapsto a)$ and push-out by $H^0(\overline{T}, \mathcal{A}) \longrightarrow \mathbf{Q}_p$ $(\bullet \mapsto -(\bullet, a))$. This will be a mixed extension E of $\Phi_T(a)$ and $(\Phi_T(b))^*(1)$.
Let y, z be two distinct closed points of $Y_0(N) \otimes_{\mathbf{Q}} F$, $\widetilde{y}$ (resp. $\widetilde{z}$) a point of $Y(N) \otimes_{\mathbf{Q}} F$ over y (resp. z) and Y (resp. Z) a cycle of dimension $r-1$ supported in the fibre $(W \otimes_{\mathbf{Q}} F)_{\widetilde{y}}$ (resp. $(W \otimes_{\mathbf{Q}} F)_{\widetilde{z}}$). If we apply the previous construction to the Tate cycles $a = a(Y), b = a(Z)$, then the same argument as in the proof of Prop. 2.4 shows that the mixed extension E will be isomorphic to the mixed extension associated to the cycles $\varepsilon_B \varepsilon Y$ and $\varepsilon_B \varepsilon Z$ as in (1.9).

(2.9) We recall a geometric realization of Hecke operators. For an integer $m \geq 1$ prime to N, let $Y(N, m)$ be the (reducible) curve over $\mathbf{Q}$ parametrizing elliptic curves E with a full level N structure C and an isogeny $\lambda : E \longrightarrow E'$ of degree m. There are two finite maps $s, t : Y(N, m) \longrightarrow Y(N)$ given by

$$s(E, C, \lambda) = (E, C), \qquad t(E, C, \lambda) = (E', \lambda_*(C))$$

In the notation of 2.1, we have a commutative diagram

$$\begin{array}{ccccccc} \mathcal{E}^{2r-2} & \xleftarrow{\phi_1} & \mathcal{E}^{2r-2} \otimes_{Y(N),s} Y(N,m) & \xrightarrow{\psi} & \mathcal{E}^{2r-2} \otimes_{Y(N),t} Y(N,m) & \xrightarrow{\phi_2} & \mathcal{E}^{2r-2} \\ \downarrow f & & \downarrow f_1 & & \downarrow f_2 & & \downarrow f \\ Y(N) & \xleftarrow{s} & Y(N,m) & = & Y(N,m) & \xrightarrow{t} & Y(N) \end{array} \qquad (2.9.1)$$

in which ψ is induced by λ.

The m-th Hecke corespondence $(\alpha_1, \alpha_2) : T_m \hookrightarrow W \times W$ is defined as the Zariski closure of the graph of

$$(\phi_1, \phi_2\psi) : \mathcal{E}^{2r-2} \otimes_{Y(N),s} Y(N,m) \longrightarrow \mathcal{E}^{2r-2} \times \mathcal{E}^{2r-2}$$

It acts on Chow groups and cohomology groups of W and commutes with the Scholl projector ε. Sometimes it is necessary to distinguish between the covariant and contravariant actions $T_m^* = \alpha_{1*}\alpha_2^*$, $T_{m*} = \alpha_{2*}\alpha_1^*$.

In the diagram (2.9.1), the first and third squares are cartesian. By the proper base change theorem, we have

$$s^*\mathcal{B} = \varepsilon s^* R^{2r-2} f_* \mathbf{Q}_p(r-1) = \varepsilon R^{2r-2} f_{1*} \mathbf{Q}_p(r-1)$$
$$t^*\mathcal{B} = \varepsilon t^* R^{2r-2} f_* \mathbf{Q}_p(r-1) = \varepsilon R^{2r-2} f_{2*} \mathbf{Q}_p(r-1)$$

and ψ induces homomorphisms

$$\psi^* : t^*\mathcal{B} \longrightarrow s^*\mathcal{B}, \qquad \psi_* : s^*\mathcal{B} \longrightarrow t^*\mathcal{B}$$

This defines operators $T_m^* = s_*\psi^*t^*$, $T_{m*} = t_*\psi_*s^*$ on $H^1(Y(N) \otimes_{\mathbf{Q}} \overline{\mathbf{Q}}, \mathcal{B})(1)$, $H^1_c(Y(N) \otimes_{\mathbf{Q}} \overline{\mathbf{Q}}, \mathcal{B})(1)$ and

$$H^1(X(N) \otimes_{\mathbf{Q}} \overline{\mathbf{Q}}, j_*\mathcal{B})(1) \xrightarrow{\sim} \mathrm{Im}\left(H^1_c(Y(N) \otimes_{\mathbf{Q}} \overline{\mathbf{Q}}, \mathcal{B})(1) \longrightarrow H^1(Y(N) \otimes_{\mathbf{Q}} \overline{\mathbf{Q}}, \mathcal{B})(1)\right)$$

The isomorphism (2.2.1) is compatible with the action of T_m^*, T_{m*} on both sides. Poincaré duality induces a non-degenerate skew-symmetric pairing

$$(\ ,\ ) : H^1(X(N) \otimes_{\mathbf{Q}} \overline{\mathbf{Q}}, j_*\mathcal{B})(1) \times H^1(X(N) \otimes_{\mathbf{Q}} \overline{\mathbf{Q}}, j_*\mathcal{B})(1) \longrightarrow \mathbf{Q}_p(1) \qquad (2.9.2)$$

which satisfies

$$(T_{m*}x, y) = (x, T_m^* y) \qquad (2.9.3)$$

The action of T_m commutes with ε_B; induced operators on $H^1(X_0(N) \otimes_{\mathbf{Q}} \overline{\mathbf{Q}}, j_*\mathcal{B})(1)$ will be denoted by the same letter. They are symmetric, i.e. satisfy $T_{m*} = T_m^*$ (cf. 2.10 below).

(2.10) Let S be a finite set of closed points on $Y(N) \otimes_{\mathbf{Q}} F$. The operators

$$T_m^* : H^0(S \otimes_F \overline{F}, \mathcal{B}) \xrightarrow{t^*} H^0(t^{-1}(S) \otimes_F \overline{F}, t^*\mathcal{B}) \xrightarrow{\psi^*} H^0(t^{-1}(S) \otimes_F \overline{F}, s^*\mathcal{B}) \xrightarrow{s_*} H^0(s(t^{-1}(S)) \otimes_F \overline{F}, \mathcal{B})$$
$$T_{m*} : H^0(S \otimes_F \overline{F}, \mathcal{B}) \xrightarrow{s^*} H^0(s^{-1}(S) \otimes_F \overline{F}, s^*\mathcal{B}) \xrightarrow{\psi_*} H^0(s^{-1}(S) \otimes_F \overline{F}, t^*\mathcal{B}) \xrightarrow{t_*} H^0(t(s^{-1}(S)) \otimes_F \overline{F}, \mathcal{B})$$

commute with the Galois action, hence operate on Tate cycles $Z(Y(N), F)$. Applying the projector ε_B, we get operators T_m^*, T_{m*} on $Z(Y_0(N), F)$. These operators are symmetric, $T_m^* = T_{m*}$. This can be seen as follows: a Tate vector v on $Y_0(N)$ is represented by a triple (E, C, b), where $C \subset E$ is a cyclic subgroup scheme of order N and $b \in \mathrm{Sym}^{2r-2}\left(H^1(E \otimes_F \overline{F}, \mathbf{Q}_p)\right)(r-1)$. By definition, $T_{m*}(v)$ is represented by

$$\sum_{\substack{\lambda : E \longrightarrow E' \\ \deg(\lambda) = m}} (E', \lambda_*(C), \lambda_*(b))$$

Writing $\mu := \widehat{\lambda} : E' \longrightarrow E$ for the dual isogeny, then $\lambda_*(C) = \mu^*(C)$ and $\lambda_*(b) = \mu^*(b)$ (as $\lambda_* = (\widehat{\lambda})^*$ on $H^1(E \otimes_F \overline{F}, \mathbf{Q}_p)$), hence

$$T_{m*}(v) = \sum_{\substack{\mu : E' \longrightarrow E \\ \deg(\mu) = m}} (E', \mu^*(C), \mu^*(b)) = T_m^*(v)$$

As Abel-Jacobi maps commute with correspondences, we have (in the notation of 2.5)

$$\Phi(T^*_m \varepsilon_B \varepsilon Y) = T^*_m \Phi(\varepsilon_B \varepsilon Y) = \Phi_T(T^*_m a(Y)) = \Phi_T(T_{m*} a(Y)) \tag{2.10.1}$$

(2.11) It is well known that T_m $(= T_{m*} = T^*_m)$ acts on

$$V = H^1(X_0(N) \otimes_{\mathbf{Q}} \overline{\mathbf{Q}}, j_{0*}\mathcal{A})(1) = \varepsilon_B \varepsilon H^{2r-1}(W \otimes_{\mathbf{Q}} \overline{\mathbf{Q}}, \mathbf{Q}_p(r))$$

as the m-th Hecke operator T_m does on $S_{2r}(\Gamma_0(N))$ ([De 1, 3.19]). Let $\mathbf{T}$ be the $\mathbf{Q}$-algebra generated by the operators T_m (for m prime to N) on $S_{2r}(\Gamma_0(N))$. As an abstract algebra, $\mathbf{T}$ is a product of finitely many number fields. As in I.2.1, fix embeddings $i_\infty : \overline{\mathbf{Q}} \hookrightarrow \mathbf{C}$, $i_p : \overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_p$. There is a bijection between $\mathbf{Q}$-algebra homomorphisms $\beta : \mathbf{T} \longrightarrow \overline{\mathbf{Q}}$ and normalized primitive forms $f_\beta \in S_{2r}(\Gamma_0(N); \overline{\mathbf{Q}})$, given by

$$\beta(T_m) = a_m(f_\beta), \qquad (m, N) = 1$$

Let $N(\beta) \mid N$ be the conductor of f_β. Then $f_\beta \in S^{\mathrm{new}}_{2r}(\Gamma_0(N(\beta)); F(\beta))$, where $F(\beta) = \mathrm{Im}(\beta)$. The image of the induced map

$$\beta_p : \mathbf{T} \otimes \mathbf{Q}_p \xrightarrow{\beta \otimes 1} \overline{\mathbf{Q}} \otimes_{\mathbf{Q}} \mathbf{Q}_p \xrightarrow{i_p} \overline{\mathbf{Q}}_p$$

is equal to the closure $\widehat{F(\beta)}$ of $i_p(F(\beta))$ in $\overline{\mathbf{Q}}_p$. For any $\mathbf{T} \otimes \mathbf{Q}_p$-module A, there is a decomposition

$$A \otimes_{\mathbf{Q}_p} \overline{\mathbf{Q}}_p \stackrel{\sim}{\longrightarrow} \bigoplus_{\beta : \mathbf{T} \longrightarrow \overline{\mathbf{Q}}} \left(A_\beta \otimes_{\widehat{F(\beta)}} \overline{\mathbf{Q}}_p \right)$$

with

$$A_\beta = A \otimes_{\mathbf{T} \otimes \mathbf{Q}_p, \beta_p} \widehat{F(\beta)}$$

For $A = V$, the β-part V_β is a vector space over $\widehat{F(\beta)}$ of dimension $2m(\beta)$, where $m(\beta)$ is the multiplicity of the eigenvalue β in $S_{2r}(\Gamma_0(N))$, and the action of $G(\overline{\mathbf{Q}}/\mathbf{Q})$ is $\widehat{F(\beta)}$-linear. Of course, f_β is a newform iff $N(\beta) = N$ iff $m(\beta) = 1$.

For any prime number $\ell \not| \, pN(\beta)$, V_β is unramified at ℓ and the characteristic polynomial of a geometric Frobenius at ℓ is equal to

$$\det_{\widehat{F(\beta)}} (1 - Fr_\ell t \mid V_\beta) = \left(1 - \beta(T_\ell)\ell^{-r} t + \ell^{-1} t^2\right)^{m(\beta)}$$

Both eigenvalues of Fr_ℓ have absolute value $\ell^{-1/2}$ ([De 1], [De 2]). For $\ell \mid N(\beta)$ (but $\ell \neq p$), V_β is ramified at ℓ and

$$\det_{\widehat{F(\beta)}} \left(1 - Fr_\ell t \mid V_\beta^{I_\ell}\right) = \left(1 - a_\ell(f_\beta)\ell^{-r} t\right)^{m(\beta)}$$

by [Ca, Thm. A]. The ℓ-th coefficient of f_β either vanishes or has absolute value ℓ^{r-1} ([At - Le, Thm. 3]).

(2.12) Lemma. *If F is a finite extension of $\mathbf{Q}_\ell$ for $\ell \neq p$, then $H^i(F, V) = 0$ for $i = 0, 1, 2$.*

Proof. It follows from what has just been said that $H^0(F, V_\beta) = 0$ for all β. As $V_\beta \stackrel{\sim}{\longrightarrow} V^*_\beta(1)$ by 2.9.2, local duality [Se, II.5.2 Thm. 2] gives $H^2(F, V_\beta) = 0$, hence $H^1(F, V_\beta) = 0$ by the formula for the local Euler characteristic [Se, II.5.7 Thm. 5].

(2.13) Suppose that the field F in 2.8 is a finite extension of $\mathbf{Q}_\ell$ with $\ell \neq p$. Since $V = H^1(X_0(N) \otimes_{\mathbf{Q}} \overline{\mathbf{Q}}, j_{0*}\mathcal{A})(1)$ satisfies, by Lemma 2.12,

$$H^i(F, V) = 0 \qquad (i = 0, 1, 2), \tag{2.13.1}$$

the mixed extension E from 2.8 splits over F into $E = V \oplus U$, as in 1.6. We show, following [ScA 2], that the class $[U] \in H^1(F, \mathbf{Q}_p(1))$ can be computed by a formula due to Beilinson [Be].

For any finite set of closed points $S \subset Y_0(N) \otimes_{\mathbf{Q}} F$, the Hochschild-Serre spectral sequence

$$H^a(F, H^b_{S\otimes_F \overline{F}}(X_0(N) \otimes_{\mathbf{Q}} \overline{F}, j_{0*}\mathcal{A}(1))) \Longrightarrow H^{a+b}_S(X_0(N) \otimes_{\mathbf{Q}} F, j_{0*}\mathcal{A}(1)) \tag{2.13.2}$$

degenerates into

$$H^q_S(X_0(N) \otimes_{\mathbf{Q}} F, j_{0*}\mathcal{A}(1)) \xrightarrow{\sim} H^{q-2}(F, H^0(S \otimes_F \overline{F}, \mathcal{A})) \tag{2.13.3}$$

We can view, therefore, the Tate classes a, b as elements of

$$a \in H^2_S(X_0(N) \otimes_{\mathbf{Q}} F, j_{0*}\mathcal{A}(1)), \quad b \in H^2_T(X_0(N) \otimes_{\mathbf{Q}} F, j_{0*}\mathcal{A}(1))$$

The sheaf $\mathcal{A}$ is self-dual and has irreducible monodromy. This implies that the analogue of (2.13.2) without supports degenerates into

$$H^q(X_0(N) \otimes_{\mathbf{Q}} F, j_{0*}\mathcal{A}(1)) \xrightarrow{\sim} H^{q-1}(F, V) = 0$$

The exact sequence

$$\ldots \longrightarrow H^1(X_0(N) \otimes_{\mathbf{Q}} F - T, j_{0*}\mathcal{A}(1)) \longrightarrow H^2_T(X_0(N) \otimes_{\mathbf{Q}} F, j_{0*}\mathcal{A}(1)) \longrightarrow H^2(X_0(N) \otimes_{\mathbf{Q}} F, j_{0*}\mathcal{A}(1)) = 0$$

shows that b can be lifted (uniquely) to

$$\widetilde{b} \in H^1(X_0(N) \otimes_{\mathbf{Q}} F - T, j_{0*}\mathcal{A}(1))$$

(2.14) Proposition. *The extension class* $[U]$ *is equal to the image of* $-a \times \widetilde{b}$ *under*

$$H^2_S(X_0(N) \otimes_{\mathbf{Q}} F, j_{0*}\mathcal{A}(1)) \times H^1(X_0(N) \otimes_{\mathbf{Q}} F - T, j_{0*}\mathcal{A}(1)) \xrightarrow{\cup} H^3_S(Y_0(N) \otimes_{\mathbf{Q}} F - T, \mathcal{A}^{\otimes 2}(2)) \xrightarrow{(\ ,\ )}$$
$$\xrightarrow{(\ ,\ )} H^3_S(Y_0(N) \otimes_{\mathbf{Q}} F - T, \mathbf{Q}_p(2)) \xrightarrow{\sim} H^1(F, H^2_S(Y_0(N) \otimes_{\mathbf{Q}} F - T, \mathbf{Q}_p(2))(2)) \xrightarrow{\mathrm{Tr}} H^1(F, \mathbf{Q}_p(1))$$

Proof. The analogous statement for cohomology with constant coefficients is proved in [ScA 2, Thm. 5.3]. Scholl's proof works in our case almost word by word.

(2.15) In fact, in [ScA 2] one works over the maximal unramified extension F^{ur} of F. In this case one assumes that $a, b \in Z(Y_0(N), F^{\mathrm{ur}})$ are Tate cycles with the following properties (which are automatically satisfied if a, b come from Tate cycles over F by a base change):

(A) $\Phi_T(a) = \Phi_T(b) = 0 \in H^1(F^{\mathrm{ur}}, V)$

(B) The image of b in $H^2(X_0(N) \otimes_{\mathbf{Q}} F^{\mathrm{ur}}, j_{0*}\mathcal{A}(1))$ vanishes.

(A) implies that the mixed extension E associated to a, b in 2.8 splits into $E = V \oplus U$ over F^{ur} and (B) guarantees that b can be lifted to $\widetilde{b} \in H^1(X_0(N) \otimes_{\mathbf{Q}} F^{\mathrm{ur}} - T, j_{0*}\mathcal{A}(1))$. The argument of [ScA 2, Thm. 5.3] then shows that the class $[U] \in H^1(F^{\mathrm{ur}}, \mathbf{Q}_p(1))$ is given by the image of $-a \times \widetilde{b}$ by a sequence of maps as in the statement of Proposition 2.14, but over F^{ur}.

(2.16) Suppose that $\mathcal{X}$ is a proper flat scheme over Λ, the ring of integers in F^{ur}, such that $\mathcal{X} \otimes_\Lambda F^{\mathrm{ur}} \xrightarrow{\sim} X_0(N) \otimes_{\mathbf{Q}} F^{\mathrm{ur}}$. Let $\mathcal{X}_s$ be the special fibre of $\mathcal{X}$ and denote by j' the inclusion $j' : Y_0(N) \otimes_{\mathbf{Q}} F^{\mathrm{ur}} \hookrightarrow \mathcal{X}$.

Let a, b be Tate cycles in $Z(Y_0(N), F^{\mathrm{ur}})$ with supports at two disjoint finite sets of closed points $S, T \subseteq Y_0(N) \otimes_{\mathbf{Q}} F^{\mathrm{ur}}$. Write $\underline{S}$ (resp. $\underline{T}$) for the Zariski closure (with the reduced scheme structure) of S (resp. T) in $\mathcal{X}$. Assume that both $\underline{S}$ and $\underline{T}$ are contained in the regular locus $\mathcal{X}^{\mathrm{reg}}$ of $\mathcal{X}$ and that a (resp. b) extends to a section $\underline{a} \in H^0(\underline{S}, j'_*\mathcal{A})$ (resp. $\underline{b} \in H^0(\underline{T}, j'_*\mathcal{A})$). If $y \in S$, $z \in T$ are two points, we define the local intersection index

$$(\underline{a} \cdot \underline{b})_x := (\underline{y} \cdot \underline{z})_x (\underline{a}_x, \underline{b}_x)$$

Here $(\underline{y} \cdot \underline{z})_x$ denotes the usual local intersection number on $\mathcal{X}^{\mathrm{reg}}$ and $\underline{a}_x$ (resp. $\underline{b}_x$) the value of $\underline{a}$ (resp. $\underline{b}$) at x.

Proposition. *Assume, furthermore, that* $H^2(\mathcal{X}, j'_*\mathcal{A}(1)) = 0$ *and that* $\Phi_T(a) = \Phi_T(b) = 0 \in H^1(F^{\mathrm{ur}}, V)$. *Then the mixed extension* E *of 2.8 splits as a representation of* $G(\overline{F}/F^{\mathrm{ur}})$ *into* $E = V \oplus U$ *and the image of* $[U]$ *under the normalized valuation* $\mathrm{ord} : H^1(F^{\mathrm{ur}}, \mathbf{Q}_p(1)) \xrightarrow{\sim} \mathbf{Q}_p$ *is equal to*

$$\mathrm{ord}([U]) = \sum_{x \in \underline{S} \cap \underline{T}} (\underline{a} \cdot \underline{b})_x$$

Proof. (cf. [ScA 2], [Br]) By [SGA $4\frac{1}{2}$, Cycle, Def. 2.1.2], there are cycle classes

$$cl(\underline{S}) \in H^2_{\underline{S}}(\mathcal{X}, \mathbf{Q}_p(1)), \qquad cl(\underline{T}) \in H^2_{\underline{T}}(\mathcal{X}, \mathbf{Q}_p(1))$$

Define $\alpha := \underline{a} \cup cl(\underline{S}) \in H^2_{\underline{S}}(\mathcal{X}, j'_*\mathcal{A}(1))$. Its restriction to

$$H^2_S(X_0(N) \otimes_{\mathbf{Q}} F^{\mathrm{ur}}, j_*\mathcal{A}(1)) \xrightarrow{\sim} H^0(F^{\mathrm{ur}}, H^0(S \otimes_{F^{\mathrm{ur}}} \overline{F}, \mathcal{A}))$$

(cf. (2.13.3)) is equal to a. Similarly, we get $\beta \in H^2_{\underline{T}}(\mathcal{X}, j'_*\mathcal{A}(1))$ restricting to b. The exact sequence

$$H^1(\mathcal{X} - \underline{T}, j'_*\mathcal{A}(1)) \longrightarrow H^2_{\underline{T}}(\mathcal{X}, j'_*\mathcal{A}(1)) \longrightarrow H^2(\mathcal{X}, j'_*\mathcal{A}(1)) = 0$$

shows that β can be lifted to $\widetilde{\beta} \in H^1(\mathcal{X} - \underline{T}, j'_*\mathcal{A}(1))$. Consider now the following commutative diagram:

$$\begin{array}{ccccc}
H^2_S(X_0(N) \otimes_{\mathbf{Q}} F^{\mathrm{ur}}, j_*\mathcal{A}(1)) \times H^1(X_0(N) \otimes_{\mathbf{Q}} F^{\mathrm{ur}} - T, j_*\mathcal{A}(1)) & \xrightarrow{\cup} & H^3_S(X_0(N) \otimes_{\mathbf{Q}} F^{\mathrm{ur}} - T, \mathbf{Q}_p(2)) & \xrightarrow{\mathrm{Tr}} & H^1(F^{\mathrm{ur}}, \mathbf{Q}_p(1)) \\
\uparrow & & \uparrow & & \\
H^2_{\underline{S}}(\mathcal{X}, j'_*\mathcal{A}(1)) \times H^1(\mathcal{X} - \underline{T}, j'_*\mathcal{A}(1)) & \xrightarrow{\cup} & H^3_{\underline{S}}(\mathcal{X} - \underline{T}, \mathbf{Q}_p(2)) & & \downarrow \delta \\
\downarrow id \otimes \delta & & \downarrow \delta & & \\
H^2_{\underline{S}}(\mathcal{X}, j'_*\mathcal{A}(1)) \times H^2_{\underline{T}}(\mathcal{X}, j'_*\mathcal{A}(1)) & \xrightarrow{\cup} & H^4_{\mathcal{X}_s}(\mathcal{X}, \mathbf{Q}_p(2)) & \xrightarrow{\mathrm{Tr}} & H^2_\bullet(\Lambda, \mathbf{Q}_p(1))
\end{array}$$

in which δ denotes boundary maps in exact cohomology sequences, and the anticommutative diagram ([SGA $4\frac{1}{2}$, Cycle, 2.1.3])

$$\begin{array}{ccc}
H^1(F^{\mathrm{ur}}, \mathbf{Q}_p(1)) & \xrightarrow{\mathrm{ord}} & \mathbf{Q}_p \\
\downarrow \delta & & \| \\
H^2_\bullet(\Lambda, \mathbf{Q}_p(1)) & \xleftarrow{cl(\bullet)} & \mathbf{Q}_p
\end{array} \tag{2.16.1}$$

Here $\bullet$ denotes the special fibre of $\mathrm{Spec}(\Lambda)$. The statement of the Proposition now follows from 2.15, if we trace down $-\alpha \times \widetilde{\beta}$ on the diagram from

$$H^2_{\underline{S}}(\mathcal{X}, j'_*\mathcal{A}(1)) \times H^1(\mathcal{X} - \underline{T}, j'_*\mathcal{A}(1))$$

to $H^2_\bullet(\Lambda, \mathbf{Q}_p(1))$ in two different ways.

Our sign disagrees with [Br, Thm. 4.13] (the local height in *loc. cit.* is defined using $a \times \widetilde{b}$, not $-a \times \widetilde{b}$). It seems that the minus sign coming from (2.16.1) has not been taken into account in [Br].

3. Heegner points and cycles

(3.1) We make a digression about cycles on powers of an elliptic curve and their intersection numbers. Consider an elliptic curve E over an algebraically closed field k and we fix a prime p different from the characteristic of k.

The simplest case is that of divisors on $E \times E$. The cycle class map

$$CH^1(E\times E)\longrightarrow H^2(E\times E,\mathbf{Q}_p)(1)$$

(from now on, we shall drop $\mathbf{Q}_p$ from the notation) takes a cycle X to its cohomology class $[X]$. Intersection of divisors is given by a non-degenerate symmetric pairing

$$(\ ,\ )_1 : H^2(E\times E)(1)\times H^2(E\times E)(1)\xrightarrow{\cup} H^4(E\times E)(2)\xrightarrow{\mathrm{Tr}_{E\times E}}\mathbf{Q}_p$$

The endomorphism ring $\mathrm{End}_k(E)$ is equipped with a canonical involution $\alpha\mapsto\overline{\alpha}$. We define

$$t(\alpha)=\alpha+\overline{\alpha},\qquad n(\alpha)=\alpha\overline{\alpha}=\overline{\alpha}\alpha$$

The degree of the isogeny $\alpha : E\longrightarrow E$ is equal to $n(\alpha)$.
A pair of endomorphisms $\alpha,\beta\in\mathrm{End}_k(E)$ defines a correspondence

$$(\alpha,\beta) : E\longrightarrow E\times E$$

Let $\Gamma_{\alpha,\beta}\in CH^1(E\times E)$ be (the class of) the graph of the correspondence (α,β), i.e. the image of E under

$$(\alpha,\beta)_* : CH^0(E)\longrightarrow CH^1(E\times E)$$

Its cohomology class is $[\Gamma_{\alpha,\beta}]=(\alpha,\beta)_*(1)$ for

$$(\alpha,\beta)_* : H^0(E)\longrightarrow H^2(E\times E)(1)$$

We have $\Gamma_{1,\beta}=\Gamma_\beta$, $\Gamma_{\alpha,1}=(\Gamma_\alpha)^t$, where Γ_α denotes the graph of α and ^{t} the transposition. In particular, $\Gamma_{0,1}=\{0\}\times E$, $\Gamma_{1,0}=E\times\{0\}$.

(3.2) Under the Künneth decomposition of $H^2(E\times E)(1)$, its two-dimensional subspace

$$(H^2(E)(1)\otimes H^0(E))\oplus(H^0(E)\otimes H^2(E)(1))=\mathbf{Q}_p\cdot[\Gamma_{0,1}]\oplus\mathbf{Q}_p\cdot[\Gamma_{1,0}]$$

is orthogonal to $H^1(E)\otimes H^1(E)(1)$ under the intersection product $(\ ,\ )_1$.

Definition. *For $\alpha,\beta\in\mathrm{End}_k(E)$, let $X_{\alpha,\beta}$ be the orthogonal projection of $[\Gamma_{\alpha,\beta}]$ on $H^1(E)\otimes H^1(E)(1)$. Write X_α for $X_{1,\alpha}$.*

As $([\Gamma_{\alpha,\beta}],[\Gamma_{0,1}])_1=n(\alpha)$, $([\Gamma_{\alpha,\beta}],[\Gamma_{1,0}])_1=n(\beta)$, we have

$$X_{\alpha,\beta}=[\Gamma_{\alpha,\beta}]-n(\alpha)[\Gamma_{1,0}]-n(\beta)[\Gamma_{0,1}]$$

(3.3) Lemma. (1) *$X_{\alpha,\beta}$ is $\mathbf{Z}$-bilinear in α,β (hence it can be extended $\mathbf{Q}$-linearly to $\mathrm{End}_k(E)\otimes\mathbf{Q}$).*
(2) $(X_{\alpha,\beta})^t=X_{\beta,\alpha};\quad X_{\alpha\gamma,\beta\gamma}=n(\gamma)X_{\alpha,\beta};\quad X_{\alpha,\beta}=X_{\beta\overline{\alpha}};\quad (X_\alpha)^t=X_{\overline{\alpha}}$
(3) $(X_\alpha,X_\beta)_1=-t(\alpha\overline{\beta})$
(4) *If $\lambda : E\longrightarrow E'$ is an isogeny, then*

$$(\lambda\times\lambda)_*(X_{\alpha,\beta})=\deg(\lambda)X_{\lambda\alpha\lambda^{-1},\lambda\beta\lambda^{-1}}$$
$$(\lambda\times\lambda)^*(X_{\alpha,\beta})=\deg(\lambda)X_{\lambda^{-1}\alpha\lambda,\lambda^{-1}\beta\lambda}$$

(5) *The orthogonal projection of $[\Gamma_{\alpha,\beta}]$ on $\left(\mathrm{Sym}^2(H^1(E))\right)(1)$ is equal to*

$$\frac{X_{\alpha,\beta}-X_{\beta,\alpha}}{2}=X_{(\beta\overline{\alpha}-\alpha\overline{\beta})/2}$$

Proof. (1) For $x\in H^1(E)$ and $y\in H^1(E)(1)$, we have

$$([\Gamma_{\alpha,\beta}], p_1^*x \otimes p_2^*y)_1 = \mathrm{Tr}_{E\times E}((\alpha,\beta)_*(1) \cup (p_1^*x \otimes p_2^*y)) = \mathrm{Tr}_{E\times E}((\alpha,\beta)_*((\alpha,\beta)^*(p_1^*x \otimes p_2^*y))) =$$
$$= \mathrm{Tr}_E((\alpha,\beta)^*(p_1^*x \otimes p_2^*y)) = \mathrm{Tr}_E(\alpha^*x \otimes \beta^*y),$$

which is $\mathbf{Z}$-bilinear in α, β. The statement follows from the fact that $(\ ,\ )_1$ is non-degenerate on $H^1(E) \otimes H^1(E)(1)$.
(2) The first equality follows from $(\Gamma_{\alpha,\beta})^t = \Gamma_{\beta,\alpha}$; the second from the fact that $\gamma_* : H^0(E) \longrightarrow H^0(E)$ is the multiplication by $n(\gamma)$. As $n(\alpha) = \alpha\overline{\alpha} \in \mathbf{Z}$, we get $n(\alpha)X_{\alpha,\beta} = X_{\alpha\overline{\alpha},\beta\overline{\alpha}} = n(\alpha)X_{1,\beta\overline{\alpha}} = n(\alpha)X_{\beta\overline{\alpha}}$. Finally, $(X_\alpha)^t = X_{\alpha,1} = X_{\overline{\alpha}}$.
(3) As $([\Gamma_\alpha],[\Gamma_\beta])_1 = n(\alpha-\beta)$, it follows that $(X_\alpha, X_\beta)_1 = n(\alpha-\beta) - n(\alpha) - n(\beta) = -\alpha\overline{\beta} - \beta\overline{\alpha} = -t(\alpha\overline{\beta})$.
(4) We have $(\lambda\times\lambda)\circ(\alpha,\beta) = (\lambda\alpha\lambda^{-1}, \lambda\beta\lambda^{-1})\circ\lambda$, hence

$$(\lambda\times\lambda)_*[\Gamma_{\alpha,\beta}] = (\lambda\alpha\lambda^{-1}, \lambda\beta\lambda^{-1})_*\mathrm{deg}(\lambda) = \mathrm{deg}(\lambda)[\Gamma_{(\lambda\alpha\lambda^{-1},\lambda\beta\lambda^{-1})}]$$

As $(\widehat{\lambda})^* = \lambda_*$ on $H^1(E)$, the second equality follows from the first one applied to the dual isogeny.
(5) Follows from (2).

(3.4) Consider now $E^{2k} = \underbrace{E\times\cdots\times E}_{2k \text{ times}}$ for arbitrary $k \geq 1$. We have the cycle class map

$$CH^k(E^{2k}) \longrightarrow H^{2k}(E^{2k})(k)$$

and the intersection pairing

$$(\ ,\ )_k : H^{2k}(E^{2k})(k) \times H^{2k}(E^{2k})(k) \xrightarrow{\cup} H^{4k}(E^{2k})(2k) \xrightarrow{\mathrm{Tr}_{E^{2k}}} \mathbf{Q}_p$$

Künneth formula and antisymmetrization give a projection (onto a direct summand of $H^{2k}(E^{2k})(k)$)

$$\varepsilon_k : H^{2k}(E^{2k})(k) \longrightarrow H^1(E)^{\otimes 2k}(k) \longrightarrow \left(\mathrm{Sym}^{2k}(H^1(E))\right)(k)$$

Note that ε_k is given by Scholl's projector ε (for $w = 2k$). In this notation, Lemma 3.3.5 says that

$$\varepsilon_1([\Gamma_{\alpha,\beta}]) = X_{(\beta\overline{\alpha}-\alpha\overline{\beta})/2} \tag{3.4.1}$$

Proposition. *For $u, v \in \left(\mathrm{Sym}^2(H^1(E))\right)(1)$ (viewed as a subspace of $H^1(E)^{\otimes 2}(1)$), we have*

$$\left(\varepsilon_k(u^{\otimes k}), \varepsilon_k(v^{\otimes k})\right)_k = \frac{2^k}{\binom{2k}{k}} \left((u,u)_1(v,v)_1\right)^{k/2} P_k\left(\frac{(u,v)_1}{\left((u,u)_1(v,v)_1\right)^{1/2}}\right),$$

where P_k is the Legendre polynomial of degree k (see I.1.9). As $P_k(-t) = (-1)^k P_k(t)$, the above expression does not, in fact, contain any square roots.

Proof. This is a question from representation theory of GL_2; we omit the calculation but refer the reader to [Br, p.24] and [Ha, pp. 552, 556].

(3.5) Let us recall basic facts about Heegner points on the curve $Y_0(N)$ (see [Gr], [Gr - Za] for more details). For a field F of characteristic zero, any F-valued point on $Y_0(N)$ can be represented by an isogeny (defined over F) $\lambda : E \longrightarrow E'$ between two elliptic curves with kernel $\mathrm{Ker}(\lambda) \simeq \mathbf{Z}/N\mathbf{Z}$. A CM-point (resp. a Heegner point) on $Y_0(N)$ is represented by an isogeny between elliptic curves with complex multiplication (resp. by an isogeny between elliptic curves with $\mathrm{End}(E) = \mathrm{End}(E') \neq \mathbf{Z}$).

We shall be interested in Heegner points of a special kind, when $\mathrm{End}(E) = \mathrm{End}(E') = \mathcal{O}_K$ is the ring of integers in an imaginary quadratic field $K = \mathbf{Q}(\sqrt{D})$ with discriminant $D < 0$ prime to N. Such points exist iff each prime number dividing N splits in K. Over the complex numbers, they are classified as follows ([Gr], [Gr - Za, II.1]).

Choose an ideal $\mathcal{N} \subset \mathcal{O}_K$ with $\mathcal{O}_K/\mathcal{N} \xrightarrow{\sim} \mathbf{Z}/N\mathbf{Z}$. For each ideal $\mathcal{A} \subseteq \mathcal{O}_K$ the isogeny $\mathbf{C}/\mathcal{A} \longrightarrow \mathbf{C}/\mathcal{A}\mathcal{N}^{-1}$ represents a Heegner point on $Y_0(N)(\mathbf{C})$ that depends only on the ideal class $[\mathcal{A}]$ of $\mathcal{A}$. This defines a bijection between Heegner points (in our restricted sense) and pairs $([\mathcal{A}], \mathcal{N})$. All Heegner points are defined over H, the Hilbert class field of K. The Galois group $G(H/K) \xrightarrow{\sim} \mathrm{Pic}(\mathcal{O}_K)$ acts on them by

$$\sigma_{\mathfrak{p}} : ([\mathcal{A}], \mathcal{N}) \mapsto ([\mathcal{A}\mathfrak{p}^{-1}], \mathcal{N})$$

($\mathfrak{p}$ is a prime ideal of $\mathcal{O}_K$, $\sigma_{\mathfrak{p}}$ the Artin symbol $(H/K, \mathfrak{p})$). The complex conjugation acts by

$$\tau : ([\mathcal{A}], \mathcal{N}) \mapsto ([\mathcal{A}]^{-1}, \overline{\mathcal{N}})$$

Atkin-Lehner involutions on $X_0(N)$ form a group $(\mathbf{Z}/2\mathbf{Z})^s$, where s is the number of primes dividing N. The action of this group permutes possible choices of $\mathcal{N}$ and $(\mathbf{Z}/2\mathbf{Z})^s \times \mathrm{Pic}(\mathcal{O}_K)$ acts simply transitively on the set of Heegner points.

(3.6) Fix one of the Heegner points y (viewed as a closed point of $Y_0(N) \otimes_{\mathbf{Q}} H$) and choose a point $\widetilde{y}$ on $Y(N) \otimes_{\mathbf{Q}} H$ over y. The fibre $E_{\widetilde{y}}$ is an elliptic curve with $\mathrm{End}(E_{\widetilde{y}}) = \mathcal{O}_K$. Let $\Gamma_{\sqrt{D}} \subset E_{\widetilde{y}} \times E_{\widetilde{y}}$ be the graph of $\sqrt{D} \in \mathcal{O}_K$ (we fix one of the square roots) and define

$$Y = \underbrace{\Gamma_{\sqrt{D}} \times \cdots \times \Gamma_{\sqrt{D}}}_{r-1 \text{ times}} \subset \underbrace{E_{\widetilde{y}} \times \cdots \times E_{\widetilde{y}}}_{2r-2 \text{ times}} = (W \otimes_{\mathbf{Q}} H)_{\widetilde{y}}$$

The Heegner cycle $\varepsilon_B \varepsilon Y$ (a cycle with rational coefficients) represents an element of $\varepsilon_B \varepsilon (CH^r(W \otimes_{\mathbf{Q}} H)_0 \otimes \mathbf{Q})$ and is a higher weight analogue of a Heegner point. Write $x = a(Y) \in Z(Y_0(N), H)$ for the corresponding Tate cycle. It follows from (3.4.1) that εY and $a(Y)$ remain unchanged if we replace $\Gamma_{\sqrt{D}}$ by $\Gamma_{\sqrt{D}} - \Gamma_{1,0} - |D|\Gamma_{0,1}$.

Let $g : E_{\widetilde{y}} \longrightarrow E'$ be an isogeny (defined over some extension of H). It induces a canonical isomorphism (i.e. not depending on g) between $\mathrm{End}(E') \otimes \mathbf{Q}$ and $\mathrm{End}(E_{\widetilde{y}}) \otimes \mathbf{Q} = K$, hence $\sqrt{D} \in \mathrm{End}(E') \otimes \mathbf{Q}$ is well-defined. Lemma 3.3.4 then gives

$$\begin{aligned} g_* \left(X_{\sqrt{D}}^{\otimes(r-1)} \right) &= (\deg(g))^{r-1} X_{\sqrt{D}}^{\otimes(r-1)} \\ g^* \left(X_{\sqrt{D}}^{\otimes(r-1)} \right) &= (\deg(g))^{r-1} X_{\sqrt{D}}^{\otimes(r-1)} \end{aligned} \tag{3.6.1}$$

(3.7) Up to now, we have been assuming that $N \geq 3$. Let us sketch an alternative definition of the sheaf $\mathcal{A}$ and the Heegner cycle $x \in Z(Y_0(N), H)$ which is valid for any integer $N \geq 1$. Let M be an arbitrary integer such that $NM \geq 3$. The projection $\pi_M : Y(NM) \longrightarrow Y_0(N)$ is the quotient map by the group

$$B_M = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbf{Z}/NM\mathbf{Z})/\{\pm 1\} \,\middle|\, c \equiv 0 \,(\mathrm{mod}\, N\,) \right\}$$

The sheaf $\mathcal{B}_M := \varepsilon R^{2r-2} f_* \mathbf{Q}_p(r-1)$ on $Y(NM)$ is equipped with an action of B_M, so it makes sense to define a sheaf on $Y_0(N)$

$$\mathcal{A}_M := (\pi_{M*} \mathcal{B}_M)^{B_M}$$

Lemma. *The sheaf $\mathcal{A} = \mathcal{A}_M$ does not depend on M.*

Proof. For $M \mid M'$ we have $\pi_{M'} = \pi_M \circ \pi_{M'/M}$, where $\pi_{M'/M} : Y(NM') \longrightarrow Y(NM)$ is a Galois covering with Galois group $B_{M'/M} := \mathrm{Ker}(B_{M'} \longrightarrow B_M)$. As $\pi_{M'/M}^* \mathcal{B}_M = \mathcal{B}_{M'}$ (proper base change), we have

$$((\pi_{M'/M})_* \mathcal{B}_{M'})^{B_{M'/M}} = \mathcal{B}_M, \tag{3.7.1}$$

hence

$$\mathcal{A}_{M'} = (\pi_{M'*}\mathcal{B}_{M'})^{B_{M'}} = \left(\pi_{M*}((\pi_{M'/M})_*\mathcal{B}_{M'})^{B_{M'/M}}\right)^{B_M} = (\pi_{M*}\mathcal{B}_M)^{B_M} = \mathcal{A}_M$$

(3.8) In order to define a Heegner cycle $x \in Z(Y_0(N), H)$, we consider only integers M such that each prime dividing NM splits in K. As in 3.6, we fix a Heegner point y on $Y_0(N) \otimes_{\mathbf{Q}} H$ and choose a point $\widetilde{y}$ on $Y(NM) \otimes_{\mathbf{Q}} H$ over y. In the fibre $E^{2r-2}_{\widetilde{y}}$ we take the cycle $Y_M = \left(\Gamma_{\sqrt{D}}\right)^{r-1}$. The cohomology class of $\varepsilon_{B_M}\varepsilon Y_M$ in the fibre defines a Tate cycle $a(Y_M) \in Z(Y_0(N), H)$. Lemma 2.5 holds true in this case and $a(Y_M)$ does not depend on the choice of M, thank to 3.7.1. We put $x = a(Y_M)$.

4. Local heights outside of p

(4.1) We are now ready to begin the computation of the p-adic height of a Heegner cycle. Recall our setup: $N \geq 1$ is an integer, $K = \mathbf{Q}(\sqrt{D})$ an imaginary quadratic field of discriminant $D < 0$ in which all primes dividing N split, H is the Hilbert class field of K, $\mathcal{O}_K$ the ring of integers in K, p a prime number, $\mathcal{A}$ the smooth p-adic sheaf on $Y_0(N)$ defined in 2.3 (resp. 3.7), $V = H^1(X_0(N) \otimes_{\mathbf{Q}} \overline{\mathbf{Q}}, j_*\mathcal{A})(1)$ the corresponding parabolic cohomology group. From now on, we assume that $p \not| N$. This implies that $X(N), X_0(N)$ and W all have good reduction at p, hence the discussion of 1.4-5 applies over any number field (V satisfies the conditions of 1.4 at places $v \not| p$ thank to Lemma 2.12). In particular, the Abel-Jacobi map over H factors through

$$\Phi : CH^r(W \otimes_{\mathbf{Q}} H)_0 \longrightarrow H^1_f(H, V)$$

The pairing 2.9.2 induces a height pairing

$$\langle\ ,\ \rangle : H^1_f(H, V) \times H^1_f(H, V) \longrightarrow \mathbf{Q}_p, \tag{4.1.1}$$

once we specify the data (A), (B) in 1.5. To do that, we fix a continuous homomorphism $\ell_K : \mathbf{A}^*_K/K^* \longrightarrow \mathbf{Q}_p$ and put $\ell_H = \ell_K \circ N_{H/K} : \mathbf{A}^*_H/H^* \longrightarrow \mathbf{Q}_p$. We also fix splittings (to be specified later in 5.10)

$$F^r \varepsilon_B \varepsilon H^{2r-1}_{\mathrm{dR}}(W \otimes_{\mathbf{Q}} \mathbf{Q}_p/\mathbf{Q}_p) \longleftrightarrow \varepsilon_B \varepsilon H^{2r-1}_{\mathrm{dR}}(W \otimes_{\mathbf{Q}} \mathbf{Q}_p/\mathbf{Q}_p)$$

and define splittings over $H \otimes_{\mathbf{Q}} \mathbf{Q}_p$ by a base change.

(4.2) In the notation of 2.11, let $\beta : \mathbf{T} \longrightarrow \overline{\mathbf{Q}}$ be a $\mathbf{Q}$-algebra homomorphism and $f_\beta \in S^{\mathrm{new}}_{2r}(\Gamma_0(N(\beta)); F(\beta))$ the corresponding normalized primitive form. We also have the β-component

$$H^1_f(H, V)_\beta = H^1_f(H, V_\beta),$$

which is a vector space over $\widehat{F(\beta)}$ (of finite dimension). The height pairing 4.1.1 defines an $\widehat{F(\beta)}$-linear pairing

$$\langle\ ,\ \rangle : H^1_f(H, V_\beta) \times H^1_f(H, V_\beta) \longrightarrow \widehat{F(\beta)}$$

In 3.6 (resp. 3.8) we defined a Heegner cycle $x \in Z(Y_0(N), H)$. Let $z = \Phi_T(x)$ be its Abel-Jacobi image. Of course, we have

$$z = \Phi(\varepsilon_B \varepsilon Y) \qquad (\text{resp. } \Phi(\varepsilon_{B_M}\varepsilon Y_M)) \in H^1_f(H, V)$$

by Lemma 2.5. Let $z_\beta \in H^1_f(H, V_\beta)$ be the β-component of z. Let $\mathcal{C}$ be a character $\mathcal{C} : G(H/K) \longrightarrow \overline{\mathbf{Q}}^*$ and L the extension of $F(\beta)$ generated by the values of $\mathcal{C}$. Then the $\mathcal{C}$-component of z_β

$$z_{\beta,\mathcal{C}} := \frac{1}{[H:K]} \sum_{\sigma \in G(H/K)} \mathcal{C}^{-1}(\sigma) z^\sigma_\beta$$

lies in $H^1_f(H, V_\beta) \otimes_{\widehat{F(\beta)}} \widehat{L}$. Our aim is to compute $\langle z_{\beta,\mathcal{C}}, z_{\beta,\overline{\mathcal{C}}}\rangle \in \widehat{L}$ (we extend $\langle\ ,\ \rangle$ $\widehat{L}$-linearly).

(4.3) Proposition. *Suppose that f_β is a newform. If ℓ_K is anticyclotomic, i.e. $\ell_K \circ \tau = -\ell_K$ for the non-trivial element $\tau \in G(K/\mathbf{Q})$, then*

$$\langle z_{\beta,\mathcal{C}}, z_{\beta,\overline{\mathcal{C}}} \rangle = 0$$

Proof. Let $\widetilde{\tau} \in G(H/\mathbf{Q})$ be a lift of τ. By [Ne 1, 6.2],

$$\widetilde{\tau}(z_\beta) = (-1)^{r-1} \lambda_N(f_\beta) z_\beta^\sigma$$

for some $\sigma \in G(H/K)$, hence

$$\widetilde{\tau}(z_{\beta,\mathcal{C}}) = (-1)^{r-1} \lambda_N(f_\beta) \mathcal{C}(\sigma)(z_{\beta,\mathcal{C}})$$

Since all splittings in 4.1 are invariant under $\widetilde{\tau}$ by construction, we have

$$\langle z_{\beta,\mathcal{C}}, z_{\beta,\overline{\mathcal{C}}} \rangle_{\ell_K \circ N_{H/K}} = \langle \widetilde{\tau}(z_{\beta,\mathcal{C}}), \widetilde{\tau}(z_{\beta,\overline{\mathcal{C}}}) \rangle_{\ell_K \circ \tau \circ N_{H/K}} =$$
$$= \lambda_N(f_\beta)^2 \langle z_{\beta,\mathcal{C}}, z_{\beta,\overline{\mathcal{C}}} \rangle_{-\ell_K \circ N_{H/K}} = -\langle z_{\beta,\mathcal{C}}, z_{\beta,\overline{\mathcal{C}}} \rangle_{\ell_K \circ N_{H/K}}$$

as $\lambda_N(f_\beta) = \pm 1$.

(4.4) Every ℓ_K can be written uniquely as a sum $\ell_K = \ell_K^+ + \ell_K^-$ with $\ell_K^\pm \circ \tau = \pm \ell_K^\pm$. In view of Prop. 4.3, the height $\langle z_{\beta,\mathcal{C}}, z_{\beta,\overline{\mathcal{C}}} \rangle_{\ell_K}$ depends only on ℓ_K^+ (recall that heights depend linearly on ℓ_K).
We can assume, therefore, that ℓ_K is cyclotomic, $\ell_K = \ell_K \circ \tau$. This determines ℓ_K up to a multiplicative constant We choose $\ell_K = \ell_{\mathbf{Q}} \circ \mathbf{N}$, where $\ell_{\mathbf{Q}} : \mathbf{A}_{\mathbf{Q}}^*/\mathbf{Q}^* \longrightarrow \mathbf{Q}_p$ is the map

$$\mathbf{A}_{\mathbf{Q}}^*/\mathbf{Q}^* \xrightarrow{\text{Artin}} G(\mathbf{Q}^{\text{ab}}/\mathbf{Q}) \longrightarrow G(\mathbf{Q}(\mu_{p^\infty})/\mathbf{Q}) \xrightarrow{\chi_{\text{cycl}}} \mathbf{Z}_p^* \xrightarrow{\log_p} \mathbf{Z}_p \hookrightarrow \mathbf{Q}_p$$

Here χ_{cycl} is the cyclotomic character defined by

$$\zeta^{\chi_{\text{cycl}}(\sigma)} = \sigma(\zeta), \qquad \zeta \in \mu_{p^\infty}$$

Another description of $\ell_{\mathbf{Q}}$ is as a composition

$$\mathbf{A}_{\mathbf{Q}}^*/\mathbf{Q}^* \simeq \mathbf{R}_+^* \times \prod_\ell \mathbf{Z}_\ell^* \longrightarrow \mathbf{Z}_p^* \xrightarrow{-\log_p} \mathbf{Z}_p \hookrightarrow \mathbf{Q}_p$$

For a prime number $\ell \neq p$, the ℓ-component $\ell_{\mathbf{Q},\ell} : \mathbf{Q}_\ell^* \longrightarrow \mathbf{Q}_p$ is trivial on $\mathbf{Z}_\ell^*$ and takes value $\log_p(\ell)$ on ℓ.

(4.5) The Heegner cycle x defined in 3.6-7 is supported at one of the Heegner points y. Fix an ideal $\mathcal{A} \subset \mathcal{O}_K$ and let σ be the corresponding element of the Galois group $G(H/K)$. According to [Gr - Za, Prop. III.4.3], Tate cycles x and $T_m x^\sigma$ have disjoint supports iff

$$r_{\mathcal{A}}(m) = \{I \subset \mathcal{O}_K \mid [I] = [\mathcal{A}],\ \mathbf{N}(I) = m\} = 0 \tag{4.5.1}$$

If 4.5.1 is satisfied, we can apply the construction of 2.8 to x and $T_m x^\sigma$, obtaining a mixed extension E of $\Phi_T(x)$ and $(\Phi_T(T_m x^\sigma))^*(1)$.
Fix a (non-archimedean) place v of H of residue characteristic $\ell \neq p$. As a representation of $G(\overline{H_v}/H_v)$, E splits into $E = V \oplus U_v$ with $[U_v] \in H^1(H_v, \mathbf{Q}_p(1))$. We shall compute the value of

$$(x, T_m x^\sigma)_v := \text{ord}_v([U_v]) \in \mathbf{Q}_p \tag{4.5.2}$$

using the intersection formula of Proposition 2.16. With our choice of ℓ_K, the local height at v is equal to

$$\langle x, T_m x^\sigma \rangle_v = -(x, T_m x^\sigma)_v \log_p(Nv)$$

The calculation will follow quite closely the article of Gross and Zagier [Gr - Za] and the reader is invited to open his or her copy of it.

(4.6) Let $\underline{X}_0(N)$ (resp. $\underline{Y}_0(N)$) be the model of $X_0(N)$ (resp. $Y_0(N)$) over $\mathbf{Z}$ defined in [Ka - Ma]. Over $\mathbf{Z}[1/N]$, $\underline{X}_0(N)\otimes\mathbf{Z}[1/N]$ is a proper and smooth curve. If ℓ is a prime dividing N, $N = \ell^n M$ with $(\ell, M) = 1$, then $\underline{X}_0(N)\otimes\mathbf{Z}/\ell\mathbf{Z}$ consists of $(n+1)$ irreducible components $\mathcal{F}_{a,b}$ $(a, b \geq 0, a + b = n)$. The reduced curve $(\mathcal{F}_{a,b})_{\mathrm{red}}$ is isomorphic to $\underline{X}_0(M)\otimes\mathbf{Z}/\ell\mathbf{Z}$. All curves $\mathcal{F}_{a,b}$ meet at supersingular points of $\underline{X}_0(N)\otimes\mathbf{Z}/\ell\mathbf{Z}$ (i.e. points represented by elliptic curves with supersingular reduction modulo ℓ).

Lemma. *For $\ell \mid N$, $(\underline{X}_0(N)^{\mathrm{reg}})\otimes\mathbf{Z}/\ell\mathbf{Z}$ contains all ordinary (i.e. not supersingular) points of $\mathcal{F}_{0,n}\cup\mathcal{F}_{n,0}$.*

Proof. [Li, Prop. 2.7] (note that [Gr - Za, Prop. III.1.4] is incorrect)

(4.7) Recall that v is a place of of H with residue characteristic $\ell \neq p$ and y a Heegner point on $Y_0(N)$. Let $\underline{y}$ be the Zariski closure of y in $\underline{X}_0(N)\otimes\mathcal{O}_{H_v}$. If $\ell \nmid N$, then $\underline{X}_0(N)\otimes\mathcal{O}_{H_v}$ is smooth over $\mathcal{O}_{H_v}$.

Lemma. *If $\ell \mid N$, then both $\underline{y}$ and $\underline{y}^\sigma$ reduce to ordinary non-cuspidal points on the component*

$$\begin{cases} \mathcal{F}_{0,n} & \text{if } v \mid \overline{\mathcal{N}} \\ \mathcal{F}_{n,0} & \text{if } v \mid \mathcal{N} \end{cases}$$

Proof. [Gr - Za, Prop. 3.1]

Corollary. *Both $\underline{y}$ and $\underline{y}^\sigma$ are contained in $(\underline{X}_0(N)\otimes\mathcal{O}_{H_v})^{\mathrm{reg}}$.*

(4.8) We want to compute $(x, T_m x^\sigma)_v$ using Proposition 2.16. In the notation of 2.16, we take $F = H_v$, $\mathcal{X} = \underline{X}_0(N)\otimes\Lambda \supset \mathcal{Y} = \underline{Y}_0(N)\otimes\Lambda$, where Λ is the ring of integers in F^{ur}.

The point $y\otimes_H F^{\mathrm{ur}}$ extends to a section $\underline{y}$ of $\mathcal{Y}\longrightarrow\mathrm{Spec}(\Lambda)$. Our aim is to extend the Heegner cycle $x \in H^0(y, \mathcal{A})$ to $\underline{x} \in H^0(\underline{y}, j'_*\mathcal{A})$. We imitate the construction of 3.7 on integral models: choose an integer M prime to ℓ such that NM is divisible by at least two odd primes. The moduli problem for elliptic curves with a Drinfeld $[\Gamma(NM)]$-structure is then representable by a regular scheme $\underline{Y}(NM)$, which is a model of $Y(NM)$ over $\mathbf{Z}$. There is a universal elliptic curve $\underline{f} : \underline{E} \longrightarrow \underline{Y}(NM)$ extending $f : E \longrightarrow Y(NM)$. The p-adic sheaf $\underline{\mathcal{B}}_M = \mathrm{Sym}^{2r-2}R^1\underline{f}_*\mathbf{Q}_p(r-1)$ is smooth on $\underline{Y}(NM)$ and satisfies $k_M^*\underline{\mathcal{B}}_M = \mathcal{B}_M$, $k_{M*}\mathcal{B}_M = \underline{\mathcal{B}}_M$ for $k_M : Y(NM) \hookrightarrow \underline{Y}(NM)$, by proper and smooth base change theorems. The finite group B_M acts on $\underline{Y}(NM)$ and $\underline{\mathcal{B}}_M$. The quotient $B_M\backslash\underline{Y}(NM)$ is equal to $\underline{Y}_0(N)$ ([Ka - Ma, Thm. 7.4.2 & Lemma 8.1.5]). Write $\underline{\pi}_M$ for the projection $\underline{Y}(NM)\longrightarrow\underline{Y}_0(N)$. The sheaf $\underline{\mathcal{A}} := (\underline{\pi}_{M*}\underline{\mathcal{B}}_M)^{B_M}$ on $\underline{Y}_0(N)$ extends $\mathcal{A} = (\pi_M\mathcal{B}_M)^{B_M}$ on $Y_0(N)$ and is independent on the choice of M, as in Lemma 3.7.

We now assume that all primes dividing M split in K. As in 3.7, we choose a point $\widetilde{y}$ on $Y(NM)\otimes_{\mathbf{Q}} H$ over y. It extends to a section $\underline{\widetilde{y}}$ of $\underline{Y}(NM)\otimes\Lambda\longrightarrow\Lambda$. Since

$$H^0(\underline{\widetilde{y}}, \underline{\mathcal{B}}_M) = H^0(\widetilde{y}, \mathcal{B}_M),$$

the Tate cycle $b(Y_M) \in H^0(\widetilde{y}, \mathcal{B}_M)$ from 3.7 (given by the cohomology class of εY_M in the fibre) extends uniquely to a section $\underline{b}(Y_M)$ of $\underline{\mathcal{B}}_M$ over $\underline{\widetilde{y}}$. The canonical map $\underline{\mathcal{A}} \longrightarrow j'_*\mathcal{A}$ takes $\varepsilon_{B_M}\underline{b}(Y_M) \in H^0(\underline{y}, (\underline{\pi}_{M*}\underline{\mathcal{B}}_M)^{B_M})$ into the sought-for section $\underline{x} \in H^0(\underline{y}, j'_*\mathcal{A})$ extending x.

It is easy to see that $\underline{x}$ does not depend on the choice of M by looking at its special fibre $\underline{x}_s \in H^0(\underline{y}_s, j'_*\mathcal{A})$. According to [Se - Ta], $\underline{y}$ can be represented by an isogeny $\lambda : E \longrightarrow E'$ between elliptic curves defined over Λ. The reduction map defines an inclusion

$$\mathcal{O}_K = \mathrm{End}_\Lambda(\underline{y}) \hookrightarrow \mathrm{End}_k(\underline{y}_s) \tag{4.8.1}$$

Here $k \simeq \overline{\mathbf{F}}_\ell$ is the residue field of Λ. This defines an element $\sqrt{D} \in \mathrm{End}(\underline{y}_s)$ and we have

$$\underline{x}_s = \frac{1}{\sharp\mathrm{Aut}(\underline{y}_s)} \sum_{\alpha\in\mathrm{Aut}(\underline{y}_s)} \alpha_*\varepsilon_{r-1}\left(X_{\sqrt{D}}^{\otimes(r-1)}\right) \in H^{2r-2}(E_s^{2r-2}, \mathbf{Q}_p)(r-1) \tag{4.8.2}$$

The same construction also applies to x^σ, yielding $\underline{x}^\sigma \in H^0(\underline{y}^\sigma, j'_*\mathcal{A})$.

(4.9) By 4.8.1, we have an inclusion $\mathcal{O}_K^* = \mathrm{Aut}(\underline{y}) \hookrightarrow \mathrm{Aut}(\underline{y}_s)$. We say that we are in the *special case* if this inclusion is strict.

Lemma. (1) *The special case occurs iff* $N = 1$, $\ell \in \{2, 3\}$, $\left(\frac{D}{\ell}\right) \neq 1$.
(2) *If we are not in the special case, then* $\mathrm{Aut}(\underline{y}_s)$ *acts trivially in (4.8.2) and* $\underline{x}_s = \varepsilon_{r-1}\left(X_{\sqrt{D}}^{\otimes(r-1)}\right)$.

Proof. (1) Suppose that we are in the special case. The well-known classification of automorphisms of elliptic curves implies that the special fibre E_s must be a supersingular elliptic curve in characteristic $\ell = 2$ or $\ell = 3$ (hence $\ell \not| N$). We have to show that $N = 1$. The level structure is given by a subgroup scheme $\mathbf{Z}/N\mathbf{Z} \hookrightarrow E_s$, as k is algebraically closed and $\ell \not| N$. The action of $A := \mathrm{Aut}(\underline{y}_s)$ on this subgroup is given by a homomorphism $\varphi : A \longrightarrow (\mathbf{Z}/N\mathbf{Z})^*$. If α lies in the commutant $[A, A]$, then $\varphi(\alpha) = 1$, hence $\deg(\alpha - 1)$ must be divisible by N. A short inspection of the lattice of subgroups between $\mathcal{O}_K^*$ and $\mathrm{Aut}(E_s)$ shows that there is always $\alpha \neq 1$ in the commutant $[A, A]$ satisfying $\alpha^\ell = 1$, hence $N \mid \deg(\alpha - 1) = 4$ (resp. 3) if $\ell = 2$ (resp. if $\ell = 3$). As $\ell \not| N$, this implies that $N = 1$. Conversely, if $N = 1$, $\ell \in \{2, 3\}$ and $\left(\frac{D}{\ell}\right) \neq 1$, then $\mathrm{Aut}(\underline{y}_s) = \mathrm{Aut}(E_s)$ is of order 12 or 24.
(2) This follows from 3.6.1.

(4.10) We now examine $T_m x^\sigma \in H^0(T_m y^\sigma, \mathcal{A})$, first in the case when $\ell \not| m$. The section $\underline{y}^\sigma$ can be represented by an elliptic curve $\underline{E}$ over Λ with a Drinfeld $[\Gamma_0(N)]$-structure. Let E be the generic fibre of $\underline{E}$. Any isogeny $E \longrightarrow E'$ of degree m is automatically defined over H_v^{ur} and lifts to an étale isogeny $g : \underline{E} \longrightarrow \underline{E}'$ over Λ. The curve $\underline{E}'$ with the induced $[\Gamma_0(N)]$-structure represents a section $\underline{y}_g^\sigma$ of $\mathcal{X} \longrightarrow \mathrm{Spec}(\Lambda)$. The divisor $T_m \underline{y}^\sigma$ (Zariski closure of $T_m y^\sigma$) decomposes into

$$T_m \underline{y}^\sigma = \sum_{\deg(g)=m} \underline{y}_g^\sigma$$

According to 3.6.1, the Tate cycle $T_m x^\sigma$ is equal to

$$T_m x^\sigma = m^{r-1} \sum_{\deg(g)=m} x_g^\sigma,$$

where $x_g^\sigma = \varepsilon_{r-1}\left(X_{\sqrt{D}}^{\otimes(r-1)}\right)$ is a Tate vector supported at y_g^σ. We extend every x_g^σ to a section $\underline{x}_g^\sigma \in H^0(\underline{y}_g^\sigma, j'_*\mathcal{A})$ as in 4.8 and define

$$T_m \underline{x}^\sigma = m^{r-1} \sum_{\deg(g)=m} \underline{x}_g^\sigma$$

If we are not in the special case of 4.9, then the special fibre $(\underline{x}_g^\sigma)_s$ is equal to $\varepsilon_{r-1}\left(X_{\sqrt{D}}^{\otimes(r-1)}\right)$, where $\sqrt{D} \in \mathrm{End}(\underline{y}_g^\sigma) \otimes \mathbf{Q} \subset \mathrm{End}((\underline{y}_g^\sigma)_s)$.

(4.11) There is another description of Hecke operators on integral models which works also for for m divisible by ℓ (but prime to N). Choose an auxiliary integer M prime to pm, such that NM is divisible by at least two odd primes. The moduli problem for elliptic curves with a combination of (Drinfeld) $[\Gamma(NM)]$ and $[\Gamma_0(m)]$ structures is representable by a curve $\underline{Y}(NM, m)$, flat over $\mathbf{Z}$, which is an integral model of $Y(NM, m)$. As in 2.9, there are two finite maps $\underline{s}, \underline{t} : \underline{Y}(NM, m) \longrightarrow \underline{Y}(NM)$ and a universal elliptic curve $\underline{f} : \underline{E} \longrightarrow \underline{Y}(NM)$. They define an integral version of the diagram 2.9.1, which gives rise to homomorphisms

$$\underline{\psi}^* : \underline{t}^*\underline{\mathcal{B}}_M \longrightarrow \underline{s}^*\underline{\mathcal{B}}_M, \qquad \underline{\psi}_* : \underline{s}^*\underline{\mathcal{B}}_M \longrightarrow \underline{t}^*\underline{\mathcal{B}}_M$$

Assume that all primes dividing M split in K. As in 4.8, we have a section $\underline{\widetilde{y}}^\sigma$ of $\underline{Y}(NM) \otimes \Lambda \longrightarrow \Lambda$ and an extended Tate cycle $\underline{b}(Y_M^\sigma) \in H^0(\underline{\widetilde{y}}^\sigma, \underline{\mathcal{B}}_M)$. Applying to $\underline{b}(Y_M^\sigma)$ operators $T_{m*} = \underline{t}_* \underline{\psi}_* \underline{s}^*$ (resp. $T_m^* = \underline{s}_* \underline{\psi}_* \underline{t}^*$; cf. 2.10) and ε_{B_M}, we obtain elements

$$T_{m*}(\varepsilon_{B_M}\underline{b}(Y_M^\sigma)) \in H^0(T_{m*}\underline{y}, \underline{\mathcal{A}})$$
$$T_m^*(\varepsilon_{B_M}\underline{b}(Y_M^\sigma)) \in H^0(T_m^*\underline{y}, \underline{\mathcal{A}})$$

which are equal by the same argument as in 2.10. Denote by $T_m\underline{x}^\sigma$ their common image in $H^0(T_m\underline{y}^\sigma, j'_*\mathcal{A})$. It extends $T_m x^\sigma$ and is equal to $T_m\underline{x}^\sigma$ constructed in 4.10 if $\ell \not| m$.

(4.12) Proposition. *Let $m \geq 1$ be prime to N such that $r_{\mathcal{A}}(m) = 0$. If $N = 1$, assume that both 2 and 3 split in K. Then*

$$(x, T_m x^\sigma)_v = \frac{1}{2} m^{r-1} \sum_{n\geq 1} \sum_g \left(\varepsilon_{r-1}\left(X^{\otimes(r-1)}_{g\sqrt{D}g^{-1}}\right), \varepsilon_{r-1}\left(X^{\otimes(r-1)}_{\sqrt{D}}\right)\right)_{r-1},$$

where the sum extends over $g \in \mathrm{Hom}_{\Lambda/\pi^n}(\underline{y}^\sigma, \underline{y})$ of degree $\deg(g) = m$.

Corollary. *If ℓ splits in K, then $(x, T_m x^\sigma)_v = 0$.*

Proof of the Corollary. According to [Gr - Za, III.7.1], the set of g in the Proposition is empty.

Proof of the Proposition. It should be noted that the symbol $\sqrt{D}$ in the above formula has a dual meaning: in the first term, it is an element of $\mathrm{End}(\underline{y}^\sigma) \hookrightarrow \mathrm{End}((\underline{y}^\sigma)_s)$ (hence $g\sqrt{D}g^{-1}$ lies in $\mathrm{End}(\underline{y}_s)$); in the second term, $\sqrt{D} \in \mathrm{End}(\underline{y}) \hookrightarrow \mathrm{End}(\underline{y}_s)$. The intersection pairing $(\ ,\ )_{r-1}$ is computed on E_s^{2r-2} (where E with a cyclic subgroup of order N represents y).

We shall apply Prop. 2.16 with $F = H_v$, $\mathcal{X} = \underline{X}_0(N)\otimes\Lambda$. By [Ka - Ma, 14.5.5.1], we have $H^2(\mathcal{X}, j'_*\mathcal{A}(1)) = 0$. Let $m = m_0\ell^t$ with $\ell \not| m_0$. As in 4.10, we have

$$T_{m_0}\underline{x}^\sigma = m_0^{r-1} \sum_g \underline{x}_g^\sigma$$

Fix one of $g \in \mathrm{Hom}_\Lambda(\underline{y}^\sigma, \underline{y}_g^\sigma)$ ($\deg(g) = m_0$) and decompose the divisor $T_{\ell^t}\underline{y}_g^\sigma$ on $\mathcal{X}$ into irreducible components $\underline{y}_g^\sigma(k)$. Then

$$T_{\ell^t}\underline{x}_g^\sigma = \ell^{t(r-1)} \sum_k \underline{x}_g^\sigma(k)$$

with $\underline{x}_g^\sigma(k)$ supported at $\underline{y}_g^\sigma(k)$. Assumptions of Prop. 2.16 are satisfied for $\underline{a} = \underline{x}$, $\underline{b} = \underline{x}_g^\sigma(k)$ (Lemma 4.6). The special fibre of $\underline{x}$ is represented by $\left(\underline{y}_s, \varepsilon_{r-1}\left(X^{\otimes(r-1)}_{\sqrt{D}}\right)\right)$. It follows from Lemma 4.13 below that the special fibre of $\underline{b}$ does not depend on k, provided the supports of $\underline{a}$ and $\underline{b}$ intersect. It follows from Prop. 2.16 that

$$(x, T_{\ell^t}x_g^\sigma)_v = \ell^{t(r-1)}(\underline{y} \cdot T_{\ell^t}\underline{y}_g^\sigma)(\underline{x}_s, \underline{x}_g^\sigma(k)_s)_{r-1}$$

Taking the sum over g, using [Gr - Za, III.6.1] and Lemma 4.13, we obtain

$$(x, T_m x^\sigma)_v = \frac{1}{2} m^{r-1} \sum_{n\geq 1} \sum_{hg} \left(\varepsilon_{r-1}\left(X^{\otimes(r-1)}_{\sqrt{D}}\right), \varepsilon_{r-1}\left(X^{\otimes(r-1)}_{hg\sqrt{D}g^{-1}h^{-1}}\right)\right)_{r-1}$$

with $hg \in \mathrm{Hom}_{\Lambda/\pi^n}(\underline{y}^\sigma, \underline{y})$, $\deg(hg) = m$ as claimed. It remains to prove the following Lemma.

(4.13) Lemma. *Suppose that $\underline{y}$ and $\underline{y}_g^\sigma(k)$ intersect. Then the special fibre of $\underline{x}_g^\sigma(k)$ is represented by*

$$\left(\underline{y}_s, \varepsilon_{r-1}\left(X^{\otimes(r-1)}_{hg\sqrt{D}g^{-1}h^{-1}}\right)\right)$$

for arbitrary isogeny $h \in \mathrm{Hom}_{\Lambda/\pi^n}(\underline{y}^\sigma, \underline{y})$ *with* $\deg(h) = \ell^t$, $n \geq 1$.

Proof. If $t = 0$, then $\underline{y}_g^\sigma$ is ireducible and the special fibre of $\underline{x}_g^\sigma$ is represented by (use 3.6.1 and 4.9.2)

$$\left((\underline{y}_g^\sigma)_s, \varepsilon_{r-1}\left(X_{\sqrt{D}}^{\otimes(r-1)}\right)\right) = \left((\underline{y}_g^\sigma)_s, \varepsilon_{r-1}\left(X_{g\sqrt{D}g^{-1}}^{\otimes(r-1)}\right)\right),$$

hence also by $\left(\underline{y}_s, \varepsilon_{r-1}\left(X_{hg\sqrt{D}g^{-1}h^{-1}}^{\otimes(r-1)}\right)\right)$, as h induces an isomorphism between $(\underline{y}_g^\sigma)_s$ and $\underline{y}_s$.

Suppose now that $t \geq 1$ (thus $\ell \not| N$ and $\mathcal{X}$ is smooth over Λ). If ℓ splits in K, then $\underline{y}$ and $\underline{y}_g^\sigma(k)$ never intersect by [Gr - Za, p. 260], so there is nothing to prove.

Assume now that ℓ is inert in K. If t is even, then the components $\underline{y}_g^\sigma(k)$ are given by $\underline{y}(2k)$ for $0 \leq k \leq t/2$, in the notation of [Gr - Za, p. 259]. For each k, the special fibre of $\underline{y}(2k)$ is isomorphic to the special fibre of $\underline{y}_g^\sigma$ (cf. the discussion after [Gr - Za, III.5.3]) and the special fibre of $\underline{x}_g^\sigma(k)$ is represented by

$$\left((\underline{y}_g^\sigma)_s, \varepsilon_{r-1}\left(X_{g\sqrt{D}g^{-1}}^{\otimes(r-1)}\right)\right)$$

Every isogeny $h \in \mathrm{Hom}_{\Lambda/\pi^n}(\underline{y}_g^\sigma, \underline{y})$ of degree $\deg(h) = \ell^t$ is of the form $h = \ell^{t/2}h_0$, where h_0 is defined over Λ/π^m for some $m \geq 1$ (possibly smaller than n) and $\deg(h_0) = 1$. This implies that $\underline{x}_g^\sigma(k)_s$ is represented by

$$\left(\underline{y}_s, \varepsilon_{r-1}\left(X_{h_0 g\sqrt{D}g^{-1}h_0^{-1}}^{\otimes(r-1)}\right)\right) = \left(\underline{y}_s, \varepsilon_{r-1}\left(X_{hg\sqrt{D}g^{-1}h^{-1}}^{\otimes(r-1)}\right)\right)$$

as claimed.

If t is odd, then the components $\underline{y}_g^\sigma(k)$ are $\underline{y}(2k+1)$ for $0 \leq k \leq (t-1)/2$. Represent $\underline{y}_g^\sigma$ by an elliptic curve E' over Λ with a cyclic subgroup of order N. Every isogeny $E' \longrightarrow E''$ of degree ℓ is defined over a totally ramified extension Λ' of Λ of degree $\ell + 1$. It lifts the Frobenius morphism $h_1 : E'_s \longrightarrow E_s'^{(\ell)}$. The special fibre of $\underline{y}(2k+1)$ is isomorphic to $(\underline{y}^\sigma)_s^{(\ell)}$ ([Gr - Za, III.5.3]) and it follows from the previous discussion and 3.6.1 that the special fibre of $\underline{x}_g^\sigma(k)$ is represented by

$$\left((\underline{y}_g^\sigma)_s^{(\ell)}, \varepsilon_{r-1}\left(X_{h_1 g\sqrt{D}g^{-1}h_1^{-1}}^{\otimes(r-1)}\right)\right)$$

Every isogeny $h \in \mathrm{Hom}_{\Lambda/\pi^n}(\underline{y}_g^\sigma, \underline{y})$ of degree $\deg(h) = \ell^t$ is of the form $h = \ell^{(t-1)/2}h_0h_1$, where h_0 is an isomorphism between $(\underline{y}^\sigma)_s^{(\ell)}$ and $\underline{y}_s$. Consequently, $\underline{x}_g^\sigma(k)_s$ is represented by

$$\left(\underline{y}_s, \varepsilon_{r-1}\left(X_{h_0 h_1 g\sqrt{D}g^{-1}h_1^{-1}h_0^{-1}}^{\otimes(r-1)}\right)\right) = \left(\underline{y}_s, \varepsilon_{r-1}\left(X_{hg\sqrt{D}g^{-1}h^{-1}}^{\otimes(r-1)}\right)\right)$$

as claimed.

For ramified ℓ the proof of the Lemma is goes along the same lines, using the description of the components and isogenies of degree ℓ^t in [Gr - Za, p. 259, 261]. We omit the details.

(4.14) We rewrite the expression in Proposition 4.12 using quaternions, as in [Gr - Za, III.7]. It suffices to treat only the case when ℓ is not split in K ($\Longrightarrow \ell \not| N$). If this is the case, then $\mathrm{End}_{\Lambda/\pi}(\underline{y}) = \mathrm{End}(\underline{y}_s) = R$ is an order in the quaternion algebra B over $\mathbf{Q}$ ramified at ℓ and ∞. Let $\mathfrak{l}$ be the unique prime dividing ℓ in K. The embedding $\mathcal{O}_K = \mathrm{End}(\underline{y}) \hookrightarrow R = \mathrm{End}(\underline{y}_s)$ extends to $K \hookrightarrow B$. There is a decomposition $B = B_+ + B_-$ with $B_+ = K$, $B_- = Kj$, where j satisfies $jaj^{-1} = a^\tau$ for all $a \in K$. Each element $b \in B$ has a unique decomposition $b = b_+ + b_-$ with $b_\pm \in B_\pm$. Write $\mathbf{N}$ for the reduced norm on B.

Recall from [Gr - Za, III.7.3] that

$$\mathrm{End}_{\Lambda/\pi^n}(\underline{y}) = \{b \in R \mid |D|\mathbf{N}(b_-) \equiv 0 \,(\mathrm{mod}\, \ell(\mathbf{N}(\mathfrak{l}))^{n-1})\}$$
$$\mathrm{Hom}_{\Lambda/\pi^n}(\underline{y}^\sigma, \underline{y}) \xrightarrow{\sim} \mathrm{End}_{\Lambda/\pi^n}(\underline{y}) \cdot \mathcal{A}$$

In the latter isomorphism, the isogeny corresponding to $b \in B$ has degree $\mathbf{N}(b)/\mathbf{N}(\mathcal{A})$. We also assume that $\ell \not| \mathbf{N}(\mathcal{A})$; we can always find $\mathcal{A}$ satisfying this condition without changing $\sigma = \sigma_{\mathcal{A}}$. Under this assumption, we have (cf. [Li])

$$\mathrm{Hom}_{\Lambda/\pi^n}(\underline{y}^\sigma, \underline{y}) = \{b \in R\mathcal{A} \mid |D|\mathbf{N}(b_-) \equiv 0 \,(\mathrm{mod}\, \ell(\mathbf{N}(\mathfrak{l}))^{n-1})\} \tag{4.14.1}$$

(4.15) Proposition. *Under the assumptions of Proposition 4.12,*

$$(x, T_m x^\sigma)_v = \frac{(4|D|m)^{r-1}}{\binom{2r-2}{r-1}} \sum_{\substack{b \in R\mathcal{A}/\{\pm 1\} \\ \mathbf{N}(b) = m\mathbf{N}(\mathcal{A})}} U(b_-) P_{r-1}\left(1 - \frac{2\mathbf{N}(b_-)}{\mathbf{N}(b)}\right),$$

where

$$U(b_-) = \begin{cases} 0 & \text{if } \ell \text{ splits in } K \\ \frac{1}{2}(1 + \mathrm{ord}_\ell(\mathbf{N}(b_-)) & \text{if } \ell \text{ is inert in } K \\ \mathrm{ord}_\ell(|D|\mathbf{N}(b_-)) & \text{if } \ell \text{ ramifies in } K \end{cases}$$

Proof. We combine Prop. 3.4, Prop. 4.12 and (4.14.1), observing that, for $u = \sqrt{D}$ and $v = b\sqrt{D}b^{-1}$, we have $\langle X_u, X_u \rangle_1 = \langle X_v, X_v \rangle_1 = 2D$, $\langle X_u, X_v \rangle_1 = t(uv) = 2D(\mathbf{N}(b_+) - \mathbf{N}(b_-))/\mathbf{N}(b)$.

(4.16) We recall some notation from [Gr - Za]: for an ideal $\mathcal{B} \subset \mathcal{O}_K$, denote by $\{\mathcal{B}\}$ the genus of $[\mathcal{B}]$. For an integer $n \geq 1$, set

$$R_{\{\mathcal{B}\}}(n) = \sum_{[\mathcal{A}] \in \{\mathcal{B}\}} r_{\mathcal{A}}(n)$$

$$\delta(n) = 2^s, \qquad s = \text{number of prime factors of } (n, |D|)$$

We shall evaluate the sum

$$\langle x, T_m x^\sigma \rangle_\ell = \sum_{v|\ell} \langle x, T_m x^\sigma \rangle_v = -\sum_{v|\ell} (x, T_m x^\sigma)_v \log_p(Nv)$$

Proposition. *Under the assumptions of Prop. 4.12,*
(1) If ℓ splits in K, then $\langle x, T_m x^\sigma \rangle_\ell = 0$.
(2) If ℓ is inert in K, then

$$\langle x, T_m x^\sigma \rangle_\ell = -u^2 \frac{(4|D|m)^{r-1}}{\binom{2r-2}{r-1}} \log_p(\ell) \sum_{\substack{0 < n < \frac{m|D|}{N} \\ \ell \not| n}} \mathrm{ord}_\ell(\ell n) r_{\mathcal{A}}(m|D| - nN) \delta(n) R_{\{\mathcal{A}\mathfrak{q}\mathcal{N}\}}(n/\ell) P_{r-1}\left(1 - \frac{2nN}{m|D|}\right)$$

(3) If ℓ is ramified in K, then

$$\langle x, T_m x^\sigma \rangle_\ell = -u^2 \frac{(4|D|m)^{r-1}}{\binom{2r-2}{r-1}} \log_p(\ell) \sum_{\substack{0 < n < \frac{m|D|}{N} \\ \ell \not| n}} \mathrm{ord}_\ell(n) r_{\mathcal{A}}(m|D| - nN) \delta(n) R_{\{\mathcal{A}\mathfrak{q}\mathfrak{l}\mathcal{N}\}}(n/\ell) P_{r-1}\left(1 - \frac{2nN}{m|D|}\right)$$

Here $u = \sharp(\mathcal{O}_K^*)/2$ and $\mathfrak{q}$ is any ideal in $\mathcal{O}_K$ with $\mathbf{N}(\mathfrak{q}) \equiv -\ell \,(\mathrm{mod}\, |D|)$.

Proof. This follows from Prop. 4.15 by the the same calculation as in [Gr - Za, III.9]; the only difference is that we use p-adic logarithms and have an extra term with the Legendre polynomial.

(4.17) Proposition. *Assume that $(m,N)=1$, $r_{\mathcal{A}}(m)=0$ and that we are not in the special case of 4.9. Then*

$$\sum_{\ell\neq p}\langle x, T_m x^\sigma\rangle_\ell = -u^2\frac{(4|D|m)^{r-1}}{\binom{2r-2}{r-1}}\sum_{0<n<\frac{m|D|}{N}}\sigma_{\mathcal{A}}(n)r_{\mathcal{A}}(m|D|-nN)P_{r-1}\left(1-\frac{2nN}{m|D|}\right),$$

where $\sigma_{\mathcal{A}}(n)$ was defined in I.6.6.

Proof. This follows from Prop. 4.16 and the equality ([Gr - Za, IV.4.6.b])

$$\sum_{\substack{\ell|n \text{ prime}\\ \ell\neq p}}\log_p(\ell)\left\{\begin{matrix}\mathrm{ord}_\ell(\ell n)r_{\mathcal{A}}(m|D|-nN)\delta(n)R_{\{\mathcal{A}\mathfrak{q}\mathcal{N}\}}(n/\ell) & \ell \text{ inert in } K\\ \mathrm{ord}_\ell(n)r_{\mathcal{A}}(m|D|-nN)\delta(n)R_{\{\mathcal{A}\mathfrak{q}\mathfrak{l}\mathcal{N}\}}(n/\ell) & \ell \text{ ramified in } K\\ 0 & \ell \text{ splits in } K\end{matrix}\right\}=$$

$$=\sum_{\substack{k|n\\ k>0}}\varepsilon_{\mathcal{A}}(n,k)\log_p\left(\frac{n}{k^2}\right)=\sigma_{\mathcal{A}}(n)$$

5. Local heights at p

(5.1) We now turn our attention to local heights at primes dividing p. The assumptions of 4.1 are in force. We first redo [PR 2, 5.2] in our setting. Let $Y_0(N)^{\mathrm{ord}}$ be the subset of $Y_0(N)(\overline{\mathbf{Q}}_p)$ represented by pairs (y, C_N), where y is an elliptic curve over $\overline{\mathbf{Q}}_p$ with good ordinary reduction and C_N a cyclic subgroup of y of order N (recall that $p\nmid N$). The p-torsion y_p contains a canonical subgroup $\mathrm{can}(y)$ of order p, namely the kernel of the reduction on y_p. The map

$$c: Y_0(N)^{\mathrm{ord}}\longrightarrow Y_0(pN)(\overline{\mathbf{Q}}_p)$$

given by

$$c(y,C_N)=(y, C_N\oplus\mathrm{can}(y))$$

is a section of the projection $Y_0(pN)\longrightarrow Y_0(N)$ over $Y_0(N)^{\mathrm{ord}}$. The absolute Frobenius on $Y_0(N)\otimes\mathbf{Z}/p\mathbf{Z}$ has a canonical lifting

$$\varphi: Y_0(N)^{\mathrm{ord}}\longrightarrow Y_0(N)^{\mathrm{ord}},$$

given by

$$\varphi(y,C_N)=(y/\mathrm{can}(y), C_N+\mathrm{can}(y)/\mathrm{can}(y))$$

We shall be interested in Tate cycles $Z(Y_0(N)^{\mathrm{ord}},\overline{\mathbf{Q}}_p)$ supported in $Y_0(N)^{\mathrm{ord}}$. Any such Tate cycle is a sum of Tate vectors, which are represented by triples $\underline{y}=(y,C_N,a)$ with $(y,C_N)\in Y_0(N)^{\mathrm{ord}}$, $a\in \mathrm{Sym}^{2r-2}(H^1(y,\mathbf{Q}_p))$. For $m\geq 1$ prime to N, the Hecke correspondence $T_m=T_{m*}=T_m^*$ acts on Tate cycles as in 2.10. There is also a correspondence T'_m acting on $Z(Y_0(pN),\overline{\mathbf{Q}}_p)$ by the following formula:

$$(T'_m)_*(y,C_{pN},a)=\sum_{\substack{\lambda:y\longrightarrow y'\\ \deg(\lambda)=m\\ \mathrm{Ker}(\lambda)\cap C_{pN}=0}}(y',\lambda_*(C_{pN}),\lambda_*(a))$$

If $p\nmid m$, then

$$c(T_m(\underline{y}))=(T'_m)_*(c(\underline{y}))\qquad(5.1.1)$$

For $k \geq 0$, denote by $S(y,k)$ the set of all subgroups of Y of order p^k. Every $C \in S(y,k)$ defines an isogeny $\lambda : y \longrightarrow y/C$ of degree p^k; we define

$$\underline{y}_C := (y/C, C + C_N/C, \lambda_*(a))$$

(5.2) Lemma. *For all $m \geq 1$ prime to N and $k \geq 0$, we have*
(1) $(T'_{mp^{k+1}})_* \circ c(\underline{y}) = c \circ T_{mp^{k+1}}(\underline{y}) - c \circ T_{mp^k} \circ \varphi(\underline{y})$
(2) $c \circ [T_{mp^{k+2}} \circ \varphi - T_{mp^{k+1}} \circ (p^{2r-2} + \varphi^2) + p^{2r-2} T_{mp^k} \circ \varphi](\underline{y}) = (T'_m)_* \circ [(T'_{p^{k+2}})_* \circ c \circ \varphi - p^{2r-2} (T'_{p^{k+1}})_* \circ c](\underline{y})$

Proof. (1) Thank to (5.1.1) and the fact that

$$T_{mp^k} = T_m \circ T_{p^k}, \qquad (T'_{mp^k})_* = (T'_m)_* \circ (T'_{p^k})_*$$

for $p \not| m$, we can assume that $m = 1$. Then

$$T_{p^{k+1}}(\underline{y}) = \sum_{\substack{C \in S(y,k+1) \\ \mathrm{can}(y) \subseteq C}} \underline{y}_C + \sum_{\substack{C \in S(y,k+1) \\ \mathrm{can}(y) \not\subseteq C}} \underline{y}_C$$

If $\mathrm{can}(y) \subseteq C$, then $C' := C/\mathrm{can}(y)$ runs through $S(\mathrm{can}(y), k)$ and $\underline{y}_C = (\varphi(\underline{y}))_{C'}$, so the first sum is equal to $T_{p^k}(\varphi(\underline{y}))$, thus

$$c \circ T_{p^{k+1}}(\underline{y}) = c \circ T_{p^k}(\varphi(\underline{y})) + (T'_{p^{k+1}})_* \circ c(\underline{y})$$

(2) This follows from (1) applied to k and $k+1$.

(5.3) Lemma. *For $(y, C_N) \in Y_0(N)^{\mathrm{ord}}$,*

$$(T'_{p^{k+2}})_* \circ c \circ \varphi(\underline{y}) - p^{2r-2}(T'_{p^{k+1}})_* \circ c(\underline{y}) = \sum_C (c \circ \varphi(\underline{y}))_C,$$

where C runs through elements $C \in S(\varphi(y), k+2)$ such that $\mathrm{can}(\varphi(y)), y_p/\mathrm{can}(y) \not\subset C$.

Proof. By definition,

$$(T'_{p^{k+1}})_* \circ c(\underline{y}) = \sum_{\substack{C \in S(y,k+1) \\ \mathrm{can}(y) \not\subseteq C}} c \circ \underline{y}_C$$

The set of the C's is in a bijection with the set of $C' := \widehat{\varphi}^{-1}(C)$ – they run through cyclic subgroups $C' \in S(\varphi(y), k+2)$ such that $C' \supset y_p/\mathrm{can}(y) = \mathrm{Ker}(\widehat{\varphi})$. As the diagram

$$\begin{array}{ccccc} y & \xrightarrow{\varphi} & \varphi(y) & \xrightarrow{\lambda_{C'}} & \varphi(y)_{C'} \\ \| & & \downarrow \widehat{\varphi} & & \| \\ y & \xrightarrow{p} & y & \xrightarrow{\lambda_C} & y_C \end{array}$$

commutes, we have $\varphi_*(a)_{C'} = [p]_* a_C = p^{2r-2} a_C$. Subtracting from the sum defining $(T'_{p^{k+2}})_* \circ c \circ \varphi(\underline{y})$ we get the statement of the Lemma.

Corollary. *If $m = m_0 p^n$ with $p \not| m_0$, then*

$$c \circ [T_{mp^{k+2}} \circ \varphi - T_{mp^{k+1}} \circ (p^{2r-2} + \varphi^2) + p^{2r-2} T_{mp^k} \circ \varphi](\underline{y}) = (T'_{m_0})_* \left[\sum_C c \circ (\varphi(\underline{y}))_C \right],$$

where C runs through cyclic subgroups of $\varphi(y)$ of order p^{n+k+2} satisfying $\mathrm{can}(\varphi(y)) \not\subset C$ *and* $y_p/\mathrm{can}(y) \not\subset C$.

(5.4) We now specialize the previous Corollary to CM points, assuming that y is an elliptic curve with a complex multiplication by an order of conductor prime to p in $K = \mathbf{Q}(\sqrt{D})$. Since y is ordinary, p splits in K, $(p) = \mathfrak{p}\overline{\mathfrak{p}}$ with $\mathfrak{p} \neq \overline{\mathfrak{p}}$. Let v be a place of H over $\mathfrak{p}$ and w a place of $H_{p^{n+k+2}}$ (the ring class field of K of conductor p^{n+k+2}) over v.

Lemma. *If $m = m_0 p^n$ with $p \not| m_0$, then*

$$T_{mp^{k+2}}(\varphi(\underline{y})) - T_{mp^{k+1}}(p^{2r-2}\underline{y} + \varphi^2(\underline{y})) + p^{2r-2} T_{mp^k}(\varphi(\underline{y})) = N_{H_{p^{n+k+2},w}/H_v}(T_{m_0}(\varphi(\underline{y})_C)),$$

where C is any subgroup of y of order p^{n+k+2} that does not contain $\mathrm{can}(\varphi(y))$ $(= \varphi(y)_{\mathfrak{p}})$ *or* $y_p/\mathrm{can}(y)$ $(= \varphi(y)_{\overline{\mathfrak{p}}})$.

Proof. This follows from Cor. 5.3 and the fact that the subgroups C form a $G(H_{p^{n+k+2},w}/H_v)$-torsor.

(5.5) Let F be a number field and v a prime of F dividing p. We denote by $Z_f(Y_0(N), F_v)$ the preimage of $H^1_f(F_v, V)$ under

$$\Phi_T : Z(Y_0(N), F_v) \longrightarrow H^1(F_v, V)$$

Suppose that $a, b \in Z_f(Y_0(N), F_v)$ are Tate cycles with disjoint supports. The mixed extension E from 2.8 can be plugged in to the height machinery of 1.8 (provided the relevant splittings and $\ell_v : F_v^* \longrightarrow \mathbf{Q}_p$ are fixed), defining thus the local height $\langle a, b\rangle_v$. If, in the notation of 2.3-5, $a = a(Y \otimes_F F_v)$, $b = a(Z \otimes_F F_v)$, then $a, b \in Z_f(Y_0(N), F_v)$ by Lemma 1.4 and $\langle a, b\rangle_v = \langle Y, Z\rangle_v$.
Denote by $\mathcal{A}_{\mathbf{Z}_p}$ the image of $R^{2r-2} f_* \mathbf{Z}_p(r-1)$ in $\mathcal{A}$. One defines integral Tate cycles $Z(Y_0(N), F)_{\mathbf{Z}_p}$ by replacing $\mathcal{A}$ by $\mathcal{A}_{\mathbf{Z}_p}$ in 2.6. The discussion of integrality properties in 1.10 carries over to integral Tate cycles. For example,

$$\Phi_T(Z_f(Y_0(N), F_v)_{\mathbf{Z}_p}) \subseteq p^{-c} \mathrm{Im}(H^1_f(F_v, T) \longrightarrow H^1_f(F_v, V)), \tag{5.5.1}$$

where T is the image of $H^1(X_0(N) \otimes_{\mathbf{Q}} \overline{\mathbf{Q}}, j_* \mathcal{A}_{\mathbf{Z}_p})(1)$ in V and p^c is the order of the finite group $H^2(X_0(N) \otimes_{\mathbf{Q}} \overline{\mathbf{Q}}, j_* \mathcal{A}_{\mathbf{Z}_p})$.
Similarly, if $a, b \in Z_f(Y_0(N), F_v)_{\mathbf{Z}_p}$ and $\Phi_T(a) = 0$, then

$$\langle a, b\rangle_v \in p^{-2c} \ell_v(F_v^*) \tag{5.5.2}$$

(5.6) We continue with the notation and assumptions of 4.1, but from now on we also assume that p splits in K, $(p) = \mathfrak{p}\overline{\mathfrak{p}}$ with $\mathfrak{p} \neq \overline{\mathfrak{p}}$. Let $\sigma_{\mathfrak{p}} \in G(H/K)$ (resp. $\sigma_{\overline{\mathfrak{p}}}$) be the Frobenius at $\mathfrak{p}$ (resp. $\overline{\mathfrak{p}}$). If A^σ_m is a function of $m \in \mathbf{Z}$ and $\sigma \in G(H/K)$, we put

$$A^\sigma_m | U_p = A^\sigma_{mp}, \qquad A^\sigma_m | \sigma' = A^{\sigma\sigma'}_m$$

Using this notation, we define, for any $m \geq 1$ prime to N and $\sigma \in G(H/K)$,

$$b^\sigma_m := T_m x^\sigma | (U_p - \sigma_{\mathfrak{p}})(U_p \sigma_{\mathfrak{p}} - p^{2r-2}) = T_{mp^2} x^{\sigma\sigma_{\mathfrak{p}}} - T_{mp}(p^{2r-2} x^\sigma + x^{\sigma\sigma^2_{\mathfrak{p}}}) + p^{2r-2} T_m x^{\sigma\sigma_{\mathfrak{p}}}$$

This is an element of $Z_f(Y_0(N), H)_{\mathbf{Z}_p}$, as the action of $\mathbf{T}$ preserves the f-subspace.

Proposition. *Let v be a place of H over $\mathfrak{p}$ and $a \in Z(Y_0(N), H_v)$ a Tate cycle with $\Phi_T(a) = 0$. Then there is an integer k_0 (depending on a) such that*

$$\langle a, b^\sigma_{mp^k}\rangle_v \in p^{k-k_0}\mathbf{Z}_p$$

for all $m \geq 1$ prime to N and $k \geq 0$, provided a and $b^\sigma_{mp^k}$ have disjoint supports.

Proof. Changing k_0, we may assume that a is an integral Tate cycle and $p \not| m$. According to Lemma 5.4, we have

$$b^\sigma_{mp^k} = N_{H_{p^{k+2},w}/H_v}(h^\sigma_{m,k})$$

for suitable $h^\sigma_{m,k} \in Z_f(Y_0(N), H_{p^{k+2},w})_{\mathbf{Z}_p}$. By 1.9.1 and 5.5.2, this implies that

$$\langle a, b^\sigma_{mp^k}\rangle_{v,\ell_v} = \langle a, h^\sigma_{m,k}\rangle_{w,\ell_w} \in p^{-2c}\ell_w(H^*_{p^{k+2},w}),$$

where $\ell_w = \ell_v \circ N_{H_{p^{k+2},w}/H_v}$. However, $H_{p^{k+2},w}$ contains the $(k+2)$-th layer $\mathbf{Q}_{p,k+2}$ of the cyclotomic $\mathbf{Z}_p$-extension $\mathbf{Q}_{p,\infty}/\mathbf{Q}_p$ (as p splits in K), hence

$$\ell_w(H^*_{p^{k+2},w}) \subseteq \ell_{\mathbf{Q},p}(N_{\mathbf{Q}_{p,k+2}/\mathbf{Q}_p}(\mathbf{Q}^*_{p,k+2})) \subseteq p^{k+2}\ell_{\mathbf{Q},p}(\mathbf{Q}^*_p) = p^{k+3}\mathbf{Z}_p$$

(5.7) Lemma. *Let $M \geq 1$ be an integer and $g = \sum_{n\geq 1} a_n q^n \in S_{2r}(\Gamma_0(N))$ such that $a_m = 0$ for all integers m prime to NM satisfying $r(m) = \sum_{[\mathcal{A}]\in \mathrm{Pic}(\mathcal{O}_K)} r_{\mathcal{A}}(m) = 0$. Then $a_n = 0$ for all $n \geq 1$ prime to N.*

Proof. We can write

$$g = \sum_{\beta:\mathbf{T}\longrightarrow\overline{\mathbf{Q}}} c_\beta \cdot f_\beta + h,$$

where $a_n(h) = 0$ for all n prime to N. Let $\ell \not| NM$ be a prime number inert in K. Then $r(\ell n) = 0$ for all n prime to ℓ, hence

$$0 = a_{\ell n}(g) = \sum_\beta c_\beta a_{\ell n}(f_\beta) = \sum_\beta c_\beta a_\ell(f_\beta) a_n(f_\beta)$$

whenever $(n, NM\ell) = 1$. This implies that $c_\beta a_\ell(f_\beta) = 0$ for all β. Assume that, for some β, $a_\ell(f_\beta) = 0$ for all primes $\ell \not| NM$ inert in K. It follows that

$$\mathrm{Tr}(Fr_\ell | V_\beta) = m(\beta) a_\ell(f_\beta)\ell^{-r} = 0$$

for such ℓ. Let $T_\beta \subset V_\beta$ be a lattice invariant under $G(\overline{\mathbf{Q}}/\mathbf{Q})$. The action of the Galois group on $T_\beta/p^n T_\beta$ factors through $G(F_n/\mathbf{Q})$, where F_n is an extension of $\mathbf{Q}$ unramified outside pN. As $(|D|, pN) = 1$, it follows from Čebotarev density theorem for $KF_n/\mathbf{Q}$ that for any prime number $q \not| NM|D|p$ there are infinitely many primes $\ell_j \not| NM|D|p$ inert in K such that

$$a_q(f_\beta) \equiv a_{\ell_j}(f_\beta) \,(\mathrm{mod}\, p^n \mathcal{O}_{\widehat{F(\beta)}})$$

Consequently, $a_q(f_\beta) = 0$ for all such q, which is impossible. This contradiction shows that for each β, one can choose ℓ in such a way that $a_\ell(f_\beta) \neq 0$. This implies that $c_\beta = 0$ for all β, hence $g = h$. Lemma is proved.

(5.8) It follows from Lemma 5.7 that there is a finite subset $S \subset \{m \mid (m, pN) = 1, r(m) = 0\}$ and constants $c_{\beta,m} \in L$ ($\beta : \mathbf{T} \longrightarrow \overline{\mathbf{Q}}$, $m \in S$, L a finite extension of $\mathbf{Q}$ containing all $F(\beta)$) such that

$$\sum_{m\in S} c_{\beta,m}\beta'(T_m) = \begin{cases} 1 & \text{if } \beta' = \beta \\ 0 & \text{if } \beta' \neq \beta \end{cases}$$

Put, for each $\beta : \mathbf{T} \longrightarrow \overline{\mathbf{Q}}$,

$$x_\beta := \sum_{m\in S} c_{\beta,m} T_m x \in Z_f(Y_0(N), H) \otimes_{\mathbf{Q}_p} \widehat{L}$$

Of course,

$$\Phi_T(x_\beta) = z_\beta \otimes 1 \in H^1_f(H, V_\beta) \otimes_{\widehat{F(\beta)}} \overline{\mathbf{Q}}_p \subset H^1_f(H, V) \otimes_{\mathbf{Q}_p} \widehat{L}$$

Let $\mathcal{F}$ be the operator $(U_p - p^{r-1}\sigma_{\mathfrak{p}})^2 (U_p - p^{r-1}\sigma_{\overline{\mathfrak{p}}})^2$. If $r_{\mathcal{A}}(m) = 0$ (for $[\mathcal{A}]$ corresponding to σ^{-1}), put

$$\langle x, T_m x^\sigma \rangle_p = \sum_{v|p} \langle x, T_m x^\sigma \rangle_v$$

The following statement will be proved in 6.2 (the reader is invited to check that our reasoning is not circular).

Lemma. *There is a modular form $H_\sigma \in \overline{M}_{2r}(\Gamma_0(Np^\infty); \mathbf{Q}_p)$ with the property that*

$$a_m(H_\sigma) = \langle x, T_m x^\sigma \rangle_p | \mathcal{F}$$

for all integers $m \geq 1$ such that $p \mid m$, (m, N) and $r_{\mathcal{A}}(m) = 0$.

Recall from I.2.3 the operator

$$e : \overline{M}_{2r}(\Gamma_0(Np^\infty); \widehat{A}) \longrightarrow M_{2r}(\Gamma_0(pN); \widehat{A})$$

Its restriction to $S_{2r}(\Gamma_0(Np^n); \widehat{A})$ is equal to

$$e_n = \lim_{k\longrightarrow\infty} U_p^{cp^k}$$

for some integer $c = c(n) \geq 1$. This formula is still valid if c is replaced by its arbitrary mutiple. This implies that

$$e = \lim_{k\longrightarrow\infty} U_p^{b(k)p^k}$$

on $\overline{M}_{2r}(\Gamma_0(Np^\infty); \widehat{A})$ with $b(k) = \prod_{n\leq k} c(k)$.

(5.9) For an integer $n \geq 1$ and $\beta : \mathbf{T} \longrightarrow \overline{\mathbf{Q}}$, we define

$$\lambda_n^\sigma(\beta) = \sum_{m\in S} c_{\beta,m} a_{mn}(H_\sigma | e)$$

Proposition. *Let $\mathcal{F}' = (U_p - \sigma_{\mathfrak{p}})(U_p\sigma_{\mathfrak{p}} - p^{2r-2})(U_p - \sigma_{\overline{\mathfrak{p}}})(U_p\sigma_{\overline{\mathfrak{p}}} - p^{2r-2})$. If $n \geq 1$ is prime to pNM, where $M = \prod_{m\in S} m$, then*
(1) $\lambda_n^\sigma(\beta)|\mathcal{F}' = \beta(T_n)\lambda_1^\sigma(\beta)|\mathcal{F}'$
(2) If $r(n) = 0$, then $\sum_\beta \lambda_n^\sigma(\beta)|\mathcal{F}' = a_n(H_\sigma | e\mathcal{F}')$

Proof. (1) Put

$$a = T_n x_\beta - \beta(T_n) x_\beta = \sum_\beta c_{\beta,m}(T_m T_n x - \beta(T_n) T_m x) \in H^1_f(H, V) \otimes_{\mathbf{Q}_p} \widehat{L}$$

As $\Phi_T(a) = 0$, we can apply Prop. 5.6. Our assumptions imply that a and $b^\sigma_{p^k}$ have disjoint supports for all $k \geq 0$, so we obtain

$$\sum_{m\in S} c_{\beta,m} \lim_{k\longrightarrow\infty} \langle T_m T_n x, x^\sigma\rangle_p | U_p^{b(k)p^k} \mathcal{F}' = \beta(T_n) \sum_{m\in S} c_{\beta,m} \lim_{k\longrightarrow\infty} \langle T_m x, x^\sigma\rangle_p | U_p^{b(k)p^k} \mathcal{F}'$$

Applying $\mathcal{F}$ to both sides and using $\langle T_{n*}x, y\rangle_p = \langle x, T_n^* y\rangle_p$, we obtain from Lemma 5.8

$$\sum_{m\in S} c_{\beta,m} a_{mn}(H_\sigma | e\mathcal{F}') = \beta(T_n) \sum_{m\in S} c_{\beta,m} a_m(H_\sigma | e\mathcal{F}'),$$

as was to be shown.
(2) This follows from Prop. 5.6 for $a = T_n(x - \sum_\beta x_\beta)$ by the same calculation, once we observe that $T_n x$ and $b^\sigma_{p^k}$ have disjoint supports for $r(n) = 0$.

(5.10) Assume that $f = f_{\beta_0}$ (for a fixed β_0) is a newform, ordinary at p (i.e. $a_p(f) = \beta_0(T_p)$ is a p-adic unit). This implies ([Wi, Thm 2]) that the Galois representation V_{β_0} is ordinary in the sense of [Gre]. In this case we have a canonical $\widehat{F(\beta_0)}$-linear splitting

$$F^r \varepsilon_B \varepsilon H^{2r-1}_{\mathrm{dR}}(W \otimes_{\mathbf{Q}} \mathbf{Q}_p/\mathbf{Q}_p)_{\beta_0} \overset{\longleftarrow}{\longrightarrow} \varepsilon_B \varepsilon H^{2r-1}_{\mathrm{dR}}(W \otimes_{\mathbf{Q}} \mathbf{Q}_p/\mathbf{Q}_p)_{\beta_0}$$

We assume that the splittings in 4.1 have been chosen in such a way that they induce this canonical splitting on the β_0-component.

Proposition. *Under the above assumptions,*
(1) $\lambda^\sigma_n(\beta_0) = 0$ *for all* n *prime to* pNM.
(2) $L_{f_0}(H_\sigma | e\mathcal{F}') = 0$

Proof. We first show that (1) implies (2). Indeed, Lemma 5.7 and Prop. 5.9 (for pN instead of N) tell us that

$$H_\sigma | e\mathcal{F}' = \sum_\beta (\lambda^\sigma_1(\beta) | \mathcal{F}') f_\beta + h_\sigma,$$

where $h_\sigma \in S_{2r}(\Gamma_0(pN); \mathbf{Q}_p)$ and $a_m(h_\sigma) = 0$ for all m prime to pN. It follows that

$$L_{f_0}(H_\sigma | e\mathcal{F}') = \lambda^\sigma_1(\beta_0) L_{f_0}(f),$$

which vanishes iff $\lambda^\sigma_1(\beta_0) = 0$.
Let us now prove (1). We have $\mathcal{F}' = \mathcal{F}'_{\mathfrak{p}} \mathcal{F}'_{\overline{\mathfrak{p}}}$ with

$$\mathcal{F}'_{\mathfrak{p}} = (U_p - \sigma_{\mathfrak{p}})(U_p \sigma_{\mathfrak{p}} - p^{2r-2}), \qquad \mathcal{F}'_{\overline{\mathfrak{p}}} = (U_p - \sigma_{\overline{\mathfrak{p}}})(U_p \sigma_{\overline{\mathfrak{p}}} - p^{2r-2})$$

Let v be a prime of H dividing $\mathfrak{p}$. For $n \geq 0$, define

$$t^\sigma_n = N_{H_{p^{k+2},w}/H_{v,n}}(h^\sigma_{1,n}),$$

where $h^\sigma_{m,n}$ was defined in 5.6 and $H_{v,n} = H_v \mathbf{Q}_{p,n}$ is the n-th layer in the cyclotomic $\mathbf{Z}_p$-extension of H_v. As in 5.6, it follows from Prop. 5.4 that

$$\langle T_m x, x^\sigma\rangle_v | \mathcal{F}'_{\mathfrak{p}} U_p^n = \langle T_m x, t^\sigma_n\rangle_{H_{v,n}}$$

whenever $r(m) = 0$, which implies that

$$\langle x_{\beta_0}, x^\sigma\rangle_v | \mathcal{F}'_{\mathfrak{p}} U_p^n = \langle x_{\beta_0}, t^\sigma_n\rangle_{H_{v,n}}$$

The representation $V \otimes_{\mathbf{Q}_p} \widehat{L}$ of $G(\overline{\mathbf{Q}}/\mathbf{Q})$ is a direct sum $V' \oplus V''$ with

$$V' = V_{\beta_0} \otimes_{\widehat{F(\beta_0)}} \widehat{L}, \qquad V'' = \bigoplus_{\beta \neq \beta_0} \left(V_\beta \otimes_{\widehat{F(\beta)}} \widehat{L}\right)^{\oplus m(\beta)}$$

The lattice $T \subset V$, defined in 5.5, determines lattices $T' \subset V'$, $T'' \subset V''$. We now want to apply Prop. 1.11 for $a_1 = \Phi_T(x_{\beta_0})$, $a_2 = t_n^\sigma$ over $H_{v,n}$. The assumptions (A)–(C) of 1.11. are all satisfied, but it is the V''-component of $a_{1,v}$, not of $a_{2,v}$, that vanishes. We have to pass, therefore, to the dual mixed extension $E^*(1)$ and use [Ne 2, 4.10.3]. As $V \xrightarrow{\sim} V^*(1)$ is self-dual (and the same is true also for V', V''), we can apply Prop. 1.11 as it stands.

Writing

$$\begin{aligned} p^{d_{n,1}} &= \sharp H^1(H_{v,n}, T''^*(1))_{\mathrm{tors}} \\ p^{d_{n,2}} &= \sharp \left(H^1_f(H_{v,n}, T')/N_\infty H^1_f(H_{v,n}, T')\right), \end{aligned}$$

we obtain from 5.5.1 and Prop. 1.11

$$p^{2c+d_0+d_{n,1}+d_{n,2}} \langle x_{\beta_0}, t_n^\sigma \rangle_{H_{v,n}} \in \ell_v \circ N_{H_{v,n}/H_v}(H^*_{v,n} \widehat{\otimes} \mathbf{Z}_p) = p^n \ell_v(H^*_v \widehat{\otimes} \mathbf{Z}_p)$$

We shall prove below that $d_{n,1}, d_{n,2}$ are bounded by some integer d. Assuming this, we have

$$\sum_{m \in S} c_{\beta_0,m} \langle T_m x, x^\sigma \rangle_v | \mathcal{F}'_{\mathfrak{p}} U_p^n = \langle x_{\beta_0}, x^\sigma \rangle_v | \mathcal{F}'_{\mathfrak{p}} U_p^n = \langle x_{\beta_0}, t_n^\sigma \rangle_{H_{v,n}} \in p^{n-2c-d_0-2d} \ell_v(H^*_v \widehat{\otimes} \mathbf{Z}_p),$$

hence

$$\lambda_1^\sigma(\beta_0) | \mathcal{F}'_{\mathfrak{p}} = \sum_{m \in S} c_{\beta_0,m} \langle T_m x, x^\sigma \rangle_p | \mathcal{F}'_{\mathfrak{p}} e = \lim_{k \longrightarrow \infty} \langle x_{\beta_0}, x^\sigma \rangle_p | \mathcal{F}'_{\mathfrak{p}} U_p^{b(k)p^k} = 0$$

Combined with Prop. 5.9.1, this implies the statement (1) of the Proposition.

It remains to show that $d_{n,1}, d_{n,2}$ are bounded. The Pančiškin condition from 1.11 means that there is an exact sequence of representations of $G(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$

$$0 \longrightarrow X \longrightarrow V' \longrightarrow Y \longrightarrow 0$$

with suitable properties. Put $T_X = T' \cap X$, $T_Y = T'/T_X$. According to [Ne 2, 6.9],

$$p^{d_{n,2}} \,\Big|\, \sharp H^0(H_{v,n}, T_X^*(1) \otimes \mathbf{Q}_p/\mathbf{Z}_p) \cdot \sharp H^0(H_{v,n}, T_Y \otimes \mathbf{Q}_p/\mathbf{Z}_p)$$

We also have

$$p^{d_{n,1}} \,\Big|\, \sharp H^0(H_{v,n}, T''^*(1) \otimes \mathbf{Q}_p/\mathbf{Z}_p)$$

The boundedness of $d_{n,1}, d_{n,2}$ then follows from the following Lemma.

(5.11) Lemma. *$H^0(H_{v,\infty}, Z) = 0$ for $Z = V''^*(1), X^*(1), Y$.*

Proof. If Z is a Hodge-Tate representation of $G(\overline{H}_v/H_v)$, then

$$H^0(H_{v,\infty}, Z) \subseteq \bigoplus_{j \in \mathbf{Z}} H^0(H_v, Z(j))(-j),$$

as $H_{v,\infty} = H_v \mathbf{Q}_{p,\infty}$. Assume, moreover, that Z is crystalline (which is true for V by [Fo - Me], as $p > 2r - 1$ by assumption) and write

$$D_v(Z) = H^0(H_v, Z \otimes_{\mathbf{Q}_p} B_{\mathrm{cris}})$$

for the corresponding filtered module. It is equipped with the action of the crystalline Frobenius f and we have

$$H^0(H_v, Z(j))(-j) \subseteq D_v(Z)^{f = p^{-j}}$$

By crystalline Weil conjectures [KaN - Me],

$$D_v(V)^{f=p^{-j}} = 0 \qquad \forall j \in \mathbf{Z}$$

This implies the vanishing of $D_v(Z)^{f=p^{-j}}$ for all subquotients Z of $V \otimes_{\mathbf{Q}_p} \widehat{L}$, which includes V'', X, Y, hence also $V''^*(1), X^*(1)$ (as $V \overset{\sim}{\longrightarrow} V^*(1)$). Lemma is proved.

6. Proof of Theorem A

(6.1) The assumptions and notation of 0.2 are in place. The newform f corresponds to a homomorphism $\beta_0 : \mathbf{T} \longrightarrow \overline{\mathbf{Q}}$. Fix $\sigma \in G(H/K)$ and define $F_\sigma \in S_{2r}(\Gamma_0(N); \mathbf{Q}_p)$ by

$$F_\sigma = \sum_\beta \langle z_\beta, z_\beta^\sigma \rangle f_\beta$$

For $m \geq 1$ prime to N, the m-th Fourier coefficient of F_σ is equal to

$$a_m(F_\sigma) = \sum_\beta \langle z_\beta, z_\beta^\sigma \rangle \beta(T_m) = \langle z, T_m z^\sigma \rangle = \langle x, T_m x^\sigma \rangle \in \mathbf{Q}_p \tag{6.1.1}$$

If, furthermore, $r_{\mathcal{A}}(m) = 0$ (where $[\mathcal{A}] \in \mathrm{Pic}(\mathcal{O}_K)$ corresponds to σ^{-1}), then we have a decomposition into local heights

$$a_m(F_\sigma) = c_m^\sigma + d_m^\sigma$$

with

$$c_m^\sigma = \sum_{v \nmid p} \langle x, T_m x^\sigma \rangle_v, \quad d_m^\sigma = \sum_{v | p} \langle x, T_m x^\sigma \rangle_v$$

(6.2) If $\mathcal{C}$ is a character of $G(H/K)$, the interpolation property of the p-adic L-function $L_p(f \otimes K, \mathcal{C}, \mathcal{W})$ stated in 0.5 follows from Thm. I.5.10. If ℓ_K is anticyclotomic, then the statement of Theorem A becomes $0 = 0$, by Cor. I.5.12 and Prop. 4.3. It is sufficient, therefore, to prove Theorem A for a single non-trivial cyclotomic ℓ_K; we take $\ell_K = \ell_{\mathbf{Q}} \circ \mathbf{N}$ as in 4.4. By definition, we have $\ell_K = \log_p \circ \lambda$, where $\lambda : G(\mathbf{Q}_\infty/\mathbf{Q}) \longrightarrow 1 + p\mathbf{Z}_p$ is given by the cyclotomic character. The corresponding homomorphism

$$\widetilde{\lambda} : \mathbf{Z}_p^* \longrightarrow 1 + p\mathbf{Z}_p,$$

defined in I.6.2, is equal to $\widetilde{\lambda}(x) = \langle x \rangle^{-1}$, where $\langle x \rangle = x\omega^{-1}(x)$ for the Teichmüller character $\omega : \mathbf{Z}_p^* \longrightarrow \mu_{p-1}(\mathbf{Q}_p)$.
According to I.6.2.1, we have

$$L_p(f \otimes K, \mathcal{C})(\lambda^s) = (-1)^r H_p(f)(1 - C\left(\frac{D}{C}\right)\langle C \rangle^{2s})^{-1} \sum_{[\mathcal{A}] \in \mathrm{Pic}(\mathcal{O}_K)} \mathcal{C}(\mathcal{A})^{-1} L_{f_0} \left[\int_{\mathbf{Z}_p^*} \langle x \rangle^{-s} \, d\widetilde{\Psi}_{\mathcal{A}}^C \right] \tag{6.2.1}$$

If $\sigma^{-1} \in G(H/K)$ corresponds to the ideal class $[\mathcal{A}]$, put

$$G_\sigma = (-1)^r \int_{\mathbf{Z}_p^*} \log_p \, d\widetilde{\Psi}_{\mathcal{A}} \in \overline{M}_{2r}(\Gamma_0(Np^\infty); \mathbf{Q}_p)$$

It follows from 6.2.1 that

$$L_p'(f\otimes K,\mathcal{C},\mathbb{1}) = -H_p(f)\sum_{\sigma\in G(H/K)}\mathcal{C}(\sigma)L_{f_0}(G_\sigma) \tag{6.2.2}$$

Combining Cor. I.6.6, I.6.7.1 and Prop. 4.17, we obtain the main identity

$$c_m^\sigma|(U_p - p^{r-1}\sigma_{\mathfrak{p}})^2(U_p - p^{r-1}\sigma_{\overline{\mathfrak{p}}})^2 = u^2(4|D|)^{r-1}a_m(G_\sigma)|(U_p^4 - p^{2r-2}U_p^2) \tag{6.2.3}$$

valid for all $m\geq 1$ satisfying $p\mid m$, $(m,N)=1$, $r_{\mathcal{A}}(m)=0$.
The modular form

$$H_\sigma = F_\sigma|(U_p - p^{r-1}\sigma_{\mathfrak{p}})^2(U_p - p^{r-1}\sigma_{\overline{\mathfrak{p}}})^2 - u^2(4|D|)^{r-1}G_\sigma|(U_p^4 - p^{2r-2}U_p^2)$$

is an element of $\overline{M}_{2r}(\Gamma_0(Np^\infty);\mathbf{Q}_p)$. According to (6.2.3),

$$d_m^\sigma|\mathcal{F} = \sum_{v|p}\langle x, T_m x^\sigma\rangle_v|\mathcal{F} = a_m(H_\sigma)$$

provided $p\mid m$, $(m,N)=1$, $r_{\mathcal{A}}(m)=0$. This proves Lemma 5.8.

(6.3) According to Prop. 5.10,

$$L_{f_0}(H_\sigma|\mathcal{F}') = \ell_{f_0}(H_\sigma|\mathcal{F}'e) = 0$$

Consequently,

$$L_{f_0}(F_\sigma|\mathcal{F}\mathcal{F}') = u^2(4|D|)^{r-1}L_{f_0}(G_\sigma|(U_p^4 - p^{2r-2}U_p^2)\mathcal{F}') \tag{6.3.1}$$

Multiplying 6.3.1 by $\mathcal{C}(\sigma)$ and summing over $\sigma\in G(H/K)$, we obtain (using 6.2.2 and the fact that $L_{f_0}\circ U_p = \alpha_p(f)L_{f_0}$)

$$(\widetilde{\mathrm{fn}})\cdot L_{f_0}(f)h\langle z_{\beta,\mathcal{C}}, z_{\beta,\overline{\mathcal{C}}}\rangle\prod_{\mathfrak{p}|p}\left(1-\frac{\mathcal{C}(\mathfrak{p})p^{r-1}}{\alpha_p(f)}\right)\left(1-\frac{\mathcal{C}(\overline{\mathfrak{p}})p^{r-1}}{\alpha_p(f)}\right) = -(\widetilde{\mathrm{fn}})\cdot u^2(4|D|)^{r-1}H_p(f)^{-1}\times$$
$$\times\left(1-\frac{p^{2r-2}}{\alpha_p(f)^2}\right)L_p'(f\otimes K,\mathcal{C},\mathbb{1}) \tag{6.3.2}$$

with

$$(\widetilde{\mathrm{fn}}) = (\alpha_p(f)-\mathcal{C}(\mathfrak{p}))(\alpha_p(f)\mathcal{C}(\mathfrak{p}) - p^{2r-2})(\alpha_p(f)-\mathcal{C}(\overline{\mathfrak{p}}))(\alpha_p(f)\mathcal{C}(\overline{\mathfrak{p}}) - p^{2r-2})$$

As neither $\alpha_p(f)$ nor $p^{2r-2}\alpha_p(f)^{-1}$ are roots of unity (by Weil's conjectures, [De 1], [De 2]), we can divide 6.3.2 by $(\widetilde{\mathrm{fn}})$. Using I.2.4.2 and the definition of $H_p(f)$, we finally obtain

$$L_p'(f\otimes K,\mathcal{C},\mathbb{1}) = -\prod_{\mathfrak{p}|p}\left(1-\frac{\mathcal{C}(\mathfrak{p})p^{r-1}}{\alpha_p(f)}\right)\left(1-\frac{\mathcal{C}(\overline{\mathfrak{p}})p^{r-1}}{\alpha_p(f)}\right)\frac{h\langle z_{\beta,\mathcal{C}}, z_{\beta,\overline{\mathcal{C}}}\rangle}{u^2(4|D|)^{r-1}}$$

Theorem A is proved.

(6.4) It seems that Thm. 1.3 of [PR 2] should also contain a minus sign, as our Thm. A does. Apparently the sign is missed in [PR 2, 2.3], where one should pass from λ to $\widetilde{\lambda}$ through the reciprocity map

$$\mathbf{Z}_p^*\hookrightarrow\mathbf{Q}_p^*\longrightarrow\mathbf{A}_{\mathbf{Q}}^*/\mathbf{Q}^*\longrightarrow G(\mathbf{Q}(\mu_{p^\infty})/\mathbf{Q}),$$

not by the cyclotomic character

$$\chi_{\mathrm{cycl}}: G(\mathbf{Q}(\mu_{p^\infty})/\mathbf{Q})\longrightarrow {\mathbf{Z}_p}^*$$

There is also a sign error in the formulas for the local maps $\ell_{\nu,v}(x)$ in [PR 2, p. 501], which disappears on the next page.

If we are unwilling to renormalize the p-adic L-function, the only way to get rid of the sign in Thm. A seems to be a sign change in the definition of the p-adic height in [Ne 2]. This would also require a sign change in [ScP 1], which perhaps would not be such a bad idea, given the main comparison result of [ScP 2].

(6.5) The proof of Theorem 0.8 in [Ne 1] works under the assumptions $p > 2r-1$, $(|D|, N) = 1$ and $p \nmid 2N$. In [Ne 1], it was required that $p \nmid 2N(2r-2)!\varphi(N)$, but a better choice of the lattice J makes the factor $(2r-2)!\varphi(N)$ superfluous. One should take for J the image of

$$\varepsilon_B \varepsilon H^1_{\text{et}}(\overline{M}_N \otimes_{\mathbf{Q}} \overline{\mathbf{Q}}, j_* \mathcal{F})$$

in

$$\varepsilon_B \varepsilon H^1_{\text{et}}(\overline{M}_N \otimes_{\mathbf{Q}} \overline{\mathbf{Q}}, j_* \mathcal{F} \otimes_{\mathbf{Z}_p} \mathbf{Q}_p)$$

The assumption $p > 2r-1$ is explained in 0.13 and should be eliminated by a further progress in p-adic Hodge theory. Note that Theorem 13.1 of [Ne 1] says that (in our language)

$$\widehat{F(f)} \cdot \text{Im}(\Phi_{K,f}) = \widehat{F(f)} \cdot z_{K,f} = H^1_f(K, V_f)$$

However, $\Phi_{K,f}$ commutes with the action of $\mathbf{T}$ and $\mathbf{T} \otimes \mathbf{Q}_p \longrightarrow \widehat{F(f)}$ is surjective, hence

$$\widehat{F(f)} \cdot \text{Im}(\Phi_{K,f}) = \mathbf{Q}_p \cdot \text{Im}(\Phi_{K,f})$$

(6.6) The formula I.5.13.4 reads in our case as follows

$$L_p(f \otimes K, \langle\,\rangle^s \circ \mathbf{N}) = \frac{\langle N \rangle^{2s} (-1)^{r-1} |D|^{1/2}}{2^{2r-1} (2\pi)^2 \langle f, f \rangle_N} L_p^{\text{MTT}}(f \otimes K, x^{r-1} \langle\,\rangle^s),$$

which proves 0.9.1.

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