On p-adic height pairings

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The aim of the present work is to construct p-adic height pairings in a sufficiently general setting, namely for Selmer groups of reasonably behaved p-adic Galois representations over number fields. These pairings should, modulo general conjectures on étale cohomology, induce height pairings between (homologically trivial) algebraic cycles on every proper smooth variety defined over a number field. For smooth projective varieties with good reduction at all primes dividing p, our construction requires only one cohomological assumption: we have to assume that the purity conjecture for certain étale cohomology group holds at all places of bad reduction of the given variety.

More specifically, let X be a proper smooth variety, of dimension d, defined over a number field K. Write $CH^i(X)_0$ for the group of homologically trivial algebraic cycles of codimension i on X, defined over K, modulo rational equivalence. If $i+j=d+1$, then there are "étale Abel-Jacobi maps"

$$\Phi_i : CH^i(X)_0 \longrightarrow H^1(K,V)$$
$$\Phi_j : CH^j(X)_0 \longrightarrow H^1(K,V^*(1)),$$

where $V = H^{2i-1}_{\text{et}}(X \otimes \overline{K}, \mathbf{Q}_p(i))$. For each p-adic representation V of $G(\overline{K}/K)$, there is a locally defined subgroup $H^1_f(K,V) \subseteq H^1(K,V)$, generalizing the classical Selmer group. We shall construct pairings

$$H^1_f(K,V) \times H^1_f(K,V^*(1)) \longrightarrow \mathbf{Q}_p$$

for 'good' p-adic Galois representations V and show that, for 'good' X, $H^{2i-1}_{\text{et}}(X \otimes \overline{K}, \mathbf{Q}_p(i))$ is good and both maps Φ factor through H^1_f.

In fact, we construct a family of height pairings, depending on the choice of a global p-adic logarithm (which amounts, essentially, to the choice of a $\mathbf{Z}_p$-extension of K) and splittings of certain Hodge filtrations associated to V at primes of K dividing p. For V satisfying "Pančiškin's condition" of [Pe-Ri 3], there is a canonical choice of these splittings, giving rise to a canonical height pairing. Previous constructions of p-adic height pairings worked mainly for abelian varieties (cf. [Co–Gr], [Ma–Ta], [Pe-Ri 1], [Sch 2,3], [Wi], [Za] – the list is by no means exhaustive), only the recent work of B. Perrin-Riou [Pe-Ri 3] treats the subject in a generality similar to ours.

The contents of the paper is the following: in sec. 1 we collect relevant facts about p-adic Galois representations and filtered modules which will be used in the sequel. Although there might be some overlap with the seminar notes [Bu], we preferred to keep the paper reasonably self-contained. In sec. 2 we define abstract height pairings, using the method of Bloch [Bl] and the ideas of Zarhin [Za]. In sec. 3 we describe the linear algebra underlying splittings of local extensions and their relationship to splittings of Hodge filtrations. In sec. 4 we define local height pairings (following closely the constructions in [Sc 2]) and prove that the height pairing constructed in sec. 2 is equal to the sum of the local height pairings. In sec. 5 we discuss the geometric situation. In sec. 6 we study another way of defining height pairings, using splittings coming from universal norms. We show that, in the good reduction case, the canonical height pairing for representations satisfying Pančiškin's condition can indeed be obtained from universal norms splittings. In sec. 7 we treat the semistable (i.e. the bad reduction) case. In this case, for representations satisfying Pančiškin's condition, universal norms define a pairing different from the canonical one, and the difference between them can be neatly described (following [Ma–Ta–Te]) in terms of an "extended height pairing". In sec. 8 we clarify the relationship of our construction to previous definitions for abelian varieties, as well to the work of B.Perrin-Riou [Pe-Ri 3].

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1. p-adic Galois representations

1.1 In this section we recall basic facts about p-adic Galois representations and associated filtered modules and discuss some complements to [Bl–Ka,ch.3]. The reader should consult [Fo–Il],[Il] for further details. Throughout the paper, all Galois cohomology groups will be continuous cohomology groups, see [Ta],[Ja 2] for further details.

1.2 Let K be a finite extension of $\mathbf{Q}_p$ with ring of integers $\mathcal{O}_K$. Denote by K_0 its maximal subfield unramified over $\mathbf{Q}_p$, by ϕ the absolute Frobenius acting on K_0 (ϕ induces the p-th power map on the residue field of K_0) and by $\mathbf{C}_p$ the completion of $\overline{K}$. Write G for the Galois group $G(\overline{K}/K)$, $I = G(\overline{K}/K_{ur})$ for the inertia group and

$$\chi_K : G \longrightarrow \mathbf{Z}_p^*$$

for the cyclotomic character giving the action of G on the Tate module $\mathbf{Z}_p(1)$ of the multiplicative group.

1.3 Denote by

$$B_{\mathrm{cris}} \subset B_{\mathrm{st}} \subset B_{\mathrm{dR}}$$

the topological rings constructed by Fontaine (see [Fo 2],[Fo–Il],[Fo–Me]). They are equipped with the following structures:

(1) A continuous action of G on all of them

(2) G-equivariant embeddings

$$K_0^{ur} = \mathbf{Q}_p^{ur} \hookrightarrow B_{\mathrm{cris}}, \qquad \overline{K} \hookrightarrow B_{\mathrm{dR}}$$

such that the induced maps

$$K \otimes_{K_0} B_{\mathrm{st}} \longrightarrow B_{\mathrm{dR}}, \qquad \overline{K} \otimes_{K_0^{ur}} B_{\mathrm{st}} \longrightarrow B_{\mathrm{dR}}$$

are injective and $B_{\mathrm{cris}}^G = B_{\mathrm{st}}^G = K_0$, $B_{\mathrm{dR}}^G = K$.

(3) A decreasing filtration of B_{dR} by $\overline{K}$-vector spaces $(F^i B_{\mathrm{dR}})_{i\in\mathbf{Z}}$ satisfying

$$\bigcup F^i B_{\mathrm{dR}} = B_{\mathrm{dR}}, \qquad \bigcap F^i B_{\mathrm{dR}} = 0.$$

(4) A G-equivariant embedding

$$\mathbf{Q}_p(1) \hookrightarrow F^1 B_{\mathrm{dR}}.$$

If t is a non-zero element in $\mathbf{Q}_p(1)$, then

$$F^i B_{\mathrm{dR}} = t^i F^0 B_{\mathrm{dR}}, \qquad F^i B_{\mathrm{dR}} / F^{i+1} B_{\mathrm{dR}} \xrightarrow{\sim} \mathbf{C}_p(i).$$

(5) A ϕ-semilinear continuous automorphism $f : B_{\mathrm{st}} \longrightarrow B_{\mathrm{st}}$, preserving B_{cris}.

(6) A B_{cris}-linear derivation $N : B_{\mathrm{st}} \longrightarrow B_{\mathrm{st}}$ satisfying $Nf = pfN$ such that

$$0 \longrightarrow B_{\mathrm{cris}} \longrightarrow B_{\mathrm{st}} \xrightarrow{N} B_{\mathrm{st}} \longrightarrow 0$$

is exact.

All these structures are canonical, except for N, which depends on the choice of a valuation $v : \overline{K}^* \longrightarrow \mathbf{Q}$, and the embedding $B_{\mathrm{st}} \hookrightarrow B_{\mathrm{dR}}$, which depends on the choice of a p-adic logarithm, i.e. of a G-equivariant homomorphism $\mathrm{Log} : \overline{K}^* \longrightarrow \overline{K}$, which coincides with the usual power series on the group of principal units. For convenience, we also use abbreviation B_{ur} for the completion of the maximal unramified extension of $\mathbf{Q}_p$. It is a subring of B_{cris}.

1.4 Let V be a p-adic representation of G, that is a vector space over $\mathbf{Q}_p$ of finite dimension with a continuous linear action of G. Define for $* \in \{\mathrm{ur}, \mathrm{cris}, \mathrm{st}, \mathrm{dR}\}$

$$D_*(V) = (V \otimes_{\mathbf{Q}_p} B_*)^G.$$

The map

$$\alpha_* : D_*(V) \otimes_{K_*} B_* \longrightarrow V \otimes_{\mathbf{Q}_p} B_*$$

(where $K_* = B_*^G$, equal to either K or K_0) is always injective, which implies that

$$\dim_{K_*} D_*(V) \leq \dim_{\mathbf{Q}_p}(V)$$

with equality iff α_* is bijective.

The representation V is called crystalline (resp. semistable, resp. de Rham) if α_{cris} (resp. α_{st}, resp. α_{dR}) is an isomorphism. Such representations form a $\mathbf{Q}_p$-linear tensor category Rep_{cris} (resp. Rep_{st}, resp. Rep_{dR}), stable under subquotients, duals and Tate twists. For $* = \mathrm{ur}$, we get the usual category of unramified representations of $G(\overline{K}/K)$, i.e. those on which I acts trivially.

1.5 The groups $D_*(V)$ inherit from B_* the following structures:

- $D_{\mathrm{cris}}(V)$ is a vector space over K_0 equipped with a ϕ-semilinear bijective map $f : D_{\mathrm{cris}}(V) \longrightarrow D_{\mathrm{cris}}(V)$
- $D_{\mathrm{st}}(V)$ is a vector space over K_0 equipped with a ϕ-semilinear bijective map $f : D_{\mathrm{st}}(V) \longrightarrow D_{\mathrm{st}}(V)$ and a K_0-linear map $N : D_{\mathrm{st}}(V) \longrightarrow D_{\mathrm{st}}(V)$ satisfying $Nf = pfN$.
- $DR(V) := D_{\mathrm{dR}}(V)$ is a vector space over K with a decreasing filtration by subspaces $(F^i DR(V))_{i\in\mathbf{Z}}$ satisfying $\bigcup F^i DR(V) = DR(V)$, $\bigcap F^i DR(V) = 0$.

They are related as follows:

- $D_{\mathrm{cris}}(V) = D_{\mathrm{st}}(V)^{N=0} = \mathrm{Ker}(N)$
- The canonical map $D_{\mathrm{st}}(V) \otimes_{K_0} K \longrightarrow DR(V)$ is injective
- $\dim_{K_0} D_{\mathrm{cris}}(V) \leq \dim_{K_0} D_{\mathrm{st}}(V) \leq \dim_K DR(V) \leq \dim_{\mathbf{Q}_p}(V)$
- V is crystalline $\Longrightarrow$ V is semistable $\Longrightarrow$ V is de Rham
- The Tate twist $D(n)$ is given by tensoring with

$$D_{\mathrm{cris}}(\mathbf{Q}_p(n)) = D_{\mathrm{st}}(\mathbf{Q}_p(n)) = (K_0, f = p^{-n}\phi, N = 0)$$

resp. $DR(\mathbf{Q}_p(n)) = (K, F^{-n}DR = K, F^{-n+1}DR = 0)$.

It will be more convenient to consider the monodromy operator N as a f-equivariant map

$$N : D_{\mathrm{st}}(V) \longrightarrow D_{\mathrm{st}}(V)(-1) = D_{\mathrm{st}}(V(-1)).$$

1.6 According to the fundamental results of Fontaine [Fo 1],[Fo-Il], a crystalline resp. semistable representation V can be recovered from $D_{\mathrm{cris}}(V)$ resp. $D_{\mathrm{st}}(V)$ as follows:

Let $MF_K(f, N)$ be the category of finite-dimensional vector spaces D over K_0 equipped with

- A ϕ-semilinear bijective map $f : D \longrightarrow D$
- A K_0-linear map $N : D \longrightarrow D$ satisfying $Nf = pfN$
- A decreasing filtration $F^i D_K$ by subspaces of $D_K = D \otimes_{K_0} K$ satisfying $\bigcup F^i D_K = D_K$, $\bigcap F^i D_K = 0$

(morphisms being $\mathbf{Q}_p$-linear maps compatible with $f, N, F^{\cdot}$).

The functor

$$D_{\mathrm{st}} : Rep_{\mathrm{st}}(G) \longrightarrow MF_K(f, N)$$

is fully faithful and induces a tensor equivalence between $Rep_{\mathrm{st}}(G)$ and its essential image $MF_K^{ad}(f, N)$ ("admissible filtered (f, N)-modules"). A quasi-inverse functor V_{st} of D_{st} is given by

$$V_{\mathrm{st}}(D) = (D \otimes_{K_0} B_{\mathrm{st}})^{f=1, N=0} \cap F^0(D_K \otimes_K B_{\mathrm{dR}}).$$

Similarly, the filtered f-modules with $N = 0$ form a category MF_K^f and

$$D_{\mathrm{cris}} : Rep_{\mathrm{cris}}(G) \longrightarrow MF_K^f$$

induces a tensor equivalence between $Rep_{\text{cris}}(G)$ and $MF_K^{ad} := MF_K^{ad}(f,N) \cap MF_K^f$, a quasi-inverse of D_{cris} being

$$V_{\text{cris}}(D) = (D \otimes_{K_0} B_{\text{cris}})^{f=1} \cap F^0(D_K \otimes_K B_{\text{dR}}).$$

1.7 The abstract correspondence between p-adic Galois representations and filtered modules is a realization of Grothendieck's "mysterious functor" between étale and de Rham cohomology.

Let $X_{/K}$ be a proper smooth K-scheme with good reduction, let $\mathcal{X}$ be a proper smooth $\mathcal{O}_K$-scheme with generic fibre $\mathcal{X} \otimes K$ isomorphic to X. Write $\mathcal{X}_s$ for the special fibre of $\mathcal{X}$. Fix an integer $m \geq 0$. Then

- $V := H^m_{\text{et}}(X \otimes \overline{K}, \mathbf{Q}_p)$ is a p-adic representation of G
- The crystalline cohomology of the special fiber $D := H^m_{\text{cris}}(\mathcal{X}_s/\mathcal{O}_{K_0}) \otimes_{\mathcal{O}_{K_0}} K_0$ admits a ϕ-semilinear bijective endomorphism f
- [Be–Og] There is a canonical isomorphism

$$D \otimes_{K_0} K \xrightarrow{\sim} H^m_{\text{dR}}(X_{/K})$$

and the Hodge filtration on H^m_{dR} makes D a filtered f-module.

1.8 Theorem. [Fa] *In the situation of 1.7, V is a crystalline representation, D is an admissible f-module and there are canonical isomorphisms*

$$D \xrightarrow{\sim} D_{\text{cris}}(V), \qquad V \xrightarrow{\sim} V_{\text{cris}}(D),$$

inducing an isomorphism

$$(DR(V), F^{\cdot}) \xrightarrow{\sim} (H^m_{\text{dR}}(X_{/K}), \text{Hodge filtration}).$$

1.9 Suppose that $\mathcal{X}$ is a proper flat $\mathcal{O}_K$-scheme with semistable reduction (i.e. the special fibre $\mathcal{X}_s$ is a reduced divisor with normal crossing) and generic fibre X. Choose a prime element $\pi \in \mathcal{O}_K$, which in turn determines a p-adic logarithm Log_π characterized by $\text{Log}_\pi(\pi) = 0$, hence an embedding $B_{\text{st}} \hookrightarrow B_{\text{dR}}$. Fix an integer $m \geq 0$. Then

- $V := H^m_{\text{et}}(X \otimes \overline{K}, \mathbf{Q}_p)$ is a p-adic representation of G
- The cohomology of the de Rham-Witt complex with logarithmic singularities along the special fiber

$$D := H^m(\mathcal{X}_s, W\omega^{\cdot}_{\mathcal{X}_s}) \otimes_{\mathcal{O}_{K_0}} K_0$$

admits a ϕ-semilinear bijective endomorphism f and a nilpotent automorphism N satisfying $Nf = pfN$.

- [Hy–Ka] There is a canonical isomorphism (depending on the choice of π)

$$D \otimes_{K_0} K \xrightarrow{\sim} H^m_{\text{dR}}(X_{/K})$$

and the Hodge filtration on the second group makes D a filtered (N, f)-module.

1.10 Theorem. [Ka] *In the situation of 1.9, suppose that $m < (p-1)/2$. Then V is a semistable representation, D is an admissible (N, f)-module and there are canonical isomorphisms*

$$D \xrightarrow{\sim} D_{\text{st}}(V), \qquad V \xrightarrow{\sim} V_{\text{st}}(D)$$

inducing an isomorphism

$$(DR(V), F^{\cdot}) \xrightarrow{\sim} (H^m_{\text{dR}}(X_{/K}), \text{Hodge filtration}).$$

It is conjectured [Fo–Il] that the statement of the theorem is valid for all m and X, after passing to a suitable finite extension L/K.

1.11 Theorem. [Fa] *Let X be a smooth K-scheme, $m \geq 0$ an integer. Then $V := H^m_{\text{et}}(X \otimes \overline{K}, \mathbf{Q}_p)$ is a de Rham representation and there is a canonical isomorphism*

$$(DR(V), F^\cdot) \xrightarrow{\sim} (H^m_{\text{dR}}(X/K), \text{Hodge filtration}).$$

It is conjectured [Fo 2] that the same should be true without the smoothness assumption.

1.12 We now recall the main points of [Bl–Ka, ch.3] and add some remarks concerning the semistable case. Let V be a p-adic representation of G. Define for $* \in \{\text{ur}, \text{cris}, \text{st}, \text{dR}\}$

$$H^1_*(K,V) = \text{Ker}(H^1(K,V) \longrightarrow H^1(K, V \otimes B_*)).$$

If V is a $*$-representation, then $H^1_*(K,V)$ classifies those extensions

$$0 \longrightarrow V \longrightarrow E \longrightarrow \mathbf{Q}_p(0) \longrightarrow 0$$

in which E is again a $*$-representation. We use abbreviations of [Bl–Ka] $H^1_f := H^1_{\text{cris}}$, $H^1_g := H^1_{\text{dR}}$ and define also

$$H^1_e(K,V) = \text{Ker}(H^1(K,V) \longrightarrow H^1(K, V \otimes B^{f=1}_{\text{cris}})).$$

We have obvious inclusions

$$H^1_{ur} \subseteq H^1_f \subseteq H^1_{\text{st}} \subseteq H^1_g$$

and $H^1_{ur}(K,V) = H^1(G/I, V^I)$ is the usual group of unramified cohomology classes (cf. [Fl]). If $T \subset V$ is a $G(\overline{K}/K)$-equivariant $\mathbf{Z}_p$-lattice in V, define $H^1_*(K,T)$ to be the inverse image of $H^1_*(K,V)$ in $H^1(K,T)$. For example, for $V = \mathbf{Q}_p(1)$ we have $H^1(K, \mathbf{Q}_p(1)) = K^* \widehat{\otimes} \mathbf{Q}_p$, where the symbol $A \widehat{\otimes} \mathbf{Q}_p$ stands for $(\varprojlim_n (A/p^n A)) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$, and

$$H^1_e(K, \mathbf{Q}_p(1)) = H^1_f(K, \mathbf{Q}_p(1)) = \mathcal{O}_K^* \widehat{\otimes} \mathbf{Q}_p \subseteq H^1(K, \mathbf{Q}_p(1)) = K^* \widehat{\otimes} \mathbf{Q}_p.$$

1.13 Recall that according to [Se, II.5.2 Th.2] the cup product

$$H^i(K,V) \times H^{2-i}(K, V^*(1)) \xrightarrow{\cup} H^2(K, \mathbf{Q}_p(1)) \xrightarrow{\sim} \mathbf{Q}_p$$

is a perfect duality for any $i = 0, 1, 2$ and that $H^i(K,V) = 0$ for $i \geq 3$.

1.14 Theorem. [Bl–Ka, 3.8] *If V is a de Rham representation, then in the perfect duality*

$$H^1(K,V) \times H^1(K, V^*(1)) \xrightarrow{\cup} H^2(K, \mathbf{Q}_p(1)) \xrightarrow{\sim} \mathbf{Q}_p$$

the annihilators of various subgroups of $H^1(K,V)$ are

$$H^1_e(V)^\perp = H^1_g(V^*(1)), \quad H^1_f(V)^\perp = H^1_f(V^*(1)), \quad H^1_g(V)^\perp = H^1_e(V^*(1))$$

($V^*(1) = \text{Hom}(V, \mathbf{Q}_p(1))$ is the Tate twist of the representation contragredient to V).

1.15 Theorem. [Bl–Ka, 3.8.4] *Let V be a de Rham representation. Then there is a commutative diagram with exact rows*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^0(K,V) & \longrightarrow & D & \xrightarrow{(1-f,i)} & D \oplus DR(V)/F^0 & \longrightarrow & H^1_f(K,V) & \longrightarrow & 0 \\
 & & \| & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & H^0(K,V) & \longrightarrow & D^{f=1} & \xrightarrow{i} & DR(V)/F^0 & \xrightarrow{exp} & H^1_e(K,V) & \longrightarrow & 0
\end{array}$$

in which $D = D_{\text{cris}}(V)$ and i is induced by the inclusion $D_{\text{cris}}(V) \hookrightarrow DR(V)$.

It comes from exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbf{Q}_p & \longrightarrow & B_{\text{cris}} & \xrightarrow{(1-f,i)} & B_{\text{cris}} \oplus B_{\text{dR}}/F^0 & \longrightarrow & 0 \\
 & & \| & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathbf{Q}_p & \longrightarrow & B^{f=1}_{\text{cris}} & \xrightarrow{i} & B_{\text{dR}}/F^0 & \longrightarrow & 0
\end{array}$$

on tensoring with V and taking cohomology.

1.16 Corollary. *There is a canonical exact sequence*

$$0 \longrightarrow H^1_e(K,V) \longrightarrow H^1_f(K,V) \longrightarrow D_{\rm cris}(V)/(f-1)D_{\rm cris}(V) \longrightarrow 0$$

1.17 Theorem. *If V is a de Rham representation, denote $D := D_{\rm st}(V)$ and define X by the cartesian square*

$$\begin{array}{ccc} X & \xrightarrow{\alpha} & D \\ \Big\downarrow{\scriptstyle\beta} & & \Big\downarrow{\scriptstyle -N} \\ D(-1) & \xrightarrow{1-f} & D(-1) \end{array}$$

Then there is a commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & 0 & & \\ & & & & \Big\downarrow & & \Big\downarrow & & \Big\downarrow & & \\ 0 & \longrightarrow & H^0(K,V) & \longrightarrow & D_{\rm cris}(V) & \xrightarrow{(1-f,i)} & D \oplus DR(V)/F^0 & \longrightarrow & H^1_f(K,V) & \longrightarrow & 0 \\ & & \Vert & & \Big\downarrow & & \Big\downarrow & & \Big\downarrow & & \\ 0 & \longrightarrow & H^0(K,V) & \longrightarrow & D & \xrightarrow{(j,i)} & X \oplus DR(V)/F^0 & \longrightarrow & H^1_{\rm st}(K,V) & \longrightarrow & 0 \\ & & & & \Big\downarrow{\scriptstyle -N} & & \Big\downarrow{\scriptstyle\beta} & & & & \\ & & & & D(-1) & = & D(-1) & & & & \end{array}$$

in which i is induced by the inclusion $D_{\rm cris}(V) \hookrightarrow DR(V)$ and $j : D \longrightarrow X$ is characterized by $\alpha \circ j = 1 - f$, $\beta \circ j = -N$.

Proof. Define Y by the cartesian square

$$\begin{array}{ccc} Y & \xrightarrow{\alpha} & B_{\rm st} \\ \Big\downarrow{\scriptstyle\beta} & & \Big\downarrow{\scriptstyle -N} \\ B_{\rm st} & \xrightarrow{1-pf} & B_{\rm st} \end{array}$$

and let $j : B_{\rm st} \longrightarrow Y$ be the unique map satisfying $\alpha \circ j = 1 - f$, $\beta \circ j = -N$. Then the exact sequence [Bl–Ka, 1.17.2] can be completed to a commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \Big\downarrow & & \Big\downarrow & & \\ 0 & \longrightarrow & \mathbf{Q}_p & \longrightarrow & B_{\rm cris} & \xrightarrow{(1-f,i)} & B_{\rm cris} \oplus B_{\rm dR}/F^0 & \longrightarrow & 0 \\ & & \Vert & & \Big\downarrow & & \Big\downarrow & & \\ 0 & \longrightarrow & \mathbf{Q}_p & \longrightarrow & B_{\rm st} & \xrightarrow{(j,i)} & Y \oplus B_{\rm dR}/F^0 & \longrightarrow & 0 \\ & & & & \Big\downarrow{\scriptstyle -N} & & \Big\downarrow{\scriptstyle\beta} & & \\ & & & & B_{\rm st} & = & B_{\rm st} & & \\ & & & & \Big\downarrow & & \Big\downarrow & & \\ & & & & 0 & & 0 & & \end{array}$$

On tensoring with V and taking cohomology we obtain the desired diagram (of course, we must check that each epimorphism in the diagram admits a continuous section; for N this follows from the explicit description of $B_{\rm st}$ as the polynomial ring in one variable over $B_{\rm cris}$ (see [Fo–Il]), for $(1-f,i)$ this is proved in [Bl–Ka, 1.18] and the claim for the remaining two maps β and (j,i) immediately follows).

1.18 Corollary. *If V is a semistable representation, then there is a canonical exact sequence*

$$0 \longrightarrow H^1_f(K,V) \longrightarrow H^1_{\mathrm{st}}(K,V) \longrightarrow (D_{\mathrm{cris}}(V^*(1))^*)^{f=1} \longrightarrow 0$$

Proof. Snake lemma applied to 1.17 gives an exact sequence

$$0 \longrightarrow H^1_f(K,V) \longrightarrow H^1_{\mathrm{st}}(K,V) \longrightarrow \mathrm{Im}(\beta)/\mathrm{Im}(N) \longrightarrow 0,$$

but the third group is equal to $(D_{\mathrm{st}}(V(-1))/ND_{\mathrm{st}}(V))^{f=1}$. Since D_{st} commutes with Tate twists and duality on $\mathit{Rep}_{\mathrm{st}}$, the group in the brackets is dual to $D_{\mathrm{st}}(V^*(1))^{N=0} = D_{\mathrm{cris}}(V^*(1))$.

1.19 Let V be a de Rham representation. Define two complexes (of K_0-vector spaces)

$$C^{\cdot}_{\mathrm{cris}}(V) := [D_{\mathrm{cris}}(V) \xrightarrow{(f-1,-i)} D_{\mathrm{cris}}(V) \oplus DR(V)/F^0]$$
$$C^{\cdot}_{\mathrm{st}}(V) := [D_{\mathrm{st}}(V) \xrightarrow{(f-1,N,-i)} D_{\mathrm{st}}(V) \oplus D_{\mathrm{st}}(V(-1)) \oplus DR(V)/F^0 \xrightarrow{(N,1-f,0)} D_{\mathrm{st}}(V(-1))]$$

in degrees $0,1$ and $0,1,2$ respectively (note the change of signs with respect to 1.15 and 1.17). We get from 1.15 and 1.17 isomorphisms

$$\alpha_V : H^0(K,V) \xrightarrow{\sim} H^0(C^{\cdot}_{\mathrm{cris}}(V)) = H^0(C^{\cdot}_{\mathrm{st}}(V))$$
$$\beta_{V,\mathrm{cris}} : H^1(C^{\cdot}_{\mathrm{cris}}(V)) \xrightarrow{\sim} H^1_f(K,V)$$
$$\beta_{V,\mathrm{st}} : H^1(C^{\cdot}_{\mathrm{st}}(V)) \xrightarrow{\sim} H^1_{\mathrm{st}}(K,V)$$

(with β's induced by boundary maps in cohomology exact sequences).
These isomorphisms are functorial (i.e. $\alpha_W \circ \phi = \phi \circ \alpha_V$, $\beta_W \circ \phi = \phi \circ \beta_V$ for each homomorphism $\phi : V \longrightarrow W$ between de Rham representations) and the map induced on H^1 by inclusion $C^{\cdot}_{\mathrm{cris}}(V) \hookrightarrow C^{\cdot}_{\mathrm{st}}(V)$ is compatible (via β's) with the natural map $H^1_f(K,V) \longrightarrow H^1_{\mathrm{st}}(K,V)$.

1.20 Let V be a semistable representation. Suppose we are given an extension

$$0 \longrightarrow D_{\mathrm{st}}(V) \longrightarrow D \longrightarrow K_0 \longrightarrow 0$$

in the category of filtered (f,N)-modules (where K_0 stands for $D_{\mathrm{st}}(\mathbf{Q}_p(0))$). Writing D_K for $D \otimes_{K_0} K$, we have a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & DR(V) & \longrightarrow & D_K & \longrightarrow & K & \longrightarrow & 0 \\
 & & \big\downarrow & & \big\downarrow{\scriptstyle \pi} & & \big\downarrow & & \\
0 & \longrightarrow & DR(V)/F^0 & \xrightarrow{\sim} & D_K/F^0 & \longrightarrow & 0 & \longrightarrow & 0.
\end{array}$$

Choose a K_0-linear section $s : K_0 \longrightarrow D$ and define a triple

$$[x,y,z] := [(f-1)s(1), Ns(1), -\pi \circ i(s(1))] \in D_{\mathrm{st}}(V) \oplus D_{\mathrm{st}}(V(-1)) \oplus DR(V)/F^0,$$

which is in fact a cycle (in degree 1) in the complex $C^{\cdot}_{\mathrm{st}}(V)$. Moreover, its class in $H^1(C^{\cdot}_{\mathrm{st}}(V))$, to be denoted $\rho(D)$, does not depend on the choice of s. It is easily seen that ρ in fact defines an *injective* $\mathbf{Q}_p$-linear map

$$\rho_{V,\mathrm{st}} : \mathrm{Ext}^1_{MF_K(f,N)}(K_0, D_{\mathrm{st}}(V)) \hookrightarrow H^1(C^{\cdot}_{\mathrm{st}}(V)).$$

Since admissible (f,N)-modules form a full subcategory, the canonical map

$$\tau_{V,\mathrm{st}} : H^1_{\mathrm{st}}(K,V) = \mathrm{Ext}^1_{MF^{ad}_K(f,N)}(K_0, D_{\mathrm{st}}(V)) \hookrightarrow \mathrm{Ext}^1_{MF_K(f,N)}(K_0, D_{\mathrm{st}}(V))$$

is injective as well.

Restricting ourselves to filtered modules with $N=0$, we obtain, for crystalline V, an injective homomorphism

$$\rho_{V,\mathrm{cris}} : \mathrm{Ext}^1_{MF_K}(K_0, D_{\mathrm{cris}}(V)) \hookrightarrow H^1(C^{\cdot}_{\mathrm{cris}}(V)).$$

1.21 Proposition. *If V is crystalline (resp. semistable), then the composite*

$$\rho_{V,\mathrm{cris}} \circ \tau_{V,\mathrm{cris}} \circ \beta_{V,\mathrm{cris}} : H^1(C^{\cdot}_{\mathrm{cris}}(V)) \longrightarrow H^1(C^{\cdot}_{\mathrm{cris}}(V))$$

resp.

$$\rho_{V,\mathrm{st}} \circ \tau_{V,\mathrm{st}} \circ \beta_{V,\mathrm{st}} : H^1(C^{\cdot}_{\mathrm{st}}(V)) \longrightarrow H^1(C^{\cdot}_{\mathrm{st}}(V))$$

is the identity map.

Proof. We shall treat only the semistable case. Let $[x, y, z] \in D_{\mathrm{st}}(V) \oplus D_{\mathrm{st}}(V(-1)) \oplus DR(V)/F^0$ represent an element in $H^1(C^{\cdot}_{\mathrm{st}}(V))$. Then, in the notation of 1.17, $[x, y] \in X = H^0(K, V \otimes Y)$, $z \in H^0(K, V \otimes B_{\mathrm{dR}}/F^0)$. Define

$$\gamma = (\gamma_1, \gamma_2) : \mathbf{Q}_p \longrightarrow (V \otimes Y) \oplus (V \otimes B_{\mathrm{dR}}/F^0)$$

by $\gamma_1(1) = [x, y]$, $\gamma_2(1) = z$. The boundary $\beta_{V,\mathrm{st}}([x, y, z])$ is then represented by the extension

$$0 \longrightarrow V \otimes \mathbf{Q}_p \longrightarrow E \longrightarrow \mathbf{Q}_p(0) \longrightarrow 0,$$

which is the pullback of

$$0 \longrightarrow V \otimes \mathbf{Q}_p \longrightarrow V \otimes B_{\mathrm{st}} \xrightarrow{id \otimes (j,i)} (V \otimes Y) \oplus (V \otimes B_{\mathrm{dR}}/F^0) \longrightarrow 0$$

via γ. Choose $u \in V \otimes B_{\mathrm{st}}$ which is mapped to $\gamma(1) = [x, y, z]$. Then

$$E = (V \otimes \mathbf{Q}_p) + \mathbf{Q}_p \cdot u \hookrightarrow V \otimes B_{\mathrm{st}}.$$

This embedding is G-equivariant and the action of $g \in G$ is given by

$$g(v \otimes 1) = gv \otimes 1, \qquad gu - u = v(g) \otimes 1,$$

where $v(g) \in V$ and $g \mapsto v(g)$ is a 1-cocycle representing $\beta_{V,\mathrm{st}}([x, y, z])$. We have

$$D_{\mathrm{st}}(E) = (E \otimes B_{\mathrm{st}})^G = (V \otimes \mathbf{Q}_p \otimes B_{\mathrm{st}} + u \otimes B_{\mathrm{st}})^G \hookrightarrow V \otimes B_{\mathrm{st}} \otimes B_{\mathrm{st}}.$$

The action of f resp. N on $D_{\mathrm{st}}(E)$ is given by $id \otimes id \otimes f$ resp. $id \otimes id \otimes N$.

Denote by $s_{23} : V \otimes B_{\mathrm{st}} \otimes B_{\mathrm{st}} \longrightarrow V \otimes B_{\mathrm{st}} \otimes B_{\mathrm{st}}$ the operator interchanging the two factors B_{st}. The canonical embedding $D_{\mathrm{st}}(V) \hookrightarrow D_{\mathrm{st}}(E)$ can be written as $d \mapsto s_{23}(d \otimes 1)$, if we consider $D_{\mathrm{st}}(V)$ as a subgroup of $V \otimes B_{\mathrm{st}}$. As for each $g \in G$

$$(g - 1)s_{23}(u \otimes 1) = v(g) \otimes 1 \otimes 1 = (g - 1)(u \otimes 1)$$

and $s_{23}(u \otimes 1) \in V \otimes \mathbf{Q}_p \otimes B_{\mathrm{st}}$, the difference $u \otimes 1 - s_{23}(u \otimes 1)$ lies in $D_{\mathrm{st}}(E)$. Its image under the map $D_{\mathrm{st}}(E) \longrightarrow D_{\mathrm{st}}(\mathbf{Q}_p)$ is equal to 1, hence we can choose a section s as in 1.20 satisfying $s(1) = u \otimes 1 - s_{23}(u \otimes 1)$. Since f, N act on $D_{\mathrm{st}}(E) \subset V \otimes B_{\mathrm{st}} \otimes B_{\mathrm{st}}$ via the third factor, we have

$$(f - 1)s(1) = s_{23}((1 - f)u \otimes 1) = s_{23}(x \otimes 1)$$
$$Ns(1) = -Ns_{23}(u \otimes 1) = -s_{23}(Nu \otimes 1) = s_{23}(y \otimes 1)$$

and analogously, using embedding $DR(E) \hookrightarrow V \otimes B_{\mathrm{st}} \otimes B_{\mathrm{dR}}$,

$$-s(1)(\mathrm{mod}\, F^0) = s_{23}(u(\mathrm{mod}\, F^0) \otimes 1) = s_{23}(z \otimes 1).$$

This proves that the triple $[x, y, z] \in Y \oplus DR(V)/F^0$ coincides with $[(f - 1)s(1), Ns(1), -i(s(1))] = \rho_{V,\mathrm{st}} \circ \tau_{V,\mathrm{st}} \circ \beta_{V,\mathrm{st}}([x, y, z])$, which was to be proved.

1.22 Corollary. *If V is semistable, then each extension*

$$0 \longrightarrow D_{\mathrm{st}}(V) \longrightarrow D \longrightarrow K_0 \longrightarrow 0$$

in the category of filtered (f, N)-modules is admissible, i.e. comes from an extension

$$0 \longrightarrow V \longrightarrow E \longrightarrow \mathbf{Q}_p(0) \longrightarrow 0$$

of semistable Galois representations.

1.23 Theorem. *Suppose that $0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$ is an exact sequence of crystalline (resp. semistable) Galois representations over K. Then $0 \longrightarrow C^{\cdot}_*(X) \longrightarrow C^{\cdot}_*(Y) \longrightarrow C^{\cdot}_*(Z) \longrightarrow 0$ is an exact sequence of complexes (for $* =$ cris resp. st) and the corresponding boundary map δ_C fits into a commutative diagram*

$$\begin{array}{ccc} H^0(K,Z) & \stackrel{\alpha_Z}{\longrightarrow} & H^0(C^{\cdot}_*(Z)) \\ \Big\downarrow \delta & & \Big\downarrow \delta_C \\ H^1_*(K,X) & \stackrel{\beta_{X,*}}{\longleftarrow} & H^1(C^{\cdot}_*(X)) \end{array}$$

(where δ is the usual boundary map in the cohomology sequence).

Proof. Exactness of the sequence of $C^{\cdot}$'s is obvious. Let $z \in H^0(K,Z)$. Define a $\mathbf{Q}_p$-linear map $\gamma : \mathbf{Q}_p \longrightarrow Z$ by $\gamma(1) = z$ (hence γ is a morphism of representations), and take the pullback of Y by γ, obtaining an extension

$$0 \longrightarrow X \longrightarrow E \longrightarrow \mathbf{Q}_p(0) \longrightarrow 0.$$

By [Bo, X.7.6.5 Cor 1.a], the class $[E] \in H^1_*(K,V)$ of this extension is equal to $\delta(z)$. We know that α is functorial, i.e. $\gamma \circ \alpha_{\mathbf{Q}_p(0)} = \alpha_Z \circ \gamma$. We are thus reduced to prove that $\beta_{X,*} \circ \delta_C \circ \alpha_{\mathbf{Q}_p(0)}(1) = [E]$, where δ_C is computed for the extension E. By definition of ρ in 1.20, we have $\delta_C \circ \alpha_{\mathbf{Q}_p(0)}(1) = \rho_{X,*} \circ \tau_{X,*}([E])$, and 1.21 tells us that $\beta_{X,*} \circ \delta_C \circ \alpha_{\mathbf{Q}_p(0)}(1) = \beta_{X,*} \circ \rho_{X,*} \circ \tau_{X,*}([E]) = [E]$ as claimed.

1.24 Proposition. *Define, for a p-adic Galois representation V of G, $h^i_*(V) := \dim_{\mathbf{Q}_p} H^i_*(K,V)$. Then*

(1) $h^1(V) = h^0(V) + h^2(V) + [K : \mathbf{Q}_p] \dim_{\mathbf{Q}_p}(V)$, $\qquad h^i(V) = h^{2-i}(V^*(1))$, $\qquad h^1_{ur}(V) = h^0(V)$

(2) *If V is a de Rham representation, then*

$$\begin{aligned} h^1_f(V) &= h^0(V) + [K : \mathbf{Q}_p] \dim_K(DR(V)/F^0) \\ h^1_e(V) &= h^1_f(V) - \dim_{\mathbf{Q}_p} D_{\mathrm{cris}}(V)^{f=1} \\ h^1_g(V) &= h^1_f(V) + \dim_{\mathbf{Q}_p} D_{\mathrm{cris}}(V^*(1))^{f=1} \end{aligned}$$

(3) *If V is a semistable representation, then*

$$H^1_{\mathrm{st}}(K,V) = H^1_g(K,V)$$

(this is proved by a different method in [Hy]*).*

Proof. (1) Follows from the local duality [Se, II.5.2 Th.2] and the formula for the local Euler characteristic [Se, II.5.7. Th.5].

(2) Follows from 1.15 and 1.14, taking into account (1) and equalities $\dim_K F^0 DR(V^*(1)) = \dim_K DR(V)/F^0$, $\dim_K(DR(V)) = \dim_{\mathbf{Q}_p}(V)$.

(3) We know that $H^1_{\mathrm{st}}(K,V)$ is a subspace of $H^1_g(K,V)$, but according to 1.18 and (2) their dimensions are equal.

1.25 Proposition. *Let* $* \in \{\mathrm{ur}, \mathrm{cris}, \mathrm{st}, \mathrm{dR}\}$. *If*

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$$

is an exact sequence of de Rham representations such that $D_*(Y) \longrightarrow D_*(Z)$ *is surjective (in particular if* X, Y, Z *are* $*$*-representations), then the cohomology sequence induces an exact sequence*

$$0 \longrightarrow H^0(K, X) \longrightarrow H^0(K, Y) \longrightarrow H^0(K, Z) \longrightarrow H^1_*(K, X) \longrightarrow H^1_*(K, Y) \longrightarrow H^1_*(K, Z).$$

If $* \in \{\mathrm{ur}, \mathrm{cris}\}$, *then this exact sequence can be extended by zero on the right.*

Proof. The first statement follows from the commutative diagram with exact rows

$$\begin{array}{ccccccc} H^0(Z) & \longrightarrow & H^1(X) & \longrightarrow & H^1(Y) & \longrightarrow & H^1(Z) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H^0(Z \otimes B_*) & \xrightarrow{0} & H^1(X \otimes B_*) & \longrightarrow & H^1(Y \otimes B_*) & \longrightarrow & H^1(Z \otimes B_*) \end{array}$$

For $* \in \{\mathrm{ur}, \mathrm{cris}\}$, the alternating sum of dimensions

$$h^0(X) - h^0(Y) + h^0(Z) - h^1_*(X) + h^1_*(Y) - h^1_*(Z) = 0$$

vanishes by 1.24, which proves the second claim.

1.26 Proposition. *Let* $* \in \{\mathrm{ur}, \mathrm{cris}, \mathrm{st}, \mathrm{dR}\}$, *let* X, Y *be* $*$*-representations. Denote by* $\mathrm{Ext}^1_{K,*}(X, Y) \subseteq \mathrm{Ext}^1_K(X, Y)$ *the* $\mathbf{Q}_p$*-vector space of those extensions*

$$0 \longrightarrow Y \longrightarrow E \longrightarrow X \longrightarrow 0$$

for which E *is a* $*$*-representation. Then there is a canonical isomorphism*

$$\mathrm{Ext}^1_{K,*}(X, Y) \xrightarrow{\sim} H^1_*(K, \mathrm{Hom}(X, Y)).$$

Proof. Since the internal Ext^1 between p-adic Galois representations vanishes, we have $\mathrm{Ext}^1_K(X, Y) \xrightarrow{\sim} H^1(K, \mathrm{Hom}(X, Y))$. This isomorphism can be described explicitly as follows: starting from

$$0 \longrightarrow Y \longrightarrow E \longrightarrow X \longrightarrow 0,$$

we first apply $\mathrm{Hom}(X, -)$ to this extension and then take the pullback via $\mathbf{Q}_p.1_X \hookrightarrow \mathrm{Hom}(X, X)$.
As the category of $*$-representations is stable under duals, tensor products and subobjects, $\mathrm{Ext}^1_{K,*}(X, Y)$ maps to $H^1_*(K, \mathrm{Hom}(X, Y))$. Conversely, suppose that

$$0 \longrightarrow Y \longrightarrow E \longrightarrow X \longrightarrow 0$$

maps to an element in $H^1_*(K, \mathrm{Hom}(X, Y))$. We must show that E is a $*$-representation. The class of E vanishes in $H^1(K, \mathrm{Hom}(X, Y) \otimes B_*)$, hence also in $H^1(K, \mathrm{Hom}(X \otimes B_*, Y \otimes B_*))$. This implies that the boundary map $H^0(K, X \otimes B_*) \longrightarrow H^1(K, Y \otimes B_*)$ is trivial, hence

$$\dim_{K_*} D_*(E) = \dim_{K_*} D_*(X) + \dim_{K_*} D_*(Y) = \dim_{\mathbf{Q}_p}(X) + \dim_{\mathbf{Q}_p}(Y) = \dim_{\mathbf{Q}_p}(E),$$

which means that E is a $*$-representation and we are done.

1.27 Corollary. *If X, Z are semistable representations, then*

(1) Every extension $0 \longrightarrow D_{\mathrm{st}}(X) \longrightarrow D \longrightarrow D_{\mathrm{st}}(Z) \longrightarrow 0$ in the category of filtered (f, N)-modules comes from an extension $0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$ of semistable Galois representations.

(2) If a de Rham representation Y sits in an exact sequence $0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$, then Y is semistable.

Proof. (1) follows from 1.22 and 1.26, while (2) from 1.24.3 and 1.26.

1.28 Proposition. *Suppose that in an exact sequence $0 \longrightarrow X \longrightarrow V \longrightarrow Y \longrightarrow 0$ of p-adic Galois representations both X and Y are de Rham and satisfy $F^0DR(X) = DR(Y)/F^0 = 0$. Then*

(1) *V is de Rham as well and the embedding $X \hookrightarrow V$ induces an isomorphism $DR(X) \xrightarrow{\sim} DR(V)/F^0$.*

(2) *If X, Y are semistable, so is V.*

(3) $D_{\mathrm{st}}(Y(-1))^{f=1} = D_{\mathrm{st}}(X^*)^{f=1} = 0$, $H^0(K, Y) = D_{\mathrm{st}}(Y)^{f=1}$, $H^0(K, X^*(1)) = D_{\mathrm{st}}(X^*(1))^{f=1}$.

(4) $H^1_{ur}(K, Y) = H^1_f(K, Y) = H^1_{\mathrm{st}}(K, Y) = D_{\mathrm{cris}}(Y)/(f-1) = D_{\mathrm{st}}(Y)/(f-1)$.

Proof. (1) Put $Z := \mathrm{Hom}(Y, X)$. This is a de Rham representation and $\mathrm{Ext}^1_K(Y, X) = H^1(K, Z)$, $\mathrm{Ext}^1_{K,\mathrm{dR}}(Y, X) = H^1_g(K, Z)$ by 1.26. As

$$F^0DR(Z) = \{z \in \mathrm{Hom}_K(DR(Y), DR(X)) \,|\, z(F^iDR(Y)) \subseteq F^iDR(X)\ \forall i\} = 0,$$

we have also $H^0(K, Z) = 0$, hence, by 1.24,

$$h^1(Z) = h^0(Z^*(1)) + [K : \mathbf{Q}_p]\dim_{\mathbf{Q}_p}(Z)$$
$$h^1_g(Z) = \dim_{\mathbf{Q}_p}(D_{\mathrm{cris}}(Z^*(1))^{f=1} + [K : \mathbf{Q}_p]\dim_K(DR(Z)/F^0).$$

Since $DR(Z^*(1))/F^0 = 0$, we have also $H^0(K, Z^*(1)) = D_{\mathrm{cris}}(Z^*(1))^{f=1}$, which implies that $H^1(K, Z) = H^1_g(K, Z)$, hence V is de Rham. The kernel resp. cokernel of $DR(X) \longrightarrow DR(V)/F^0$ is then $F^0DR(X) = 0$ resp. $DR(Y)/F^0 = 0$.

(2) Follows from (1) and 1.27.2.

(3) Suppose that $0 \neq x \in D_{\mathrm{st}}(Y(-1))^{f=1}$. Then $N^ix \neq 0$, $N^{i+1}x = 0$ for some $i \geq 0$. As $DR(Y(-i-1)) = F^{i+1}DR(Y(-i-1))$, N^ix lies in $D_{\mathrm{st}}(Y(-i-1))^{f=1,N=0} \cap F^0 = H^0(K, Y(-i-1))$, defining an injective map $\mathbf{Q}_p(i+1) \hookrightarrow Y$, in contradiction to our assumption $DR(Y)/F^0 = 0$. This proves that $D_{\mathrm{st}}(Y(-1))^{f=1} = 0$, hence N vanishes on $D_{\mathrm{st}}(Y)^{f=1}$. Consequently, $H^0(K, Y) = D_{\mathrm{st}}(Y)^{f=1,N=0} \cap F^0 = D_{\mathrm{st}}(Y)^{f=1}$. The same arguments applied to $0 \longrightarrow Y^*(1) \longrightarrow V^*(1) \longrightarrow X^*(1) \longrightarrow 0$ prove the statements about X.

(4) We have $H^1_{ur} \subseteq H^1_f \subseteq H^1_g$. By 1.24, $h^1_{ur}(Y) = h^1_f(Y)$. The formulas in (3) imply that $(D_{\mathrm{st}}(Y(-1))/ND_{\mathrm{st}}(Y))^{f=1} = 0$, hence $H^1_f(K, Y) = H^1_{\mathrm{st}}(K, Y)$ by 1.18 and $H^1_{ur}(K, Y) = H^1_{\mathrm{st}}(K, Y)$ by counting dimensions. By 1.16, $H^1_f(K, Y) = D_{\mathrm{cris}}(Y)/(f-1)$, and the canonical map $D_{\mathrm{cris}}(Y)/(f-1) \longrightarrow D_{\mathrm{st}}(Y)/(f-1)$ is an isomorphism: it is injective, as $D_{\mathrm{st}}(Y(-1))^{f=1} = 0$, and both groups have the same dimension equal to $h^0(Y)$.

1.29 Definition. A p-adic Galois representation V is called *ordinary*, if it admits a decreasing filtration by subrepresentations $\cdots F^iV \supseteq F^{i+1}V \cdots$ such that $\bigcup F^iV = V$, $\bigcap F^iV = 0$ and all graded factors are of the form

$$F^iV/F^{i+1}V = A_i(i)$$

with A_i unramified, i.e. $A_i = A_i^I$.

1.30 Theorem. *Every ordinary representation is semistable.*

Proof. Suppose that $V = F^iV$ and $F^jV = 0$ for some $i < j$. If $j - i = 1$, then $V = A_i(i)$ is a Tate twist of an unramified representation, hence crystalline. Suppose the claim has been proved for all ordinary representations with $j - i \leq k$. If, for our V, $j - i = k + 1$, write $V(-i)$ as an extension

$$0 \longrightarrow (F^{i+1}V)(-i) \longrightarrow V(-i) \longrightarrow A_i(0) \longrightarrow 0.$$

By induction hypothesis, both extreme terms are semistable. Since $DR(A_l(l))$ has pure filtration $-l$, $DR((F^{i+1}V)(-i))$, which is a consecutive extension of terms $DR(A_l(l-i))$ with $l \geq i+1$, satisfies $F^0 DR((F^{i+1}V)(-i)) = 0$. As $DR(A_i(0))/F^0 = 0$, $V(-i)$ is semistable by 1.28.2., and so must be V.

1.31 In fact, the proof of 1.30 tells us that for each ordinary representation V, the exact sequence

$$0 \longrightarrow F^1V \longrightarrow V \longrightarrow V/F^1V \longrightarrow 0$$

satisfies the assumptions of 1.28 and gives an isomorphism $DR(F^1V) \xrightarrow{\sim} DR(V)/F^0$.

1.32 Proposition. *Suppose that, in the situation of 1.28, V is a semistable representation satisfying $D_{\mathrm{cris}}(V)^{f=1} = D_{\mathrm{cris}}(V^*(1))^{f=1} = 0$. Then*
(1) $H^1_{\mathrm{st}}(K,X) = H^1(K,X)$
(2) *N induces an isomorphism $H^0(K,Y) = D_{\mathrm{st}}(Y)^{f=1} \xrightarrow{\sim} D_{\mathrm{st}}(X(-1))^{f=1}$*
(3) *The cohomology sequence*

$$0 \longrightarrow H^0(K,Y) \longrightarrow H^1(K,X) \longrightarrow H^1_{\mathrm{st}}(K,V) \longrightarrow 0$$

is exact.

Proof. (1) As $DR(X^*(1))/F^0 = 0$, $H^1_e(K,X^*(1)) = 0$, hence $H^1(K,X) = H^1_g(K,X) = H^1_{\mathrm{st}}(K,X)$ by 1.14 and 1.24.3.
(2) By 1.28.3, there is a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & 0 & \longrightarrow & D_{\mathrm{st}}(V)^{f=1} & \xrightarrow{\sim} & D_{\mathrm{st}}(Y)^{f=1} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow N & & \downarrow N & & \\ 0 & \longrightarrow & D_{\mathrm{st}}(X(-1))^{f=1} & \longrightarrow & D_{\mathrm{st}}(V(-1))^{f=1} & \longrightarrow & 0 & \longrightarrow & 0. \end{array}$$

Our assumption on V implies that the only non-trivial vertical map is an isomorphism, hence we conclude by snake lemma.
(3) The exactness, possibly without the zero on the right, follows from 1.25 and (1). We get from 1.24, 1.28.3 and vanishing of $H^0(K,V) \subseteq D_{\mathrm{cris}}(V)^{f=1} = 0$

$$\begin{aligned} h^1_{\mathrm{st}}(V) &= h^1_g(V) = [K:\mathbf{Q}_p]\dim_K DR(V)/F^0 = [K:\mathbf{Q}_p]\dim_{\mathbf{Q}_p}(X) \\ h^1(X) &= h^0(X^*(1)) + [K:\mathbf{Q}_p]\dim_{\mathbf{Q}_p}(X). \end{aligned}$$

According to 1.28.3 and (2), $h^0(Y) = \dim_{\mathbf{Q}_p} D_{\mathrm{st}}(X(-1))^{f=1} = \dim_{\mathbf{Q}_p} D_{\mathrm{st}}(X^*(1))^{f=1} = h^0(X^*(1))$, hence $h^1(X) = h^1_{\mathrm{st}}(V) + h^0(Y)$ and we are done by counting dimensions.

1.33 Lemma. *If V is a de Rham representation, then $(V/V^I)^I = 0$.*

Proof. Let $0 \neq x \in (V/V^I)^I$. Let $A \subseteq (V/V^I)^I$ be the $G(\overline{K}/K)$-submodule generated by $\mathbf{Q}_p \cdot x$ and take the pullback

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V^I & \longrightarrow & V & \longrightarrow & V/V^I & \longrightarrow & 0 \\ & & \| & & \cup & & \cup & & \\ 0 & \longrightarrow & V^I & \longrightarrow & E & \longrightarrow & A & \longrightarrow & 0. \end{array}$$

As V is de Rham, so is E. Applying 1.28.4 and 1.26, we see that E is unramified, i.e. $E \subseteq V^I$ – a contradiction.

1.34 Recall from [Bl–Ka 3.10-11] the relation between 1.15 and classical exponential maps. If A is an abelian variety over K and $T = T_p(A)$, then $H^1_e(K,T) = H^1_f(K,T) = A(K)\widehat{\otimes}\mathbf{Z}_p$, $DR(V)/F^0$ is canonically identified with the tangent space Lie(A) of A at the origin and the map $\exp : DR(V)/F^0 \longrightarrow H^1_e(K,V)$ is the composition of the usual exponential map $\mathrm{Lie}(A) \longrightarrow A(K)\otimes\mathbf{Q}$ with the "Kummer map" $A(K)\widehat{\otimes}\mathbf{Z}_p \longrightarrow H^1(K,T)$, which is obtained as a projective limit of boundary maps arising from

$$0 \longrightarrow A(\overline{K})_{p^n} \longrightarrow A(\overline{K}) \xrightarrow{p^n} A(\overline{K}) \longrightarrow 0.$$

Similarly, for $T = \mathbf{Z}_p(1)$, $DR(\mathbf{Q}_p(1))/F^0 = K(1)$ is the tangent space to the formal multiplicative group at the origin and $\exp : K(1) \longrightarrow H^1_e(K,\mathbf{Q}_p(1)) = \mathcal{O}^*_K\widehat{\otimes}\mathbf{Q}_p$ is the usual exponential, provided we identify $K(1)$ with K by means of the canonical invariant differential dz/z. We shall keep track of the Tate twist, however. As a result, we shall consider both the standard logarithm (= inverse of exp) $\log : \mathcal{O}^*_K\widehat{\otimes}\mathbf{Q}_p \xrightarrow{\sim} K(1)$ and its extension $\mathrm{Log} : K^*\widehat{\otimes}\mathbf{Q}_p \longrightarrow K(1)$ as $K(1)$-valued. Later on, we shall encounter algebraic logarithms, i.e. homomorphisms $l : K^*\widehat{\otimes}\mathbf{Q}_p \longrightarrow \mathbf{Q}_p$. We say that l is *ramified*, if its restriction to $\mathcal{O}^*_K\widehat{\otimes}\mathbf{Q}_p$ does not vanish. If this is the case, then there is a *unique* analytic logarithm Log and a unique $\mathbf{Q}_p$-linear map $\lambda : K(1) \longrightarrow \mathbf{Q}_p$ such that $l = \lambda \circ \mathrm{Log}$.

1.35 Let us recall explicit description of D_{st} for Kummer extensions: if

$$0 \longrightarrow \mathbf{Q}_p(1) \longrightarrow V \longrightarrow \mathbf{Q}_p(0) \longrightarrow 0$$

is an extension corresponding to $[V] = u \in K^*\widehat{\otimes}\mathbf{Q}_p = H^1(K,\mathbf{Q}_p(1))$, then there is a unique f-equivariant decomposition

$$D_{\mathrm{st}}(V) = D_{\mathrm{st}}(\mathbf{Q}_p(1)) \oplus D_{\mathrm{st}}(\mathbf{Q}_p(0)) = K_0(1) \oplus K_0,$$

inducing $DR(V) = K(1) \oplus K$. The Hodge filtration is determined by $F^0DR(V) = K \cdot (\mathrm{Log}(u), 1)$ and monodromy operator is given by

$$N : K(1) \oplus K \longrightarrow K \xrightarrow{v(u)} K \hookrightarrow K \oplus K(-1),$$

where v, Log are the valuation resp. analytic logarithm used for the definition of $B_{\mathrm{st}} \hookrightarrow B_{\mathrm{dR}}$.
The complex $C^{\cdot}_{\mathrm{st}}(\mathbf{Q}_p(1))$ is quasi-isomorphic to $[K_0 \oplus K(1) \xrightarrow{(f-1,0)} K_0][-1]$ and the isomorphism

$$\beta_{\mathbf{Q}_p(1),st} : H^1(C^{\cdot}_{\mathrm{st}}(\mathbf{Q}_p(1))) = \mathbf{Q}_p \oplus K(1) \xrightarrow{\sim} H^1(K,\mathbf{Q}_p(1))$$

is given by

$$\beta^{-1}(u) = (v(u), \mathrm{Log}(u)).$$

2. General p-adic pairings for Galois representations

2.1.1 We start with the following data:

- K – a finite extension of $\mathbf{Q}$
- V – a p-adic Galois representation over K, i.e. a finite dimensional $\mathbf{Q}_p$-vector space with a continuous linear action of $G(\overline{K}/K)$.

2.1.2 We assume that the following conditions are satisfied:

(A) V is unramified outside a finite set S_0 of primes of K
(B) $L_v(V,0)L_v(V^*(1),0) \neq 0 \qquad (\forall v \nmid p)$
(C) $V_\wp$ is a crystalline representation of $G_\wp = G(\overline{K_\wp}/K_\wp) \qquad (\forall \wp | p)$
(D) $L_\wp(V,0)L_\wp(V^*(1),0) \neq 0 \qquad (\forall \wp | p)$

Here V_v denotes V considered as a representation of $G_v = G(\overline{K_v}/K_v)$ and the local L-factors are

$$\begin{aligned} L_v(V,s) &= \det(1 - Fr(v)(Nv)^{-s} \mid V^{I_v}) \qquad (v \not| p) \\ L_\wp(V,s) &= \det(1 - f^{[K_\wp^0 : \mathbf{Q}_p]}(N\wp)^{-s} \mid D_{\mathrm{cris}}(V_\wp)) \quad (\wp | p), \end{aligned}$$

where $Fr(v)$ is a representative of the geometric Frobenius element at v.

2.1.3 Note the following consequences of (B)–(D):

$$\begin{aligned} &(B) \quad \Rightarrow \quad H^i(K_v, V) = H^i(K_v, V^*(1)) = 0 \qquad (\forall v \not| p,\ i = 0,1,2) \\ &(D) \quad \Rightarrow \quad D_{\mathrm{cris}}(V_\wp)^{f=1} = D_{\mathrm{cris}}(V^*(1)_\wp)^{f=1} = 0 \qquad (\forall \wp | p), \end{aligned}$$

by local duality and the formula for the local Euler charakteristic.

2.1.4 Denote by $\mathcal{O}_K$ the ring of integers in K. Let us fix a finite set $S \supseteq S_0 \cup \{\wp | p\}$ of primes of K and denote by G_S the Galois group of the maximal extension of K unramified outside S. Define for $v \not| p$

$$H^1_f(K_v, V) = H^1_{ur}(K_v, V)$$

and put

$$H^1_f(G_S, V) = \{x \in H^1(G_S, V) \mid x_v \in H^1_f(K_v, V) \quad \forall v \in S\}$$

and similarly for $V^*(1)$.
It follows from the condition (B) that the groups $H^1(G_S, V), H^1(G_S, V^*(1))$ do not depend on S (cf. [Ja 1, Lemma 4]) and the same is true for H^1_f. In fact, they are equal to $H^1(K, -)$ and

$$H^1_f(K, -) = \{x \in H^1(K, -) \mid x_v \in H^1_f(K_v, -) \quad \forall v\}$$

respectively. We may now formulate our main result.

2.2 Theorem. *Suppose we are given a non-trivial continuous homomorphism* $l : \mathbf{A}_K^*/K^* \longrightarrow \mathbf{Q}_p$. *One may then define a family of continuous bilinear pairings*

$$h : \quad H^1_f(K, V) \times H^1_f(K, V^*(1)) \longrightarrow \mathbf{Q}_p$$

depending on the choice of splittings of Hodge filtrations $0 \longrightarrow F^0 DR(V_\wp) \longrightarrow DR(V_\wp) \longrightarrow DR(V_\wp)/F^0 \longrightarrow 0$ *for primes* $\wp | p$.

Proof. Assume that $a \in H^1_f(K,V) = H^1_f(G_S, V)$, $a' \in H^1_f(K, V^*(1)) = H^1_f(G_S, V^*(1))$. Then a represents an extension

$$0 \longrightarrow V \longrightarrow E \longrightarrow \mathbf{Q}_p(0) \longrightarrow 0$$

of continuous G_S-modules with $E_\wp$ ($= E$ as a module over $G_\wp$) crystalline for all $\wp | p$. Dualizing, we get an extension

$$0 \longrightarrow \mathbf{Q}_p(1) \longrightarrow E^*(1) \longrightarrow V^*(1) \longrightarrow 0$$

with $E^*(1)_\wp$ crystalline as well.

2.3 Lemma. *There is an exact sequence*

$$0 \longrightarrow H^1_f(G_S, \mathbf{Q}_p(1)) \longrightarrow H^1_f(G_S, E^*(1)) \longrightarrow H^1_f(G_S, V^*(1)) \longrightarrow 0,$$

in which $H^1_f(G_S, \mathbf{Q}_p(1)) = \mathcal{O}_K^* \otimes \mathbf{Q}_p$.

Proof. According to [Sch 1], $H^1(G_S, \mathbf{Q}_p(1)) = \mathcal{O}_S^* \otimes \mathbf{Q}_p$ and by 1.12 $H^1_f(K_\wp, \mathbf{Q}_p(1)) = \mathcal{O}_\wp^* \widehat{\otimes} \mathbf{Q}_p$, which proves that $H^1_f(G_S, \mathbf{Q}_p(1)) = \mathcal{O}_K^* \otimes \mathbf{Q}_p$.
Consider the commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_S^* \otimes \mathbf{Q}_p & \longrightarrow & H^1(G_S, E^*(1)) & \longrightarrow & H^1(G_S, V^*(1)) & \longrightarrow & {\mathbf{Q}_p}^{\sharp S-1} \\ & & \downarrow & & \downarrow & & \downarrow & & \cap\downarrow \\ 0 & \longrightarrow & \bigoplus_{v \in S} H^1(K_v, \mathbf{Q}_p(1)) & \longrightarrow & \bigoplus_{v \in S} H^1(K_v, E^*(1)) & \longrightarrow & \bigoplus_{v \in S} H^1(K_v, V^*(1)) & \stackrel{\delta}{\longrightarrow} & {\mathbf{Q}_p}^{\sharp S} \end{array}$$

We know that $H^1(K_v, V^*(1)) = H^1(K_v, V) = 0$ for all $v \not| p$ as a consequence of the condition (B). It follows that

$$E^*(1)_v = V^*(1)_v \oplus \mathbf{Q}_p(1)$$

as a representation of $G(\overline{K_v}/K_v)$, hence

$$H^1_f(K_v, E^*(1)) = H^1_f(K_v, \mathbf{Q}_p(1)) = 0.$$

Let $\wp | p$. As

$$H^0(K_\wp, V^*(1)) \subseteq D_{\mathrm{cris}}(V^*(1)_\wp)^{f=1} = 0,$$

we get from 1.23 an exact sequence

$$0 \longrightarrow H^1_f(K_\wp, \mathbf{Q}_p(1)) \longrightarrow H^1_f(K_\wp, E^*(1)) \longrightarrow H^1_f(K_\wp, V^*(1)) \longrightarrow 0.$$

An easy diagram chase then shows that the global sequence in question is exact if the zero on the right is removed. To verify the surjectivity it suffices to check that the composition

$$H^1_f(G_S, V^*(1)) \longrightarrow H^1(K_\wp, V^*(1)) \stackrel{\delta}{\longrightarrow} \mathbf{Q}_p$$

vanishes for all $\wp | p$. The boundary map δ is given by the cup product with the extension class $a_\wp$ lying in $H^1_f(K_\wp, V)$, hence vanishes on $H^1_f(K_\wp, V^*(1))$ by 1.14. Lemma is proved.

2.4 It follows from the previous discusion that the following commutative diagram has exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_K^* \otimes \mathbf{Q}_p & \longrightarrow & H^1_f(G_S, E^*(1)) & \stackrel{\pi}{\longrightarrow} & H^1_f(G_S, V^*(1)) & \longrightarrow & 0 \\ & & \downarrow i & & \downarrow j & & \downarrow & & \\ 0 & \longrightarrow & \bigoplus_{\wp | p} \mathcal{O}_\wp^* \widehat{\otimes} \mathbf{Q}_p & \longrightarrow & \bigoplus_{\wp | p} H^1_f(K_\wp, E^*(1)) & \longrightarrow & \bigoplus_{\wp | p} H^1_f(K_\wp, V^*(1)) & \longrightarrow & 0 \\ & & \downarrow l & & \vdots\downarrow l' & & & & \\ & & \mathbf{Q}_p & = & \mathbf{Q}_p & & & & \end{array}$$

and $l \circ i = 0$. Suppose we are given a splitting of this diagram, i.e. a homomorphism l' making the lower square commutative. The symbol

$$h(a, a') := l' \circ j \circ \pi^{-1}(a') \in \mathbf{Q}_p$$

is then well defined and is linear in a'. We are looking for l' "functorial" in a, so that $h(a, a')$ becomes linear in a as well. To do so, fix $\wp | p$ and look at the sequence

$$0 \longrightarrow H^1_f(K_\wp, \mathbf{Q}_p(1)) \longrightarrow H^1_f(K_\wp, E^*(1)) \longrightarrow H^1_f(K_\wp, V^*(1)) \longrightarrow 0.$$

2.5 Lemma. *The exponential maps of 1.15 define a canonical isomorphism between two exact sequences*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^1_f(K_\wp, \mathbf{Q}_p(1)) & \longrightarrow & H^1_f(K_\wp, E^*(1)) & \longrightarrow & H^1_f(K_\wp, V^*(1)) & \longrightarrow & 0 \\ & & \uparrow\wr & & \uparrow\wr & & \uparrow\wr & & \\ 0 & \longrightarrow & DR(\mathbf{Q}_p(1)) & \longrightarrow & DR(E^*(1)_\wp)/F^0 & \longrightarrow & DR(V^*(1)_\wp)/F^0 & \longrightarrow & 0 \end{array}$$

Proof. As $D_{\mathrm{cris}}(\mathbf{Q}_p(1))^{f=1} = D_{\mathrm{cris}}(V^*(1)_\wp)^{f=1} = 0$, $D_{\mathrm{cris}}(E^*(1)_\wp)^{f=1}$ must vanish as well. It follows then from 1.15, 1.16 that

$$\exp : DR(-)/F^0 \xrightarrow{\sim} H^1_e(K_\wp, -) = H^1_f(K_\wp, -)$$

for each of the modules $\mathbf{Q}_p(1), E^*(1), V^*(1)$.

2.6 Define, for a $K_\wp$-vector space A, $\mathcal{D}(A) = \mathrm{Hom}_{K_\wp}(A, K_\wp(1))$. Applying $\mathcal{D}$ to the bottom row of the diagram in Lemma 2.5, we get an exact sequence

$$0 \longrightarrow F^0DR(V_\wp) \longrightarrow F^0DR(E_\wp) \longrightarrow F^0DR(\mathbf{Q}_p(0)) = K_\wp \longrightarrow 0,$$

which will be denoted by $\sigma(e)$. Recall that $H^1_f(K_\wp, V_\wp)$ is equal to the extension group $\mathrm{Ext}^1_{\mathrm{cris}}(\mathbf{Q}_p(0), V_\wp)$ in the category of crystalline representations of $G_\wp$. Define the set of *rigidified extensions* to be

$$\mathrm{Extrig}^1_{\mathrm{cris}}(\mathbf{Q}_p(0), V_\wp) := \{(e, w) \,|\, e \in \mathrm{Ext}^1_{\mathrm{cris}}(\mathbf{Q}_p(0), V_\wp)\,,\ w \text{ a splitting of } \sigma(e)\},$$

where by a splitting we mean a $K_\wp$-linear retraction $F^0DR(V_\wp) \overset{w}{\underset{\longrightarrow}{\longleftarrow}} F^0DR(E_\wp)$. The set $\mathrm{Extrig}^1_{\mathrm{cris}}(\mathbf{Q}_p(0), V_\wp)$ becomes an abelian group under a slightly modified Baer sum (see [Za]).

2.7 Lemma. *The forgetful functor induces an exact sequence*

$$0 \longrightarrow F^0DR(V_\wp) \longrightarrow \mathrm{Extrig}^1_{\mathrm{cris}}(\mathbf{Q}_p(0), V_\wp) \longrightarrow \mathrm{Ext}^1_{\mathrm{cris}}(\mathbf{Q}_p(0), V_\wp) \longrightarrow 0,$$

which is canonically isomorphic to the sequence

$$0 \longrightarrow F^0DR(V_\wp) \longrightarrow DR(V_\wp) \longrightarrow DR(V_\wp)/F^0 \longrightarrow 0.$$

Proof. Exactness of the first sequence follows easily from the definitions and the fact that rigidifications on $V_\wp \oplus \mathbf{Q}_p$ correspond to $\mathrm{Hom}_{K_\wp}(K_\wp, F^0DR(V_\wp)) = F^0DR(V_\wp)$. In order to identify the two exact sequences we recall a down-to-earth description of the exponential morphism

$$DR(-)/F^0 \longrightarrow H^1_e(K_\wp, -)$$

(equal to β_{cris} in our case) given in 1.21.
We define an isomorphism $\mathrm{Extrig}^1_{\mathrm{cris}}(\mathbf{Q}_p(0), V_\wp) \xrightarrow{\sim} DR(V_\wp)$ as follows: as $D_{\mathrm{cris}}(V)^{f=1} = 0$, for each crystalline extension

$$0 \longrightarrow V_\wp \longrightarrow E \longrightarrow \mathbf{Q}_p(0) \longrightarrow 0$$

the corresponding extension of filtered modules

$$0 \longrightarrow D_{\mathrm{cris}}(V) \longrightarrow D(E) \longrightarrow K^0_\wp \longrightarrow 0$$

admits a unique section $s : K^0_\wp \longrightarrow D_{\mathrm{cris}}(E)$ satisfying $(f-1)s(1) = 0$. A splitting

$$0 \longrightarrow F^0DR(V_\wp) \overset{w}{\underset{\longrightarrow}{\longleftarrow}} F^0DR(E) \longrightarrow K \longrightarrow 0$$

determines in turn a section $t : K \longrightarrow F^0 DR(E)$. The difference $x(E) := t(1) - i(s(1))$ then lies in $DR(V_\wp)$ and one checks, using 1.21, that the map $E \longmapsto x(E)$ defines a homomorphism $\mathrm{Extrig}^1_{\mathrm{cris}}(\mathbf{Q}_p(0), V_\wp) \longrightarrow DR(V_\wp)$ making the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F^0 DR(V_\wp) & \longrightarrow & \mathrm{Extrig}^1_{\mathrm{cris}}(\mathbf{Q}_p(0), V_\wp) & \longrightarrow & \mathrm{Ext}^1_{\mathrm{cris}}(\mathbf{Q}_p(0), V_\wp) & \longrightarrow & 0 \\ & & \| & & \downarrow & & \uparrow \wr exp & & \\ 0 & \longrightarrow & F^0 DR(V_\wp) & \longrightarrow & DR(V_\wp) & \longrightarrow & DR(V_\wp)/F^0 & \longrightarrow & 0 \end{array}$$

commutative.

2.8 We can now conclude the proof of Theorem 2.2: suppose that we are given for each $\wp|p$ a $\mathbf{Q}_p$-linear splitting of

$$0 \longrightarrow F^0 DR(V_\wp) \longrightarrow DR(V_\wp) \overset{\longleftarrow}{\longrightarrow} DR(V_\wp)/F^0 \longrightarrow 0.$$

By lemma 2.4, it can be identified with a splitting $r_\wp$ of

$$0 \longrightarrow F^0 DR(V_\wp) \longrightarrow \mathrm{Extrig}^1_{\mathrm{cris}}(\mathbf{Q}_p(0), V_\wp) \overset{\overset{r_\wp}{\longleftarrow}}{\longrightarrow} \mathrm{Ext}^1_{\mathrm{cris}}(\mathbf{Q}_p(0), V_\wp) \longrightarrow 0$$

and one defines the map

$$l' : \bigoplus_{\wp|p} H^1_f(K_\wp, E^*(1)) \longrightarrow \mathbf{Q}_p$$

to be

$$l' = l \circ \bigoplus_{\wp|p} w_\wp,$$

where $w_\wp$ is the splitting

$$0 \longrightarrow H^1_f(K_\wp, \mathbf{Q}_p(1)) \overset{\overset{w_\wp}{\longleftarrow}}{\longrightarrow} H^1_f(K_\wp, E^*(1)) \longrightarrow H^1_f(K_\wp, V^*(1)) \longrightarrow 0$$

corresponding to the splitting $r_\wp(a_\wp)$ of the dual exact sequence. It follows from the previous discussion that $h(a, a') \in \mathbf{Q}_p$ is additive in the variable a, which concludes the proof of Theorem 2.2.

2.9 In fact, it suffices to have splittings $r_\wp$ for those primes $\wp|p$ in which l is ramified (= does not vanish on $\mathcal{O}^*_\wp$). It may be surprising that the data required for the construction of the pairings are $\mathbf{Q}_p$-linear and not $K_\wp$-linear splittings of Hodge filtrations.

3. Splittings of local extensions

3.1 This section deals with relations between splittings of Hodge filtrations and splittings of certain cohomology sequences. We shall work with p-adic Galois representations over a field $K_\wp$ of finite degree over $\mathbf{Q}_p$. In 3.2-4 we retain the assumptions (C),(D) of 2.1.2, namely that V is a crystalline p-adic Galois representation over $K_\wp$ satisfying

$$D_{\mathrm{cris}}(V)^{f=1} = D_{\mathrm{cris}}(V^*(1))^{f=1} = 0,$$

which implies that we have isomorphisms

$$\exp : DR(V)/F^0 \overset{\sim}{\longrightarrow} H^1_e(K_\wp, V) = H^1_f(K_\wp, V)$$

and similarly for $V^*(1)$.

3.2 If A is a $\mathbf{Q}_p$-vector space of finite dimension, write $A(n) = A \otimes \mathbf{Q}_p(n)$ for A as a Galois representation, on which $G_\wp$ acts by the n-th power of the cyclotomic character χ_K. Suppose we are given an extension of crystalline representations

$$0 \longrightarrow V \longrightarrow E_1 \longrightarrow A(0) \longrightarrow 0.$$

Its class $[E_1]$ lies in

$$\mathrm{Ext}^1_{K_\wp,f}(A(0), V) = \mathrm{Hom}(A, H^1_f(K_\wp, V)) \xrightarrow{\sim} \mathrm{Hom}(A, DR(V)/F^0).$$

Denote by $\alpha : A \otimes K_\wp \longrightarrow DR(V)/F^0$ its $K_\wp$-linear extension and by $\mu : DR(V) \longrightarrow DR(V)/F^0$ the canonical projection. As $D_{\mathrm{cris}}(V)^{f=1} = 0$, the exact sequence

$$0 \longrightarrow D_{\mathrm{cris}}(V) \longrightarrow D_{\mathrm{cris}}(E_1) \longrightarrow D_{\mathrm{cris}}(A(0)) = A \otimes K_\wp^0 \longrightarrow 0$$

admits a unique f-equivariant splitting

$$D_{\mathrm{cris}}(E_1) = D_{\mathrm{cris}}(V) \oplus (A \otimes K_\wp^0),$$

inducing

$$DR(E_1) = DR(V) \oplus (A \otimes K_\wp).$$

According to 1.21,

$$F^0 DR(E_1) = \{(x, a) \,|\, x \in DR(V),\, a \in A \otimes K_\wp,\, \mu(x) = \alpha(a)\},$$

i.e. the exact sequence

$$0 \longrightarrow F^0 DR(V) \longrightarrow F^0 DR(E_1) \longrightarrow A \otimes K_\wp \longrightarrow 0$$

is the pullback of

$$0 \longrightarrow F^0 DR(V) \longrightarrow DR(V) \longrightarrow DR(V)/F^0 \longrightarrow 0$$

via $\alpha : A \otimes K_\wp \longrightarrow DR(V)/F^0$.

3.3 Suppose that we are given an extension of crystalline representations

$$0 \longrightarrow B(1) \longrightarrow E_2 \longrightarrow V \longrightarrow 0,$$

where B is another $\mathbf{Q}_p$-vector space of finite dimension. Applying 3.2 to $E_2^*(1)$, we get a unique f-equivariant splitting

$$D_{\mathrm{cris}}(E_2^*(1)) = D_{\mathrm{cris}}(V^*(1)) \oplus (B^* \otimes K_\wp^0),$$

inducing

$$D_{\mathrm{cris}}(E_2) = D_{\mathrm{cris}}(V) \oplus (B \otimes K_\wp^0(1))$$
$$DR(E_2) = DR(V) \oplus (B \otimes K_\wp(1)).$$

Write $\mu' : DR(V^*(1)) \longrightarrow DR(V^*(1))/F^0$ for the canonical projection and $\beta' : B^* \otimes K_\wp \longrightarrow DR(V^*(1))/F^0$ for the $K_\wp$-linear extension of $[E_2^*(1)] \in \mathrm{Hom}(B^*, DR(V^*(1))/F^0)$. Let $\beta : F^0 DR(V) \longrightarrow B \otimes K_\wp(1)$ be the map obtained from β by applying the functor $\mathcal{D}$ of 2.6.
According to 3.2,

$$F^0 DR(E_2^*(1)) = \{(x', b') \,|\, x' \in DR(V^*(1)),\, b' \in B^* \otimes K_\wp,\, \mu'(x') = \beta'(b')\},$$

which implies that

$$F^0DR(E_2) = (F^0DR(E_2^*(1)))^\perp = \{(x, -\beta(x)) \,|\, x \in F^0DR(V)\}.$$

In analogy to 3.2, this means that there is a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F^0DR(V) & \longrightarrow & DR(V) & \longrightarrow & DR(V)/F^0 & \longrightarrow & 0 \\
 & & \downarrow \beta & & \downarrow & & \| & & \\
0 & \longrightarrow & B \otimes K_\wp(1) & \longrightarrow & DR(E_2)/F^0 & \longrightarrow & DR(V)/F^0 & \longrightarrow & 0 \\
 & & \downarrow \wr\, exp & & \downarrow \wr\, exp & & \downarrow \wr\, exp & & \\
0 & \longrightarrow & H^1_f(K_\wp, B(1)) & \longrightarrow & H^1_f(K_\wp, E_2) & \longrightarrow & H^1_f(K_\wp, V) & \longrightarrow & 0
\end{array}$$

It follows that any splitting of the Hodge filtration

$$0 \longrightarrow F^0DR(V) \longrightarrow DR(V) \overset{\longleftarrow}{\longrightarrow} DR(V)/F^0 \longrightarrow 0$$

induces a splitting of

$$0 \longrightarrow H^1_f(K_\wp, B(1)) \longrightarrow H^1_f(K_\wp, E_2) \overset{\longleftarrow}{\longrightarrow} H^1_f(K_\wp, V) \longrightarrow 0.$$

In down-to-earth terms, it is simply the composition

$$DR(V)/F^0 \longrightarrow DR(V) \hookrightarrow DR(E_2) \longrightarrow DR(E_2)/F^0.$$

3.4 Suppose that V is merely semistable, but still satisfies $D_{\mathrm{cris}}(V)^{f=1} = D_{\mathrm{cris}}(V^*(1))^{f=1} = 0$. If

$$\begin{array}{ccccccccc}
0 & \longrightarrow & V & \longrightarrow & E_1 & \longrightarrow & A(0) & \longrightarrow & 0 \\
0 & \longrightarrow & B(1) & \longrightarrow & E_2 & \longrightarrow & V & \longrightarrow & 0
\end{array}$$

are extensions of semistable representations, then our condition on V implies that there are unique (f, N)-equivariant decompositions

$$\begin{aligned}
D_{\mathrm{st}}(E_1) &= D_{\mathrm{st}}(V) \oplus (A \otimes K_\wp^0) \\
D_{\mathrm{st}}(E_2) &= D_{\mathrm{st}}(V) \oplus (B \otimes K_\wp^0(1)).
\end{aligned}$$

The extension classes $[E_1] \in \mathrm{Hom}(A, H^1_{\mathrm{st}}(K_\wp, V)) = \mathrm{Hom}(A, DR(V)/F^0)$ and $[E_2^*(1)]$ define, as before, $K_\wp$-linear maps

$$\alpha : A \otimes K_\wp \longrightarrow DR(V)/F^0, \qquad \beta : F^0DR(V) \longrightarrow B \otimes K_\wp(1)$$

and $F^0DR(E_1), F^0DR(E_2)$ are given by the same formulas as in 3.2, 3.3.

3.5 The rest of sec.3 collects some preliminary material for sec.7 and should be skipped in the first reading. Recall from 1.23 that the complex

$$C^{\cdot}_{\mathrm{st}}(W) = [D_{\mathrm{st}}(W) \xrightarrow{(f-1, N, -i)} D_{\mathrm{st}}(W) \oplus D_{\mathrm{st}}(W(-1)) \oplus DR(W)/F^0 \xrightarrow{(N, 1-f, 0)} D_{\mathrm{st}}(W(-1))]$$

computes functorially $H^0(K_\wp, W)$ and $H^1_{\mathrm{st}}(K_\wp, W)$ for a semistable representation W. An exact sequence of complexes

$$\begin{aligned}
0 \longrightarrow [D_{\mathrm{st}}(W(-1)) \oplus DR(W)/F^0 \xrightarrow{(1-f, 0)} D_{\mathrm{st}}(W(-1))][-1] \longrightarrow C^{\cdot}_{\mathrm{st}}(W) \longrightarrow \\
\longrightarrow [D_{\mathrm{st}}(W) \xrightarrow{f-1} D_{\mathrm{st}}(W)] \longrightarrow 0
\end{aligned}$$

induces a long exact sequence

$$0 \longrightarrow H^0(K_\wp, W) \longrightarrow D_{\rm st}(W)^{f=1} \xrightarrow{(N,-i)} D_{\rm st}(W(-1))^{f=1} \oplus DR(W)/F^0 \longrightarrow H^1_{\rm st}(K_\wp, W) \longrightarrow$$
$$\longrightarrow D_{\rm st}(W)/(f-1)D_{\rm st}(W) \xrightarrow{N} D_{\rm st}(W(-1))/(f-1)D_{\rm st}(W(-1))$$

Let us apply this machinery to all terms in the exact sequence

$$0 \longrightarrow B(1) \longrightarrow E_2 \longrightarrow V \longrightarrow 0$$

of 3.4. Using the (f, N)-equivariant decomposition, we get a commutative diagram with exact rows and columns:

$$\begin{array}{ccccccc}
 & & & 0 & & 0 & \\
 & & & \downarrow & & \downarrow & \\
 & & & D_{\rm st}(E_2)^{f=1} & \simeq & D_{\rm st}(V)^{f=1} & \\
 & & & \downarrow{\scriptstyle (0,N,-i)} & & \downarrow{\scriptstyle (N,-i)} & \\
0 \to & B \oplus B \otimes K_\wp(1) & \to & B \oplus D_{\rm st}(V(-1))^{f=1} \oplus DR(E_2)/F^0 & \to & D_{\rm st}(V(-1))^{f=1} \oplus DR(V)/F^0 & \to 0 \\
 & \downarrow \wr & & \downarrow & & \downarrow & \\
0 \to & H^1_{\rm st}(K_\wp, B(1)) & \to & H^1_{\rm st}(K_\wp, E_2) & \to & H^1_{\rm st}(K_\wp, V) & \to 0 \\
 & & & \downarrow & & \downarrow & \\
 & & & 0 & & 0 &
\end{array}$$

As $D_{\rm cris}(E_2)^{f=1} = 0$, there are, as in 3.3, isomorphisms

$$\begin{array}{ccccccc}
0 & \longrightarrow & B \otimes K_\wp(1) & \longrightarrow & DR(E_2)/F^0 & \longrightarrow & DR(V)/F^0 & \longrightarrow 0 \\
 & & \downarrow \wr exp & & \downarrow \wr exp & & \downarrow \wr exp & \\
0 & \longrightarrow & H^1_f(K_\wp, B(1)) & \longrightarrow & H^1_f(K_\wp, E_2) & \longrightarrow & H^1_{\rm st}(K_\wp, V) & \longrightarrow 0
\end{array}$$

It follows that any splitting of the Hodge filtration

$$0 \longrightarrow F^0 DR(V) \longrightarrow DR(V) \longrightarrow DR(V)/F^0 \longrightarrow 0$$

again induces a splitting

$$DR(V)/F^0 \longrightarrow DR(V) \hookrightarrow DR(E_2) \longrightarrow DR(E_2)/F^0$$

of $H^1_f(K_\wp, E_2) \overset{\longleftarrow}{\longrightarrow} H^1_{\rm st}(K_\wp, V)$, hence also of $H^1_{\rm st}(K_\wp, E_2) \overset{\longleftarrow}{\longrightarrow} H^1_{\rm st}(K_\wp, V)$.

4. Local height pairings

4.1 Following closely the construction of archimedean height pairings in [Sc 2], we shall decompose the pairing (depending on the choice of splittings)

$$h: \quad H^1_f(K, V) \times H^1_f(K, V^*(1)) \longrightarrow \mathbf{Q}_p$$

constructed in sec. 2 into a sum of local pairings

$$h = \sum_v h_v,$$

the sum extending over all non-archimedean places v of K. The decomposition is not unique and depends on the choice of an auxiliary object, a certain "mixed extension".

4.2 We retain the assumptions of 2.1. Let there be given two extensions of p-adic representations of $G(\overline{K}/K)$

$$\begin{aligned} e_1 : \quad & 0 \longrightarrow V \longrightarrow E_1 \longrightarrow A(0) \longrightarrow 0 \\ e_2 : \quad & 0 \longrightarrow B(1) \longrightarrow E_2 \longrightarrow V \longrightarrow 0 \end{aligned}$$

in which the action of $G(\overline{K}/K)$ on A, B is trivial and $E_{1,\wp}$, $E_{2,\wp}$ are de Rham representations for all $\wp|p$ (hence crystalline, by 1.14, 1.16).

4.3 Definition. A *mixed extension* of e_1, e_2 is a p-adic Galois representation E equipped with an increasing weight filtration by subrepresentations

$$0 = W_{-3}E \subseteq W_{-2}E \subseteq W_{-1}E \subseteq W_0E = E$$

and isomorphisms

$$W_{-1}E \xrightarrow{\sim} E_2, \qquad W_0E/W_{-2}E \xrightarrow{\sim} E_1.$$

Such an object (called "extension panachée" in [SGA 7/I, IX.9.3]) lives in a commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & B(1) & \longrightarrow & E_2 & \longrightarrow & V & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & B(1) & \longrightarrow & E & \longrightarrow & E_1 & \longrightarrow & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & A(0) & = & A(0) & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0 & & \end{array}$$

There is a natural action of $\mathrm{Ext}^1_K(A(0), B(1)) = \mathrm{Hom}(A, B) \otimes H^1(K, \mathbf{Q}_p(1))$ on the set of mixed extensions E: for $X \in \mathrm{Ext}^1_K(A(0), B(1))$ the Baer sum of its image in $\mathrm{Ext}^1_K(A(0), E_2)$ with

$$0 \longrightarrow E_2 \longrightarrow E \longrightarrow A(0) \longrightarrow 0$$

is again a mixed extension in a natural way. Similarly, we can take the Baer sum of

$$0 \longrightarrow B(1) \longrightarrow E \longrightarrow E_1 \longrightarrow 0$$

with the image of X in $\mathrm{Ext}^1_K(E_1, B(1))$, but the resulting mixed extensions will be canonically isomorphic. If E is a mixed extension of e_1, e_2, then $E^*(1)$ is a mixed extension of

$$e_2^*(1) : 0 \longrightarrow V^*(1) \longrightarrow E_2^*(1) \longrightarrow B^*(0) \longrightarrow 0$$

$$e_1^*(1) : 0 \longrightarrow A^*(1) \longrightarrow E_1^*(1) \longrightarrow V^*(1) \longrightarrow 0.$$

Let v be a non-archimedean place of K not dividing p. As $H^1(K_v, V) = H^1(K_v, V^*(1)) = 0$, we have a splitting of the $G(\overline{K_v}/K_v)$-module $E_v = V_v \oplus N$, where N is an extension

$$0 \longrightarrow B(1) \longrightarrow N \longrightarrow A(0) \longrightarrow 0.$$

We say that E is *essentially unramified at* v, if N is unramified.

4.4 Proposition. *Suppose we are given two extensions e_1, e_2 as in 4.2. Then*

(1) *The set of isomorphism classes of mixed extensions E of e_1, e_2 is a principal homogeneous space of*

$$\mathrm{Ext}^1_K(A(0), B(1)) = \mathrm{Hom}(A, B) \otimes H^1(K, \mathbf{Q}_p(1))$$

(under the action described in 4.3)

(2) *The set of isomorphism classes of those mixed extensions E for which E_v is essentially unramified at all $v\not| p$ and $E_\wp$ is crystalline for all $\wp | p$, is a principal homogeneous space of*

$$\mathrm{Ext}^1_{K,f}(A(0), B(1)) = \mathrm{Hom}(A, B) \otimes H^1_f(K, \mathbf{Q}_p(1))$$

(3) *The automorphism group of every mixed extension E is trivial.*

(4) *If E is any mixed extension of e_1, e_2, then $E_\wp$ is de Rham for each $\wp | p$.*

Proof. (1) According to [SGA 7/I, IX.9.3.8], the obstruction for existence of E is given by the class of the Yoneda composition

$$e_2 \circ e_1 = \quad 0 \longrightarrow B(1) \longrightarrow E_2 \longrightarrow E_1 \longrightarrow A(0) \longrightarrow 0$$

in $\mathrm{Ext}^2_K(A(0), B(1)) = \mathrm{Hom}(A, B) \otimes H^2(K, \mathbf{Q}_p(1))$. This class is, up to a sign, equal to the cup product of

$$[e_1] \in A^* \otimes H^1_f(K, V) \quad \text{and} \quad [e_2^*(1)] \in B \otimes H^1_f(K, V^*(1)).$$

Since $H^2(K, \mathbf{Q}_p(1))$ injects into $\prod_v H^2(K_v, \mathbf{Q}_p(1))$, this cup product vanishes by 1.14 and 2.1.3. The claim now follows from [SGA 7/I, 9.3.8.b].

(2) There is a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Ext}^1_K(A(0), B(1)) & \longrightarrow & \mathrm{Ext}^1_K(A(0), E_2) & \xrightarrow{\pi} & \mathrm{Ext}^1_K(A(0), V) & \longrightarrow & 0 \\
 & & \cup\!\uparrow & & \cup\!\uparrow & & \cup\!\uparrow & & \\
0 & \longrightarrow & \mathrm{Ext}^1_{K,f}(A(0), B(1)) & \longrightarrow & \mathrm{Ext}^1_{K,f}(A(0), E_2) & \longrightarrow & \mathrm{Ext}^1_{K,f}(A(0), V) & \longrightarrow & 0
\end{array}$$

(exactness of the bottom row follows from the same argument as in 2.3). The assertion is equivalent to the fact that the fibre $\pi^{-1}([e_1])$ meets $\mathrm{Ext}^1_{K,f}(A(0), E_2)$ and this is indeed the case, as $[e_1] \in \mathrm{Ext}^1_{K,f}(A(0), V)$.

(3) According to [SGA 7/I, 9.3.8.a], the set of automorphisms of any E is equal to $\mathrm{Ext}^0_K(A(0), B(1)) = 0$.

(4) Follows from (1),(2) and the fact that $H^1(K_\wp, \mathbf{Q}_p(1)) = H^1_g(K_\wp, \mathbf{Q}_p(1))$.

4.5 We are now ready to define local pairings h_v. Suppose we are given the same data as in Theorem 2.2: local splittings $r_\wp$ of Hodge filtrations

$$F^0 DR(V_\wp) \overset{r_\wp}{\underset{\longrightarrow}{\longleftarrow}} DR(V_\wp)$$

and a non-trivial continuous homomorphism

$$l : \mathbf{A}_K^*/K^* \longrightarrow \mathbf{Q}_p.$$

Such a homomorphism can be written as a sum of local terms

$$l_v : K_v^* \longrightarrow \mathbf{Q}_p,$$

which are continuous and satisfy the reciprocity formula

$$\sum_v l_v(a) = 0 \qquad (\forall a \in K^*).$$

Note that for $v \not| p$ the local map l_v is unramified, i.e. factors through $\mathrm{ord}_v : K^* \longrightarrow \mathbf{Z}$. We shall consider all maps l_v as continuous homomorphisms

$$l_v : H^1(K_v, \mathbf{Q}_p(1)) \longrightarrow \mathbf{Q}_p.$$

Let there be given extensions e_1, e_2 as in 4.2. We fix one of their mixed extensions E. The data $(r_\wp), E$ define the same objects for $V^*(1)$: the dual splittings $\mathcal{D}(r_\wp)$ and the mixed extension $E^*(1)$.

4.6 Local pairings at $v \not| p$

Recall from 4.3 the decomposition $E_v = V_v \oplus N$, where N is an extension

$$0 \longrightarrow B(1) \longrightarrow N \longrightarrow A(0) \longrightarrow 0.$$

Its class lies in

$$[N] \in \mathrm{Ext}^1_{K_v}(A(0), B(1)) = \mathrm{Hom}(A, B) \otimes H^1(K_v, \mathbf{Q}_p(1)).$$

Equivalently, cohomology sequences of the first row and column give

$$\begin{array}{ccc} & & A \\ & & \big\downarrow {\scriptstyle [E_v]} \\ \phi : B \otimes H^1(K_v, \mathbf{Q}_p(1)) & \stackrel{\sim}{\longrightarrow} & H^1(K_v, E_2), \end{array}$$

where $[E_v] \in \mathrm{Hom}(A, H^1(K_v, E_2))$ is the extension class of E_v, and

$$\phi^{-1} \circ [E_v] \in \mathrm{Hom}(A, B) \otimes H^1(K_v, \mathbf{Q}_p(1))$$

is equal (by [Bo, X.7.6.5 Cor.1.a]) to $[N]$.
The valuation $\mathrm{ord}_v : K_v^* \longrightarrow \mathbf{Z}$ induces an isomorphism

$$(\mathrm{ord}_v)_* : H^1(K_v, \mathbf{Q}_p(1)) \stackrel{\sim}{\longrightarrow} K_v \widehat{\otimes} \mathbf{Q}_p \stackrel{\sim}{\longrightarrow} \mathbf{Q}_p.$$

Since l_v factors through ord_v, we may define

$$h_{v,E} := -l_v([N]) \in \mathrm{Hom}(A, B),$$

in other words a pairing

$$h_{v,E} : A \otimes B^* \longrightarrow \mathbf{Q}_p.$$

We have obvious implications

$$E \text{ is unramified at } v \Longrightarrow E \text{ is essentially unramified at } v \Longrightarrow h_{v,E} = 0.$$

4.7 Local pairings at $\wp | p$ (crystalline case)

Suppose that $E_\wp$ is crystalline. We shall follow the construction for $v \not| p$. According to 1.25, the extension class $[E_\wp]$ lies in

$$[E_\wp] \in \mathrm{Hom}(A, H^1_f(E_{2,\wp}))$$

(all cohomology groups in 4.7 and 4.8 are over $K_\wp$) and the splitting $r_\wp$ induces by 3.3 a splitting of

$$0 \longrightarrow B \otimes H^1_f(\mathbf{Q}_p(1)) \overset{\longleftarrow}{\longrightarrow} H^1_f(E_{2,\wp}) \longrightarrow H^1_f(V_\wp) \longrightarrow 0.$$

Composing $[E_\wp]$ with this splitting, we obtain an element

$$(r_\wp)_*[E_\wp] \in \mathrm{Hom}(A, B \otimes H^1_f(\mathbf{Q}_p(1))) = \mathrm{Hom}(A, B) \otimes H^1_f(\mathbf{Q}_p(1)).$$

Using

$$l_\wp : H^1_f(\mathbf{Q}_p(1)) = \mathcal{O}^*_\wp \widehat{\otimes} \mathbf{Q}_p \longrightarrow \mathbf{Q}_p$$

we define $h_{\wp,E} \in \mathrm{Hom}(A, B)$ by

$$h_{\wp,E} := -l_\wp((r_\wp)_*[E_\wp]) \in \mathrm{Hom}(A, B).$$

Note that the only data needed to this definition is an extension $l'_\wp : H^1_f(E_{2,\wp}) \longrightarrow B$ of $l_\wp : B \otimes H^1_f(\mathbf{Q}_p(1)) \longrightarrow B$. Given such $l'_\wp$, we define $h_{\wp,E} := -l'_\wp \circ [E_\wp]$. Such $l'_\wp$ will arise as “universal norm splittings”, cf. 6.11.

4.8 Local pairings at $\wp|p$ (general case)

Denote by π the map $H^1(E_{2,\wp}) \longrightarrow H^1(V_\wp)$. There is a commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & B \otimes H^1_f(\mathbf{Q}_p(1)) & \overset{\overset{w}{\longleftarrow}}{\longrightarrow} & H^1_f(E_{2,\wp}) & \longrightarrow & H^1_f(V_\wp) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & B \otimes H^1(\mathbf{Q}_p(1)) & \longrightarrow & \pi^{-1}(H^1_f(V_\wp)) & \longrightarrow & H^1_f(V_\wp) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle 1\otimes \mathrm{ord}_\wp} & & \downarrow{\scriptstyle \phi} & & & & \\
 & 1 \otimes l_\wp & B & = & B & & & & \\
 & & \downarrow & & \downarrow & & & & \\
B & & 0 & & 0 & & & &
\end{array}$$

equipped with a splitting w of the first row by 3.3.

We have again the extension class $[E_\wp] : A \longrightarrow H^1(E_{2,\wp})$. As $E_{1,\wp}$ is crystalline, $\pi(\mathrm{Im}([E_\wp]))$ lies in $H^1_f(V_\wp)$ by 1.25, hence

$$[E_\wp] \in \mathrm{Hom}(A, \pi^{-1}(H^1_f(V_\wp))).$$

We distinguish two cases:

(a) $l_\wp$ *is unramified:* we have $1 \otimes l_\wp = a \cdot (1 \otimes \mathrm{ord}_\wp)$ for some $a \in K_\wp$. We then define

$$h_{\wp,E} = -a \cdot (\phi \circ [E_\wp]) \in \mathrm{Hom}(A, B).$$

Note that this element does not depend on the splitting w.

(b) $l_\wp$ *is ramified:* this means that

$$l_\wp(H^1(\mathbf{Q}_p(1))) = l_\wp(H^1_f(\mathbf{Q}_p(1))) = \mathbf{Q}_p,$$

whence a map

$$\sigma : \frac{\pi^{-1}(H^1_f(V_\wp))}{B \otimes \mathrm{Ker}(l_\wp)} \xrightarrow{\sim} \frac{H^1_f(E_{2,\wp})}{B \otimes \mathrm{Ker}(l_{\wp,f})} \xrightarrow{(1\otimes l_{\wp,f})\circ w} B,$$

where $l_{\wp,f}$ is the restriction of $l_\wp$ to $H^1_f(\mathbf{Q}_p(1))$ and w is the splitting of the first row in the diagram. We then define

$$h_{\wp,E} = -\sigma \circ [E_\wp] \in \mathrm{Hom}(A, B).$$

It is easily checked that this definition is compatible with 4.7 for crystalline $E_\wp$.

Note that for each v, the local height pairing $h_{v,E}$ depends only on the local representation E_v, considered as a mixed extension of $(e_1)_v$ and $(e_2)_v$. It makes sense, therefore, to speak about the local height pairing associated to E_v, l_v and (if $v|p$) r_v.

4.9 We shall now consider in more detail the local case, i.e. we shall work with Galois representations over $K_\wp$. Suppose we are given two extensions of p-adic Galois representations over $K_\wp$

$$e_1 : 0 \longrightarrow V \longrightarrow E_1 \longrightarrow A(0) \longrightarrow 0$$
$$e_2 : 0 \longrightarrow B(1) \longrightarrow E_2 \longrightarrow V \longrightarrow 0.$$

Let E be a crystalline mixed extension of e_1, e_2. An argument similar to that in the proof of 4.4.2 shows that the set of all such E is a principal homogeneous space of

$$\mathrm{Ext}^1_{K_\wp,f}(A(0), B(1)) = \mathrm{Hom}(A, B \otimes K_\wp(1)).$$

As in 3.2-3, there is a canonical f-equivariant decomposition

$$D_{\mathrm{cris}}(E) = D_{\mathrm{cris}}(V) \oplus (A \otimes K^0_\wp) \oplus (B \otimes K^0_\wp(1))$$

and corresponding decompositions

$$\begin{array}{rcccccc} DR(E) & = & DR(V) & \oplus & (A \otimes K_\wp) & \oplus & (B \otimes K_\wp(1)) \\ D_{\mathrm{cris}}(E_1) & = & D_{\mathrm{cris}}(V) & \oplus & (A \otimes K^0_\wp) & \oplus & 0 \\ D_{\mathrm{cris}}(E_2) & = & D_{\mathrm{cris}}(V) & \oplus & 0 & \oplus & (B \otimes K^0_\wp(1)) \end{array}$$

and 3.2-3 give us formulas for $F^0DR(E_1), F^0DR(E_2)$ in terms of the maps $\alpha : A \otimes K_\wp \longrightarrow DR(V)/F^0$, $\beta : F^0DR(V) \longrightarrow B \otimes K_\wp(1)$, $\mu : DR(V) \longrightarrow DR(V)/F^0$. The projection on the first two factors induces an isomorphism $F^0DR(E) \xrightarrow{\sim} F^0DR(E_1)$, hence

$$F^0DR(E) = \{(x, a, \nu(x, a)) \,|\, \mu(x) = \alpha(a)\}$$

for some $K_\wp$-linear map $\nu : F^0DR(E_1) \longrightarrow B \otimes K_\wp(1)$. Since $F^0DR(E_2)$ is the kernel of the projection $F^0DR(E) \longrightarrow A \otimes K_\wp$, the restriction of ν to $F^0DR(V)$ must be equal to $-\beta$.

If we let E vary, the set of possible ν forms a principal homogeneous space of

$$\mathrm{Hom}(A, B \otimes K_\wp(1)) = \mathrm{Ext}^1_{K_\wp,f}(A(0), B(1)),$$

in agreement with the local version of 4.4.2 alluded to before.

Consider now the exact sequence

$$0 \longrightarrow B \otimes K_\wp(1) \longrightarrow DR(E_2)/F^0 \xrightarrow{\pi} DR(V)/F^0 \longrightarrow 0.$$

The class of the extension E is a map $e = [E] : A \longrightarrow H^1_f(K_\wp, E_2) = DR(E_2)/F^0$ such that $\pi \circ e(a) = \alpha(a \otimes 1)$.

Let $DR(V) \overset{w}{\longleftrightarrow} DR(V)/F^0$ be a splitting of μ. According to 3.3, it defines a splitting $DR(E_2)/F^0 \overset{r}{\longleftrightarrow} DR(V)/F^0$ given by the formula

$$r(x) = (w(x), 0, 0) \bmod F^0DR(E_2).$$

Write $h = -(1-r\pi)\circ e : A \longrightarrow B\otimes K_\wp(1)$. Then, by definition of the local height pairing in 4.8, $h_{\wp,E} = l_\wp \circ h$. Let us compute h explicitly: for $a \in A$, choose $x \in DR(V)$ with $\mu(x) = \alpha(a\otimes 1)$. Then $(x, a\otimes 1, \nu(x,a)) \in F^0DR(E)$. By 1.20-1,

$$e(a) \equiv (0, -a\otimes 1, 0) \equiv (x, 0, \nu(x,a)) \bmod F^0DR(E).$$

Taking $x = w\circ\alpha(a)$, we get

$$h(a) = (r\circ\alpha)(a\otimes 1) - e(a) = -\nu(w\circ\alpha(a\otimes 1), a) \in B\otimes K_\wp(1)$$

Similarly, the dual mixed extension $E^*(1)$ together with a splitting w' of $DR(V^*(1)) \xrightarrow{\mu'} DR(V^*(1))/F^0$ defines a map

$$h' = -(1 - r'\pi')\circ e' : B^* \longrightarrow A^*\otimes K_\wp(1).$$

Let $x \in DR(V)$, $x' \in DR(V^*(1))$, $a \in A\otimes K_\wp$, $b' \in B^*\otimes K_\wp$ with $\mu(x) = \alpha(a)$, $\mu'(x') = \beta'(b')$. Since $F^0DR(E) \perp F^0DR(E^*(1))$, we have

$$\langle x, x'\rangle + \langle a, \nu'(x', b')\rangle + \langle \nu(x,a), b'\rangle = 0.$$

In particular, for $x = w\circ\alpha(a\otimes 1)$, $x' = w'\circ\beta'(b'\otimes 1)$ $(a \in A,\ b' \in B^*)$, we obtain

$$\langle w\circ\alpha(a\otimes 1), w'\circ\beta'(b'\otimes 1)\rangle = \langle a, h'(b')\rangle + \langle h(a), b'\rangle \in K_\wp(1).$$

If the splitting w' is dual to the projection $u = 1 - w\circ\mu$, then

$$\langle w(a), w'(b')\rangle = \langle u\circ w(a), b'\rangle = 0,$$

hence $h' = -h^*$. Applying $l_\wp$ to both sides, we get

$$h_{\wp,E^*(1)} = -(h_{\wp,E})^*.$$

The formula for $h_{\wp,E}$ we have obtained can be summarized by the following commutative diagram, in which both squares are cartesian:

$$\begin{array}{ccccccccc}
 & & & & & & A & & \\
 & & & & & & \downarrow & & \\
 & & & & F^0DR(E_{1,\wp}) & \overset{w'}{\rightleftarrows} & A\otimes K_\wp & & \\
 & & \nu & & \downarrow & & \downarrow{\scriptstyle\alpha} & & \\
0 & \longrightarrow & F^0DR(V_\wp) & \longrightarrow & DR(V_\wp) & \overset{w}{\rightleftarrows} & DR(V_\wp)/F^0 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\beta} & & \downarrow & & & & \\
 & & B\otimes K_\wp(1) & \longrightarrow & DR(E_{2,\wp})/F^0 & & & & \\
 & & \downarrow{\scriptstyle -l_\wp} & & & & & & \\
 & & B & & & & & & ,
\end{array}$$

where w' is induced by w and ν controls $F^0DR(E_\wp)$.

4.10 Proposition. *Under the hypotheses of 4.2, fix a mixed extension E of e_1,e_2 and a non-archimedean place v of K. Then*

(1) *If E'_v is a translation of E_v by*

$$x_v \in \mathrm{Ext}^1_{K_v}(A(0), B(1)) = \mathrm{Hom}(A, B) \otimes H^1(K_v, \mathbf{Q}_p(1)),$$

then

$$h_{v,E'} = h_{v,E} - (1 \otimes l_v)(x_v).$$

(2) *If $\alpha : A \longrightarrow A'$, $\beta : B' \longrightarrow B$ are homomorphisms and E' is obtained from E by pushout and pullback via α, β, then*

$$h_{v,E} = \beta \circ h_{v,E'} \circ \alpha \in \mathrm{Hom}(A, B).$$

(3) $h_{v,E^*(1)} = -(h_{v,E})^*$ *(with the dual splittings described in 4.5)*

Proof. (1) By [Bo, X.7.6.5. Cor.1.b], the extension classes satisfy $[E'_v] - [E_v] = x_v$, whence the claim.
(2) Follows immediately from the definitions of $h_{E,v}$.
(3) Supose first that $v\not|p$. Then we are reduced to prove that for an extension

$$0 \longrightarrow \mathbf{Q}_p(1) \longrightarrow N \longrightarrow \mathbf{Q}_p(0) \longrightarrow 0,$$

the class $[N^*(1)]$ of the dual extension is equal to $-[N] \in H^1(K_v, \mathbf{Q}_p(1))$. This follows from the relations between extension classes and induced boundary maps [Bo, X.7.6.5. Cor.1].
For $v = \wp|p$, we first note that the statement is invariant under translations by elements in $\mathrm{Ext}^1_{K_v}(A(0), B(1))$ (by (1) and the claim used in the proof of the case $v\not|p$). According to 4.4.2 we can, therefore, assume that E_v is crystalline. In this case the statement has been proved in 4.9.

4.11 Theorem. *Under the assumptions of 4.2, choose a mixed extension E of e_1,e_2. Then*

(1) *The sum over all non-archimedean places*

$$h := \sum_v h_{v,E} \ \in \mathrm{Hom}(A, B)$$

is convergent and does not depend on the choice of E. If E is unramified outside a finite set of places, then the sum is finite.

(2) *Take for e_1,e_2 the universal extensions*

$$0 \longrightarrow V \longrightarrow E_1 \longrightarrow H^1_f(K, V)(0) \longrightarrow 0$$

$$0 \longrightarrow H^1_f(K, V^*(1))^*(1) \longrightarrow E_2 \longrightarrow V \longrightarrow 0.$$

Then the pairing

$$h_V = \sum_v h_{v,E} \ : \ H^1_f(K, V) \times H^1_f(K, V^*(1)) \longrightarrow \mathbf{Q}_p$$

coincides with the pairing constructed in sec. 2.

(3) *This pairing is functorial with respect to isomorphisms: if $j : V \xrightarrow{\sim} W$ is one, then*

$$h_W(j(x), y) = h_V(x, \hat{\jmath}(y)), \qquad x \in H^1_f(K, V), y \in H^1_f(K, W^*(1))$$

with $\hat{\jmath}$ being induced by $j^*(1)$ and splittings compatible via j.

(4) *Suppose there exists a skew-symmetric non-degenerate pairing of p-adic Galois representations*

$$[\ ,\ ] : \ V \times V \longrightarrow \mathbf{Q}_p(1).$$

Define an isomorphism $j : V \xrightarrow{\sim} V^*(1)$ by $[v_1, v_2] = (j(v_1))(v_2)$. The induced pairing

$$h : \quad H^1_f(K, V) \ \times \ H^1_f(K, V) \longrightarrow \mathbf{Q}_p$$

defined by $h(x,y) := h_V(x, j(y))$ is symmetric, provided the following condition is satisfied at all $\wp|p$: the image of $DR(V_\wp)/F^0$ under the splitting $w_\wp : DR(V_\wp)/F^0 \longrightarrow DR(V_\wp)$ is isotropic (in fact, Lagrangean) with respect to the symplectic form $DR(V_\wp) \times DR(V_\wp) \longrightarrow K_\wp(1)$ induced by $[\ ,\ ]$.

Proof. (1) According to 4.4, E is equal to some translate of E' by $x \in H^1(K, \mathbf{Q}_p(1))$, where E' is essentially unramified at all $v\nmid p$ and crystalline at all $\wp|p$. The claim follows from the fact that the sum

$$l = \sum_v l_v : K^*\widehat{\otimes}\mathbf{Q}_p \longrightarrow \mathbf{Q}_p$$

is convergent to zero, which is obvious, since $l : \mathbf{A}_K^* \longrightarrow \mathbf{Q}_p$ extends uniquely to a continuous map $l' : \mathbf{A}_K^*\widehat{\otimes}\mathbf{Q}_p \longrightarrow \mathbf{Q}_p$ trivial on $K^*\widehat{\otimes}\mathbf{Q}_p$.
(2) We can perform the construction of sec. 1 also in the universal situation, i.e. considering

$$0 \longrightarrow V \longrightarrow E_1 \longrightarrow H^1_f(K,V)(0) \longrightarrow 0$$

instead of an extension of $\mathbf{Q}_p(0)$. The diagram thus obtained is

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^1_f(G_S, A^*(1)) & \longrightarrow & H^1_f(G_S, E_1^*(1)) & \overset{\delta}{\rightleftarrows} & B^* & \longrightarrow & 0 \\
 & & \big\downarrow & & \big\downarrow{\scriptstyle \oplus\rho_\wp} & & \big\downarrow & & \\
0 & \longrightarrow & \bigoplus_{\wp|p} H^1_f(K_\wp, A^*(1)) & \longrightarrow & \bigoplus_{\wp|p} H^1_f(K_\wp, E_1^*(1)) & \longrightarrow & \bigoplus_{\wp|p} H^1_f(K_\wp, V^*(1)) & \longrightarrow & 0 \\
 & & \big\downarrow{\scriptstyle \sum l_\wp\otimes 1} & & & & & & \\
 & & A^* & & & & & &
\end{array}$$

(with $A = H^1_f(K,V)$, $B^* = H^1_f(G_S, V^*(1))$). Any splitting δ is an element of

$$\mathrm{Hom}(B^*, H^1_f(G_S, E_1^*(1))) = \mathrm{Ext}^1_{G_S}(B^*, E_1^*(1)),$$

hence defines an extension $0 \longrightarrow E_1^*(1) \longrightarrow E^*(1) \longrightarrow B^* \longrightarrow 0$, which makes $E^*(1)$ a mixed extension of the universal extensions

$$0 \longrightarrow V^*(1) \longrightarrow E_2^*(1) \longrightarrow B^*(0) \longrightarrow 0$$
$$0 \longrightarrow A^*(1) \longrightarrow E_1^*(1) \longrightarrow V^*(1) \longrightarrow 0,$$

essentially unramified at all $v\nmid p$ and crystalline at all $\wp|p$. The given splittings of Hodge filtrations define splittings

$$H^1_f(K_\wp, A^*(1)) \overset{w_\wp}{\rightleftarrows} H^1_f(K_\wp, E_1^*(1))$$

and the global height pairing of 2.2 is defined as

$$h_{\text{global}} = \sum_\wp (l_\wp \otimes 1) \circ w_\wp \circ \rho_\wp \circ \delta \in \mathrm{Hom}(B^*, A^*) = \mathrm{Hom}(A, B).$$

At the same time,

$$\begin{aligned} h_{v,E^*(1)} &= 0 && (v\nmid p) \\ h_{\wp,E^*(1)} &= -(l_\wp \otimes 1) \circ w_\wp \circ \rho_\wp \circ \delta && (\wp|p), \end{aligned}$$

which proves that

$$h_{\text{global}} = -\sum h_{v,E^*(1)} = \sum h_{v,E}$$

by 4.10.2.
(3) is obvious.
(4) The condition on $w_\wp$ simply means that j carries $w_\wp$ into the dual splitting $\mathcal{D}(w_\wp)$. Consequently, using $\hat{j} = -j$ and (3), we have

$$h(x,y) = h_V(x, j(y)) = -h_V(x, \hat{j}(y)) = h_{V^*(1)}(j(x), y) = h_V(y, j(x)) = h(y,x).$$

4.12 One is tempted to speculate about a possible analytic definition of the local pairings at those primes $\wp|p$ where $l_\wp$ ramifies. Perhaps there is a p-adic Arakelov theory on a regular model $\mathcal{X}$ of X over the ring of integers $\mathcal{O}_K$ in K, i.e. an intersection theory for "enhanced algebraic cycles" on $\mathcal{X}$. We have no candidate for such an object, but the corresponding enhanced Chow group $\widehat{CH}^j(\mathcal{X})$ (or, rather, its subgroup of homologically trivial enhanced cycles) could fit into an exact sequence

$$H^{2j-1}_M(X, \mathbf{Q}_p(j)) \longrightarrow H^{2j-1}_{\mathcal{D}}(X, \mathbf{Q}_p(j)) \longrightarrow \widehat{CH}^j(\mathcal{X})_0 \otimes \mathbf{Q}_p \longrightarrow CH^j(\mathcal{X})_0 \otimes \mathbf{Q}_p \longrightarrow 0$$

with p-adic Deligne cohomology equal to

$$H^{2j-1}_{\mathcal{D}}(X, \mathbf{Q}_p(j)) = \bigoplus_{\wp|p} H^1_f(K_\wp, H^{2j-2}_{\mathrm{et}}(X \otimes \overline{K})(j))$$

and "motivic" cohomology $H^{2j-1}_M(X, \mathbf{Q}_p(j))$ being the j-th graded piece of $\mathrm{Im}(K_1'(\mathcal{X}) \otimes \mathbf{Q}_p \longrightarrow K_1(X) \otimes \mathbf{Q}_p)$ with respect to the Adams filtration.

5. Geometric p-adic height pairings

5.1 Let K be a finite extension of $\mathbf{Q}$ and $X_{/K}$ a proper smooth K-scheme, equidimensional of dimension d. Denote by $CH^i(X_{/K})$ the group of algebraic cycles of codimension i on X defined over K, modulo rational equivalence, and by $CH^i(X_{/K})_0$ its subgroup of homologically trivial cycles (say, in $H^{2i}_{\mathrm{B}}(X(\mathbf{C}), \mathbf{Z}(i))$ for some embedding $K \hookrightarrow \mathbf{C}$). Write $\overline{X}$ for $X \otimes \overline{K}$.
One can define étale version of the Abel-Jacobi map

$$\Phi_i : CH^i(X_{/K})_0 \otimes \mathbf{Q}_p \longrightarrow H^1(K, V)$$

for $V = H^{2i-1}_{\mathrm{et}}(\overline{X}, \mathbf{Q}_p(i))$ as follows:
If Z is a homologically trivial cycle defined over K of codimension i, then the relative cohomology sequence

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^{2i-1}_{\mathrm{et}}(\overline{X}, \mathbf{Q}_p(i)) & \longrightarrow & H^{2i-1}_{\mathrm{et}}(\overline{X} - \overline{Z}, \mathbf{Q}_p(i)) & \longrightarrow & H^{2i}_{|\overline{Z}|}(\overline{X}, \mathbf{Q}_p(i)) & \longrightarrow & H^{2i}_{\mathrm{et}}(\overline{X}, \mathbf{Q}_p(i)) \\
 & & \| & & \uparrow & & \uparrow & & \uparrow \\
0 & \longrightarrow & V & \longrightarrow & E & \longrightarrow & \mathbf{Q}_p \cdot cl(\overline{Z}) & \longrightarrow & 0
\end{array}$$

defines by pullback an extension of p-adic representations of $G(\overline{K}/K)$

$$0 \longrightarrow V \longrightarrow E \longrightarrow \mathbf{Q}_p(0) \longrightarrow 0,$$

whose class in $H^1(K, V)$ is $\Phi_i(Z)$. See [Ja 2,3] for further details.

5.2 Let $\mathcal{X}_{/\mathcal{O}}$ be a proper flat model of X over the ring of integers of K. Let S_0 be the set of primes of bad reduction of $\mathcal{X}$. We assume that $S_0 \cap \{\wp|p\} = \emptyset$, i.e. that $\mathcal{X}$ has good reduction at all primes above p. Fix an integer i, $1 \le i \le d$ and put $V = H^{2i-1}_{\mathrm{et}}(\overline{X}, \mathbf{Q}_p(i))$. Then, by Poincaré duality, $V^*(1) = H^{2j-1}_{\mathrm{et}}(\overline{X}, \mathbf{Q}_p(j))$, where $i + j = d + 1$.

5.3 Consider now the conditions (A)–(D) of 2.1.2 for V:
(A) follows from smooth and proper base change theorems for étale cohomology
(B) follows for $v \notin S_0$ from Weil's conjectures proved by Deligne
(C) is proved in [Fa]
(D) is proved in [Ka–Me] for smooth *projective* varieties as a consequence of Weil's conjectures

(B) for $v \in S_0$ follows from the purity conjecture for the monodromy filtrations on V and $V^*(1)$ (cf.[Ja 1]), which has been proved so far only for certain classes of varieties; these include e.g. abelian varieties and some surfaces (see [Ra–Zi]). It is also known for motives corresponding to elliptic modular forms; they arise as factors of cohomology of non-singular compactifications of fibre products of the universal elliptic curve over a modular curve (see 8.3 for further details).

5.4 Proposition. *Assuming that the cohomology groups $V, V^*(1)$ satisfy (B),(D) (in particular if X is projective and the monodromy filtration on $V, V^*(1)$ is pure for each place v of bad reduction), then the Abel-Jacobi maps Φ_i, Φ_j factor through*

$$\Phi_i : CH^i(X_{/K})_0 \otimes \mathbf{Q}_p \longrightarrow H^1_f(K, V)$$
$$\Phi_j : CH^j(X_{/K})_0 \otimes \mathbf{Q}_p \longrightarrow H^1_f(K, V^*(1)).$$

Proof. Let $\wp|p$. According to [Fa], $V_\wp$ and $V^*(1)_\wp$ both satisfy (C). The de Rham conjecture for open varieties 1.11, also proved in [Fa], implies that the representation E defined in 5.1 is de Rham at $\wp$, hence the $\wp$-localization of $\mathrm{Im}(\Phi_i)$ lies in $H^1_g(K_\wp, V)$. We know that (C),(D) imply $D_{\mathrm{cris}}(V_\wp)^{f=1} = D_{\mathrm{cris}}(V^*(1)_\wp)^{f=1} = 0$ and this means that $H^1_e = H^1_f = H^1_g$ for both $V_\wp$ and $V^*(1)_\wp$, by 1.14, 1.16. Thank to (B), $H^1(K_v, V) = 0$ for all $v \not| p$, which means that there is no condition at these primes and $\mathrm{Im}(\Phi_i)$ lies indeed in $H^1_f(K, V)$. Replacing V by $V^*(1)$, we get the statement for Φ_j.

5.5 Theorem. *Under the same assumptions as in 5.4, the choice of a non-trivial continuous homomorphism $l : \mathbf{A}^*_K/K^* \longrightarrow \mathbf{Q}_p$ and of $\mathbf{Q}_p$-linear splittings of Hodge filtrations $F^i H^{2i-1}_{\mathrm{dR}}(X_{/K_\wp}) \overset{\longleftarrow}{\longrightarrow} H^{2i-1}_{\mathrm{dR}}(X_{/K_\wp})$ for primes $\wp|p$ ramified with respect to l defines a bilinear pairing*

$$CH^i(X_{/K})_0 \otimes CH^j(X_{/K})_0 \longrightarrow \mathbf{Q}_p$$

which factors through Abel-Jacobi maps. If $i = j = (d+1)/2$ and the condition of 4.11.4 is satisfied at all $\wp|p$, then the pairing is symmetric.

Proof. Combine 4.11 with 5.4.

5.6 Mixed extensions arise in the geometric situation as follows (cf. [Sc 2, 7.7]): Let $A \subseteq H^1_f(K, V)$, $B^* \subseteq H^1_f(K, V^*(1))$ be $\mathbf{Q}_p$-vector subspaces contained in the $\mathrm{Im}(\Phi_i)$ resp. $\mathrm{Im}(\Phi_j)$ (according to general conjectures, p-adic Abel-Jacobi maps should be isomorphisms on H^1_f in our situation). By the moving lemma, there are disjoint closed subschemes Y, Z of X (of codimensions i resp. j) such that any element of A (resp. B^*) can be represented by a cycle with support in Y (resp. Z).

Write $\mathcal{Z}^i_Y(X)^0$ for the group of cycles of codimension i with coefficients in $\mathbf{Q}_p$ which are supported on Y and homologically trivial in $H^{2i}_{\mathrm{et}}(\overline{X}, \mathbf{Q}_p(i))$, and likewise for Z. Relative cohomology sequences

$$0 \longrightarrow H^{2i-1}_{\mathrm{et}}(\overline{X}, \mathbf{Q}_p(i)) \longrightarrow H^{2i-1}_{\mathrm{et}}(\overline{X} - \overline{Y}, \mathbf{Q}_p(i)) \longrightarrow H^{2i}_{|\overline{Y}|}(\overline{X}, \mathbf{Q}_p(i)) \longrightarrow H^{2i}_{\mathrm{et}}(\overline{X}, \mathbf{Q}_p(i))$$

$$0 \longrightarrow H^{2i-1}_{\mathrm{et}}(\overline{X} rel\, \overline{Z}, \mathbf{Q}_p(i)) \longrightarrow H^{2i-1}_{\mathrm{et}}(\overline{X} - \overline{Y} rel\, \overline{Z}, \mathbf{Q}_p(i)) \longrightarrow H^{2i}_{|\overline{Y}|}(\overline{X}, \mathbf{Q}_p(i)) \longrightarrow H^{2i}_{\mathrm{et}}(\overline{X} rel\, \overline{Z}, \mathbf{Q}_p(i))$$

$$H^{2i-2}_{\mathrm{et}}(\overline{X}, \mathbf{Q}_p(i)) \longrightarrow H^{2i-2}_{\mathrm{et}}(\overline{Z}, \mathbf{Q}_p(i)) \longrightarrow H^{2i-1}_{\mathrm{et}}(\overline{X} rel\, \overline{Z}, \mathbf{Q}_p(i)) \longrightarrow H^{2i-1}_{\mathrm{et}}(\overline{X}, \mathbf{Q}_p(i)) \longrightarrow 0$$

$$H^{2i-2}_{\mathrm{et}}(\overline{X}-\overline{Y},\mathbf{Q}_p(i)) \longrightarrow H^{2i-2}_{\mathrm{et}}(\overline{Z},\mathbf{Q}_p(i)) \longrightarrow H^{2i-1}_{\mathrm{et}}(\overline{X}-\overline{Y} rel\, \overline{Z},\mathbf{Q}_p(i)) \longrightarrow H^{2i-1}_{\mathrm{et}}(\overline{X}-\overline{Y},\mathbf{Q}_p(i)) \longrightarrow 0$$

then define, via maps

$$\mathcal{Z}^i_{\overline{Y}}(\overline{X})^0 \xrightarrow{\sim} \mathrm{Ker}(H^{2i}_{|\overline{Y}|}(\overline{X},\mathbf{Q}_p(i)) \longrightarrow H^{2i}_{\mathrm{et}}(\overline{X},\mathbf{Q}_p(i)))$$

$$\mathrm{Coker}(H^{2i-2}_{\mathrm{et}}(\overline{X},\mathbf{Q}_p(i)) \longrightarrow H^{2i-2}_{\mathrm{et}}(\overline{Z},\mathbf{Q}_p(i))) \xrightarrow{\sim} (\mathcal{Z}^j_{\overline{Z}}(\overline{X})^0)^*(1)$$

(given by the cycle class and $(-1)\times$ the trace map respectively), a mixed extension

$$E' = H^{2i-1}_{\mathrm{et}}(\overline{X}-\overline{Y} rel\, \overline{Z},\mathbf{Q}_p(i))$$

of

$$0 \longrightarrow V \longrightarrow E'_1 \longrightarrow \mathcal{Z}^i_{\overline{Y}}(\overline{X})^0(0) \longrightarrow 0$$

$$0 \longrightarrow (\mathcal{Z}^j_{\overline{Z}}(\overline{X})^0)^*(1) \longrightarrow E'_2 \longrightarrow V \longrightarrow 0$$

with

$$E'_1 = H^{2i-1}_{\mathrm{et}}(\overline{X}-\overline{Y},\mathbf{Q}_p(i))$$
$$E'_2 = H^{2i-1}_{\mathrm{et}}(\overline{X} rel\, \overline{Z},\mathbf{Q}_p(i)).$$

Choose splittings

$$s_1 : A \hookrightarrow \mathcal{Z}^i_Y(X)^0 \subseteq \mathcal{Z}^i_{\overline{Y}}(\overline{X})^0$$
$$s_2 : B^* \hookrightarrow \mathcal{Z}^j_Z(X)^0 \subseteq \mathcal{Z}^j_{\overline{Z}}(\overline{X})^0.$$

Applying to E' pullback by s_1 and pushout by $s_2^*(1)$, we obtain a mixed extension E of

$$0 \longrightarrow B(1) \longrightarrow E_2 \longrightarrow V \longrightarrow 0$$
$$0 \longrightarrow V \longrightarrow E_1 \longrightarrow A(0) \longrightarrow 0.$$

This mixed extension is unramified outside a finite set of primes (depending on Y, Z) by the generic base change theorem ([SGA $4\frac{1}{2}$, Théorème de finitude]). The extension class of E_1 resp. $E_2^*(1)$ in $\mathrm{Hom}(A, H^1(K,V))$ resp. $\mathrm{Hom}(B^*, H^1(K,V^*(1))$ is the inclusion map, by [Sc 2,7.5] (this explains why the inverse of the trace map has been used in the definition of E').

6. Universal norms

6.0 In this section we investigate the behavior of universal norms from a $\mathbf{Z}_p$-extension for H^1_f of certain local Galois representations. In some favorable cases, universal norms will define splittings of local cohomology sequences, giving rise to local height pairings (as in the classical approach of Schneider [Sch 2]).

6.1 Suppose that we are in the local situation of 1.2. Let X be a de Rham represantation of $G(\overline{K}/K)$ satisfying $F^0DR(X)=0$. Choose a $G(\overline{K}/K)$-stable $\mathbf{Z}_p$-lattice $T_X \subset X$. Define, for any extension L/K,

$$X_n(L) := H^1(L, T_X \otimes \mathbf{Z}/p^n\mathbf{Z}), \quad Y_n(L) := H^0(L, T_X \otimes \mathbf{Z}/p^n\mathbf{Z}),$$

$$X(L) := \varprojlim_n X_n(L), \quad Y(L) := \varprojlim_n Y_n(L).$$

Of course, $Y(L) = H^0(L, T_X)$ for all L and $X(L) = H^1(L, T_X)$ for L of finite degree over K.

6.2 Let K_∞/K be a $\mathbf{Z}_p$-extension, i.e $K_\infty = \bigcup K_n$ with $G(K_n/K) = \mathbf{Z}/p^n\mathbf{Z}$. Fix an isomorphism $l : G(K_\infty/K) = \Gamma \xrightarrow{\sim} \mathbf{Z}_p$ (i.e. a topological generator $\gamma = l^{-1}(1)$ of Γ). Via reciprocity map, l defines an "algebraic logarithm" (to be denoted by the same letter) $l : K^* \longrightarrow \mathbf{Z}_p \hookrightarrow \mathbf{Q}_p$. Write

$$N_\infty X(K) := \bigcap_{n=1}^{\infty} \mathrm{cor}_{K_n/K} X(K_n) \subseteq X(K)$$

for the group of universal norms from K_∞/K. In view of 6.3 below, it is also equal to

$$\bigcap_{n=1}^{\infty} N_n X(K_n) \subseteq \varinjlim_n X(K_n)^\Gamma = X(K),$$

where $N_n = \mathrm{res}_{K_n/K} \circ \mathrm{cor}_{K_n/K}$ is the norm from K_n/K. Define $N_\infty H^1(K,X) := (N_\infty X(K)) \otimes \mathbf{Q}$ – this is independent of the choice of T_X.

6.3 Lemma. *Let L be an extension of K. Then*
(1) *If L/K is finite, then $H^0(L_{ur}, X) = 0$.*
(2) *If L/K is Galois extension and $H^0(L,X) = 0$, then the restriction map $X(K) \longrightarrow X(L)^{G(L/K)}$ is an isomorphism. For $L = K_\infty$, there is an exact sequence*

$$0 \longrightarrow Y(K_\infty)_\Gamma \longrightarrow X(K) \longrightarrow X(K_\infty)^\Gamma \longrightarrow 0,$$

in which the first term is finite.
(3) *There is a canonical isomorphism*

$$X(K)/N_\infty X(K) \xrightarrow{\sim} H^1(\Gamma, H^0(K_\infty, T_X^*(1) \otimes \mathbf{Q}_p/\mathbf{Z}_p))^\vee$$

(where $M^\vee = \mathrm{Hom}_{\mathrm{cont}}(M, \mathbf{Q}_p/\mathbf{Z}_p)$ for any $\mathbf{Z}_p$-module M).
(4) *If $H^0(K, X^*(1)) = 0$, then $X(K)/N_\infty X(K)$ is finite of order equal to*

$$\sharp(X(K)/N_\infty X(K)) = \sharp H^0(K, T_X^*(1) \otimes \mathbf{Q}_p/\mathbf{Z}_p)/\sharp(\mathrm{Div}(H^0(K_\infty, T_X^*(1) \otimes \mathbf{Q}_p/\mathbf{Z}_p)))^\Gamma.$$

Proof. (1) $W = H^0(L_{ur}, X)$ is an unramified subrepresentation of X (over L). If nontrivial, it would contribute by $F^0DR(W_L) \neq 0$ to $F^0DR(X_L) = 0$.
(2) In the inf–res exact sequence

$$0 \longrightarrow H^1(L/K, Y_n(L)) \longrightarrow X_n(K) \longrightarrow X_n(L)^{G(L/K)} \longrightarrow H^2(L/K, Y_n(L))$$

the groups $Y_n(L)$ form a ML-zero system if $H^0(L,X) = 0$ (see [Ja 2] for terminology concerning projective systems). Passing to the projective limit $\varprojlim_n$ (which is an exact functor on projective systems of finite abelian groups) we get the isomorphism $X(K) \xrightarrow{\sim} X(L)^{G(L/K)}$. If $L = K_\infty$, then $H^2(L/K, Y_n(K_\infty)) = 0$,

$$H^1(L/K, Y_n(K_\infty)) \xrightarrow{\sim} Y_n(K_\infty)_\Gamma$$

and we again conclude by passing to the projective limit (the finiteness of $Y(K_\infty)_\Gamma$ follows from $Y(K_\infty)^\Gamma = 0$).
(3) By local duality, the corestriction $\mathrm{cor}_{K_n/K} : X(K_n) \longrightarrow X(K)$ is dual to $\mathrm{res}_{K_n/K} : H^1(K, T_X^*(1) \otimes \mathbf{Q}_p/\mathbf{Z}_p) \longrightarrow H^1(K_n, T_X^*(1) \otimes \mathbf{Q}_p/\mathbf{Z}_p)$, hence

$$\mathrm{Coker}(\mathrm{cor}_{K_n/K}) \xrightarrow{\sim} \mathrm{Ker}(\mathrm{res}_{K_n/K})^\vee = H^1(K_n/K, (T_X^*(1) \otimes \mathbf{Q}_p/\mathbf{Z}_p)^{\Gamma_n})^\vee.$$

Passing to the limit, we get the assertion.
(4) Put $M = H^0(K_\infty, T_X^*(1) \otimes \mathbf{Q}_p/\mathbf{Z}_p)$. By assumption, M^Γ is finite, hence $\mathrm{Div}(M)_\Gamma$ vanishes, being simultaneously divisible and finite (as M has finite corank). Using the multiplicativity of the Herbrand quotient and finiteness of $M/\mathrm{Div}(M)$,

$$\sharp X(K)/N_\infty X(K) = \sharp H^1(\Gamma, M) = \sharp H^0(\Gamma, M)/\sharp H^0(\Gamma, \mathrm{Div}(M))$$

as claimed.

6.4 Put $A := H^0(K, X^*(1))^*$. There is a canonical surjective morphism of representations $X \longrightarrow A(1)$. Let $T_A(1)$ be the image of T_X. The isomorphism of 6.3.3 being canonical, we have a commutative diagram

$$\begin{array}{ccc} X(K)/N_\infty X(K) & \xrightarrow{\sim} & (H^0(K_\infty, T_X^*(1) \otimes \mathbf{Q}_p/\mathbf{Z}_p)_\Gamma)^\vee \\ \downarrow & & \downarrow \\ H^1(K, T_A(1))/N_\infty H^1(K, T_A(1)) & \xrightarrow{\sim} & T_A, \end{array}$$

in which the bottom arrow is induced by $l \otimes 1 : K^* \widehat{\otimes} T_A \longrightarrow T_A$.

Tensoring with $\mathbf{Q}$, we get an isomorphism

$$H^1(K, X)/N_\infty H^1(K, X) \xrightarrow{\sim} (H^0(K_\infty, X^*(1))_\Gamma)^*.$$

6.5 Lemma. *Suppose that one of the following two conditions is satisfied:*
(a) *K_∞/K is ramified*
(b) *K_∞/K is unramified and the canonical map $D_{\mathrm{st}}(X^*(1))^{f=1} \longrightarrow D_{\mathrm{st}}(X^*(1))/(f-1)$ is an isomorphism.*
Then the canonical map $H^0(K_\infty, X^*(1))^\Gamma \longrightarrow H^0(K_\infty, X^*(1))_\Gamma$ is an isomorphism.

Proof. (a) Suppose first that K_∞/K is totally ramified. Set $E = H^0(K_\infty, X^*(1))$ and $W = X^*(1)^I$. Consider the exact sequence

$$0 \longrightarrow H^0(K_\infty, W) \longrightarrow E = H^0(K_\infty, X^*(1)) \longrightarrow H^0(K_\infty, X^*(1)/W).$$

By 1.33, the third group has no Γ-invariants, hence

$$A^* = E^\Gamma = H^0(K_\infty, W)^\Gamma = H^0(K, W) = H^0(K_\infty, W)$$

(the last equality follows from the fact that K_∞/K is totally ramified). This implies that $(E/E^\Gamma)^\Gamma \subseteq H^0(K_\infty, X^*(1)/W)^\Gamma = 0$, hence $E^\Gamma \xrightarrow{\sim} E_\Gamma$ as claimed.

In general, K_∞/K_m is totally ramified for some $m < \infty$, and by the previous argument $E^{\Gamma_m} \xrightarrow{\sim} E_{\Gamma_m}$. These two spaces, as well as E, are semisimple as representations of the finite group Γ/Γ_m, which proves that $E^\Gamma \xrightarrow{\sim} E_\Gamma$.

(b) Put $B := X^*(1)^I$. Then, by 1.28.3-4, $D_{\mathrm{st}}(X^*(1))^{f=1} = B^{\langle \phi_K \rangle}$, $D_{\mathrm{st}}(X^*(1))/(f-1) = B_{\langle \phi_K \rangle}$. Since $G(K_{ur}/K_\infty)$ has (pro) order prime to p, its action on B factors through a finite group, say, H. We have then $H^0(K_\infty, X^*(1))^\Gamma = B^{\langle \phi_K \rangle}$, $H^0(K_\infty, X^*(1))_\Gamma = (B^H)_\Gamma \xrightarrow{\sim} (B_H)_\Gamma = B_{\langle \phi_K \rangle}$ by the same semisimplicity argument used in (a) above.

6.6 Theorem. *Under the assumptions of 6.5, the homomorphism $X \longrightarrow A(1)$ induces isomorphisms*

$$H^1(K, X)/N_\infty H^1(K, X) \xrightarrow{\sim} H^1(K, A(1))/N_\infty H^1(K, A(1)) \overset{l\otimes 1}{\xrightarrow{\sim}} A.$$

If $A = 0$, then the order of $X(K)/N_\infty X(K)$ divides $\sharp H^0(K, T_X^*(1) \otimes \mathbf{Q}_p/\mathbf{Z}_p)$, with equality if $H^0(K_\infty, X^*(1)) = 0$.

Proof. Combine 6.4, 6.5 and 6.3.4.

6.7 Let V be a p-adic Galois representation over K. We say that V satisfies Pančiškin's condition (cf. [Pe-Ri 3]), if it sits in an exact sequence of p-adic Galois representations (with X, Y de Rham)

$$0 \longrightarrow X \longrightarrow V \longrightarrow Y \longrightarrow 0$$

such that $F^0DR(X) = DR(Y)/F^0 = 0$ (V is then also de Rham, by 1.28.1). Note that this exact sequence is uniquely determined by V: if X', Y' have the same property, then the induced map $\alpha : X \hookrightarrow V \longrightarrow Y'$ satisfies $DR(Y') = F^0DR(Y') = F^0DR(\mathrm{Coker}(\alpha)) \subseteq DR(\mathrm{Coker}(\alpha))$, hence $DR(Y') = DR(\mathrm{Coker}(\alpha))$, which implies that $\alpha = 0$ and $X \subseteq X'$. By symmetry, $X = X'$.

Choose a $G(\overline{K}/K)$-invariant $\mathbf{Z}_p$-lattice $T \subset V$ and put $T_X = T \cap X$, $T_Y = T/T_X$. Basic example the reader should keep in mind is provided by an ordinary representation V, when $X = F^1V$, $Y = V/F^1V$.

6.8 Lemma. *Suppose that V is a crystalline representation satisfying the Pančiškin condition and such that $D_{\mathrm{cris}}(V)^{f=1} = D_{\mathrm{cris}}(V^*(1))^{f=1} = 0$. Then the following sequence is exact:*

$$0 \longrightarrow H^1(K,T_X) \longrightarrow H^1_f(K,T) \longrightarrow H^0(K, T_Y \otimes \mathbf{Q}_p/\mathbf{Z}_p) \longrightarrow H^2(K,T_X)_{\mathrm{tors}}$$

Proof. By 1.32.3 and 1.28.4, $H^1_f(K,Y) = 0$ and $H^1_f(K,X) = H^1(K,X) \xrightarrow{\sim} H^1_f(K,V)$. Consider the following commutative diagram with exact rows (the exactness of the bottom row follows from the argument used in 1.25):

$$\begin{array}{ccccccc}
H^1(K,T_X) & \longrightarrow & H^1(K,T) & \xrightarrow{\alpha} & H^1(K,T_Y) & \xrightarrow{\delta} & H^2(K,T_X) \\
\| & & \bigcup\!\!\uparrow & & \uparrow{\scriptstyle\gamma} & & \\
H^1_f(K,T_X) & \longrightarrow & H^1_f(K,T) & \xrightarrow{\beta} & H^1_f(K,T_Y) & &
\end{array}$$

The map $\phi : H^1(K,T)/H^1_f(K,T) \longrightarrow H^1(K,T_Y)/H^1_f(K,T_Y)$ induced by α has finite kernel, as $H^1_f(K,T_Y) = H^1(K,T_Y)_{\mathrm{tors}}$ is finite. On the other hand, $H^1(K,T)/H^1_f(K,T)$ is torsion free, hence $\mathrm{Ker}(\phi) = 0$. An easy diagram chase then shows that $\mathrm{Ker}(\delta\gamma) = \mathrm{Im}(\beta)$, which proves the lemma, if we take into account isomorphisms

$$H^1_f(K,T_Y) = H^1(K,T_Y)_{\mathrm{tors}} \xrightarrow{\sim} H^0(K, T_Y \otimes \mathbf{Q}_p/\mathbf{Z}_p)$$

(the last one follows from [Ta] and the fact that $H^0(K,Y) = 0$, by 1.28.3).

6.9 Theorem. *Suppose that V is a crystalline representation satisfying Pančiškin's condition, $D_{\mathrm{cris}}(V)^{f_m=1} = D_{\mathrm{cris}}(V^*(1))^{f_m=1} = 0$ for all $m < \infty$ for which K_∞/K is unramified (where $f_m = f^{p^m}$) and, if K_∞/K is unramified, that X satisfies the condition 6.5.b. Then*

1. *$H^1_f(K,T)/N_\infty H^1_f(K,T)$ is finite.*
2. *If $H^0(K_\infty, X^*(1)) = H^0(K_\infty, Y) = 0$, then there is an exact sequence*

$$0 \longrightarrow H^1(K,T_X)/N_\infty H^1(K,T_X) \longrightarrow H^1_f(K,T)/N_\infty H^1_f(K,T) \longrightarrow E \longrightarrow 0,$$

where E is the image of $H^1_f(K,T)$ in $H^1(K,T_Y)_{\mathrm{tors}} \xrightarrow{\sim} H^0(K, T_Y \otimes \mathbf{Q}_p/\mathbf{Z}_p)$.

3. *E contains $H^0(K, T_Y^I \otimes \mathbf{Q}_p/\mathbf{Z}_p)$.*

Proof. (1) Under our assumptions Lemma 6.8 applies over each K_n. Snake lemma then gives an exact sequence

$$H^1(K,T_X)/N_\infty H^1(K,T_X) \longrightarrow H^1_f(K,T)/N_\infty H^1_f(K,T) \longrightarrow E \longrightarrow 0,$$

where E is a subfactor of the finite group $H^0(K, T_Y \otimes \mathbf{Q}_p/\mathbf{Z}_p)$. According to 6.6, the first group is finite as well, since $A = 0$ by 1.28.3.

(2) Applying lemma 6.8 over K_n, we get exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^1(K_n, T_X) & \longrightarrow & H^1_f(K_n, T) & \longrightarrow & E_n & \longrightarrow & 0 \\
0 & \longrightarrow & E_n & \longrightarrow & H^0(K_n, T_Y \otimes \mathbf{Q}_p/\mathbf{Z}_p) & \longrightarrow & F_n & \longrightarrow 0 & \\
0 & \longrightarrow & F_n & \longrightarrow & H^2(K_n, T_X)_{\mathrm{tors}} & & & &
\end{array}$$

with some finite groups E_n, F_n. Write E, F for the corresponding objects over K. Snake lemma implies that $E_n \subseteq E_{n+1}^{\Gamma_n}$ and $E_n^\Gamma/E \xrightarrow{\sim} \mathrm{Ker}(F \longrightarrow F_n^\Gamma)$. As $H^0(K_\infty, Y) = 0$, $E_\infty = \varinjlim E_n$ is finite and the sequence of E_n stabilizes. The map $F \longrightarrow F_n^\Gamma$ is induced by the restriction $\mathrm{res}_{K_n/K}$, which is dual to $\mathrm{cor}_{K_n/K} : H^0(K_n, T_X^*(1) \otimes \mathbf{Q}_p/\mathbf{Z}_p) \longrightarrow H^0(K, T_X^*(1) \otimes \mathbf{Q}_p/\mathbf{Z}_p)$. This map is trivial for sufficiently big n, as $H^0(K_\infty, T_X^*(1) \otimes \mathbf{Q}_p/\mathbf{Z}_p)$ is finite by our assumptions. For such n, therefore, $E_n^\Gamma/E = F$ and $E_n^\Gamma = H^0(K, T_Y \otimes \mathbf{Q}_p/\mathbf{Z}_p)$. We know by 6.3.2 that $H^1(K, T_X) \xrightarrow{\sim} H^1(K_n, T_X)^\Gamma$. Snake lemma then gives an exact sequence

$$0 \longrightarrow H^1_f(K, T) \longrightarrow H^1_f(K_n, T)^\Gamma \longrightarrow \mathrm{Ker}(E_n^\Gamma/E \xrightarrow{\alpha} H^1(\Gamma/\Gamma_n, H^1(K_n, T_X))) \longrightarrow 0.$$

For big n, the composite map

$$F = E_n^\Gamma/E \xrightarrow{\alpha} H^1(\Gamma/\Gamma_n, H^1(K_n, T_X)) \hookrightarrow H^2(K, T_X)$$

is the inclusion, hence $\mathrm{Ker}(\alpha) = 0$ and $H^1_f(K, T) \xrightarrow{\sim} H^1_f(K_n, T)^\Gamma$.
Set

$$M_1 = \varinjlim H^1(K_n, T_X), \qquad M_2 = \varinjlim H^1_f(K_n, T).$$

We now apply the trick of [Jo 2] to the exact sequence

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow E_\infty \longrightarrow 0,$$

which generalizes (4) of *loc. cit.* Its cohomology sequence becomes

$$0 \longrightarrow F \longrightarrow H^1(\Gamma, M_1) \longrightarrow H^1(\Gamma, M_2) \longrightarrow H^1(\Gamma, E_\infty) \longrightarrow 0$$

(apply 6.6 and [Jo 2, 1.1.2]). All of the groups in this sequence are finite: this is trivially true for the first and the fourth group, and

$$H^1(\Gamma, M_1) = \varinjlim_n H^1(\Gamma/\Gamma_n, M_1^{\Gamma_n}) = \varinjlim_n H^1(\Gamma/\Gamma_n, H^1(K_n, T_X)) \subseteq H^2(K, T_X)$$

is finite as well.
Snake lemma gives an exact sequence

$$\frac{H^1(K, T_X)}{\mathrm{cor}_{K_n/K}(H^1(K_n, T_X))} \longrightarrow \frac{H^1_f(K, T)}{\mathrm{cor}_{K_n/K}(H^1_f(K_n, T))} \longrightarrow \frac{E}{\mathrm{cor}_{K_n/K}(E_n)} \longrightarrow 0,$$

which is for big n nothing else than

$$\widehat{H}^0(\Gamma/\Gamma_n, H^1(K_n, T_X)) \longrightarrow \widehat{H}^0(\Gamma/\Gamma_n, H^1_f(K_n, T)) \longrightarrow E \longrightarrow 0.$$

Multiplicity of the Herbrand quotient then gives

$$\frac{\sharp \widehat{H}^0(\Gamma/\Gamma_n, H^1_f(K_n, T))}{\sharp \widehat{H}^0(\Gamma/\Gamma_n, H^1(K_n, T_X)) \sharp E} = \frac{\sharp H^1(\Gamma/\Gamma_n, H^1_f(K_n, T))}{\sharp H^1(\Gamma/\Gamma_n, H^1(K_n, T_X)) \sharp E} = \frac{\sharp H^1(\Gamma, M_1)}{\sharp H^1(\Gamma, M_2) \sharp E} =$$

$$= \frac{\sharp E_\infty^\Gamma}{\sharp E \sharp F} = 1,$$

hence the sequence above is exact even with 0 added to the left. Passing to the limit then concludes the proof.
(3) is proved in [Pe-Ri 3, 2.3.2].

6.10 If $T = T_p(A)$ is the Tate module of an abelian variety $A_{/K}$ with ordinary good reduction, then $T_X = F^1T$, $T_Y = T/F^1T$ with $T_X^*(1), T_Y$ unramified. In the exact sequence

$$0 \longrightarrow H^1(K, F^1T) \longrightarrow H^1_f(K,T) \longrightarrow E \longrightarrow 0,$$

the first group is equal to the $\mathcal{O}_K$-points of the formal group of A, the second is equal to $A(K)\widehat{\otimes}\mathbf{Z}_p$ and $E = H^0(K, (T/F^1T) \otimes \mathbf{Q}_p/\mathbf{Z}_p) = \tilde{A}(k)_{p^\infty}$ is the p-primary part of the points of the reduction $\tilde{A}$ of A with values in the residue field k of $\mathcal{O}_K$. As $T^*(1)$ is the Tate module of the dual abelian variety B, 6.9 gives the well-known formula (under the assumptions of 6.9.2)

$$\sharp A(K)/N_\infty A(K) = p - \text{part of } \sharp\tilde{A}(k)\sharp\tilde{B}(k) = p - \text{part of } (\sharp\tilde{A}(k))^2.$$

6.11 Theorem. *Suppose V satisfies the assumptions of 6.9 and sits in an exact sequence $0 \longrightarrow B(1) \longrightarrow E_2 \longrightarrow V \longrightarrow 0$ with crystalline E_2. Then*

(1) *The canonical map*

$$B \overset{\sim}{\longleftarrow} H^1_f(K, B(1))/N_\infty \longrightarrow H^1_f(K, E_2)/N_\infty$$

is an isomorphism, defining thus an extension $l' : H^1_f(K, E_2) \longrightarrow B$ of $l : H^1_f(K, B(1)) \longrightarrow B$.

(2) *If $0 \longrightarrow V \longrightarrow E_1 \longrightarrow A(0) \longrightarrow 0$ is an exact sequence with crystalline E_1 and E a crystalline mixed extension of E_1, E_2 (as Galois representations over K), then the local height pairing $h_E^{norm} : A \longrightarrow B$ defined via l' coincides with the height pairing $h_E^{can} : A \longrightarrow B$ obtained from the splitting $DR(V)/F^0 \overset{\sim}{\longleftarrow} DR(X) \hookrightarrow DR(V)$.*

Proof. Using 1.32 and 1.28 applied to $V^*(1)$,

$$\mathrm{Hom}_{G(\overline{K}/K)}(X, B(1)) = B \otimes H^0(K, X^*(1)) = 0$$
$$\mathrm{Ext}^1_{K,f}(X, B(1)) = B \otimes H^1_f(K, X^*(1)) = 0.$$

This implies that the pullback of E_2 via $j : X \hookrightarrow V$ admits a unique splitting $j^*E_2 = X \oplus B(1)$ as a representation of $G(\overline{K}/K)$. In view of the formulas in 3.3, this decomposition is compatible with the unique f-equivariant splitting $D_{\mathrm{cris}}(E_2) = D_{\mathrm{cris}}(V) \oplus D_{\mathrm{cris}}(B(1))$. The same is true over all K_n. By 1.32 and 1.28, $H^1(X) \overset{\sim}{\longrightarrow} H^1_f(V)$, $H^1_f(j^*E_2) \overset{\sim}{\longrightarrow} H^1_f(E_2)$ over all K_n. According to 6.6, $N_\infty H^1(K, X) = H^1(K, X)$. This means that under the isomorphism

$$H^1_f(K, E_2) \overset{\sim}{\longleftarrow} H^1_f(K, j^*E_2) = H^1(K, X) \oplus H^1_f(K, B(1))$$

the group of universal norms $N_\infty H^1_f(K, E_2)$ corresponds to

$$N_\infty H^1(K, X) \oplus N_\infty H^1_f(K, B(1)) = H^1(K, X) \oplus \mathrm{Ker}(1 \otimes l)$$

(where $1 \otimes l : H^1_f(K, B(1)) = B\widehat{\otimes}\mathcal{O}_K^* \longrightarrow B$), proving thus (1). Let $w : H^1_f(K, V) \simeq DR(V)/F^0 \longrightarrow DR(E_2)/F^0 \simeq H^1_f(K, E_2)$ be the splitting induced by the splitting $DR(V)/F^0 \overset{\sim}{\longleftarrow} DR(X) \hookrightarrow DR(V)$ (see 3.3). Denote by $\pi : H^1_f(K, E_2) \longrightarrow H^1_f(K, V)$ the canonical projection. Then the map $1 - w\pi$ is equal to the composition

$$H^1_f(K, E_2) \overset{\sim}{\longleftarrow} H^1_f(K, j^*E_2) \longrightarrow H^1_f(K, j^*E_2)/H^1(K, X) = H^1_f(K, B(1)).$$

Consequently, $l \circ (1 - w\pi) = l'$, which proves (2).

7. Semistable representations

7.1 In this section we extend previous constructions to representations which are no longer crystalline, only semistable at all places over p. We also investigate the behavior of local universal norms for representations satisfying Pančiškin's condition and define for them an “extended height”, following [Ma–Ta–Te].

7.2 Supose we are in the situation of 2.1, with the condition (C) replaced by
(C') $V_\wp$ is a semistable representation of $G(\overline{K_\wp}/K_\wp)$ for all $\wp|p$.
In particular, we assume that $D_{\rm cris}(V_\wp)^{f=1} = D_{\rm cris}(V^*(1)_\wp)^{f=1} = 0$. Define for any p-adic representation W of $G(\overline{K}/K)$

$$H^1_{\rm st}(K,W) = \{x \in H^1(K,W) \,|\, x_\wp \in H^1_{\rm st}(K_\wp,W)\ \forall \wp|p\}.$$

We have, as in 2.1, $H^1(K_v,V) = H^1(K_v,V^*(1)) = 0$ for all $v\nmid p$ and $H^1_{\rm st}(K,-) = H^1_f(K,-)$ for both $V, V^*(1)$.

7.3 As in 4.4, $H^1_{\rm st}(K,V) = H^1_f(K,V)$ and $H^1_{\rm st}(K,V^*(1)) = H^1_f(K,V^*(1))$ are orthogonal with respect to the cup product $H^1(K,V)\times H^1(K,V^*(1)) \longrightarrow H^2(K,\mathbf{Q}_p(1))$. Consequently, given a pair of extensions of p-adic representations of $G(\overline{K}/K)$

$$\begin{array}{lccccccccc} e_1: & 0 & \longrightarrow & V & \longrightarrow & E_1 & \longrightarrow & A(0) & \longrightarrow & 0 \\ e_2: & 0 & \longrightarrow & B(1) & \longrightarrow & E_2 & \longrightarrow & V & \longrightarrow & 0 \end{array}$$

with $E_{1,\wp}, E_{2,\wp}$ de Rham (hence semistable, by 1.27), the Yoneda composition $e_2\circ e_1 = 0$ vanishes. This implies that there exists a mixed extension E of e_1, e_2 and that the set of such E is a principal homogeneous space under

$$\mathrm{Ext}^1_{K,\rm st}(A(0),B(1)) = \mathrm{Hom}(A,B)\otimes H^1(K,\mathbf{Q}_p(1)).$$

The argument of 4.4.4 shows that $E_\wp$ is de Rham, hence semistable, for all $\wp|p$. As in 4.3, each E splits as a representation of $G(\overline{K_v}/K_v)$ at $v\nmid p$: $E_v = V_v \oplus N$ for some

$$0 \longrightarrow B(1)_v \longrightarrow N \longrightarrow A(0)_v \longrightarrow 0.$$

The set of E for which E is essentially unramified (i.e. N is unramified) at all $v\nmid p$ forms a pricipal homogeneous space under

$$\mathrm{Hom}(A,B)\otimes \mathcal{O}_K[1/p]^*.$$

7.4 We are now ready to define local height pairings, in analogy to 4.6-7: fix a continuous homomorphism $l : \mathbf{A}_K^*/K^* \longrightarrow \mathbf{Q}_p$ and splittings of the Hodge filtration $DR(V_\wp) \overset{\longleftarrow}{\longrightarrow} DR(V_\wp)/F^0$ at all $\wp|p$.
(i) *local pairing at* $v\nmid p$:
The definition of 4.6 works word by word, giving

$$h_{v,E} = -l_v([N]) \in \mathrm{Hom}(A,B)$$

(ii) *local pairing at* $\wp|p$: The extension class of $E_\wp$ gives a map $[E_\wp] : A \longrightarrow H^1_{\rm st}(K_\wp,E_2)$ and the chosen splitting of the Hodge filtration determines, by 3.5, a splitting $w_\wp$ of

$$0 \longrightarrow H^1_{\rm st}(K_\wp,B(1)) \overset{w_\wp}{\overset{\longleftarrow}{\longrightarrow}} H^1_{\rm st}(K_\wp,E_2) \longrightarrow H^1_{\rm st}(K_\wp,V) \longrightarrow 0.$$

We define

$$h_{\wp,E} := -l_\wp\circ w_\wp \circ [E_\wp] : A \longrightarrow B.$$

7.5 Fix $\wp|p$. As in 4.9, there is a unique decomposition

$$D_{\rm st}(E_\wp) = D_{\rm st}(V_\wp)\oplus D_{\rm st}(A(0)_\wp)\oplus D_{\rm st}(B(1)_\wp)$$

which is f-equivariant and "almost N-equivariant":

$$N(x,a,b) = (Nx, 0, \gamma_0(a)),$$

where $\gamma_0 : D_{\rm st}(A(0)_\wp) \longrightarrow D_{\rm st}(B(0)_\wp)$ is the $K_\wp^0$-linear extension of some $\gamma : A \longrightarrow B$. We say that E is essentially crystalline at $\wp$ if $\gamma = 0$. If this is the case, then $[E_\wp] \in \mathrm{Hom}(A, H^1_f(K_\wp, E_2))$ by 1.25 and $h_{\wp,E}$ can be defined via splitting of the sequence

$$0 \longrightarrow H^1_f(K_\wp, B(1)) \longrightarrow H^1_f(K_\wp, E_2) \longrightarrow H^1_f(K_\wp, V) \longrightarrow 0.$$

If we change E by $x \in \mathrm{Hom}(A,B) \otimes H^1(K, \mathbf{Q}_p(1))$, γ changes by $(1 \otimes v_\wp)(x)$, where $v_\wp : \overline{K_\wp}^* \longrightarrow \mathbf{Q}_p$ is the valuation used in the definition of $B_{\rm st}$ for $K_\wp$. As

$$\bigoplus_{\wp | p} v_\wp : K^* \widehat{\otimes} \mathbf{Q}_p \longrightarrow \bigoplus_{\wp | p} \mathbf{Q}_p$$

is surjective, there exists E which is essentially crystalline at all $\wp | p$ and essentially unramified at all $v \nmid p$ (E comes, therefore, from a "mixed motive over $\mathcal{O}_K$", in the terminology of A.J.Scholl [Sc 2]) and the set of such E is a principal homogeneous space under $\mathrm{Hom}(A,B) \otimes \mathcal{O}_K^*$.

7.6 In the situation of 7.5, the sum

$$h = \sum_v h_{v,E} \in \mathrm{Hom}(A,B)$$

is convergent and does not depend on the choice of E. If E is essentially unramified at all $v \nmid p$ and essentially crystalline at all $\wp | p$, then only terms $h_{\wp,E}$ with $\wp | p$ and ramified $l_\wp$ contribute. Similarly, other statements of 4.11 carry over to the present situation: the universal extensions give rise to

$$h_V : H^1_f(K,V) = H^1_{\rm st}(K,V) \longrightarrow H^1_{\rm st}(K, V^*(1))^* = H^1_f(K, V^*(1))^*$$

satisfying $h_{V^*(1)} = -(h_V)^*$ (indeed, the calculation of 4.9 is valid also for essentially crystalline $E_\wp$).
In the geometric situation of 5.1, the proof of 5.4 implies that the Abel-Jacobi maps factor through $H^1_f(K,V)$ resp. $H^1_f(K, V^*(1))$, once $V = H^{2i-1}_{\rm et}(\overline{X}, \mathbf{Q}_p(i))$ is known to satisfy (B),(C') and (D). It is expected that examples of such V should be provided by representations corresponding to modular forms of level divisible by p. Despite great recent progress, none of the conditions (C'),(D) seems to have been verified for such V.
On the other hand, in an ideal world, general cohomological conjectures (cf. [Fo–Il]) predict that for any proper and smooth X, V should satisfy all of the conditions (A),(B),(C'),(D), possibly after a finite extension of the base field K. If true, then the previous construction covers the most general geometric situation.

7.7 We shall now investigate the behavior of universal norms for $H^1_{\rm st}(K_\wp, V)$. For notational convenience, we shall write K instead of $K_\wp$ throughout 7.7-11. Let K_∞/K be a Γ-extension. Fix an isomorphism $\Gamma = G(K_\infty/K) \xrightarrow{\sim} \mathbf{Z}_p$. It induces, via reciprocity map, an algebraic logarithm $l : H^1(K, \mathbf{Q}_p(1)) = K^* \widehat{\otimes} \mathbf{Q}_p \longrightarrow \mathbf{Q}_p$. Suppose that V is a semistable representation of $G(\overline{K}/K)$ satisfying one of the conditions 6.5.a/b and $D_{\rm cris}(V)^{f_m=1} = D_{\rm cris}(V^*(1))^{f_m=1} = 0$ whenever K_m/K is unramified. We further assume that V satisfies Pančiškin's condition 6.7, i.e. it is an extension

$$0 \longrightarrow X \longrightarrow V \longrightarrow Y \longrightarrow 0$$

of semistable representations X, Y with $F^0 DR(X) = DR(Y)/F^0 = 0$.
Taking pullback (resp. pushout) of V via $H^0(K,Y)(0) \hookrightarrow Y$ (resp. $X \longrightarrow H^0(K, X^*(1))^*(1)$), we get an extension

$$0 \longrightarrow H^0(K, X^*(1))^*(1) \longrightarrow W \longrightarrow H^0(K,Y)(0) \longrightarrow 0.$$

In our situation 1.28.3 implies that

$$H^0(K,Y) = D_{\mathrm{st}}(V)^{f=1}, \qquad H^0(K,X^*(1)) = D_{\mathrm{st}}(V^*(1))^{f=1},$$

thus W is an extension of $(D_{\mathrm{st}}(V)^{f=1})(0)$ by $(D_{\mathrm{st}}(V^*(1))^{f=1})^*(1)$.

Thank to 1.32.3, we have exact sequences

$$0 \longrightarrow H^0(K_n,Y) \longrightarrow H^1(K_n,X) \longrightarrow H^1_{\mathrm{st}}(K_n,V) \longrightarrow 0$$

over all K_n ($n < \infty$). The induced exact sequence

$$H^0(K,Y)/N_\infty \longrightarrow H^1(K,X)/N_\infty \longrightarrow H^1_{\mathrm{st}}(K,V)/N_\infty \longrightarrow 0$$

is canonically isomorphic, by 6.6, to

$$H^0(K,Y) \overset{\delta}{\longrightarrow} H^0(K,X^*(1))^* \longrightarrow H^1_{\mathrm{st}}(K,V)/N_\infty \longrightarrow 0.$$

The map δ is the same as for the extension W. Write $[W] \in \mathrm{Hom}(H^0(K,Y), H^0(K,X^*(1))^*) \otimes H^1(K,\mathbf{Q}_p(1))$ for the extension class of W; then $\delta = l([W])$. Since $h^0(Y) = h^0(X^*(1))$ by 1.28.3 and 1.32.2, we see that

$$H^1_{\mathrm{st}}(K,V)/N_\infty = 0 \iff l([W]) \text{ is surjective} \iff l([W]) \text{ is an isomorphism}$$

7.8 Let $0 \longrightarrow B(1) \longrightarrow E_2 \longrightarrow V \longrightarrow 0$ be an extension of representations of $G(\overline{K}/K)$ with E_2 de Rham (hence semistable, by 1.27). Let $\beta : F^0DR(V) \longrightarrow B \otimes K(1)$ be the K-linear map defined in 3.4. From 3.5, we get a unique (f,N)-equivariant splitting

$$D_{\mathrm{st}}(E_2) = D_{\mathrm{st}}(V) \oplus D_{\mathrm{st}}(B(1))$$

inducing $DR(E_2) = DR(V) \oplus (B \otimes K(1))$ with $F^0DR(E_2) = \{(x, -\beta(x)) \mid x \in F^0DR(V)\}$. Write $j : X \hookrightarrow V$ for the inclusion map. The induced decomposition

$$D_{\mathrm{st}}(j^*E_2) = D_{\mathrm{st}}(X) \oplus D_{\mathrm{st}}(B(1))$$

is in fact a splitting in $MF_K(f,N)$, i.e. respects filtrations on both sides of $DR(j^*E_2) = DR(X) \oplus DR(B(1))$. It comes, therefore, from an isomorphism $j^*E_2 \overset{\sim}{\longrightarrow} X \oplus B(1)$ of Galois representations (the fact that j^*E_2 splits also follows from 1.32.3 applied to $V^*(1)$). Note that all possible splittings $j^*E_2 \overset{\longleftarrow}{\longrightarrow} X$ form a principal homogeneous space under $H^0(K,X^*(1)) \otimes B$, whereas we have just constructed a *canonical* one.

Consider the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & H^1(K,B(1)) & \longrightarrow & H^1_{\mathrm{st}}(K,E_2) & \longrightarrow & H^1_{\mathrm{st}}(K,V) & \longrightarrow & 0 \\
 & & \| & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & H^1(K,B(1)) & \longrightarrow & H^1(K,j^*E_2) & \overset{\longleftarrow}{\longrightarrow} & H^1(K,X) & \longrightarrow & 0 \\
 & & & & & & \uparrow & & \\
 & & & & & & H^0(K,Y) & & \\
 & & & & & & \uparrow & & \\
 & & & & & & 0 & &
\end{array}$$

Using 3.5, it can be written as

$$
\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \\
B \oplus B \otimes K(1) & \rightarrow & B \oplus DR(E_2)/F^0 & \rightarrow & DR(V)/F^0 & \rightarrow 0 \\
\| & & \uparrow & & \uparrow & & \\
B \oplus B \otimes K(1) & \rightarrow & D_{\mathrm{st}}(X(-1))^{f=1} \oplus B \oplus DR(X) \oplus B \otimes K(1) & \leftrightarrows & D_{\mathrm{st}}(X(-1))^{f=1} \oplus DR(X) & \rightarrow 0 \\
 & & & & \uparrow & & \\
 & & & & H^0(K,Y) & & \\
 & & & & \uparrow & & \\
 & & & & 0 & &
\end{array}
$$

(as $D_{\mathrm{st}}(E_2(-1))^{f=1} = D_{\mathrm{st}}(V(-1))^{f=1} \oplus B$, $D_{\mathrm{st}}(V(-1))^{f=1} = D_{\mathrm{st}}(X(-1))^{f=1}$). The interesting maps in this diagram are given by the following formulas (cf. 3.5 and 1.23):

$$
\begin{array}{ccc}
[a, (i \circ \varepsilon \circ N_V^{-1}(x) + (y,b)) \bmod F^0 DR(E_2)] & \longmapsto & \mathrm{can}(y) + \pi \circ i \circ \varepsilon \circ N_V^{-1}(x) \\
\uparrow & & \uparrow \\
(x,y,a,b) & \longmapsto & (x,y) \\
 & & (N_V(z), -\mathrm{spl} \circ \pi \circ i \circ \varepsilon(z)) \\
 & & \uparrow \\
 & & z
\end{array}
$$

($x \in D_{\mathrm{st}}(X(-1))^{f=1}$, $y \in DR(X)$, $z \in H^0(K,Y)$, $a \in B$, $b \in B \otimes K(1)$), where

$$N_V : H^0(K,Y) = D_{\mathrm{st}}(V)^{f=1} = D_{\mathrm{st}}(E_2)^{f=1} \xrightarrow{\sim} D_{\mathrm{st}}(V(-1))^{f=1} = D_{\mathrm{st}}(X(-1))^{f=1}$$

is induced by N on $D_{\mathrm{st}}(V)$,

$$\varepsilon : H^0(K,Y) = D_{\mathrm{st}}(E_2)^{f=1} \hookrightarrow D_{\mathrm{st}}(E_2), \quad i : D_{\mathrm{st}}(E_2) \longrightarrow DR(E_2)$$

are canonical maps, π is induced by $E_2 \longrightarrow V$, $\mathrm{can} : DR(X) \xrightarrow{\sim} DR(V)/F^0$ is the canonical isomorphism and $\mathrm{spl} = \mathrm{can}^{-1} : DR(V)/F^0 \xrightarrow{\sim} DR(X)$ the canonical splitting. We shall use later on the maps p_0, p_X, which are given by the first (resp. second) projection in $DR(V) \xrightarrow{\sim} F^0 DR(V) \oplus DR(X)$.

7.9 We are now ready to describe $N_\infty H^1_{\mathrm{st}}(E_2)$. Write $\rho : X \longrightarrow H^0(K, X^*(1))^*(1)$ for the canonical (surjective) map and also for the induced homomorphisms

$$D_{\mathrm{st}}(X(-1))^{f=1} \longrightarrow H^0(K, X^*(1))^*, \quad DR(X) \longrightarrow H^0(K, X^*(1))^* \otimes K(1).$$

We have $N_\infty H^1(K, B(1)) = B \otimes \mathrm{Ker}(l)$, if we view l as a homomorphism $l : \mathbf{Q}_p \oplus K(1) \longrightarrow \mathbf{Q}_p$ (cf. 1.34-5). Using $H^1(K,X) \xrightarrow{\sim} D_{\mathrm{st}}(X(-1))^{f=1} \oplus DR(X)$, the projection

$$H^1(K,X) \longrightarrow H^1(K,X)/N_\infty \xrightarrow{\sim} H^0(K, X^*(1))^*$$

(the last isomorphism by 6.6) is given by the formula

$$(x,y) \longmapsto l \circ \rho(y) \qquad (x \in D_{\mathrm{st}}(X(-1))^{f=1},\ y \in DR(X)).$$

It follows that

$$N_\infty H^1(K,X) = D_{\mathrm{st}}(X(-1))^{f=1} \oplus \mathrm{Ker}(l\circ\rho)$$

and that the map $\delta : H^0(K,Y) \longrightarrow H^0(K,X^*(1))^*$ from 7.7 is equal to $\delta = -l\circ\rho\circ\mathrm{spl}\circ\pi\circ i\circ\varepsilon$:

$$\delta : H^0(K,Y) \xrightarrow{\pi\circ i\circ\varepsilon} DR(V)/F^0 \xrightarrow{spl} DR(X) \xrightarrow{\rho} H^0(K,X^*(1))^*\otimes K(1) \xrightarrow{-l} H^0(K,X^*(1))^*.$$

We now distinguish two cases

(i) l *is unramified*:

Then $\mathrm{Ker}(l) = K(1)$, $N_\infty H^1(K,X) = D_{\mathrm{st}}(X(-1))^{f=1} \oplus DR(X) = H^1(K,X)$, hence $H^1_{\mathrm{st}}(K,V)/N_\infty = 0$, $N_\infty H^1_{\mathrm{st}}(K,E_2) = 0 \oplus DR(E_2)/F^0$ and the map

$$l' : H^1_{\mathrm{st}}(K,E_2) = B \oplus DR(E_2)/F^0 \longrightarrow H^1_{\mathrm{st}}(K,E_2)/N_\infty \xrightarrow{\sim} B$$

(given by the projection to the first factor) extends $l : H^1(K,B(1)) \longrightarrow H^1(K,B(1))/N_\infty \xrightarrow{\sim} B$.

(ii) l *is ramified*:

By 1.34, there is a unique pair of $\mathbf{Q}_p$-linear maps $\mathrm{Log} : K^*\widehat{\otimes}\mathbf{Q}_p \longrightarrow K(1)$, $\lambda : K(1) \longrightarrow \mathbf{Q}_p$ satisfying $l = \lambda\circ\mathrm{Log}$. Use this analytic logarithm Log for the embedding $B_{\mathrm{st}} \hookrightarrow B_{\mathrm{dR}}$. Then, via $H^1(K,\mathbf{Q}_p(1)) \xrightarrow{\sim} \mathbf{Q}_p \oplus K(1)$, $\mathrm{Ker}(l) = \mathbf{Q}_p \oplus \mathrm{Ker}(\lambda)$ (cf. 1.35). This implies that

$$N_\infty H^1_{\mathrm{st}}(K,V) = \pi\circ i\circ\varepsilon(H^0(K,Y)) + \mathrm{can}(\mathrm{Ker}(l\circ\rho)),$$

hence (as $l\circ\rho : DR(X) \longrightarrow H^0(K,X^*(1))^*$ is surjective)

$$\begin{aligned} H^1_{\mathrm{st}}(K,V)/N_\infty = 0 \quad &\Longleftrightarrow \quad DR(X) = \mathrm{spl}\circ\pi\circ i\circ\varepsilon(H^0(K,Y)) + \mathrm{Ker}(l\circ\rho) \\ &\Longleftrightarrow \quad w_V := l\circ\rho\circ\mathrm{spl}\circ\pi\circ i\circ\varepsilon \quad \text{is surjective} \\ &\Longleftrightarrow \quad w_V : H^0(K,Y) \longrightarrow H^0(K,X^*(1))^* \quad \text{is an isomorphism,} \end{aligned}$$

in perfect agreement with 7.7, as $w_V = -\delta = -l([W])$ by the formula for δ.

Similarly, we have

$$N_\infty H^1_{\mathrm{st}}(K,E_2) = B \oplus \frac{i\circ\varepsilon(H^0(K,Y)) + (\mathrm{Ker}(\lambda\circ\rho)\oplus B\otimes\mathrm{Ker}(\lambda))}{F^0DR(E_2)} \subset B \oplus DR(E_2)/F^0.$$

Suppose that $H^1_{\mathrm{st}}(K,V)/N_\infty = 0$. Then $w_V : H^0(K,Y) \longrightarrow H^0(K,X^*(1))^*$ is an isomorphism. For $x \in DR(X)$, put $y := w_V^{-1}\circ\lambda\circ\rho(x)$. Then

$$x - p_X\circ\pi\circ i\circ\varepsilon(y) \in \mathrm{Ker}(\lambda\circ\rho), \quad (\pi\circ i\circ\varepsilon(y),0) \in N_\infty H^1_{\mathrm{st}}(K,E_2),$$

hence

$$(x,0) \equiv (0, -p_0\circ\pi\circ i\circ\varepsilon(y), 0) \bmod N_\infty H^1_{\mathrm{st}}(K,E_2).$$

As $(z,-\beta(z)) \in F^0DR(E_2)$ for all $z \in F^0DR(V)$,

$$(x,0) \equiv (0, -\beta\circ p_0\circ\pi\circ i\circ\varepsilon(y)) \bmod N_\infty H^1_{\mathrm{st}}(K,E_2).$$

This proves that the map

$$l' : [a,(v,b)F^0DR(E_2)] \longmapsto l(b + \beta\circ p_0(1 - \pi\circ i\circ\varepsilon\circ w_V^{-1}\circ\lambda\circ\rho\circ p_X)v) \in B$$

($a \in B$, $v \in DR(V)$, $b \in B\otimes K(1)$) defines a homomorphism

$$l' : H^1_{\mathrm{st}}(K,E_2) \longrightarrow H^1_{\mathrm{st}}(K,E_2)/N_\infty \longrightarrow B$$

extending $l : H^1(K, B(1)) \longrightarrow H^1(K, B(1))/N_\infty \xrightarrow{\sim} B$ and it is easy to see that it in fact induces an isomorphism $H^1_{\rm st}(K, E_2)/N_\infty \xrightarrow{\sim} B$.

The canonical splitting $DR(V)/F^0 \xleftarrow{\sim} DR(X) \hookrightarrow DR(V)$ defines, by 3.5, a splitting $H^1_{\rm st}(K, E_2) \overset{r}{\underset{}{\longleftrightarrow}} H^1_{\rm st}(K, V)$, given by

$$r(v) = [0, ({\rm spl}(v), 0)\, F^0DR(E_2)] \in B \oplus DR(E_2)$$

($v \in DR(V)/F^0$). It follows from the previous discussion that $l' \circ r = 0$ if l is unramified and

$$l' \circ r(v) = -\lambda \circ \beta \circ p_0 \circ \pi \circ i \circ \varepsilon \circ w_V^{-1} \circ \lambda \circ \rho \circ {\rm spl}(v)$$

if l is ramified.

7.10 Suppose we are also given an extension of representations of $G(\overline{K}/K)$

$$0 \longrightarrow V \longrightarrow E_1 \longrightarrow A(0) \longrightarrow 0$$

with de Rham (hence semistable) E_1. Choose a mixed extension E of E_1, E_2 (it exists, as $H^1_{\rm st} = H^1_f$ for $V, V^*(1)$). Let $\alpha : A \otimes K \longrightarrow DR(V)/F^0$ be the map defined in 3.4. The canonical splitting r defines a local height pairing (cf. 7.4)

$$h^{\rm can}_E : A \xrightarrow{[E]} H^1_{\rm st}(K, E_2) \xrightarrow{1-r\pi} H^1(K, B(1)) \xrightarrow{-l} B.$$

Suppose that $H^1_{\rm st}(K, V)/N_\infty = 0$. Then the map $l' : H^1_{\rm st}(K, E_2) \longrightarrow B$ defined in 7.9 defines another pairing

$$h^{\rm norm}_E : A \xrightarrow{[E]} H^1_{\rm st}(K, E_2) \xrightarrow{-l'} B$$

("norm adapted height"). As $\pi \circ [E](a) = \alpha(a \otimes 1)$, we have

$$(h^{\rm norm}_E - h^{\rm can}_E)(a) = (l \circ (1 - r\pi) \circ [E] - l' \circ [E])(a) = -l' \circ r \circ \pi \circ [E](a) = -l' \circ r \circ \alpha(a \otimes 1).$$

Define maps u_V, v_V as compositions

$$u_V : A \longrightarrow A \otimes K \xrightarrow{\alpha} DR(V)/F^0 \xrightarrow{spl} DR(X) \xrightarrow{\rho} H^0(K, X^*(1))^* \otimes K(1) \xrightarrow{\lambda} H^0(K, X^*(1))^*$$

$$v_V : H^0(K, Y) \xrightarrow{\pi \circ i \circ \varepsilon} DR(V) \xrightarrow{p_0} F^0DR(V) \xrightarrow{\beta} B \otimes K(1) \xrightarrow{\lambda} B$$

for ramified l and as zero maps for unramified l.

7.11 Proposition. *Under the assumptions of 7.10,*

(1) $h^{\rm norm}_E - h^{\rm can}_E = v_V \circ w_V^{-1} \circ u_V$

(2) $v_{V^*(1)} = (u_V)^*$

(3) $w_{V^*(1)} = -(w_V)^*$

(4) $h^{\rm norm}_{E^*(1)} = -(h^{\rm norm}_E)^*$

Proof. (1) Follows from the formulas for $l' \circ r$.
(2) Consider the "K-linear part" of u_V:

$$\overline{u} : A \otimes K \xrightarrow{\alpha} DR(V)/F^0 \xrightarrow{spl} DR(X) \xrightarrow{\rho} H^0(K, X^*(1))^* \otimes K(1).$$

Applying to $\overline{u}$ the functor $\mathcal{D}$ of 2.6 and replacing V by $V^*(1)$, we get

$$\overline{v} : H^0(K, Y) \otimes K \hookrightarrow DR(Y) \xrightarrow{\sim} F^0DR(V) \xrightarrow{\beta} B \otimes K(1).$$

The statement now follows from the fact that v_V is equal to the composition

$$H^0(K,Y) \longrightarrow H^0(K,Y)\otimes K \xrightarrow{\overline{v}} B\otimes K(1) \xrightarrow{\lambda} B.$$

(3) As $w_V = -l([W])$, it suffices to recall the fact used in the proof of 4.10.3.
(4) Follows from (1)–(3) and $h^{\rm can}_{E^*(1)} = -(h^{\rm can}_E)^*$ (cf. 7.6).

7.12 Let us return to the global situation of 7.2. Suppose we are given a global Γ-extension K_∞/K. As before, fixing a topological generator $\gamma \in \Gamma$ defines a global algebraic logarithm $l : \mathbf{A}_K^*/K^* \longrightarrow G(K_\infty/K) \xrightarrow{\sim} \mathbf{Z}_p \hookrightarrow \mathbf{Q}_p$. Let P be the set of primes $\wp|p$ for which the induced local logarithm $l_\wp : K_\wp\widehat{\otimes}\mathbf{Q}_p \longrightarrow \mathbf{Q}_p$ is non-trivial (i.e. $\wp$ does not split completely in K_∞/K). Suppose, furthermore, that for all $\wp \in P$ the local representation $V_\wp$ satisfies the assumptions of 7.7. In particular, we have

$$0 \longrightarrow X_\wp \longrightarrow V_\wp \longrightarrow Y_\wp \longrightarrow 0$$

with $F^0DR(X_\wp) = DR(Y_\wp)/F^0 = 0$ and also the induced extension

$$0 \longrightarrow H^0(K_\wp, X_\wp^*(1))^*(1) \longrightarrow W_\wp \longrightarrow H^0(K_\wp, Y_\wp)(0) \longrightarrow 0.$$

Consider the universal extensions (over K)

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V & \longrightarrow & E_1 & \longrightarrow & A(0) & \longrightarrow & 0 \\ 0 & \longrightarrow & B(1) & \longrightarrow & E_2 & \longrightarrow & V & \longrightarrow & 0 \end{array}$$

with $A = H^1_{\rm st}(K,V)$, $B = H^1_{\rm st}(K,V^*(1))^*$ (of course, $H^1_{\rm st} = H^1_f$ for both $V, V^*(1)$). Restricting to $G(\overline{K_\wp}/K_\wp)$, we define, following 7.10, maps

$$\begin{aligned} u_{\wp,V} &: H^1_f(K,V) = A \longrightarrow H^0(K_\wp, X_\wp^*(1))^* = (D_{\rm st}(V^*(1)_\wp)^{f=1})^* \\ v_{\wp,V} &: D_{\rm st}(V_\wp)^{f=1} = H^0(K_\wp, Y_\wp) \longrightarrow B = H^1_f(K,V^*(1))^* \\ w_{\wp,V} &= -l_\wp([W_\wp]) : D_{\rm st}(V_\wp)^{f=1} = H^0(K_\wp,Y_\wp) \longrightarrow H^0(K_\wp, X_\wp^*(1))^* = (D_{\rm st}(V^*(1)_\wp)^{f=1})^* \end{aligned}$$

for all $\wp \in P$.
Choose a mixed extension E of E_1, E_2, which is essentially unramified at all $v\nmid p$. Canonical splittings

$$DR(V_\wp)/F^0 \xleftarrow{\sim} DR(X_\wp) \hookrightarrow DR(V_\wp)$$

at all $\wp \in P$ then define a global canonical height

$$h^{\rm can}_V = \sum_{\wp\in P} h^{\rm can}_{\wp,E} \quad : \quad H^1_f(K,V) \longrightarrow H^1_f(K,V^*(1))^*.$$

If $H^1_{\rm st}(K_\wp,V)/N_\infty = 0$ (i.e. $w_{\wp,V}$ is an isomorphism) for all $\wp \in P$, then a global "norm adapted height"

$$h^{\rm norm}_V = \sum_{\wp\in P} h^{\rm norm}_{\wp,E} \quad : \quad H^1_f(K,V) \longrightarrow H^1_f(K,V^*(1))^*$$

is defined as well (and does not depend on the choice of E, obviously).

7.13 Theorem. *In the notation of 7.12, suppose that V satisfies the hypotheses of 7.7 at all primes $\wp \in P$. Then*

(1) *For $\wp \in P$, $H^1_{\rm st}(K_\wp,V)/N_\infty = 0$ iff $w_{\wp,V} = -l_\wp([W_\wp])$ is an isomorphism (which is automatically true for $l_\wp$ unramified).*
(2) *If $H^1_{\rm st}(K_\wp,V)/N_\infty = 0$ for all $\wp \in P$, then*

$$h^{\rm norm}_V \ : \quad H^1_f(K,V) \longrightarrow H^1_f(K,V^*(1))^*$$

is defined and is equal to

$$h_V^{\mathrm{norm}} = h_V^{\mathrm{can}} + \sum_{\wp\in P^{\mathrm{ram}}} v_{\wp,V}\circ w_{\wp,V}^{-1}\circ u_{\wp,V},$$

where the summation extends over those $\wp|p$ for which $l_\wp$ ramifies.

(3) *Define extended height*

$$h_V^{\mathrm{ext}} \;:\quad H^1_f(K,V)\oplus\bigoplus_{\wp\in P} D_{\mathrm{st}}(V_\wp)^{f=1}\longrightarrow H^1_f(K,V^*(1))^*\oplus\bigoplus_{\wp\in P}(D_{\mathrm{st}}(V^*(1)_\wp)^{f=1})^*$$

by the matrix

$$h_V^{\mathrm{ext}} = \begin{pmatrix} h_V^{\mathrm{can}} & \oplus - v_{\wp,V} \\ \oplus u_{\wp,V} & \oplus w_{\wp,V}\end{pmatrix}$$

Then h_V^{ext} is an isomorphism iff h_V^{norm} is and

$$\det(h_V^{\mathrm{ext}}) = \det(h_V^{\mathrm{norm}})\prod_{\wp\in P}\det(-l_\wp([W_\wp]))$$

$$\text{in}\quad \det(H^1_f(K,V^*(1)))^*\otimes\det(H^1_f(K,V))^*\otimes\bigotimes_{\wp\in P}\det(D_{\mathrm{st}}(V^*(1)_\wp)^{f=1})^*\otimes\det(D_{\mathrm{st}}(V_\wp)^{f=1})^*$$

Proof. (1),(2) In view of 7.9 and 7.11, we only have to show that our assumptions imply that $w_{\wp,V} = -l_\wp([W_\wp])$ is an isomorphism for unramified (and non-trivial) $l_\wp$. In this case, $w_{\wp,V}$ is proportional to the monodromy operator on $W_\wp$, which sits in a commutative diagram

$$\begin{array}{llcll} N_{V_\wp}: & D_{\mathrm{st}}(Y_\wp)^{f=1} & \xrightarrow{\sim} & D_{\mathrm{st}}(X_\wp(-1))^{f=1} & \\ & \| & & \downarrow\alpha & \\ N_{W_\wp}: & H^0(K_\wp,Y_\wp) & \longrightarrow & D_{\mathrm{st}}(X_\wp(-1))/(f-1) & = H^0(K_\wp,X_\wp^*(1))^* \end{array}$$

By our assumptions, α is an isomorphism, hence $N_{W_\wp}$ is, too.

(3) Follows from the formula

$$\begin{pmatrix} A & B\\ C & D\end{pmatrix} = \begin{pmatrix} 1 & 0\\ 0 & D\end{pmatrix}\begin{pmatrix} A-BD^{-1}C & B\\ 0 & 1\end{pmatrix}\begin{pmatrix} 1 & 0\\ D^{-1}C & 1\end{pmatrix}.$$

7.14 Suppose that we are given a symplectic form on V, i.e. an isomorphism $j: V\xrightarrow{\sim} V^*(1)$ satisfying $j^*(1) = -j$. It induces, then, isomorphisms $H^1_f(K,V)\xrightarrow{\sim} H^1_f(K,V^*(1))$, $D_{\mathrm{st}}(V_\wp)^{f=1}\xrightarrow{\sim} D_{\mathrm{st}}(V^*(1)_\wp)^{f=1}$, to be denoted by the same letter. Writing $\langle.,.\rangle$ for the canonical pairing between a vector space and its dual, we may define various pairings (depending on the choice of j) $h^?(x,x) = \langle j(x), h_V^?(x)\rangle$ (for $? = \mathrm{can},\mathrm{norm},\mathrm{ext}$) and similarly $w_\wp(y,y) = \langle j(y), w_{\wp,V}(y)\rangle$ (for $y\in D_{\mathrm{st}}(V_\wp)^{f=1} = H^0(K_\wp,Y_\wp)$).

In our situation, the condition of 4.11.4 is satisfied, hence all of the pairings are symmetric and

$$\det(h^{\mathrm{ext}}) = \det(h^{\mathrm{norm}})\prod_{\wp\in P}\det(w_\wp).$$

It is instructive to give explicit formulas for elliptic curves: suppose that $V = T_p(E)\otimes\mathbf{Q}$, where E is an elliptic curve over K. We further assume that E has ordinary reduction at all $\wp\in P$. Let $P_T\subseteq P$ be the subset of primes in which E has split multiplicative reduction. For $\wp\in P_T$, we have $X_\wp = \mathbf{Q}_p(1)$, $Y_\wp = \mathbf{Q}_p(0)$ and the extension class of $W_\wp = V_\wp$ is equal to the Tate parameter $q_\wp\otimes 1\in H^1(K_\wp,\mathbf{Q}_p(1))$, hence $w_{\wp,V} = -l_\wp(q_\wp)$. The map $u_{\wp,V}: H^1_f(K,V)\longrightarrow\mathbf{Q}_p$ is given as follows: for an element $x\in H^1_f(K,V)$, its local component $x_\wp$ lies in

$$E(K_\wp)\otimes\mathbf{Q}_p = (K_\wp^*/q_\wp^{\mathbf{Z}})\widehat{\otimes}\mathbf{Q}_p\xrightarrow{\sim}\mathcal{O}_\wp^*\widehat{\otimes}\mathbf{Q}_p.$$

Taking the unit representative $t_\wp(x) \in \mathcal{O}_\wp^* \widehat{\otimes} \mathbf{Q}_p$ of $x_\wp$, we have $u_{\wp,V}(x) = l_\wp(t_\wp(x))$. There is a canonical pair of isomorphisms j, which differ by a sign, given by Weil's pairing. The corresponding pair of symmetrized extended heights

$$h^{\rm ext} : \left[H^1_f(K,V) \oplus \bigoplus_{\wp \in P_T} \mathbf{Q}_p \right] \times \left[H^1_f(K,V) \oplus \bigoplus_{\wp \in P_T} \mathbf{Q}_p \right] \longrightarrow \mathbf{Q}_p$$

is given by the following formula:

$$\pm h^{\rm ext}([x,(a_\wp)],[y,(b_\wp)]) = h^{\rm can}(x,y) + \sum_{\wp \in P_T} (a_\wp l_\wp(t_\wp(y)) + b_\wp l_\wp(t_\wp(x)) + l_\wp(q_\wp) a_\wp b_\wp)$$

In matrix form,

$$\pm h^{\rm ext} = \begin{pmatrix} h^{\rm can} & \oplus l_\wp \circ t_\wp \\ \oplus l_\wp \circ t_\wp & \oplus l_\wp(q_\wp) \end{pmatrix}$$

and the norm adapted height, defined iff $l_\wp(q_\wp) \neq 0$ for all $\wp \in P_T$, is equal to (cf. 7.13.2)

$$h^{\rm norm}(x,y) = h^{\rm can}(x,y) - \sum_{\wp \in P_T} \frac{l_\wp(t_\wp(x)) l_\wp(t_\wp(y))}{l_\wp(q_\wp)}.$$

The formula given in [Ma–Ta–Te, II.6] (without a proof) is different. In fact, Mazur-Tate-Teitelbaum give an ad hoc definition of $h^{\rm ext}$ (which is not to say that our definition in 7.13 is any more conceptual than theirs) which contains an extra term $\mathrm{ord}_\wp(q_\wp)$ in the denominator. There is no justification for such a term in our framework, as the computation of universal norms in 7.9 is independent on the choice of any particular valuation $v_\wp$ (and hence of the normalization of the monodromy operator N at $\wp$). One can, therefore, expect that the full $\mathcal{L}$-invariant of [Ma–Ta–Te], equal to the product of $l_\wp(q_\wp)/\mathrm{ord}_\wp(q_\wp)$, appears only in a more detailed study of the arithmetic of the whole situation (cf. [Jo 1]).

8. Comparisons and open questions

8.1 Suppose that $X_{/K}$ is a proper smooth variety over a number field K. Let $A = \mathrm{Pic}^0(X_{/K})$ be its Picard variety. Then $V = H^1(\overline{X}, \mathbf{Q}_p(1))$ is equal to $T_p(A) \otimes \mathbf{Q}$, where $T_p(A)$ is the Tate module of the abelian variety A and the Abel-Jacobi map

$$\Phi_1 : CH^1(X_{/K})_0 \longrightarrow H^1(K,V)$$

is, essentially, nothing else then the Kummer map $A(K) \longrightarrow H^1(K, T_p(A) \otimes \mathbf{Q})$ (cf. 1.34). The subgroup $H^1_f(K,V) \subseteq H^1(K,V)$ is equal to the classical Selmer group (more precisely, to the projective limit of Selmer groups of levels p^n, tensored with $\mathbf{Q}$), hence equal to $A(K) \otimes \mathbf{Q}_p$ if the p-primary part of the Tate-Šafarevič group of $A_{/K}$ is finite. Bearing this in mind, then the construction of sec. 2 for this particular V is easily seen to be identical to that in [Za]. Similarly, our $h_V^{\rm norm}$, if defined, coincides with Schneider's height [Sch 2], hence also with the canonical height of Mazur–Tate [Ma–Ta] in the case of good ordinary reduction at all primes dividing p. By the same token, for general V satisfying the assumptions of 6.11, $h_V^{\rm norm}$ coincides with the height defined by Perrin-Riou [Pe-Ri 3].

8.2 It seems very likely that the canonical height of Mazur–Tate coincides with our $h_V^{\rm can}$ even for abelian varieties with bad ordinary reduction. It would also be interesting to compare our construction to [Co–Gr], where the authors define p-adic height pairings on a curve, which also depend on the choice of splittings of Hodge filtrations. Their approach is analytic and uses integration of p-adic Green currents. Is it possible to give a similar analytic interpretation of our construction in the higher-dimensional geometric case as well?

8.3 Suppose that $f \in S_{2r}^{new}(\Gamma_0(N))$ is a normalized newform of weight $2r \geq 4$ with rational coefficients. Let Y be the $(2r-1)$-dimensional Kuga-Sato variety over $Y(N)$, i.e. the $(2r-2)$-fold fibre product of the universal elliptic curve over the open modular curve $Y(N)$ with itself. There is a canonical nonsingular compactification X of Y (see [Sc 1]) and a 'motive' $M(f)$ of rank 2 over $\mathbf{Q}$ associated to f, sitting inside the 'motive' of X ([Ja 3], [Sc 1]). Its p-adic realization $V = M(f)_p$ is a factor of the cohomology group $H^{2r-1}_{\text{et}}(\overline{X}, \mathbf{Q}_p(r))$. In particular, we have, over any number field K, the Abel-Jacobi map

$$\Phi_{f,K} : CH^r(X_{/K})_0 \longrightarrow H^1(K, H^{2r-1}_{\text{et}}(\overline{X}, \mathbf{Q}_p(r))) \longrightarrow H^1(K, V).$$

For every quadratic field K in which all primes dividing N split completely, it is possible to define a 'Heegner cycle'

$$y_K \in CH^r(X_{/K})_0 \otimes \mathbf{Q}.$$

Write $y = \Phi_{f,K}(y_K) \in H^1(K, V)$. Using a Kolyvagin-type descent, it was proved in [Ne] (for p not dividing $2(2r-2)!N\varphi(N)$) that $\text{Im}(\Phi_{f,K}) \otimes \mathbf{Q}_p$ is equal to $\mathbf{Q}_p \cdot y$, provided $y \neq 0$. Unfortunately, there does not seem to be any known criterion for deciding whether y vanishes or not in a given situation. Note, however, that V satisfies all conditions (A)–(D) of 2.1 (see [Ca], [Sc 1]; in fact, monodromy conjecture is automatic in this case and does not require deep results of [Ca]: it follows from the fact that V is two-dimensional and its determinant is equal to $\mathbf{Q}_p(1)$, as was pointed out to me by A.J.Scholl) and therefore it can be plugged into our height machinery. In particular, if p is ordinary for f, then there is a canonical symmetric height pairing

$$h : H^1_f(K, V) \times H^1_f(K, V) \longrightarrow \mathbf{Q}_p$$

and it is quite natural to expect a formula of Gross-Zagier type

$$h(y, y) = \Omega_{f \otimes K, p} L'_p(f \otimes K, r)$$

for the height of y as a multiple of the central value of the derivative of the p-adic L-function associated to f (over K). If true, it would provide a sought for criterion for non-vanishing of y.

8.4 Let us finally make a few remarks about height pairings for representations which do not satisfy Pančiškin's condition. If $\wp$ is unramified in $K/\mathbf{Q}$ and $V_\wp$ is a crystalline representation of $G(\overline{K_\wp}/K_\wp)$, then there is a canonical splitting of the Hodge filtration

$$0 \longrightarrow F^0 DR(V_\wp) \longrightarrow DR(V_\wp) \underset{\longrightarrow}{\longleftarrow} DR(V_\wp)/F^0 \longrightarrow 0$$

constructed in [Wt] (provided the span of the filtration on $DR(V_\wp)$ is less than p). Consequently, if V satisfies (A)–(D) of 2.1 and p is unramified in $K/\mathbf{Q}$, these splittings define a height pairing

$$h_V : H^1_f(K, V) \times H^1_f(K, V^*(1)) \longrightarrow \mathbf{Q}_p,$$

which one may be tempted to call canonical. Some degree of caution might not be quite out of place at this point: B.Perrin-Riou gave in [Pe-Ri 2] another construction of p-adic heights for supersingular elliptic curves, which is based on a generalized notion of universal norms, first proposed by P.Schneider. Although Wintenberger's splitting appears in her construction, the height pairing also depends on the choice of one of the roots of the local Euler factor at p. The referee informs me, however, that such a choice is artifical and should be ultimately eliminated, as the recent progress in the theory of p-adic L-functions suggests.

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