Higher Heisenberg Lie algebras and metaplectic representations

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0. Introduction

Theta functions and, more generally, modular forms of half-integral weight can be interpreted as automorphic forms on the metaplectic group, a two-fold covering of the adelic symplectic group. Representation theoretical foundations of this theory are provided by the Weil representation [We], a unitary representation of a semi-direct product of the metaplectic group with an adelic Heisenberg group. This representation can be constructed from scratch using the Stone-von Neumann uniqueness theorem for unitary representations of the Heisenberg group.

Higher metaplectic groups are non-trivial central extensions

$$1 \longrightarrow \mu_n \longrightarrow \widetilde{G} \longrightarrow G(\mathbb{A}_K) \longrightarrow 1,$$

where G is a classical group such as $GL(r)$ or $SL(r)$ and K a number field containing μ_n. The extension $\widetilde{G}$ splits over $G(K)$; this fact is equivalent to the n-th power reciprocity law over K. It is known that the case $r = n$ is somewhat special: there is a representation of $\widetilde{G}$ analogous to (the even part of) the Weil representation, which was studied by Kazhdan-Patterson [KP]. One of the main open questions of the theory is whether this representation (more precisely, its direct sum with $n-1$ other representations, which are analogues of the odd part of the Weil representation) admits an action of a suitable higher Heisenberg group.

In this article we show that this is true on the level of Lie algebras. We define the "Heisenberg Lie algebra of level n" $h(n)$ to be the quotient of the free Lie algebra on an n-dimensional vector space V by relations

$$(\mathrm{ad}(x))^n y = 0 \qquad\qquad (x, y \in V);$$

for $n = 2$ this is the usual Heisenberg Lie algebra with basis $P, Q, 1$. Our main result states that the Weil-like representations of the Lie algebra $sl(n)$ extends naturally to a representation of the semi-direct product of $h(n)$ with $sl(n)$ (if we work with Lie algebras over a field F of characteristic zero, then the action of $h(n)$ depends on a parameter in F^*/F^{*n}). For $n = 3$, $h(3)$ acts through its finite-dimensional quotient G_2, as observed first by Bernstein (unpublished).

1. Representations of $sl(2)$

(1.1) The Weil representation of $sl(2)$

Let F be a field of characteristic zero. The operators $Q = t$ and $P = d/dt$ acting on the polynomial ring $F[t]$ generate the Heisenberg Lie algebra $h(2)$ with basis $P, Q, 1 = [P, Q]$. The Weil representation of $sl(2) = sl(2, F)$ in its simplest form acts on $F[t]$ by the following formulas:

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \mapsto -\frac{Q^2}{2}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \mapsto \frac{P^2}{2}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mapsto \frac{PQ + QP}{2}$$

Taken together, the six operators $1, P, Q, -Q^2/2, P^2/2, (PQ + QP)/2$ define a representation of the semi-direct product $h(2) \rtimes sl(2)$ on $F[t]$ ($sl(2)$ acts trivially on 1 and by the standard two-dimensional representation on the span of P, Q). The action of $h(2)$ on $F[t]$ is irreducible. As an $sl(2)$-module, $F[t]$ is a direct sum

$$F[t] = F[t^2] \oplus tF[t^2],$$

where $F[t^2]$ (resp. $tF[t^2]$) is an irreducible $sl(2)$-module of lowest weight $\frac{1}{2}$ (resp. $\frac{3}{2}$) with a lowest weight vector 1 (resp. t). This follows from $Y1 = Yt = 0$ and $H1 = \frac{1}{2}$, $Ht = \frac{3}{2}t$.

(1.2) The space $V(\mu) \oplus V(\mu+1)$

Let us consider slightly more general representations of $sl(2)$. For $\mu \in F$, denote by $V(\mu)$ the lowest weight Verma module (not necessarily irreducible) with lowest weight $\mu + 1$. Then

$$V(\mu) = \bigoplus_{n=0}^{\infty} F \cdot X^n u, \quad Yu = 0, \quad Hu = (\mu+1)u$$
$$V(\mu+1) = \bigoplus_{n=0}^{\infty} F \cdot X^n v, \quad Yv = 0, \quad Hv = (\mu+2)v.$$

Consider the direct sum $U(\mu) = V(\mu) \oplus V(\mu+1)$. We are going to construct non-zero operators $A, B : U(\mu) \longrightarrow U(\mu)$ analogous to $P, Q : U(-\frac{1}{2}) \longrightarrow U(-\frac{1}{2})$ in the Weil representation:

(i) A (resp. B) should increase (resp. decrease) weights by one, i.e.

$$A\langle X^n u\rangle \subseteq \langle X^n v\rangle, \quad A\langle X^n v\rangle \subseteq \langle X^{n+1} u\rangle, \quad B\langle X^n u\rangle \subseteq \langle X^{n-1} v\rangle, \quad B\langle X^n v\rangle \subseteq \langle X^n u\rangle.$$

(ii) The vector space $V = \langle A, B\rangle \subset \mathrm{End}(U(\mu))$ is stable under the adjoint action of the image of $sl(2)$ in $\mathrm{End}(U(\mu))$ and defines the standard two-dimensional representation of $sl(2)$:

$$\begin{aligned} &[X, A] = 0, && [H, A] = A, && [Y, A] = B \\ &[X, B] = A, && [H, B] = -B, && [Y, B] = 0. \end{aligned}$$

These conditions imply that

(1.2.1) $$A : X^n u \mapsto a_1 X^n v, \qquad X^n v \mapsto a_2 X^{n+1} u \qquad (n \geq 0)$$

for some $a_1, a_2 \in F$. The standard formulas for the action of $sl(2)$ on $U(\mu)$

$$\begin{aligned} &X : X^n u \mapsto X^{n+1} u && X : X^n v \mapsto X^{n+1} v \\ &Y : X^n u \mapsto -n(\mu+n) X^{n-1} u && Y : X^n v \mapsto -n(\mu+n+1) X^{n-1} v \\ &H : X^n u \mapsto (\mu+2n+1) X^n u && H : X^n v \mapsto (\mu+2n+2) X^n v \end{aligned}$$

together with $[Y, A] = B$ then imply

(1.2.2) $$B : X^n u \mapsto -n a_1 X^{n-1} v, \qquad X^n v \mapsto -(\mu+n+1) a_2 X^n u \qquad (n \geq 0).$$

A short calculation shows that A and B defined by the formulas (1.2.1–2) satisfy (i) and (ii). We can renormalize the lowest weight vectors by $u \mapsto t_0 u$, $v \mapsto t_1 v$ and the operator $A \mapsto t_2 A$ ($t_i \in F^*$); the pair (a_1, a_2) is then replaced by $(a_1 t_0^{-1} t_1 t_2, a_2 t_0 t_1^{-1} t_2)$. Assuming that $a_i \neq 0$, the renormalization leaves invariant $a_1 a_2 \pmod{F^{*2}} \in F^*/F^{*2}$. Replacing F by $F(\sqrt{a_1 a_2})$ and choosing suitable t_i, we can (and will) assume that $a_1 = a_2 = 1$, i.e.

$$\begin{aligned} &A : X^n u \mapsto X^n v && X^n v \mapsto X^{n+1} u \\ &B : X^n u \mapsto -n X^{n-1} v && X^n v \mapsto -(\mu+n+1) X^n u \end{aligned}$$

Note that $A^2 = X, B^2 = -Y, AB + BA = -H$.

(1.3) The Lie algebra $\mathcal{L}_\mu$

Denote by $\mathcal{L}_\mu \subset \mathrm{End}(U(\mu))$ the Lie subalgebra generated by A and B. One computes

$$
\begin{aligned}
[A,B] &: X^n u \mapsto (\mu+1)X^n u & X^n v &\mapsto -\mu X^n v\\
[A,[A,B]] &: X^n u \mapsto (2\mu+1)X^n v & X^n v &\mapsto -(2\mu+1)X^{n+1}u\\
[B,[A,B]] &: X^n u \mapsto -(2\mu+1)nX^{n-1}v & X^n v &\mapsto (2\mu+1)(\mu+n+1)X^n u
\end{aligned}
$$

These formulas show a dichotomy in the behaviour of $\mathcal{L}_\mu$:

Special case $\mu = -\frac{1}{2}$ (corresponding to the Weil representation): $[A,B] = \frac{1}{2}\cdot I$ acts on $U(-\frac{1}{2})$ as a scalar and $\mathcal{L}_{-\frac{1}{2}} = h(2)$ is the Heisenberg Lie algebra.

Generic case $\mu \neq -\frac{1}{2}$: denote by π_i $(i = 0,1)$ the projection of $U(\mu)$ on $V(\mu+i)$ along $V(\mu+1-i)$. We have $[A,B] = (\mu+1)\pi_0 - \mu\pi_1$, hence

$$\pi_0 = \frac{\mu I + [A,B]}{2\mu+1}, \qquad \pi_1 = \frac{(\mu+1)I - [A,B]}{2\mu+1}.$$

A short calculation shows that operators $\pi_i A, \pi_i B, \pi_i X, \pi_i Y, \pi_i H$ $(i = 0,1)$ are all contained in $\mathcal{L}_\mu$ and that $(\mathrm{ad}(\pi_0 X))^n(\pi_0 B) \neq 0$ for all $n \geq 0$. This implies that $\dim(\mathcal{L}_\mu) = \infty$ for $\mu \neq -\frac{1}{2}$.

(1.4) The Casimir operator

What is so special about the weight $\mu = -\frac{1}{2}$? The Casimir operator

$$\Omega = XY + YX + \frac{H^2}{2} = 2XY + \frac{H^2}{2} - H = 2YX + \frac{H^2}{2} + H$$

acts on $V(\mu)$ (resp. $V(\mu+1)$) by the scalar

$$\frac{(\mu+1)^2}{2} - (\mu+1) = \frac{(\mu+1)(\mu-1)}{2} \qquad \left(\text{resp. } \frac{(\mu+2)\mu}{2}\right).$$

As

$$\frac{(\mu+2)\mu}{2} - \frac{(\mu+1)(\mu-1)}{2} = \mu + \frac{1}{2},$$

we see that

$$\Omega \text{ acts on } U(\mu) \text{ as a scalar} \iff \mu = -\frac{1}{2}$$

(in which case $\Omega = -\frac{3}{8}\cdot I$).

This observation gives an alternative approach to the study of $\mathcal{L}_{-\frac{1}{2}}$ that does not involve computation of iterated commutators in $\mathrm{End}(U(-\frac{1}{2}))$.

(1.5) A calculation in $U(L(V) \rtimes sl(2))$

Let $L(V)$ be the free Lie algebra over $V = \langle A, B\rangle$. The action of A, B and $sl(2)$ extends uniquely to a representation of the semi-direct product $L(V) \rtimes sl(2)$ on $U(\mu)$. Assume, until the end of Sect. 1, that $\mu = -\frac{1}{2}$.

As Ω acts on $U(-\frac{1}{2})$ as a scalar, the representation of the universal enveloping algebra $U(L(V) \rtimes sl(2))$ on $U(-\frac{1}{2})$ factors through the quotient of $U(L(V) \rtimes sl(2))$ by the bilateral ideal I generated by $[\Omega, A]$.

The ideal I contains the following elements:

$$
\begin{aligned}
[\Omega, A] &= \left(H - \frac{3}{2}\right)A + 2XB\\
[\Omega, B] &= 2YA - \left(H + \frac{3}{2}\right)B,
\end{aligned}
$$

hence also

$$\begin{aligned} a = & \frac{1}{2}[A,[\Omega,A]] = -\frac{A^2}{2} + X[A,B] \\ b = -&\frac{1}{2}[B,[\Omega,B]] = \frac{B^2}{2} + Y[A,B] \\ c = -& [B,[\Omega,A]] = \frac{AB+BA}{2} + H[A,B] \end{aligned}$$

and

$$\begin{aligned} &[A,a] = X[A,[A,B]] && [B,a] = -\frac{1}{2}[A,[A,B]] + X[B,[A,B]] \\ &[A,b] = Y[A,[A,B]] - \frac{1}{2}[B,[A,B]] && [B,b] = Y[B,[A,B]] \\ &[A,c] = \left(H - \frac{1}{2}\right)[A,[A,B]] && [B,c] = \left(H + \frac{1}{2}\right)[B,[A,B]] \end{aligned}$$

It follows that

$$\begin{aligned} -Y[A,a] + X[A,b] - [A,c] + \frac{1}{2}[B,a] = & \frac{1}{4}[A,[A,B]] \\ -Y[B,a] + X[B,b] - [B,c] - \frac{1}{2}[A,b] = -& \frac{1}{4}[B,[A,B]] \end{aligned}$$

both lie in I, hence $[A,B]$ is a central element of $\mathcal{L}_{-\frac{1}{2}}$. As $2X[A,B] = A^2 \neq 0$ in $\mathrm{End}(U(-\frac{1}{2}))$, the formula defining a implies that $[A,B] \neq 0$ in $\mathcal{L}_{-\frac{1}{2}}$ and $\mathcal{L}_{-\frac{1}{2}} = h(2)$.

2. Representations of $sl(n)$

(2.1) The Lie algebra $sl(n)$ - standard notation

F is a field of characteristic zero.
$\mathfrak{g} = sl(n) = sl(n,F) = \{X \in M_n(F) \mid Tr(X) = 0\}$
$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$, where
$\mathfrak{h} = \{\text{diagonal matrices}\} \cap \mathfrak{g}$ = a Cartan subalgebra of $\mathfrak{g}$.
$\mathfrak{n}_+$ (resp. $\mathfrak{n}_-$) is the set of upper (resp. lower) triangular matrices with zeros on the main diagonal.
$\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}_+$ = a Borel subalgebra of $\mathfrak{g}$.
The positive roots of $\mathfrak{g}$ (with respect to $\mathfrak{b}$) are $\alpha = e_i - e_j \in \mathfrak{h}^*$ $(1 \le i < j \le n)$, where $e_i(\mathrm{diag}(d_1,\ldots,d_n)) = d_i$. The corresponding $sl(2)$-triple in $\mathfrak{g}$ consists of: $X_\alpha \in \mathfrak{n}_+$ with entries $(X_\alpha)_{kl} = \delta_{ik}\delta_{jl}$, $Y_\alpha = X_{-\alpha} = (X_\alpha)^T \in \mathfrak{n}_-$ and $H_\alpha = [X_\alpha, Y_\alpha] \in \mathfrak{h}$.
The simple roots are $\alpha_i = e_i - e_{i+1}$ $(1 \le i \le n-1)$; put $X_i = X_{\alpha_i}$, $Y_i = Y_{\alpha_i}$, $H_i = H_{\alpha_i}$. Each positive root is of the form $e_i - e_j = \alpha_i + \cdots + \alpha_{j-1}$ $(i < j)$; write $X_{i\cdots(j-1)}$ (resp. $Y_{i\cdots(j-1)}$) for $X_{e_i-e_j}$ (resp. $Y_{e_i-e_j}$). For example, for $n = 3$, we have

$$X_1 = \begin{pmatrix} 0&1&0\\0&0&0\\0&0&0 \end{pmatrix}, X_2 = \begin{pmatrix} 0&0&0\\0&0&1\\0&0&0 \end{pmatrix}, X_{12} = \begin{pmatrix} 0&0&1\\0&0&0\\0&0&0 \end{pmatrix}, H_1 = \begin{pmatrix} 1&0&0\\0&-1&0\\0&0&0 \end{pmatrix}$$
$$Y_1 = \begin{pmatrix} 0&0&0\\1&0&0\\0&0&0 \end{pmatrix}, Y_2 = \begin{pmatrix} 0&0&0\\0&0&0\\0&1&0 \end{pmatrix}, Y_{12} = \begin{pmatrix} 0&0&0\\0&0&0\\1&0&0 \end{pmatrix}, H_2 = \begin{pmatrix} 0&0&0\\0&1&0\\0&0&-1 \end{pmatrix}$$

Commutation relations in $\mathfrak{g}$ are as follows: for every pair α, β of positive roots,

$$[X_\alpha, X_\beta] \quad (\text{resp. } [Y_\alpha, Y_\beta]) = \begin{cases} X_{\alpha+\beta} \ (\text{resp. } -Y_{\alpha+\beta}) & \text{if } \alpha+\beta \text{ is a root} \\ 0 & \text{otherwise} \end{cases}$$
$$[X_\alpha, Y_\alpha] = H_\alpha, \quad [H_\alpha, X_\beta] = \beta(H_\alpha)X_\beta, \quad [H_\alpha, Y_\beta] = -\beta(H_\alpha)Y_\beta$$

Restriction of the standard scalar product $(e_i, e_j) = \delta_{ij}$ on the span of $e_1, \ldots, e_n$ defines a scalar product on $\mathfrak{h}^*$ satisfying

$$(\alpha_i, \alpha_j) = \begin{cases} 2 & \text{if } i = j \\ -1 & \text{if } i = j \pm 1 \\ 0 & \text{otherwise} \end{cases}$$

Under the induced isomorphism $\mathfrak{h}^* \xrightarrow{\sim} \mathfrak{h}$, α corresponds to H_α, i.e. $(\alpha, \beta) = \beta(H_\alpha)$ for all roots α, β. In particular, $(\alpha, \alpha_i) = \alpha(H_i)$ and

$$[H_i, X_\alpha] = (\alpha, \alpha_i)X_\alpha, \quad [H_i, Y_\alpha] = -(\alpha, \alpha_i)Y_\alpha \qquad (1 \le i \le n-1).$$

The induced scalar product on $\mathfrak{h}$ satisfies $(H_\alpha, H_\beta) = (\alpha, \beta)$, in particular $(H_i, H_j) = (\alpha_i, \alpha_j)$.
The fundamental weights $\lambda_i \in \mathfrak{h}^*$ $(1 \le i \le n-1)$ are highest weights of $\bigwedge^i V$, where V is the standard n-dimensional representation of $\mathfrak{g}$. They are characterized by

$$(\lambda_i, \alpha_j) = \lambda_i(H_j) = \delta_{ij}$$

and satisfy

$$\begin{aligned} \lambda_1 &= \left(1 - \frac{1}{n}\right)\alpha_1 + \left(1 - \frac{2}{n}\right)\alpha_2 + \cdots + \frac{1}{n}\alpha_{n-1} \\ \lambda_i &= i\lambda_1 - (i-1)\alpha_1 - (i-2)\alpha_2 - \cdots - \alpha_{i-1} = \\ &= \left(1 - \frac{i}{n}\right)(\alpha_1 + 2\alpha_2 + \cdots + i\alpha_i) + \left(\left(1 - \frac{i+1}{n}\right)\alpha_{i+1} + \cdots + \frac{1}{n}\alpha_{n-1}\right). \end{aligned}$$

The corresponding elements of $\mathfrak{h}$ are $H_i^* \in \mathfrak{h}$ $(1 \le i \le n-1)$ such that $(H_i, H_j^*) = \delta_{ij}$ and

$$H_i^* = \left(1 - \frac{i}{n}\right)(H_1 + 2H_2 + \cdots + iH_i) + \left(\left(1 - \frac{i+1}{n}\right)H_{i+1} + \cdots + \frac{1}{n}H_{n-1}\right).$$

The half sum of positive roots is equal to

$$\rho = \frac{1}{2}\sum_{\alpha>0}\alpha = \lambda_1 + \cdots + \lambda_{n-1} = \sum_{i=1}^{n-1}\frac{i(n-i)}{2}\alpha_i \in \mathfrak{h}^*.$$

The following scalar products will appear later on:

$$(\lambda_i, \lambda_j) = \left(1 - \frac{\max(i,j)}{n}\right)\min(i,j)$$
$$(\lambda, 2\rho) = i(n-i) = n(\lambda_i, \lambda_i)$$

The Casimir operator

$$\Omega = \sum_{\alpha>0}(X_\alpha Y_\alpha + Y_\alpha X_\alpha) + \sum_{i=1}^{n-1} H_i^* H_i$$

lies in the centre $Z(\mathfrak{g})$ of the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$. We have

$$\frac{1}{2}\sum_{\alpha>0} H_\alpha = \sum_{i=1}^{n-1}\frac{i(n-i)}{2}H_i,$$

hence

$$\Omega = 2\sum_{\alpha>0} X_\alpha Y_\alpha + \sum_{i=1}^{n-1} H_i(H_i^* - i(n-i)) = 2\sum_{\alpha>0} Y_\alpha X_\alpha + \sum_{i=1}^{n-1} H_i(H_i^* + i(n-i)).$$

The Weyl group $W \subset \mathrm{Aut}(\mathfrak{h}^*)$ is generated by reflections $s_i : x \mapsto (\alpha_i, x)\alpha_i - x$ ($1 \le i \le n-1$); it is isomorphic to S_n generated by transpositions $s_i : i \leftrightarrow i+1$. The Coxeter element $c = s_1 \cdots s_{n-1} \in W$ has order n. Conjugation by

$$w_0 = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 1 & 0 \\ \vdots & .\cdot^{\cdot} & \vdots & \vdots \\ 1 & \cdots & 0 & 0 \end{pmatrix}$$

is the longest element in the Weyl group.

(2.2) Lowest weight Verma modules

For $\mu \in \mathfrak{h}^*$, denote by $V(\mu)$ the lowest weight Verma module of $U(\mathfrak{g})$ (not necessarily irreducible) with lowest weight $\mu + \rho$. It is free of rank one over $U(\mathfrak{n}_+)$ with a basis vector (= a lowest weight vector) v satisfying

$$U(\mathfrak{n}_-)v = 0, \quad H_i v = (\mu_i + 1)v \qquad (\mu_i = \mu(H_i), 1 \le i \le n-1).$$

The Casimir operator Ω acts on $V(\mu)$ by the scalar

$$\sum_{i=1}^{n-1} (\mu+\rho)(H_i)((\mu+\rho)(H_i^*) - i(n-i)) = (\mu+\rho, \mu-\rho) = (\mu,\mu) - (\rho,\rho).$$

Conjugation by w_0 interchanges $\mathfrak{n}_-$ with $\mathfrak{n}_+$ and transforms $V(\mu)$ into $M(w_0 \cdot \mu)$ (the usual Verma module with highest weight $w_0 \cdot \mu - \rho$).

The module

$$U(\mu) = V(\mu) \oplus V(\mu + \lambda_1) \oplus \cdots \oplus V(\mu + \lambda_{n-1})$$

is an analogue of $V(\mu) \oplus V(\mu+1)$ from 1.2 for $sl(n)$.

(2.3) Action of $L(V)$ on $U(\mu)$

Denote by $V = \oplus_{i=1}^n F \cdot A_i$ the standard n-dimensional representation of $sl(n)$, where A_1 is a highest weight vector (of weight λ_1) and $Y_i \cdot A_i = A_{i+1}$ ($1 \le i \le n-1$). Let $L(V)$ be the free Lie algebra on V. We want to extend the action of $sl(n)$ on $U(\mu)$ (for every $\mu \in \mathfrak{h}^*$) to an action of the semi-direct product $L(V) \rtimes sl(n)$. All we have to do is to construct operators

$$A_1 : V(\mu + \lambda_k) \longrightarrow V(\mu + \lambda_{k+1}) \qquad (k \in \mathbb{Z}/n\mathbb{Z})$$

(where $\lambda_0 = 0$ and $\lambda_{n+i} = \lambda_i$) satisfying

$$\text{(2.3.1)} \qquad \begin{aligned} [X_i, A_1] &= 0 \\ [H_i, A_1] &= \delta_{i1} A_1 \qquad (1 \le i \le n-1) \\ (\mathrm{ad}(Y_i))^{1+\delta_{i1}} A_1 &= 0 \end{aligned}$$

and define inductively

$$A_{j+1} = [Y_j, A_j] : V(\mu + \lambda_k) \longrightarrow V(\mu + \lambda_{k+1}) \qquad (k \in \mathbb{Z}/n\mathbb{Z}, \quad 1 \le j \le n-1).$$

For each $k = 0, \ldots, n-1$ fix a lowest weight vector $u_k \in V(\mu + \lambda_k)$. If the operators A_j exist they must be of the form

$$A_{j+1} : u_k \mapsto P_{j,k} \cdot u_{k+1}, \qquad (0 \le j, k \le n-1)$$

where $P_{j,k}=0$ for $j>k$ and $P_{j,k}\in U(\mathfrak{n}_+)$ is a homogeneous element of degree

$$\deg(P_{j,k})=(\lambda_1-\alpha_1-\cdots-\alpha_j)+(\mu+\lambda_k)-(\mu+\lambda_{k+1})=\alpha_{j+1}+\cdots+\alpha_k$$

for $j\le k$. Here $U(\mathfrak{n}_+)$ is graded by the root lattice $\oplus\mathbb{Z}\alpha_i$ and $\deg(X_\alpha)=\alpha$. This implies that, for every $1\le j\le n-1$ and $0\le k\le n-1$,

$$\begin{aligned}P_{j,k}\cdot u_{k+1}=[Y_j,A_j](u_k)=Y_jA_j(u_k)-A_jY_j(u_k)=Y_jA_j(u_k)=Y_jP_{j-1,k}\cdot u_{k+1}=\\=(Y_jP_{j-1,k}-P_{j-1,k}Y_j)\cdot u_{k+1}=[Y_j,P_{j-1,k}]\cdot u_{k+1}.\end{aligned}$$

The commutator $[Y_j,P_{j-1,k}]$ does not necessarily lie in $U(\mathfrak{n}_+)$; it may contain H_j. As A_1 commutes with all X_i, we have

$$A_1:P\cdot u_k\mapsto PP_{0,k}\cdot u_{k+1}\qquad\qquad (k\in\mathbb{Z}/n\mathbb{Z})$$

for all $P\in U(\mathfrak{n}_+)$.

(2.4) Proposition. *For every $\mu\in\mathfrak{h}^*$ there are $P_{j,k}\in U(\mathfrak{n}_+)$ of degree $\alpha_{j+1}+\cdots+\alpha_k$ $(0\le j\le k\le n-1)$, unique up to a scalar multiple $P_{j,k}\mapsto t_kP_{j,k}$ depending on k, such that operators*

$$A_{j+1}:u_k\mapsto\begin{cases}P_{j,k}\cdot u_{k+1} & \text{if } j\le k\\ 0 & \text{if } j>k\end{cases}$$

(for $0\le j,k\le n-1$) satisfy (2.3.1) and define an action of $L(V)\rtimes sl(n)$ on $U(\mu)$.

(2.5) Remark. For the first few values of j,k we have

$$\begin{aligned}
&P_{0,0}=a_0\cdot 1\\
&P_{0,1}=a_1\cdot X_1\\
&P_{0,2}=a_2\cdot(X_1X_2-(1+\mu_2)X_{12})\\
&P_{0,3}=a_3\cdot(X_1X_2X_3-(1+\mu_2+\mu_3)X_{12}X_3-(1+\mu_3)X_1X_{23}+(1+\mu_2+\mu_3)(1+\mu_3)X_{123})\\
&P_{1,1}=-a_1\cdot(1+\mu_1)\\
&P_{1,2}=-a_2\cdot(1+\mu_1+\mu_2)X_2\\
&P_{1,3}=-a_3\cdot(1+\mu_1+\mu_2+\mu_3)(X_2X_3-(1+\mu_3)X_{23})\\
&P_{2,2}=a_2\cdot(1+\mu_2)(1+\mu_1+\mu_2)\\
&P_{2,3}=a_3\cdot(1+\mu_2+\mu_3)(1+\mu_1+\mu_2+\mu_3)X_3\\
&P_{3,3}=-a_3\cdot(1+\mu_3)(1+\mu_2+\mu_3)(1+\mu_1+\mu_2+\mu_3)
\end{aligned}$$

Here $\mu_i=\mu(H_i)$ and $a_0,\ldots,a_{n-1}\in F$ are arbitrary constants.

(2.6) *Proof of Proposition.* We begin by fixing a notation. For integers $1\le a\le b\le n-1$ put $I=[a,b]=\{a,a+1,\ldots,b\}$ and $X_I=X_{a\cdots b}$ $(=(\mathrm{ad}X_a)(\mathrm{ad}X_{a+1})\cdots(\mathrm{ad}X_{b-1})X_b)$. If $I_1=[a_1,b_1]$ and $I_2=[a_2,b_2]$ with $a_2=b_1+1$, we write $(I_1)(I_2)$ for the partition of $[a_1,b_2]$ into I_1 and I_2 (and similarly for $(I_1)\cdots(I_r)$). We allow $a=b+1$, in which case $I=\emptyset$ and $X_\emptyset=1$.

Proof of uniqueness. Assume that $P_{j,k}$ with the required properties exist. For $0\le j\le k\le n-1$ a basis of $\{P\in U(\mathfrak{n}_+)\,|\,\deg(P)=\alpha_{j+1}+\cdots+\alpha_k\}$ is given by $X_{I_1}\cdots X_{I_\ell}$, for all partitions $[j+1,k]=(I_1)\cdots(I_\ell)$. This means that

$$P_{j,k}=\sum_{[j+1,k]=(I_1)\cdots(I_\ell)}c_{j,k}(I_1|\cdots|I_\ell)X_{I_1}\cdots X_{I_\ell}$$

for some $c_{j,k}(I_1|\cdots|I_\ell)\in F$.

Consider first the case $j = 0$. Fix a partition $[1,k] = (I_1)\cdots(I_\ell)$ with $I_r = [a_r, b_r]$, $a_1 = 1$, $b_\ell = k$, $b_r + 1 = a_{r+1}$. It follows from $[Y_i, X_i] = -H_i$ and

$$\begin{aligned} &[Y_i, X_{i\cdots j}] = X_{(i+1)\cdots j} \qquad && (i < j) \\ &[Y_i, X_{k\cdots i}] = -X_{k\cdots(i-1)} \qquad && (i > k) \end{aligned}$$

that

$$[Y_i, X_{I_1}\cdots X_{I_\ell}] = \begin{cases} 0 & \text{if } (\forall r)\ i \neq a_r, b_r \\ X_{I_1}\cdots X_{I_r\setminus\{i\}}\cdots X_{I_\ell} & \text{if } i = a_r \neq b_r \\ -X_{I_1}\cdots X_{I_r\setminus\{i\}}\cdots X_{I_\ell} & \text{if } i = b_r \neq a_r \\ X_{I_1}\cdots X_{I_{r-1}}X_{I_{r+1}}\cdots X_{I_\ell}(1-H_i) & \text{if } i = a_r = b_r,\ r < \ell \\ X_{I_1}\cdots X_{I_{\ell-1}}(-H_i) & \text{if } i = a_r = b_r,\ r = \ell \end{cases}$$

Using

$$H_i(u_{k+1}) = (\mu + \rho + \lambda_{k+1})(H_i)u_{k+1} = (\mu_i + 1 + \delta_{i,k+1})u_{k+1},$$

we obtain

$$[Y_i, A_1](u_k) = [Y_i, P_{0,k}]\cdot u_{k+1} = 0 \qquad (i > k)$$

and

$$\begin{aligned} [Y_i, A_1](u_k) = \sum [(-\delta_{ik} - \mu_i)c_{0,k}(J_1|\cdots|J_a|i|K_1|\cdots|K_b) + (1-\delta_{ik})c_{0,k}(J_1|\cdots|J_a|iK_1|\cdots|K_b) - \\ -(1-\delta_{i1})c_{0,k}(J_1|\cdots|J_ai|K_1|\cdots|K_b)]\, X_{J_1}\cdots X_{J_a}X_{K_1}\cdots X_{K_b}\cdot u_{k+1} \qquad (i \leq k), \end{aligned}$$

where the sum is taken over all partitions $[1, i-1] = (J_1)\cdots(J_a)$ and $[i+1, k] = (K_1)\cdots(K_b)$. The conditions $[Y_i, A_1] = 0$ $(2 \leq i \leq n-1)$ are then equivalent to

(2.6.1)

$$\begin{aligned} -(1+\mu_k)c_{0,k}(J_1|\cdots|J_a|k) = c_{0,k}(J_1|\cdots|J_ak) \qquad (1 < i = k) \\ -\mu_i c_{0,k}(J_1|\cdots|J_a|i|K_1|\cdots|K_b) + c_{0,k}(J_1|\cdots|J_a|iK_1|\cdots|K_b) = c_{0,k}(J_1|\cdots|J_ai|K_1|\cdots|K_b) \quad (1 < i < k) \end{aligned}$$

for all partitions $[1, i-1] = (J_1)\cdots(J_a)$, $[i+1, k] = (K_1)\cdots(K_b)$. These equations determine recursively all coefficients $c_{0,k}(I_1|\cdots|I_\ell)$ as unique multiples of $a_k = c_{0,k}(1|2|\cdots|k)$. This proves the uniqueness of $P_{0,k}$ (up to scalar multiples $P_{0,k} \mapsto t_kP_{0,k}$). The formulas $[Y_j, A_j] = A_{j+1}$ then determine all $P_{j,k}$ (again up to scalar multiples $P_{j,k} \mapsto t_kP_{j,k}$).

Proof of existence. One can verify by direct calculation that

$$c_{0,k}(I_1|\cdots|I_\ell) = a_k \prod_{\substack{i=1 \\ i,i+1\in\text{same } I_r}}^{k-1} (-(1 + \mu_{i+1} + \cdots + \mu_k))$$

(for all partitions $(I_1)\cdots(I_\ell) = [1,k]$) are solutions of the equations (2.6.1). This implies that the corresponding operators

$$A_1 : u_k \mapsto P_{0,k}\cdot u_{k+1}$$

satisfy $[Y_i, A_1] = 0$ for all $i = 2, \ldots, n-1$. We compute

$$\begin{aligned} A_2(u_k) = [Y_1, A_1](u_k) = \\ = \sum_{[2,k]=(I_1)\cdots(I_\ell)} ((-\mu_1)c_{0,k}(1|I_1|\cdots|I_\ell) + c_{0,k}(1I_1|\cdots|I_\ell))\, X_{I_1}\cdots X_{I_\ell}\cdot u_{k+1}, \qquad (k \geq 1) \end{aligned}$$

hence

$$c_{1,k}(I_1|\cdots|I_\ell) = -a_k(1+\mu_1+\cdots+\mu_k) \prod_{\substack{i=2\\ i,i+1\in \text{same } I_r}}^{k-1} (-(1+\mu_{i+1}+\cdots+\mu_k)). \qquad (k\geq 1)$$

Continuing this process, we obtain by induction (for $k \geq j$)

$$(2.6.2)\quad P_{j,k} = a_k \prod_{i=1}^{j}(-(1+\mu_i+\cdots+\mu_k)) \sum_{[j+1,k]=(I_1)\cdots(I_\ell)} X_{I_1}\cdots X_{I_\ell} \prod_{\substack{i=j+1\\ i,i+1\in \text{same } I_r}}^{k-1} (-(1+\mu_{i+1}+\cdots+\mu_k)).$$

As $P_{1,k}$ is a polynomial in $X_2,\ldots,X_{n-1}$ but not in X_1, we have $(\mathrm{ad}(Y_1))^2 A_1 = [Y_1, A_2] = 0$. This means that A_1 satisfies (2.3.1). Proposition is proved.

(2.7) Assume that all $a_k \neq 0$. One can reparametrize $u_k \mapsto t_k u_k$ and $A_1 \mapsto t_n A_1$ with $t_i \in F^*$; this changes $(a_0, a_1, \ldots, a_{n-1})$ to $(a_0 t_n t_0^{-1} t_1, a_1 t_n t_1^{-1} t_2, \ldots, a_{n-1} t_n t_{n-1}^{-1} t_0)$. This transformation leaves the expression $(a_0 \cdots a_{n-1}) \pmod{F^{*n}} \in F^*/F^{*n}$ invariant; replacing F by $F(\sqrt[n]{a_0\cdots a_{n-1}})$ and choosing suitable t_i we can (and will) assume that $a_0 = \cdots = a_{n-1} = 1$.

One can write $P_{j,k}$ in a more compact form as follows. We have

$$(2.7.1)\qquad \begin{aligned} [X_i, X_{i+1}^a] &= a X_{i+1}^{a-1} X_{i(i+1)} \\ [X_i^a, X_{i+1}] &= a X_i^{a-1} X_{i(i+1)} \end{aligned} \qquad (a = 0,1,2,\cdots)$$

We shall formally use these formulas for all $a \in F$, in the form

$$X_{i+1}^{1-a} X_i X_{i+1}^a = X_i^a X_{i+1} X_i^{1-a} = X_{i+1} X_i + a X_{i(i+1)}.$$

We claim that, with the normalization $a_0 = \cdots = a_{n-1} = 1$, we have

$$(2.7.2)\qquad \begin{aligned} P_{0,k} &= X_k^{1+\mu_k} X_{k-1}^{1+\mu_{k-1}+\mu_k} \cdots X_2^{1+\mu_2+\cdots+\mu_k} X_1 X_2^{-\mu_2-\cdots-\mu_k} \cdots X_{k-1}^{-\mu_{k-1}-\mu_k} X_k^{-\mu_k} \\ P_{j,k} &= \prod_{i=1}^{j}(-(1+\mu_i+\cdots+\mu_k)) X_k^{1+\mu_k} \cdots X_{j+2}^{1+\mu_{j+2}+\cdots+\mu_k} X_{j+1} X_{j+2}^{-\mu_{j+2}-\cdots-\mu_k} \cdots X_k^{-\mu_k} \end{aligned}$$

Indeed, induction based on (2.7.1) shows that all $P_{j,k}$ defined by (2.7.2) are in fact elements of $U(\mathfrak{n}_+)$. As $[Y_i, P_{j,k}] = 0$ for $i \leq j$ and

$$[Y_{j+1}, P_{j,k}] = P_{j+1,k} \frac{(-H_{j+1} - \mu_{j+2} - \cdots - \mu_k)}{(-1-\mu_{j+1}-\cdots-\mu_k)},$$

we have indeed $[Y_{j+1}, A_{j+1}] = A_{j+2}$. This shows that $P_{j,k}$ satisfy (2.3.1) and gives an alternative proof of the existence statement of Proposition 2.4.

(2.8) Proposition. *For $\mu \in \mathfrak{h}^*$, the following conditions are equivalent:*
(i) Every $z \in Z(\mathfrak{g})$ acts on $U(\mu)$ as a scalar.
(ii) Ω acts on $U(\mu)$ as a scalar.
(iii) $\mu = -\frac{\rho}{n}$.
(iv) $\mu, \mu+\lambda_1, \ldots, \mu+\lambda_{n-1}$ lie in one W-orbit.

Proof. (i) $\Longrightarrow$ (ii) is automatic.

(iv) $\iff$ (i) follows from the fact that, for every $w \in W$, z acts on $V(\mu)$ and $V(w \cdot \mu)$ by the same scalar (Harish-Chandra's theorem).
(ii) $\iff$ (iii): the condition (ii) is equivalent to

$$(\forall i = 1, \dots, n-1) \quad (\mu, \mu) = (\mu + \lambda_i, \mu + \lambda_i) \iff (\forall i) \quad -2(\mu, \lambda_i) = (\lambda_i, \lambda_i) = (\frac{2\rho}{n}, \lambda_i) \iff$$
$$\iff (\forall i) \quad (2\mu + \frac{2\rho}{n}, \lambda_i) = 0 \iff \mu = -\frac{\rho}{n}.$$

(iii) $\implies$ (iv): the Coxeter element $c = s_1 \cdots s_{n-1} \in W$ satisfies

$$c^i \left(-\frac{\rho}{n}\right) = -\frac{\rho}{n} + \lambda_i \qquad (1 \le i \le n-1).$$

Proposition is proved.

(2.9) This important property of the weight $-\frac{\rho}{n}$ was discovered by Bernstein (unpublished), who also posed the question of determining the structure of the Lie algebra $\mathcal{L}(n)$ defined in 3.1 below. Another approach can be found in an unpublished part of the author's thesis [Ne].

The weight $-\frac{\rho}{n}$ also corresponds to "exceptional characters" ω introduced by Kazhdan-Patterson [KP, p. 71] (the apparent discrepancy of signs is caused by our convention of using lowest weight, rather than highest weight, vectors). The significance of exceptional characters (for n-fold coverings of $GL(n)$) is explained by Cor. I.3.6 and Thm. II.2.5(b) of [KP], where it is shown that local components $V_0(\omega)$ of the corresponding metaplectic representations have unique Whittaker models.

(2.10) It seems appropriate to call $U(-\frac{\rho}{n})$ the *Weil representation of $sl(n)$*. For this value of $\mu = -\frac{\rho}{n}$ we have $\mu_i = -\frac{1}{n}$ for all i, hence

$$P_{0,k} = X_k^{1-\frac{1}{n}} X_{k-1}^{1-\frac{2}{n}} \cdots X_2^{1-\frac{k-1}{n}} X_1 X_2^{\frac{k-1}{n}} \cdots X_{k-1}^{\frac{2}{n}} X_k^{\frac{1}{n}}$$
$$P_{j,k} = \left(\prod_{i=1}^{j} (\frac{k-i+1}{n} - 1) \right) X_k^{1-\frac{1}{n}} X_{k-1}^{1-\frac{2}{n}} \cdots X_{j+2}^{1-\frac{k-1-j}{n}} X_{j+1} X_{j+2}^{\frac{k-1-j}{n}} \cdots X_{k-1}^{\frac{2}{n}} X_k^{\frac{1}{n}}.$$

3. Higher Heisenberg Lie algebras

(3.1) The Lie algebra $h(n)$

From now on, we restrict our attention to the space $U(-\frac{\rho}{n})$ as a representation of $L(V) \rtimes sl(n)$. Denote by $\mathcal{L}(n)$ the image of $L(V)$ in $\mathrm{End}(U(-\frac{\rho}{n}))$. We know from 1.1 that $\mathcal{L}(2) = h(2)$ is the Heisenberg Lie algebra.

Definition. *The Heisenberg Lie algebra $h(n)$ of level n is the quotient of $L(V)$ by the Lie ideal generated by $(\mathrm{ad}(x))^n y$, for all $x, y \in V$.*

For $n = 2$ this gives the usual Heisenberg Lie algebra: if A, B is a basis of V, then we are dividing $L(V)$ by relations $[A,[A,B]] = [B,[A,B]] = 0$, i.e. we are forcing $[A,B]$ to be a central element of $h(2)$.

(3.2) Theorem. *The canonical surjection $L(V) \longrightarrow \mathcal{L}(n)$ factors through $L(V) \longrightarrow h(n) \longrightarrow \mathcal{L}(n)$, i.e. the operators $A_1, \dots, A_n$ define a representation of $h(n)$ on $U(-\frac{\rho}{n})$.*

Proof. We prove a slightly stronger statement, with $\mathcal{L}(n)$ replaced by another Lie algebra $\widetilde{\mathcal{L}}(n)$, intermediate between $L(V)$ and $\mathcal{L}(n)$. Let $\chi : Z(sl(n)) \longrightarrow F$ be the ring homomorphism given by the action on $U(-\frac{\rho}{n})$ (using Proposition 2.8) and denote by J_χ the bilateral ideal of $U(L(V) \rtimes sl(n))$ generated by $z - \chi(z)1$ (for all

$z \in Z(sl(n)))$. Let $\widetilde{\mathcal{L}}(n)$ be the image of $L(V)$ in $U(L(V) \rtimes sl(n))/J_\chi$; then $L(V) \longrightarrow \mathcal{L}(n)$ factors through $L(V) \longrightarrow \widetilde{\mathcal{L}}(n) \longrightarrow \mathcal{L}(n)$. In order to prove the theorem, it is enough to show that $(\mathrm{ad}(A_1))^n A_2 \in J_\chi$.

As a first step, note that $[z,x] = [z-\chi(z)1, x]$ lies in J_χ for all $z \in Z(sl(n))$ and $x \in L(V)$. In particular, $[\Omega, v] \in J_\chi$ for all $v \in V$. The commutators are computed in $U(L(V) \rtimes sl(n))$; one must be careful to distinguish between $X * v$ (the action of $X \in sl(n)$ on $v \in V$) and Xv (multiplication in $U(L(V) \rtimes sl(n))$). The basic commutation rule is $[X, v] = Xv - vX = X * v$ ($X \in sl(n)$, $v \in V$). With this notation we have

$$[\Omega, A_1] = \sum_{\alpha>0}(X_\alpha(Y_\alpha * A_1) + (X_\alpha * A_1)Y_\alpha + Y_\alpha(X_\alpha * A_1) + (Y_\alpha * A_1)X_\alpha) + \sum_{i=1}^{n-1}(\lambda_1(H_i)A_1H_i^* + \lambda_1(H_i^*)H_iA_1) =$$

$$= \sum_{i=1}^{n-1}(X_{1\cdots i}A_{i+1} + A_{i+1}X_{1\cdots i}) + A_1H_1^* + \sum_{i=1}^{n-1}\left(1 - \frac{i}{n}\right)H_iA_1 = \sum_{i=1}^{n-1}(2X_{1\cdots i}A_{i+1} - A_1) + (2H_1^* - \lambda_1(H_1^*))A_1 =$$

$$= 2\left((H_1^* - \frac{n^2-1}{2n})A_1 + \sum_{i=1}^{n-1} X_{1\cdots i}A_{i+1}\right).$$

Applying successively $\mathrm{ad}(Y_1), \ldots, \mathrm{ad}(Y_{n-1})$ we obtain formulas for all $[\Omega, A_i]$. They can be written in matrix form as

$$\frac{1}{2}[\Omega, A] = (M - cI)A,$$

where $c = \frac{n^2-1}{2n}$, $A = (A_1, \ldots, A_n)^T$ and

$$\text{(3.2.1)} \qquad M = \begin{pmatrix} H_1^* & X_1 & X_{12} & \cdots & X_{1\cdots(n-1)} \\ Y_1 & H_2^* - H_1^* & X_2 & \cdots & X_{2\cdots(n-1)} \\ Y_{12} & Y_2 & H_3^* - H_2^* & \cdots & X_{3\cdots(n-1)} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ Y_{1\cdots(n-1)} & Y_{2\cdots(n-1)} & \cdots & Y_{n-1} & -H_{n-1}^* \end{pmatrix} \in M_n(U(sl(n))).$$

Note that $H_{i+1}^* - H_i^* = H_1^* - H_1 - \cdots - H_i \qquad (H_0^* = H_n^* = 0)$. Taking commutators with A_1, we obtain equalities in $M_n(U(L(V) \rtimes sl(n)))$

$$[A_1, M] = \frac{A_1}{n}I + \begin{pmatrix} -A_1 & 0 & \cdots & 0 \\ -A_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -A_n & 0 & \cdots & 0 \end{pmatrix}$$

$$(\mathrm{ad}(A_1))^i M = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ -(\mathrm{ad}(A_1))^{i-1}A_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -(\mathrm{ad}(A_1))^{i-1}A_n & 0 & \cdots & 0 \end{pmatrix} \qquad (i > 1)$$

It follows from

$$(\mathrm{ad}(A_1))^j A = \begin{pmatrix} 0 \\ (\mathrm{ad}(A_1))^j A_2 \\ \vdots \\ (\mathrm{ad}(A_1))^j A_n \end{pmatrix} \qquad (j > 0)$$

that

$$((\mathrm{ad}(A_1))^i M)((\mathrm{ad}(A_1))^j A) = 0 \qquad (i > 1, j > 0).$$

This implies that

$$\frac{1}{2}(\mathrm{ad}(A_1))^n[\Omega, A] = (\mathrm{ad}(A_1))^n((M-cI)A) = \sum_{i+j=n} \binom{n}{i} ((\mathrm{ad}(A_1))^i(M-cI))((\mathrm{ad}(A_1))^j A) =$$

$$= ((\mathrm{ad}(A_1))^n M)A + n[A_1, M](\mathrm{ad}(A_1))^{n-1}A + (M-cI)(\mathrm{ad}(A_1))^n A =$$

$$\begin{pmatrix} 0 \\ -(\mathrm{ad}(A_1))^{n-1}A_2A_1 \\ \vdots \\ -(\mathrm{ad}(A_1))^{n-1}A_nA_1 \end{pmatrix} + \begin{pmatrix} 0 \\ A_1(\mathrm{ad}(A_1))^{n-1}A_2 \\ \vdots \\ A_1(\mathrm{ad}(A_1))^{n-1}A_n \end{pmatrix} + (M-cI)(\mathrm{ad}(A_1))^n A = (M-(c-1)I)(\mathrm{ad}(A_1))^n A.$$

It will be shown in Corollary 3.12 below that $M - \lambda I$ is invertible in $M_n(U(L(V) \rtimes sl(n))/J_\chi)$ for all $\lambda \notin \{c, c-\frac{1}{n}, \dots, c-\frac{n-1}{n}\}$. As $[\Omega, A_i] \in J_\chi$, it follows that

$$(\mathrm{ad}(A_1))^n A = (M-(c-1)I)^{-1}(M-(c-1)I)(\mathrm{ad}(A_1))^n A = 0$$

in $(U(L(V) \rtimes sl(n))/J_\chi)^n$, hence $(\mathrm{ad}(A_1))^n A_i \in J_\chi$ for all i. This implies that $L(V) \longrightarrow \widetilde{\mathcal{L}}(n)$ factors through $L(V) \longrightarrow h(n) \longrightarrow \widetilde{\mathcal{L}}(n)$ as claimed.

(3.3) About the matrix M

The matrix $M \in M_n(U(sl(n)))$ from (3.2.1) has very interesting properties. In the simplest case $n = 2$,

$$M = \begin{pmatrix} \frac{H}{2} & X \\ Y & -\frac{H}{2} \end{pmatrix}, \qquad M^2 = \begin{pmatrix} \frac{H^2}{4} + XY & X \\ Y & \frac{H^2}{4} + YX \end{pmatrix},$$

hence

$$M^2 - M - \frac{\Omega}{2} I = 0 \qquad \text{in } M_2(U(sl(2))).$$

For every ring homomorphism $\varphi : Z(sl(2)) \longrightarrow F$ put

$$U(sl(2))_\varphi = U(sl(2)) \otimes_{Z(sl(2)),\varphi} F.$$

If $\varphi = \chi_\mu$ is given by the action on $V(\mu)$, then $\varphi(\Omega) = \frac{\mu^2-1}{2}$ and

$$\left(M - \frac{1+\mu}{2}\right)\left(M - \frac{1-\mu}{2}\right) = M^2 - M - \frac{\mu^2-1}{4} I = 0 \qquad \text{in } M_2(U(sl(2))_{\chi_\mu}).$$

In particular, for $\mu = -\frac{1}{2}$,

$$\left(M - \frac{3}{4}\right)\left(M - \frac{1}{4}\right) = 0 \qquad \text{in } M_2(U(L(V) \rtimes sl(2))/J_\chi).$$

We are going to prove similar identities for $\mathfrak{g} = sl(n)$. It will be convenient to consider the Casimir element Ω as a $\mathfrak{g}$-invariant element of $\mathfrak{g} \otimes \mathfrak{g}$:

$$\Omega = \sum_{\alpha > 0} (X_\alpha \otimes Y_\alpha + Y_\alpha \otimes X_\alpha) + \sum_{i=1}^{n-1} \frac{H_i \otimes H_i^* + H_i^* \otimes H_i}{2} \in S^2(\mathfrak{g})^{\mathfrak{g}} \subset (\mathfrak{g} \otimes \mathfrak{g})^{\mathfrak{g}}.$$

For every pair of representations $\rho_{V_i} : \mathfrak{g} \longrightarrow \mathrm{End}_F(V_i) \quad (i = 1, 2)$ consider elements

$$(1 \otimes \rho_{V_2})(\Omega) \in (\mathfrak{g} \otimes_F \mathrm{End}_F(V_2))^{\mathfrak{g}}$$
$$(\rho_{V_1} \otimes \rho_{V_2})(\Omega) \in (\mathrm{End}_F(V_1) \otimes_F \mathrm{End}_F(V_2))^{\mathfrak{g}} \xrightarrow{\sim} \mathrm{End}_F(V_1 \otimes_F V_2)^{\mathfrak{g}}.$$

If $V_2 = V$, then $\mathrm{End}_F(V) = M_n(F)$ and

$$(1 \otimes \rho_V)(\Omega) = \begin{pmatrix} H_1^* & Y_1 & Y_{12} & \cdots & Y_{1\cdots(n-1)} \\ X_1 & H_2^* - H_1^* & Y_2 & \cdots & Y_{2\cdots(n-1)} \\ X_{12} & X_2 & H_3^* - H_2^* & \cdots & Y_{3\cdots(n-1)} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ X_{1\cdots(n-1)} & X_{2\cdots(n-1)} & \cdots & X_{n-1} & -H_{n-1}^* \end{pmatrix} = M^T$$

(as an element of $\mathfrak{g} \otimes_F M_n(F) \subset U(\mathfrak{g}) \otimes_F M_n(F)$). We want to show that M^T satisfies a polynomial equation of degree n with coefficients in $Z(\mathfrak{g}) \cdot I$, by considering the action on $V(\mu)$ for variable $\mu \in \mathfrak{h}^*$.

(3.4) Lemma. *Assume that V_i ($i = 0, 1, 2$) are $\mathfrak{g}$-modules such that $V_0 \subset V_1 \otimes V_2$ and Ω acts on each V_i by a scalar d_i. Then the restriction of $(\rho_{V_1} \otimes \rho_{V_2})(\Omega)$ to $\mathrm{End}(V_0)^{\mathfrak{g}}$ is equal to $\frac{1}{2}(d_0 - d_1 - d_2) \cdot I$.*

Proof. As explained to us by A. Wassermann, the formula

$$\begin{gathered}(X \otimes 1 + 1 \otimes X)(Y \otimes 1 + 1 \otimes Y) + (X \otimes 1 + 1 \otimes X)(Y \otimes 1 + 1 \otimes Y) - \\ -(XY + YX) \otimes 1 - 1 \otimes (XY + YX) = 2(X \otimes Y + Y \otimes X)\end{gathered}$$

implies

$$\Omega_{V_1 \otimes V_2} - \Omega_{V_1} \otimes I_{V_2} - I_{V_1} \otimes \Omega_{V_2} = 2(\rho_{V_1} \otimes \rho_{V_2})(\Omega);$$

restricting to V_0 we get the required statement.

(3.5) Corollary. *For every $V(\nu) \subset V(\mu) \otimes V(\lambda)$, the restriction of $(\rho_{V(\mu)} \otimes \rho_{V(\lambda)})(\Omega)$ to $\mathrm{End}(V(\nu))$ is equal to*

$$\frac{1}{2}\left((\nu, \nu) - (\mu, \mu) - (\lambda, \lambda) + (\rho, \rho)\right) \cdot I.$$

(3.6) For every $\mu \in \mathfrak{h}^*$ we have

$$V(\mu) \otimes V = \bigoplus_{i=1}^{n} V(\mu - (\lambda_i - \lambda_{i-1})) \qquad (\lambda_0 = \lambda_n = 0)$$

(note that this formula can be used to give a conceptual proof of Proposition 2.4). Applying Corollary 3.5 to $V(\mu)$ and $V \subset V(-\lambda_{n-1} - \rho)$, we obtain

$$(\rho_{V(\mu)} \otimes \rho_V)(\Omega) = \bigoplus_{i=1}^{n} C_i \cdot I_{V(\mu - (\lambda_i - \lambda_{i-1}))}$$

with

$$\begin{aligned} C_i = C_i(\mu) &= \frac{1}{2}\left((\mu - (\lambda_i - \lambda_{i-1}), \mu - (\lambda_i - \lambda_{i-1})) - (\mu, \mu) - (-\lambda_{n-1} - \rho, -\lambda_{n-1} - \rho) + (\rho, \rho)\right) = \\ &= -(\mu, \lambda_i - \lambda_{i-1}) - (\lambda_{n-1}, \rho) = -(\mu, \lambda_i - \lambda_{i-1}) - \frac{n-1}{2}. \end{aligned}$$

The coefficients q_i of the polynomial

$$\sum_{i=0}^{n} q_i X^i = \prod_{i=1}^{n} (X - C_i(\mu)) = \prod_{i=1}^{n} \left(X + (\mu, \lambda_i - \lambda_{i-1}) + \frac{n-1}{2}\right)$$

are polynomial functions of $\mu \in \mathfrak{h}^*$, i.e. $q_i \in S(\mathfrak{h})$. The set of weights $\{\lambda_i - \lambda_{i-1}\}$ forms a W-orbit, hence all $q_i \in S(\mathfrak{h})^W$ are W-invariant.

Recall the definition of the Harish-Chandra isomorphism, slightly modified as we work with *lowest weight* Verma modules: the algebra $U(\mathfrak{g})$ is graded by the root lattice $\oplus\mathbb{Z}\alpha_i$ and its component of degree zero $U(\mathfrak{g})_0$ is equal to the centralizer of $U(\mathfrak{h}) = S(\mathfrak{h})$. The subspace $L = U(\mathfrak{g})\mathfrak{n}_- \cap U(\mathfrak{g})_0 = \mathfrak{n}_+ U(\mathfrak{g}) \cap U(\mathfrak{g})_0$ is a bilateral ideal in $U(\mathfrak{g})_0$ satisfying $U(\mathfrak{g})_0 = L \oplus S(\mathfrak{h})$. The Harish-Chandra isomorphism

$$\omega : Z(\mathfrak{g}) \xrightarrow{\sim} S(\mathfrak{h})^W$$

is given by the composition

$$Z(\mathfrak{g}) \hookrightarrow U(\mathfrak{g})_0 = L \oplus S(\mathfrak{h}) \xrightarrow{\mathrm{pr}} S(\mathfrak{h}) \xrightarrow{\gamma} S(\mathfrak{h}),$$

where $\gamma(\lambda \mapsto P(\lambda)) = (\lambda \mapsto P(\lambda + \rho))$ $(\lambda \in \mathfrak{h}^*)$. For $\mu \in \mathfrak{h}^*$, any $z \in Z(\mathfrak{g})$ acts on $V(\mu)$ by the scalar $\omega(z)(\mu)$.

Define

$$Q(X) = \sum_{i=0}^{n} \omega^{-1}(q_i) X^i \in Z(sl(n))[X].$$

For $n = 2$ the scalar product is normalized by $(1,1) = \frac{1}{2}$, $\lambda_1 = 1$, $C_1 = -\frac{\mu+1}{2}$, $C_2 = \frac{\mu-1}{2}$,

$$(X - C_1)(X - C_2) = X^2 + X + \frac{1-\mu^2}{4}, \qquad Q(X) = X^2 + X - \frac{\Omega}{2}.$$

(3.7) Proposition. *The matrix M^T satisfies $Q(M^T) = 0$ in $M_n(U(sl(n))) = U(sl(n)) \otimes_F M_n(F)$.*

Proof. An element $x \in U(sl(n))$ is equal to zero if and only if its image in $\mathrm{End}(V(\mu))$ is zero for all $\mu \in \mathfrak{h}^*$. This means that it is enough to check that

$$(\rho_{V(\mu)} \otimes 1)(Q(M^T)) = Q((\rho_{V(\mu)} \otimes 1)(M^T)) = Q((\rho_{V(\mu)} \otimes 1)(1 \otimes \rho_V)\Omega)) =$$
$$= Q((\rho_{V(\mu)} \otimes \rho_V)(\Omega)) \stackrel{?}{=} 0,$$

which follows from the definition of Q.

(3.8) In order to pass from M^T to M we need some notation. If R is a ring, denote the opposite multiplication on R by $a \underset{\mathrm{op}}{\cdot} b = ba$; the ring R with opposite multiplication will be denoted by R^{op}. With this notation, three matrices $A, B, C \in M_n(R)$ satisfy

(3.8.1) $$C = AB \text{ in } M_n(R) \iff C^T = B^T A^T \text{ in } M_n(R^{\mathrm{op}}).$$

As $(-X) \underset{\mathrm{op}}{\cdot} (-Y) - (-Y) \underset{\mathrm{op}}{\cdot} (-X) = -[X,Y]$ in $U(\mathfrak{g})$ for all $X, Y \in \mathfrak{g}$, the map $X \mapsto -X$ extends to an isomorphism of rings

$$\theta : U(\mathfrak{g}) \longrightarrow U(\mathfrak{g})^{\mathrm{op}}, \qquad \theta(X_1 \cdots X_k) = (-1)^k X_k \cdots X_1.$$

The Harish-Chandra isomorphism satisfies $\omega(\theta(\mu)) = \omega(-\mu)$.

After all these preliminaries, put

$$\sum_{i=0}^{n} p_i X^i = \prod_{i=1}^{n} \left(X + (\mu, \lambda_i - \lambda_{i-1}) - \frac{n-1}{2} \right) \in S(\mathfrak{h})^W[X]$$
$$P(X) = \sum_{i=0}^{n} \omega^{-1}(p_i) X^i \in Z(sl(n))[X].$$

For $n = 2$, $\sum_{i=0}^{2} p_i X^i = \left(X + \frac{\mu-1}{2}\right)\left(X - \frac{\mu+1}{2}\right) = X^2 - X + \frac{1-\mu^2}{4}$ and $P(X) = X^2 - X - \frac{\Omega}{2}$.

(3.9) Proposition. *The matrix* M *satisfies* $P(M) = 0$ *in* $M_n(U(sl(n))) = U(sl(n)) \otimes_F M_n(F)$.

Proof. We have $Q(M^T) = 0$ in $M_n(U(sl(n)))$ by Proposition 3.7. It follows from (3.8.1) that $Q(M) = 0$ in $M_n(U(sl(n))^{\mathrm{op}})$, hence $(\theta(Q))(\theta(M)) = 0$ in $M_n(U(sl(n)))$. As $\theta(M) = -M$, this implies that $P(M) = 0$ for

$$\omega(P)(X) = \sum_{i=0}^{n} \omega(\theta \circ \omega^{-1}(q_i))(-1)^{n-i} X^i = (-1)^n \prod_{i=1}^{n}(-X - C_i(-\mu)) = \prod_{i=1}^{n}(X + C_i(-\mu)) =$$

$$= \prod_{i=1}^{n} \left(X + (\mu, \lambda_i - \lambda_{i-1}) - \frac{n-1}{2} \right).$$

(3.10) Corollary. *For every* $\mu \in \mathfrak{h}^*$,

$$\prod_{i=1}^{n} \left(M + \left((\mu, \lambda_i - \lambda_{i-1}) - \frac{n-1}{2} \right) I \right) = 0 \qquad \text{in } M_n(U(sl(n))_{\chi_\mu}).$$

(3.11) Proposition. *The matrix* M *satisfies the equation*

$$\prod_{j=0}^{n-1} \left(M - \left(c - \frac{j}{n} \right) I \right) = 0 \qquad \left(c = \frac{n^2-1}{2n} \right)$$

in $M_n(U(L(V) \rtimes sl(n))/J_\chi)$.

Proof. For $\mu = -\frac{\rho}{n}$ and $j = 0, \ldots, n-1$ we have

$$-C_{j+1}(-\mu) = \left(\frac{\rho}{n}, \lambda_{j+1} - \lambda_j \right) + \frac{n-1}{2} = \frac{n-1-2j}{2n} + \frac{n-1}{2} = \frac{n^2-1}{2n} - \frac{j}{n} = c - \frac{j}{n},$$

hence

$$\omega(P)\left(-\frac{\rho}{n} \right) = \prod_{j=0}^{n-1} \left(X - \left(c - \frac{j}{n} \right) \right).$$

(3.12) Corollary. $M - \lambda I$ *is invertible in* $M_n(U(L(V) \rtimes sl(n))/J_\chi)$ *for every* $\lambda \neq c, c - \frac{1}{n}, \cdots, c - \frac{n-1}{n}$.

4. Further properties of $U(-\frac{\rho}{n})$ and $h(n)$

(4.1) The operators $\Delta, \widetilde{\Delta}$

Let $\mu \in \mathfrak{h}^*$ be arbitrary. We are interested in the action of operators

$$\Delta = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) A_{\sigma(1)} \cdots A_{\sigma(n)}, \qquad \widetilde{\Delta} = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) A_{\sigma(n)} \cdots A_{\sigma(1)} = (-1)^{n(n-1)/2} \Delta$$

on $U(\mu)$. We have

$$\Delta(u_k) = d_k(\mu) u_k, \qquad \widetilde{\Delta}(u_k) = \widetilde{d}_k(\mu) u_k \qquad (0 \leq k \leq n-1)$$

for some constants $\widetilde{d}_k(\mu) = (-1)^{n(n-1)/2} d_k(\mu) \in F$. As $[X, \Delta] = 0$ for every $X \in \mathfrak{g}$, it follows that

$$\Delta(P \cdot u_k) = d_k(\mu) P \cdot u_k \qquad P \in U(\mathfrak{n}_+).$$

(4.2) Lemma. *For every $\mu \in \mathfrak{h}^*$,*

$$\widetilde{d}_0(\mu) = \prod_{1\le i\le j\le n-1} (-(1+\mu_i+\cdots+\mu_j)) = \prod_{\alpha>0} (-(1+(\mu,\alpha)))$$
$$d_0(\mu) = \prod_{1\le i\le j\le n-1} (1+\mu_i+\cdots+\mu_j) = \prod_{\alpha>0} (1+(\mu,\alpha))\,.$$

Proof. For $\sigma \neq 1$, $A_{\sigma(n)}\cdots A_{\sigma(1)}(u_0) = 0$, hence $\widetilde{\Delta}(u_0) = A_n\cdots A_1(u_0)$. The result follows from the formulas (2.6.2), which give

$$A_{j+1}(u_j) = \prod_{i=1}^{j}(-(1+\mu_i+\cdots+\mu_j))u_{j+1}.$$

(4.3) For $\mu = -\frac{\rho}{n}$, $\mu_i = -\frac{1}{n}$ for all $i = 1,\ldots,n-1$, hence

(4.3.1) $$d_0\left(-\frac{\rho}{n}\right) = \prod_{j=1}^{n-1}\left(\frac{j}{n}\right)^j.$$

For $n = 3$, an easy calculation shows that

$$\begin{aligned}\widetilde{d}_0(\mu) &= -(1+\mu_1)(1+\mu_2)(1+\mu_1+\mu_2)\\ \widetilde{d}_1(\mu) &= -\mu_1(1+\mu_2)(\mu_1+\mu_2)\\ \widetilde{d}_2(\mu) &= -(1+\mu_1)\mu_2(\mu_1+\mu_2)\end{aligned} \qquad \Longrightarrow \qquad \Delta = \frac{4}{27}\cdot I \quad \text{on } U(-\frac{\rho}{3}).$$

For bigger n explicit computations become very complicated.

(4.4) Conjecture. *For every $k = 0,\ldots,n-1$,*

$$\widetilde{d}_k(\mu) = (-1)^{k(n-k)}\widetilde{d}_0(\mu-\lambda_k) = (-1)^{k(n-k)}\prod_{\alpha>0}(-(1+(\mu-\lambda_k,\alpha)))\,.$$

(4.5) A short calculation shows that

$$\widetilde{d}_0\left(-\frac{\rho}{n}-\lambda_k\right) = (-1)^{k(n-k)}\widetilde{d}_0\left(-\frac{\rho}{n}\right).$$

Conjecture 4.4 would then imply that Δ acts on $U(-\frac{\rho}{n})$ by the scalar (4.3.1).

(4.6) The Lie algebra $\mathcal{L}(3) = G_2$

For $n = 3$ and $\mu = -\frac{\rho}{3}$ we have

$$\begin{array}{lll} A_1 : u_0 \mapsto u_1 & A_2 : u_0 \mapsto 0 & A_3 : u_0 \mapsto 0\\ A_1 : u_1 \mapsto X_1u_2 & A_2 : u_1 \mapsto -\frac{2}{3}u_2 & A_3 : u_1 \mapsto 0\\ A_1 : u_2 \mapsto (X_1X_2 - \frac{2}{3}X_{12})u_0 & A_2 : u_2 \mapsto -\frac{1}{3}X_2u_0 & A_3 : u_2 \mapsto \frac{2}{9}u_0 \end{array}$$

This implies that

$$[A_1, A_2] : u_0 \mapsto \frac{2}{3} u_2 \qquad [A_1, [A_1, A_2]] : u_0 \mapsto -\frac{2}{9} X_{12} u_0$$
$$[A_1, A_2] : u_1 \mapsto \frac{2}{3} (X_1 X_2 - \frac{1}{3} X_{12}) u_0 \qquad [A_1, [A_1, A_2]] : u_1 \mapsto -\frac{2}{9} X_{12} u_1$$
$$[A_1, A_2] : u_2 \mapsto \frac{2}{3} X_2 u_1 \qquad [A_1, [A_1, A_2]] : u_2 \mapsto -\frac{2}{9} X_{12} u_2$$

Both $[A_1, [A_1, A_2]]$ and X_{12} are highest weight vectors of 8-dimensional irreducible representations of $sl(3)$. This implies that we have equalities in $\mathrm{End}(U(-\frac{\rho}{3}))$

$$[A_1, [A_1, A_2]] = -\frac{2}{9} X_{12} \qquad [A_2, [A_3, A_2]] = -\frac{2}{9} (H_2^* - H_1^*)$$
$$[A_2, [A_1, A_2]] = -\frac{2}{9} X_2 \qquad [A_2, [A_2, A_3]] = -\frac{2}{9} Y_1$$
$$[A_1, [A_3, A_1]] = -\frac{2}{9} X_1 \qquad [A_3, [A_3, A_1]] = -\frac{2}{9} Y_2$$
$$[A_1, [A_2, A_3]] = -\frac{2}{9} H_1^* \qquad [A_3, [A_2, A_3]] = -\frac{2}{9} Y_{12}$$

It follows that $\mathcal{L}(3) \subset \mathrm{End}(U(-\frac{\rho}{3}))$ is spanned by A_1, A_2, A_3, $[A_1, A_2]$, $[A_1, A_3]$, $[A_2, A_3]$ and the eight double commutators listed above. This is a 14-dimensional simple Lie algebra of type G_2. Moreover, $\mathcal{L}(3)$ contains the image of $sl(3)$ in $\mathrm{End}(U(-\frac{\rho}{3}))$; this seems to be a specific feature of the case $n = 3$.

This representation of G_2 on $U(-\frac{\rho}{3}))$ appeared in the work of S. Gelfand [Ge] and also in [Ne]. On the group level, the corresponding representation of a 3-fold covering of G_2 was studied by Savin [Sa].

(4.7) Proposition. dim $h(n) = \infty$ *for* $n > 2$.

Proof. We argue by induction. Write V_n for V and V_{n-1} for the subspace spanned by A_i with $1 \leq i \leq n-1$. By definition, $I(n) := \mathrm{Ker}(L(V_n) \longrightarrow h(n))$ is the Lie ideal generated by $(\mathrm{ad}(x))^n y$ $(x, y \in V_n)$; this implies that $L(V_{n-1}) \cap I(n)$ is equal to the Lie ideal generated by $(\mathrm{ad}(x))^n y$ $(x, y \in V_{n-1})$. It follows that $L(V_{n-1}) \cap I(n) \subset I(n-1)$, hence $h(n-1)$ is a subquotient of $h(n)$ and $\dim h(n) \geq \dim h(n-1)$. It remains to show that $\dim h(3) = \infty$. Put $a = [A_1, [A_2, A_3]]$, $b = [A_1, [A_1, A_3]] \in L(V_3)$. It is enough to show that the images of $(\mathrm{ad}(a))^k b$ $(k = 1, 2, \ldots)$ are linearly independent in $h(3)$. Suppose that $\sum_{k \geq 1} c_k (\mathrm{ad}(a))^k b \in I(3)$ for some $c_k \in F$. As $I(n)$ is a homogeneous ideal of $L(V_n)$ for every n (with respect to the grading induced from the tensor algebra $T(V_n)$), it follows that, for all $k \geq 1$, $c_k (\mathrm{ad}(a))^k b \in I(3)$, hence $c_k (\mathrm{ad}(a))^k b = 0$ in $h(3)$. However, the image of $(\mathrm{ad}(a))^k b$ in $\mathrm{End}(U(-\frac{\rho}{3}))$ is non-zero for all $k \geq 1$, as $(\mathrm{ad}(H_1^*))^k X_1 = X_1$ in $sl(3)$. This shows that all $c_k = 0$ as claimed.

5. Open questions

(5.1) Can $h(n)$ be related to some known class of infinite-dimensional Lie algebras?

(5.2) Can the representation $U(-\frac{\rho}{n})$ of $h(n)$ be characterized in a manner similar to the Stone-von Neumann theorem?

(5.3) The Kazhdan-Patterson space $V_0(\omega)$ mentioned in 2.9 is a representation space of the n-fold covering of $SL(n, K)$ analogous to $V(-\frac{\rho}{n})$ (here K is a local field containing μ_n). Is there a group-version of $h(n)$ defined over K and acting on the space obtained from $V_0(\omega)$ by adding $n-1$ representations analogous to $V(-\frac{\rho}{n} + \lambda_i)$?

(5.4) The representation of $h(n) \rtimes sl(n)$ on $U(-\frac{\rho}{n})$ depends on a parameter $a \in F^*/F^{*n}$ (equal to $a_0 \cdots a_{n-1}$ (mod F^{*n})), i.e. on the form $x \mapsto ax^n$ of degree n on a one-dimensional vector space over F. It would be interesting to construct natural representations of $sl(n, F)$ corresponding to other forms of degree n, such as norm forms for extensions E/F of degree n or reduced norms for division algebras of degree n^2 over F (in the well-known case $n = 2$ everything works even on the level of groups; for the case $n = 3$ see [Ka]).

(5.5) Is there an infinite-dimensional analogue of Howe's dual reductive pairs in the context of 5.4?

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