Semisimplicity of certain Galois representations occurring in étale cohomology of unitary Shimura varieties

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Abstract: we show that a certain part of the middle degree ℓ-adic cohomology of an arbitrary simple unitary Shimura variety is a semisimple Galois representation.

Introduction

A new method for proving semisimplicity of a certain class of ℓ-adic representations of profinite groups was developed by the second author in [N], where it was applied to the Galois representations occurring in étale cohomology of quaternionic Shimura varieties. The method combined several abstract semisimplicity criteria with Eichler-Shimura relations for partial Frobenii and with the existence of Galois representations attached to Hilbert modular forms.

In the present article, which grew out of the first author's thesis [F] at Université Pierre et Marie Curie (Paris 6), we improve one of the abstract criteria of [N]. Combining it with Eichler-Shimura relations and the existence of Galois representations attached to self-dual cuspidal automorphic representations of $GL(n)$ over CM fields, we show that the Galois action on a certain part of the middle degree étale cohomology of an arbitrary simple unitary Shimura variety is semisimple (Theorems 2.20 and 2.25 below). The most satisfactory results are obtained for unitary groups whose signatures at infinity are contained in the set $\{(n,0),(n-1,1),(1,n-1),(0,n)\}$.

Our results are more general than those of [F]. Firstly, we do not require the Shimura varieties in question to be compact. Secondly, we work with the "usual" group of unitary similitudes G^* rather than with the bigger group G considered in [F] (see 2.2 below), which allows us to treat more general coefficient systems. We use the Eichler-Shimura relations in the form proved by Wedhorn [W, §5-6] (see also [Mo, §4]), with the sign conventions corrected as in [N, Appendix].

The same method should apply to Shimura varieties attached to groups $GSpin(n)$ over totally real number fields (with some restrictions if n is even).

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General conventions and notation. As in [N], all representations and characters are assumed to be continuous with respect to the natural topologies involved. By an automorphic representation we mean an irreducible automorphic representation.

The characteristic polynomial of an endomorphism u of a finite-dimensional vector space over a field k will be denoted by $P_u(X)=\det(X\cdot\mathrm{id}-u)\in k[X]$. If $k\subset K$ are fields and X is a k-vector subspace of a K-vector space Y, we denote by $K\cdot X$ the K-vector subspace of Y generated by X. We abbreviate $\otimes_{\mathbf{Z}}$ as $\otimes$. For an abelian group A we let $\widehat{A}=A\otimes\widehat{\mathbf{Z}}$. We denote by $\mathbf{A}$ and $\mathbf{A}_k$, respectively, the ring of adeles of $\mathbf{Q}$ and of a number field k, and by ${}_kT$ the algebraic torus ${}_kT=R_{k/\mathbf{Q}}(\mathbf{G}_{m,k})$. Let $\overline{\mathbf{Q}}$ be the algebraic closure of $\mathbf{Q}$ in $\mathbf{C}$. The absolute Galois group of a subfield $K\subset\overline{\mathbf{Q}}$ will be denoted by $\Gamma_K=\mathrm{Gal}(\overline{\mathbf{Q}}/K)$. The reciprocity map of class field theory will be normalised by letting uniformisers correspond to geometric Frobenius elements $\mathrm{Fr}(v)$.

1. An abstract semisimplicity criterion

(1.1) We need to recall several general facts about Lie algebras, as stated in [N, 1.7-1.9, 2.6]. Let V be a finite-dimensional vector space over a field $k\supset\mathbf{Q}$ and $\mathfrak{g}\subset\mathrm{End}_k(V)$ a k-Lie subalgebra. As in [LIE, Ch. VII, §5, no. 3], denote by $\mathfrak{n}_V(\mathfrak{g})$ the set of all elements of the radical of $\mathfrak{g}$ that are nilpotent in $\mathrm{End}_k(V)$. It is a nilpotent ideal of $\mathfrak{g}$ containing the intersection of the radical with $\mathscr{D}\mathfrak{g}$.

Recall that $\mathfrak{g}$ is called a *decomposable linear Lie algebra* [LIE, Ch. VII, §5, Def. 1] if both the semisimple and the nilpotent part of every element of $\mathfrak{g}$ belong to $\mathfrak{g}$.

(1.2) Proposition. (1) *[LIE, Ch. VII, §5, Thm. 2] The Lie algebra $\mathfrak{g} \subset \mathrm{End}_k(V)$ is decomposable $\Longleftrightarrow$ some ($\Longleftrightarrow$ each) Cartan subalgebra of $\mathfrak{g}$ is decomposable $\Longleftrightarrow$ the radical of $\mathfrak{g}$ is decomposable.*
(2) *[LIE, Ch. VII, §5, Thm. 1] If $\mathfrak{g}$ is generated as a k-Lie algebra by a subset $S \subset \mathfrak{g}$ such that every $X \in S$ is either semisimple or nilpotent in $\mathrm{End}_k(V)$, then $\mathfrak{g}$ is decomposable.*
(3) *[LIE, Ch. VII, §5, Prop. 7] If $\mathfrak{g}$ is decomposable, then there exists a Lie subalgebra $\mathfrak{m} \subset \mathfrak{g}$, reductive in $\mathrm{End}_k(V)$ (in particular, $\mathfrak{m}$ is a reductive Lie algebra acting semisimply on V), such that $\mathfrak{g} = \mathfrak{m} \ltimes \mathfrak{n}_V(\mathfrak{g})$.*
(4) *[LIE, Ch. VII, §3, Ex. 16] The set of elements of $\mathfrak{g}$ that are semisimple in $\mathrm{End}_k(V)$ is Zariski dense in $\mathfrak{g}$ $\Longleftrightarrow$ some ($\Longleftrightarrow$ each) Cartan subalgebra of $\mathfrak{g}$ is commutative and consists of elements that are semisimple in $\mathrm{End}_k(V)$.*

(1.3) Corollary. *If the set of elements of $\mathfrak{g}$ that are semisimple in $\mathrm{End}_k(V)$ is Zariski dense in $\mathfrak{g}$, then:*
(1) *$\mathfrak{g}$ is decomposable;*
(2) *$\mathfrak{g} = \mathfrak{m} \ltimes \mathfrak{n}_V(\mathfrak{g})$ for a suitable reductive Lie subalgebra $\mathfrak{m} \subset \mathfrak{g}$ acting semisimply on V;*
(3) *There exists a flag $\{0\} = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_s = V$ of $\mathfrak{g}$-submodules such that $\mathfrak{n}_V(\mathfrak{g})$ acts trivially (and $\mathfrak{m}$ semisimply) on $\mathrm{gr}(V) = \bigoplus_{i=1}^s V_i/V_{i-1}$ (one such flag is $V_0 = \{0\}$ and $V_{i+1} = \{v \in V \mid \mathfrak{n}_V(\mathfrak{g})\, v \in V_i\}$). The isomorphism class of the semisimple $\mathfrak{g}/\mathfrak{n}_V(\mathfrak{g})$-module $\mathrm{gr}(V)$ does not depend on the choice of $\{V_i\}$.*

(1.4) Theorem [N, Thm. 2.6]. *Let V be a non-zero vector space of finite dimension over an algebraically closed field $k \supset \mathbf{Q}$. If $\overline{\mathfrak{g}} \subset \mathrm{End}_k(V)$ is a k-Lie subalgebra such that*
(H1-ZAR) *$\overline{\mathfrak{g}}$ contains a Zariski dense set of elements that are semisimple in $\mathrm{End}_k(V)$,*
then:
(1) *$\overline{\mathfrak{g}} \subset \mathrm{End}_k(V)$ is a decomposable linear Lie algebra.*
(2) *Let $\overline{\mathfrak{n}} = \mathfrak{n}_V(\overline{\mathfrak{g}})$ and fix $\overline{\mathfrak{m}} \subset \overline{\mathfrak{g}}$ ($\overline{\mathfrak{m}} \simeq \overline{\mathfrak{g}}/\overline{\mathfrak{n}}$), a Cartan subalgebra $\overline{\mathfrak{h}} \subset \overline{\mathfrak{m}}$ and $\{0\} = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_s = V$ as in Corollary 1.3. Assume that the following condition holds:*
(H2) *all weights of $\overline{\mathfrak{h}}$ occurring in $\mathrm{gr}(V)$ lie in one coset of the root lattice of $(\overline{\mathfrak{m}}, \overline{\mathfrak{h}})$.*
Then $\overline{\mathfrak{n}} = 0$, $\overline{\mathfrak{g}} = \overline{\mathfrak{m}}$ is a reductive Lie algebra and V is a semisimple $\overline{\mathfrak{g}}$-module.

(1.5) Let Γ be a profinite group, $V \neq 0$ a finite-dimensional vector space over $\overline{\mathbf{Q}}_\ell$ and $\rho : \Gamma \longrightarrow \mathrm{Aut}_{\overline{\mathbf{Q}}_\ell}(V) \simeq GL_n(\overline{\mathbf{Q}}_\ell)$ a (continuous) representation. The image $\rho(\Gamma)$ is a compact Lie group of finite dimension over $\mathbf{Q}_\ell$. Its Lie algebra $\mathrm{Lie}(\rho(\Gamma)) \subset \mathrm{End}_{\overline{\mathbf{Q}}_\ell}(V)$ is a $\mathbf{Q}_\ell$-Lie algebra of finite dimension. The corresponding $\overline{\mathbf{Q}}_\ell$-Lie algebra $\overline{\mathbf{Q}}_\ell \cdot \mathfrak{g} \subset \mathrm{End}_{\overline{\mathbf{Q}}_\ell}(V)$ is a quotient of $\overline{\mathfrak{g}} = \mathfrak{g} \otimes_{\mathbf{Q}_\ell} \overline{\mathbf{Q}}_\ell$.

It is well-known that the representation ρ is semisimple iff the restriction $\rho|_{\Gamma'}$ to each ($\Longleftrightarrow$ to some) open subgroup $\Gamma' \subset \Gamma$ is semisimple, which is in turn equivalent to V being a semisimple $\overline{\mathbf{Q}}_\ell \cdot \mathfrak{g}$-module.

Recall that ρ is called *strongly irreducible* if $\rho|_{\Gamma'}$ is irreducible for every open subgroup $\Gamma' \subset \Gamma$, which is equivalent to V being a simple $\overline{\mathbf{Q}}_\ell \cdot \mathfrak{g}$-module. If this is the case, then $\overline{\mathbf{Q}}_\ell \cdot \mathfrak{g}$ is a reductive Lie algebra over $\overline{\mathbf{Q}}_\ell$. This implies that $\mathfrak{g}$ (resp. $\overline{\mathfrak{g}}$) is a reductive Lie algebra over $\mathbf{Q}_\ell$ (resp. over $\overline{\mathbf{Q}}_\ell$) and that $\mathfrak{z}(\overline{\mathfrak{g}})$ acts on V through a map $\mathfrak{z}(\overline{\mathfrak{g}}) \longrightarrow \overline{\mathbf{Q}}_\ell \cdot \mathrm{id}_V \subset \mathrm{End}_{\overline{\mathbf{Q}}_\ell}(V)$.

(1.6) In [N, Thm. 3.3] it was deduced from Theorem 1.4 above that $\rho : \Gamma \longrightarrow \mathrm{Aut}_{\overline{\mathbf{Q}}_\ell}(V)$ is semisimple, provided there exists an open subgroup $\Gamma' \subset \Gamma$ satisfying the following two conditions:

(1.6.1) the semisimplification $\rho|_{\Gamma'}^{\mathrm{ss}} = \rho^{\mathrm{ss}}|_{\Gamma'}$ is a subrepresentation of $(\rho_1 \otimes \cdots \otimes \rho_r)^{\oplus m}$ (for some $m \geq 1$), where each ρ_i is a strongly irreducible representation of Γ';
(1.6.2) $\rho(g)$ is a semisimple element of $\mathrm{Aut}_{\overline{\mathbf{Q}}_\ell}(V)$, for all g lying in a dense subset $\Sigma \subset \Gamma'$.

In the geometric context considered in [N, §5] the representations ρ_i were suitable twists of the Galois representation attached to a cuspidal Hilbert eigenform without complex multiplication and V was the corresponding Hecke eigenspace of the middle degree étale cohomology of a quaternionic Shimura variety. The condition (1.6.1) (resp. (1.6.2)) was deduced there from the Eichler-Shimura relation for the usual Frobenius (resp. for partial Frobenii) at split primes. The verification of (1.6.1) based on [N, Prop. 3.10] crucially relied on the fact that $\overline{\mathbf{Q}}_\ell \cdot \mathrm{Lie}(\rho_i(\Gamma')) = gl(2)$ acts on $\overline{\mathbf{Q}}_\ell^2$ by its standard representation, which is minuscule.

This minuscule property does not hold for general n-dimensional irreducible representations of Lie algebras if $n > 2$, nor for many representations occurring in étale cohomology of unitary Shimura varieties. This is the principal motivation behind the following abstract result, in which we replace (1.6.1) by a weaker assumption and (1.6.2) by a stronger one.

(1.7) Theorem. *Let Γ be a profinite group, $V, W_1, \ldots, W_r$ non-zero vector spaces of finite dimension over $\overline{\mathbf{Q}}_\ell$, $\rho : \Gamma \longrightarrow \mathrm{Aut}_{\overline{\mathbf{Q}}_\ell}(V)$ and $\rho_i : \Gamma \longrightarrow \mathrm{Aut}_{\overline{\mathbf{Q}}_\ell}(W_i)$ representations with Lie algebras $\mathfrak{g}_i = \mathrm{Lie}(\rho_i(\Gamma))$, $\mathfrak{g} = \mathrm{Lie}(\rho(\Gamma))$, $\overline{\mathfrak{g}}_i = \mathfrak{g}_i \otimes_{\mathbf{Q}_\ell} \overline{\mathbf{Q}}_\ell$, $\overline{\mathfrak{g}} = \mathfrak{g} \otimes_{\mathbf{Q}_\ell} \overline{\mathbf{Q}}_\ell$. If the following three conditions hold, then the representation $\rho = \rho^{\mathrm{ss}}$ is semisimple.*
(A) *Each ρ_i is strongly irreducible ($\Longrightarrow$ each $\overline{\mathfrak{g}}_i$ is a reductive $\overline{\mathbf{Q}}_\ell$-Lie algebra and each element of its centre acts on W_i by a scalar).*
(B) *For each $i = 1, \ldots, r$, every ($\Longleftrightarrow$ some) Cartan subalgebra $\overline{\mathfrak{h}}_i$ of $\overline{\mathfrak{g}}_i$ acts on W_i without multiplicities (i.e., all weight spaces of $\overline{\mathfrak{h}}_i$ on W_i are one-dimensional).*
(C) *There exists an open subgroup $\Gamma' \subset \Gamma$ and a dense subset $\Sigma \subset \Gamma'$ such that for each $g \in \Sigma$ there exists a finite dimensional vector space over $\overline{\mathbf{Q}}_\ell$ (depending on g) $V(g) \supset V$ and elements $u_1, \ldots, u_r \in \mathrm{Aut}_{\overline{\mathbf{Q}}_\ell}(V(g))$ such that $u_i u_j = u_j u_i$, $P_{\rho_i(g)}(u_i) = 0$ for all $i, j = 1, \ldots, r$, V is stable under $u_1 \cdots u_r$ and $u_1 \cdots u_r|_V = \rho(g)$.*

Proof. We can replace Γ by Γ' and assume that $\Gamma' = \Gamma$. We are going to apply Theorem 1.4 to the $\overline{\mathbf{Q}}_\ell$-Lie algebra $\overline{\mathbf{Q}}_\ell \cdot \mathfrak{g} \subset \mathrm{End}_{\overline{\mathbf{Q}}_\ell}(V)$. According to [N, Prop. 3.6], assumptions (A) and (B) imply that there exists an open subgroup $\Gamma_0 \subset \Gamma$ and an open dense subset $U_0 \subset \Gamma_0$ such that the polynomials $P_{\rho_i(g)}(X)$ do not have multiple roots, for any $g \in U_0$ and $i = 1, \ldots, r$. The intersection $\Sigma \cap U_0$ is then dense in Γ_0 and for each $g \in \Sigma \cap U_0$ the mutually commuting elements $u_i \in \mathrm{End}_{\overline{\mathbf{Q}}_\ell}(V(g))$ are semisimple, hence so is $\rho(g) = u_1 \cdots u_r|_V$. Therefore (H1-ZAR) is satisfied for $\overline{\mathbf{Q}}_\ell \cdot \mathfrak{g}$.

We are now going to check (H2). After replacing Γ by an open subgroup we can assume that the filtration $\{V_i\}$ is Γ-stable; then $\mathrm{gr}(V) = \rho^{\mathrm{ss}}$ and $\mathfrak{m} = \overline{\mathbf{Q}}_\ell \cdot \mathrm{Lie}(\rho^{\mathrm{ss}}(\Gamma))$.

Step 1: It follows from (C) that

$$P_{(\rho_1 \otimes \cdots \otimes \rho_r)(g)}(\rho(g)) = 0 \in \mathrm{End}_{\overline{\mathbf{Q}}_\ell}(V) \tag{1.7.1}$$

holds for all $g \in \Sigma$ (cf. [N, Thm. 3.12(1)]), hence for all $g \in \Gamma$, by continuity. Consider the representation $\rho_0 = \rho_1 \oplus \cdots \oplus \rho_r : \Gamma \longrightarrow \mathrm{Aut}_{\overline{\mathbf{Q}}_\ell}(W_0)$, where $W_0 = W_1 \oplus \cdots \oplus W_r$. The formula (1.7.1) implies that $(\rho(g) - 1)^N = 0$ for all $g \in \mathrm{Ker}(\rho_0)$, where $N = \dim(W_1 \otimes \cdots \otimes W_r)$. It follows that the Lie ideal $\mathfrak{a} = \overline{\mathbf{Q}}_\ell \cdot \mathrm{Lie}(\rho^{\mathrm{ss}}(\mathrm{Ker}(\rho_0)))$ in $\mathfrak{m} = \overline{\mathbf{Q}}_\ell \cdot \mathfrak{g}/\mathfrak{n}_V(\overline{\mathbf{Q}}_\ell \cdot \mathfrak{g}) = \overline{\mathbf{Q}}_\ell \cdot \mathrm{Lie}(\rho^{\mathrm{ss}}(\Gamma)) \subset \mathrm{End}_{\overline{\mathbf{Q}}_\ell}(\mathrm{gr}(V))$ consists of nilpotent elements, hence is a nilpotent Lie ideal [LIE, Ch. I, §4, no. 2, Cor. 3], and so $\mathfrak{a} = 0$, since $\mathfrak{m}$ is reductive. Therefore $\rho^{\mathrm{ss}}(\mathrm{Ker}(\rho_0))$ is a finite subgroup of $\mathrm{Aut}_{\overline{\mathbf{Q}}_\ell}(\mathrm{gr}(V))$, hence $\mathrm{Ker}(\rho_0) \cap \mathrm{Ker}(\rho^{\mathrm{ss}})$ is an open subgroup of $\mathrm{Ker}(\rho_0)$. The surjection $\Gamma/(\mathrm{Ker}(\rho_0) \cap \mathrm{Ker}(\rho^{\mathrm{ss}})) \twoheadrightarrow \Gamma/\mathrm{Ker}(\rho^{\mathrm{ss}})$ yields canonical surjections $f : \mathfrak{g}_0 = \mathrm{Lie}(\rho_0(\Gamma)) \twoheadrightarrow \mathrm{Lie}(\rho^{\mathrm{ss}}(\Gamma))$ and $\overline{f} : \overline{\mathfrak{g}}_0 = \mathfrak{g}_0 \otimes_{\mathbf{Q}_\ell} \overline{\mathbf{Q}}_\ell \twoheadrightarrow \mathfrak{m}$. In particular, $\mathfrak{g}_0$ acts on $\mathrm{gr}(V)$ via $\overline{f}$.

Step 2: After replacing Γ by an open subgroup we can assume that we can apply the ℓ-adic logarithm to $\rho_i(g)$ and $\rho(g)$, for all $g \in \Gamma$. The formula (1.7.1) then implies that

$$\forall X = (X_1, \ldots, X_r) \in \overline{\mathfrak{g}}_0 \subset \overline{\mathfrak{g}}_1 \times \cdots \times \overline{\mathfrak{g}}_r \qquad P_{(X_1, \ldots, X_r)|W_1 \otimes \cdots \otimes W_r}(\overline{f}(X)) = 0 \in \mathrm{End}_{\overline{\mathbf{Q}}_\ell}(\mathrm{gr}(V))$$

(above, X acts on $W_1 \otimes \cdots \otimes W_r$ by $\sum_i 1 \otimes \cdots \otimes 1 \otimes X_i \otimes 1 \otimes \cdots \otimes 1$). Restricting this formula to a Cartan subalgebra $\overline{\mathfrak{h}}_0$ of the reductive Lie algebra $\overline{\mathfrak{g}}_0$ we see that each weight of $\overline{\mathfrak{h}}_0$ occurring in $\mathrm{gr}(V)$ must occur in $W_1 \otimes \cdots \otimes W_r$.

Step 3: Recall that $\overline{\mathfrak{g}}_0$ projects surjectively on each $\overline{\mathfrak{g}}_1, \ldots, \overline{\mathfrak{g}}_r$ and that each W_i is a simple $\overline{\mathfrak{g}}_i$-module. This implies, by [N, Prop. 2.2], that all weights of $\overline{\mathfrak{h}}_0$ occurring in $W_1 \otimes \cdots \otimes W_r$ lie in one coset modulo the root lattice of $(\overline{\mathfrak{g}}_0, \overline{\mathfrak{h}}_0)$. By Step 2, the same holds for the weights of $\overline{\mathfrak{h}}_0$ occurring in $\mathrm{gr}(V)$. Writing $\overline{\mathfrak{g}}_0 = \mathfrak{m} \times \mathrm{Ker}(\overline{f})$ and choosing $\overline{\mathfrak{h}}_0$ of the form $\mathfrak{h} \times \mathfrak{h}'$, where $\mathfrak{h} \subset \mathfrak{m}$ and $\mathfrak{h}' \subset \mathrm{Ker}(\overline{f})$ are Cartan subalgebras, we deduce that all weights of $\mathfrak{h}$ occurring in $\mathrm{gr}(V)$ lie in one coset modulo the root lattice of $(\mathfrak{m}, \mathfrak{h})$, which is precisely (H2).

Having checked (H1-ZAR) and (H2) for the linear Lie algebra $\overline{\mathbf{Q}}_\ell \cdot \mathfrak{g} \subset \mathrm{End}_{\overline{\mathbf{Q}}_\ell}(V)$, we can apply Theorem 1.4, which tells us that $\overline{\mathbf{Q}}_\ell \cdot \mathfrak{g} = \mathfrak{m}$ is reductive and V is a semisimple $\overline{\mathbf{Q}}_\ell \cdot \mathfrak{g}$-module, which is equivalent to ρ being semisimple. Theorem is proved.

2. Unitary Shimura varieties

(2.1) Let $(B, *, V, \langle\ ,\ \rangle)$ be a simple PEL datum of type (A): B is a finite-dimensional simple $\mathbf{Q}$-algebra, $*$ is a positive involution of second kind on B, V is a non-zero left B-module of finite type and $\langle\ ,\ \rangle : V \times V \longrightarrow \mathbf{Q}$ is a non-degenerate alternating $\mathbf{Q}$-bilinear form such that $\langle bv, v'\rangle = \langle v, b^* v'\rangle$ for all $b \in B$ and $v, v' \in V$.

The centre $Z(B) = F_c$ is then a CM field, with maximal real subfield $F = F_c^{*=\mathrm{id}}$ of degree $r = [F : \mathbf{Q}]$. There is a unique (non-degenerate) alternating F-bilinear form $\langle\ ,\ \rangle_F : V \times V \longrightarrow F$ such that $\langle\ ,\ \rangle = \mathrm{Tr}_{F/\mathbf{Q}} \circ \langle\ ,\ \rangle_F$. The centraliser $C = \mathrm{End}_B(V)$ is again a simple $\mathbf{Q}$-algebra with centre F_c; let $\# : C \longrightarrow C$ be the F-linear involution given by $\langle cv, v'\rangle = \langle v, c^{\#} v'\rangle$.

(2.2) These data define the following algebraic groups and morphisms: $H = GSp_B(V, \langle\ ,\ \rangle_F) \xrightarrow{\nu} \mathbf{G}_{m,F}$ (over F), its restriction of scalars $G = R_{F/\mathbf{Q}}(H)$ and the closed subgroup $G^* \subset G$ defined by a cartesian diagram of connected reductive groups

$$\begin{tikzcd} G^* \arrow[r] \arrow[d, "\nu"] & G \arrow[d, "\nu"] \\ \mathbf{G}_{m,\mathbf{Q}} \arrow[r] & R_{F/\mathbf{Q}}(\mathbf{G}_{m,F}). \end{tikzcd}$$

In concrete terms

$$\begin{aligned} H(S) &= \{h \in GL_B(V \otimes_F S) \mid \exists\, \nu(h) \in S^\times\ \forall v, v' \in V\ \langle hv, hv'\rangle_F = \nu(h)\langle v, v'\rangle_F\} \\ &= \{h \in (C \otimes_F S)^\times \mid hh^{\#} = \nu(h) \in S^\times\}, \\ G^*(R) &= \{g \in (C \otimes R)^\times \mid gg^{\#} = \nu(h) \in R^\times\}, \end{aligned}$$

for all F-algebras S and $\mathbf{Q}$-algebras R.

(2.3) If S is an F_c-algebra, then $C \otimes_F S \xrightarrow{\sim} (C \otimes_{F_c} S) \times (C \otimes_{F_c,*} S)$ and $\# \otimes \mathrm{id}$ interchanges the two factors, which gives an isomorphism [HT, p. 47]

$$\alpha : H(S) = \{h = (u, \lambda(u^{\#})^{-1}) \mid u \in (C \otimes_{F_c} S)^\times,\ \lambda = \nu(h) \in S^\times\} \xrightarrow{\sim} (C \otimes_{F_c} S)^\times \times S^\times, \qquad \alpha(h) = (u, \lambda).$$

If $S = S^+ \otimes_F F_c$ for an F-algebra S^+, then the complex conjugation $* = \mathrm{id}_{S^+} \otimes * : S \longrightarrow S$ acts on $(C \otimes_F S^+)^\times \times (S^+ \otimes_F F_c)^\times$ by $\alpha \circ (\mathrm{id}_{S^+} \otimes *) \circ \alpha^{-1} : (A, b) \mapsto ((b^*)^{-1} A^{\#}, b^*)$.

(2.4) There exists a morphism of $\mathbf{R}$-algebras $h : \mathbf{C} \longrightarrow C \otimes \mathbf{R}$, unique up to conjugation by an element $c \in (C \otimes \mathbf{R})^\times$ with $cc^{\#} = 1$ [K, Lemma 4.3], such that $h(\overline{z}) = h(z)^{\#}$ and for which the symmetric $\mathbf{R}$-bilinear form $\langle v, h(i)v'\rangle : V_{\mathbf{R}} \times V_{\mathbf{R}} \longrightarrow \mathbf{R}$ is positive definite. This gives rise to a Shimura datum $(G^*, \mathscr{X}^*)$ (resp. $(G, \mathscr{X})$), where $\mathscr{X}^*$ (resp. $\mathscr{X}$) is the conjugacy class of $h : \mathbf{S} \longrightarrow G^*_{\mathbf{R}}$ under $G^*(\mathbf{R})$ (resp. under $G(\mathbf{R})$). The corresponding Shimura varieties $Sh(G^*, \mathscr{X}^*)$ and $Sh(G, \mathscr{X})$ have a common reflex field $E \subset F_c^{gal}$, where F_c^{gal} is the Galois closure of F_c in $\overline{\mathbf{Q}}$.

The action of $h(i)$ defines a complex structure on $V_{\mathbf{R}}$, hence a Hodge decomposition $V_{\mathbf{C}} = V^{-1,0} \oplus V^{0,-1}$ of weight -1, with $h(z) \otimes \mathrm{id}$ acting as z (resp. $\overline{z}$) on $V^{-1,0}$ (resp. on $V^{0,-1}$). The cocharacter $\mu_h : \mathbf{G}_{m,\mathbf{C}} \longrightarrow G_{\mathbf{C}}$ attached to h acts on $V_{\mathbf{C}}$ as follows: $\mu_h(z)$ acts as $z \cdot \mathrm{id}$ (resp. as id) on $V^{-1,0}$ (resp. on $V^{0,-1}$).

(2.5) Let $X = \{x : F \hookrightarrow \mathbf{R}\}$ be the set of infinite primes of F. Fix a CM type $\Phi = \{\sigma_x : F_c \hookrightarrow \mathbf{C}\}_{x \in X}$ $(\sigma_x|_F = x)$ of F_c. The choice of Φ induces an isomorphism

$$B \otimes \mathbf{R} = \prod_{x \in X} B \otimes_{F,x} \mathbf{R} = \prod_{x \in X} B \otimes_{F_c,\sigma_x} \mathbf{C} \xrightarrow{\sim} \prod_{x \in X} M_N(\mathbf{C})$$

under which $V^{-1,0} \xrightarrow{\sim} \bigoplus_{x \in X} \left((\mathbf{C}^N)^{a_x} \oplus (\overline{\mathbf{C}}^N)^{b_x}\right)$, where $a_x + b_x = n$ and $C \otimes \mathbf{R} \xrightarrow{\sim} \prod_{x \in X} M_n(\mathbf{C})$. The common dimension of the Shimura varieties $Sh_{K^*}(G^*, \mathscr{X}^*)$ and $Sh_K(G, \mathscr{X})$ is equal to $\dim = \sum_{x \in X} a_x b_x$.

The isomorphism α from 2.3 is functorial in S; it defines an isomorphism of algebraic groups over F_c

$$H \otimes_F F_c \xrightarrow{\sim} C^\times \times \mathbf{G}_{m,F_c},$$

hence an isomorphism $H \otimes_F k \xrightarrow{\sim} (C^\times)_k \times \mathbf{G}_{m,k}$, for any field embedding $F_c \hookrightarrow k$. In particular, Φ gives rise to an isomorphism

$$G_{\mathbf{C}} = \prod_{x \in X} (H \otimes_F F_c) \otimes_{F_c, \sigma_x} \mathbf{C} \xrightarrow{\sim} \prod_{x \in X} (GL(n)_{\mathbf{C}} \times \mathbf{G}_{m,\mathbf{C}}) = (GL(n)_{\mathbf{C}} \times \mathbf{G}_{m,\mathbf{C}})^X \tag{2.5.1}$$

under which $G^*_{\mathbf{C}}$ corresponds to the subgroup $GL(n)^X_{\mathbf{C}} \times \mathbf{G}_{m,\mathbf{C}}$ (with $\mathbf{G}_{m,\mathbf{C}}$ embedded diagonally into $\mathbf{G}^X_{m,\mathbf{C}}$) and μ_h is given by $\mu_h(z) = ((zI_{a_x}, I_{b_x})_{x \in X}, z)$.

(2.6) Assume that ξ (resp. ξ^*) is an irreducible algebraic representation of $G_{\mathbf{C}}$ (resp. of $G^*_{\mathbf{C}}$). Under (2.5.1),

$$\xi = \bigotimes_{x \in X} \xi_x, \quad \xi_x = \eta_x \otimes \xi_{0,x}, \qquad \xi^* = \Big(\bigotimes_{x \in X} \eta^*_x\Big) \otimes \xi^*_0,$$

where η_x (resp. η^*_x) is an irreducible algebraic representation of $GL(n)_{\mathbf{C}}$ with highest weight (with respect to the upper Borel subgroup) equal to $diag(t_1, \ldots, t_n) \mapsto \prod t_i^{m_{i,x}}$, $m_{1,x} \geq \cdots \geq m_{n,x}$ (resp. $\prod t_i^{m^*_{i,x}}$, $m^*_{1,x} \geq \cdots \geq m^*_{n,x}$) and $\xi_{0,x}, \xi^*_0 : \mathbf{G}_{m,\mathbf{C}} \longrightarrow \mathbf{G}_{m,\mathbf{C}}$ given by $t \mapsto t^{m_{0,x}}$ and $t \mapsto t^{m^*_0}$, respectively.

The weight morphism attached to h is defined as

$$wt : \mathbf{G}_{m,\mathbf{R}} \longrightarrow \mathbf{S} \longrightarrow G^*_{\mathbf{R}}, \qquad t \mapsto h(t^{-1}),$$

which means that

$$\xi^*(wt(t)) = t^{w^*}, \quad w^* = -\sum_{x \in X} \sum_{i=1}^n m^*_{i,x} - 2m^*_0, \quad \xi(wt(t)) = t^{\sum w_x}, \quad w_x = -\sum_{i=1}^n m_{i,x} - 2m_{0,x}.$$

For $j \in \mathbf{Z}$, the Tate twist $\xi^*(j)$ is defined as follows: the components η^*_x do not change, but m^*_0 is replaced by $m^*_0 + j$.

(2.7) Proposition. (1) *The respective centres of H and G are equal to $Z_H = R_{F_c/F}(\mathbf{G}_{m,F_c})$ and $Z_G = {}_{F_c}T \supset Z^+_G = {}_FT$, where the map $(Z^+_G)_{\mathbf{C}} \hookrightarrow G_{\mathbf{C}}$ is given by*

$$(\mathbf{G}_{m,\mathbf{C}} \hookrightarrow GL(n)_{\mathbf{C}} \times \mathbf{G}_{m,\mathbf{C}})^X, \qquad \forall x \in X \;\; t \mapsto (t \cdot I_n, t^2).$$

(2) *Given ξ as in 2.6, the following two conditions are equivalent: $\xi(Z_G(\mathbf{Q}) \cap K) = \xi(F_c^\times \cap K) = \{1\}$ for every neat open compact subgroup $K \subset G(\widehat{\mathbf{Q}}) = H(\widehat{F})$ $\iff$ the integers $w_x = w_x(\xi)$ do not depend on $x \in X$.*

(3) *Given ξ^* as in 2.6, the following two conditions are equivalent: there exist $j \in \mathbf{Z}$ and ξ satisfying (2) such that $\xi^*(j) = \xi|_{G^*_{\mathbf{C}}}$ $\iff$ $\sum_{i=1}^n m^*_{i,x} \pmod 2 \in \mathbf{Z}/2\mathbf{Z}$ does not depend on $x \in X$.*

Proof. (1) This follows from the definitions. (2) If K is neat, then $\Delta = F_c^\times \cap K$ is a subgroup of finite index in $O^\times_{F,+}$ and $\xi^*(\beta) = \prod_{x \in X} x(\beta)^{w_x}$ for $\beta \in \Delta$, which identically vanishes on Δ $\iff$ all w_x are equal to each other. (3) If j and ξ exist, then $m_{i,x} = m^*_{i,x}$ for all $x \in X$ and $1 \leq i \leq n$. We need to choose $j \in \mathbf{Z}$ and $m_{0,x}$ so that $\sum_{i=1}^n m^*_{i,x} + 2m_{0,x}$ do not depend on $x \in X$ and $\sum_{i=1}^n m_{0,x} = m^*_0 + j$, which is possible $\iff$ $\sum_{i=1}^n m^*_{i,x} \pmod 2 \in \mathbf{Z}/2\mathbf{Z}$ does not depend on $x \in X$.

(2.8) Any $\xi^* : G^*_{\mathbf{C}} \longrightarrow GL(N^*)_{\mathbf{C}}$ from 2.6 (resp. any $\xi : G_{\mathbf{C}} \longrightarrow GL(N)_{\mathbf{C}}$ satisfying the equivalent conditions from Proposition 2.7(2)) gives rise, for small enough open compact subgroups $K^* \subset G^*(\widehat{\mathbf{Q}})$ (resp. $K \subset G(\widehat{\mathbf{Q}})$), to a locally constant sheaf of complex vector spaces $\mathscr{L}_{\xi^*}$ (resp. $\mathscr{L}_\xi$) on $Sh^{an}_{K^*} = G^*(\mathbf{Q}) \backslash (\mathscr{X}^* \times G^*(\widehat{\mathbf{Q}})/K)$ (resp. on $Sh^{an}_K = G(\mathbf{Q}) \backslash (\mathscr{X} \times G(\widehat{\mathbf{Q}})/K)$). Denote by $j : Sh_{K^*} \hookrightarrow Sh_{K^*,BB}$ (resp. $j : Sh_K \hookrightarrow Sh_{K,BB}$) the Baily-Borel compactification. The respective analytic intersection cohomology groups

$$H^i(Sh^{an}_{*,BB}, j_{!*}\mathscr{L}_{\xi^*}) = \varinjlim_{K^*} H^i(Sh^{an}_{K^*,BB}, j_{!*}\mathscr{L}_{\xi^*}), \qquad H^i(Sh^{an}_{BB}, j_{!*}\mathscr{L}_{\xi}) = \varinjlim_{K} H^i(Sh^{an}_{K,BB}, j_{!*}\mathscr{L}_{\xi})$$

have an automorphic description which generalises Matsushima's formula [BW, Thm. VII.5.2] in the compact case, thanks to [BC, Th. A] combined with [BG, Prop. 5.6] and the proof of Zucker's conjecture [Lo], [SS]:

$$H^i(Sh^{an}_{BB}, j_{!*}\mathscr{L}_{\xi}) = \bigoplus_{\pi\in L^2_{disc}(G)} m_{disc}(\pi)\, H^i(\mathfrak{g}, K_\infty; \pi_\infty\otimes\xi)\otimes\pi^\infty \tag{2.8.1}$$

(and similarly for G^*), where $\pi = \pi_\infty\otimes\pi^\infty$ runs through the discrete automorphic representations of $G(\mathbf{A}) = G(\mathbf{R})\times G(\widehat{\mathbf{Q}})$ whose infinite component π_∞ is cohomological in degree i with respect to ξ ($H^i(\mathfrak{g}, K_\infty; \pi_\infty\otimes\xi)\neq 0$, where $\mathfrak{g} = \mathrm{Lie}(G(\mathbf{R}))$ and $K_\infty\subset G(\mathbf{R})$ is the stabiliser of h) and $m_{disc}(\pi)$ denotes the multiplicity of π in the discrete part of the automorphic spectrum of G.

The restriction of the canonical map

$$H^i(Sh^{an}_{BB}, j_{!*}\mathscr{L}_{\xi}) \longrightarrow H^i(Sh^{an}, \mathscr{L}_{\xi}) = \varinjlim_{K} H^i(Sh^{an}_{K}, \mathscr{L}_{\xi})$$

to the cuspidal subspace

$$\bigoplus_{\pi\in L^2_{cusp}(G)} m_{cusp}(\pi)\, H^i(\mathfrak{g}, K_\infty; \pi_\infty\otimes\xi)\otimes\pi^\infty$$

of (2.8.1) is injective, by [B, Cor. 5.5].

(2.9) Fix a prime number ℓ and an isomorphism $\mathbf{C}\xrightarrow{\sim}\overline{\mathbf{Q}}_\ell$. For any algebraic object $(-)$ defined over a subfield of $\mathbf{C}$ we denote by $(-)_\ell$ its base change to $\overline{\mathbf{Q}}_\ell$.

The algebraic representation $\xi_\ell : G_{\overline{\mathbf{Q}}_\ell}\longrightarrow GL(N)_{\overline{\mathbf{Q}}_\ell}$ defines, for small enough K, a lisse $\overline{\mathbf{Q}}_\ell$-sheaf $\mathscr{L}_{\xi,\ell}$ on $Sh_K(G,\mathscr{X})_{et}$. The decomposition (2.8.1) yields, thanks to the comparison isomorphism between Betti and ℓ-adic cohomology, a $G(\widehat{\mathbf{Q}})$-equivariant isomorphism

$$H^i_{BB} = H^i_{et}(Sh_{BB}\otimes_E\overline{\mathbf{Q}}, j_{!*}\mathscr{L}_{\xi,\ell}) = \varinjlim_{K} H^i_{et}(Sh_{K,BB}\otimes_E\overline{\mathbf{Q}}, j_{!*}\mathscr{L}_{\xi,\ell}) = \bigoplus_{\pi^\infty} V^i(\pi^\infty)\otimes\pi^\infty, \tag{2.9.1}$$

where π^∞ is an irreducible smooth representation of $G(\widehat{\mathbf{Q}})$ over $\overline{\mathbf{Q}}_\ell$ and $V^i(\pi^\infty) = \mathrm{Hom}_{G(\widehat{\mathbf{Q}})}(\pi^\infty, H^i_{BB})$ is a finite-dimensional $\overline{\mathbf{Q}}_\ell$-vector space. The action of the Galois group Γ_E on étale cohomology defines an ℓ-adic representation

$$\Gamma_E\longrightarrow \mathrm{Aut}_{G(\widehat{\mathbf{Q}})}(V^i(\pi^\infty)\otimes\pi^\infty) = \mathrm{Aut}_{\overline{\mathbf{Q}}_\ell}(V^i(\pi^\infty)),$$

under which (2.9.1) becomes $\Gamma_E\times G(\widehat{\mathbf{Q}})$-equivariant. The only π^∞ contributing to (2.9.1) are those for which there exists π_∞ cohomological in degree i with respect to ξ such that $\pi = \pi_\infty\otimes\pi^\infty$ is a discrete automorphic representation of $G(\mathbf{A})$, since

$$\dim V^i(\pi^\infty) = \sum_{\pi=\pi_\infty\otimes\pi^\infty} m_{disc}(\pi)\,\dim H^i(\mathfrak{g}, K_\infty; \pi_\infty\otimes\xi).$$

Similarly, there is a decomposition

$$H^i_{*,BB} = H^i_{et}(Sh_{*,BB}\otimes_E\overline{\mathbf{Q}}, j_{!*}\mathscr{L}_{\xi^*,\ell}) = \varinjlim_{K^*} H^i_{et}(Sh_{K^*,BB}\otimes_E\overline{\mathbf{Q}}, j_{!*}\mathscr{L}_{\xi^*,\ell}) = \bigoplus_{(\pi^*)^\infty} V^i((\pi^*)^\infty)\otimes(\pi^*)^\infty, \tag{2.9.2}$$

$\pi^* = \pi^*_\infty\otimes(\pi^*)^\infty\in L^2_{disc}(G^*)$.

(2.10) Chevalley's theorem on units [C] implies [D, Cor. 2.0.12] (see also [TX, Lemma 2.5] and [N, Prop. 5.7]) that the canonical map $i : Sh(G^*, \mathscr{X}^*) \longrightarrow Sh(G, \mathscr{X})$ induces an isomorphism between the topological connected components containing $(h, 1)$:

$$Sh(G^*, \mathscr{X}^*)^{an,+} = Sh(G, \mathscr{X})^{an,+}. \tag{2.10.1}$$

It follows that there is a $G(\widehat{\mathbf{Q}})$-equivariant isomorphism $Sh(G^*, \mathscr{X}^*) \times^U G(\widehat{\mathbf{Q}}) \simeq Sh(G, \mathscr{X})$, where $U \subset G(\widehat{\mathbf{Q}})$ is the stabiliser of the subset $Sh(G^*, \mathscr{X}^*) \subset Sh(G, \mathscr{X})$, which yields an isomorphism of $\Gamma_E \times G(\widehat{\mathbf{Q}})$-modules

$$H^i_{BB} \simeq \mathrm{Ind}_U^{G(\widehat{\mathbf{Q}})} H^i_{*,BB} \tag{2.10.2}$$

(with smooth induction on the R.H.S.), provided that $\xi^* = \xi|_{G^*_{\mathbf{C}}}$.

(2.11) From now on until the end of §2 we assume that ξ^* is as in 2.6 and π^* is an automorphic representation of $G^*(\mathbf{A})$ for which π^*_∞ is cohomological in degree i with respect to ξ^*.

Recall that the fixed CM type Φ of F_c gives rise to identifications $F_c \otimes \mathbf{R} \simeq \mathbf{C}^X$ and

$$Z(\mathbf{R}) = (F_c \otimes \mathbf{R})^\times \simeq (\mathbf{C}^\times)^X, \qquad Z^*(\mathbf{R}) \simeq \{z = (z_x) \in (\mathbf{C}^\times)^X \mid \lambda = z_x \overline{z_x} \in \mathbf{R} \text{ is independent of } x \in X\}.$$

The infinitesimal characters of π^*_∞ and $(\xi^*)^\vee$ coincide. In particular, the infinity type of the central character ω_{π^*} of π^* is equal to

$$\omega_{\pi^*_\infty} = \omega^{-1}_{\xi^*}|_{Z^*(\mathbf{R})} : z = (z_x) \mapsto \lambda^u \prod_{x \in X} z_x^{u_x} \qquad (\lambda = z_x\overline{z_x}), \qquad u = -m_0^* \in \mathbf{Z}, \qquad u_x = -\sum_{i=1}^n m^*_{i,x} \in \mathbf{Z}. \tag{2.11.1}$$

There is a unique character $Z(\mathbf{R}) \simeq (\mathbf{C}^\times)^X \longrightarrow \mathbf{C}^\times$ trivial on $O^\times_{F,+}$ and extending $\omega_{\pi^*_\infty}$, namely,

$$z = (z_x) \mapsto \prod_{x \in X} z_x^{u_x} |z_x|^{t - u_x}, \qquad t = (2u + \sum_{x \in X} u_x)/r.$$

This character is of the form

$$z = (z_x) \mapsto \prod_{x \in X} z_x^{c_x} \overline{z_x}^{d_x} |z_x|^s \qquad (c_x, d_x \in \mathbf{Z}, s \in \mathbf{R})$$

(is "algebraic up to a twist") $\iff$ $u_x \pmod 2 \in \mathbf{Z}/2\mathbf{Z}$ does not depend on $x \in X$ $\iff$ ξ^* is as in Proposition 2.7(3).

(2.12) Assume, in addition, that π^* is **cuspidal**. Every (cuspidal) automorphic form on $G^*(\mathbf{A})$ extends to a (cuspidal) automorphic form on $G(\mathbf{A})$ (cf. [LSc, Prop. 3.5]), which implies that there exists a cuspidal automorphic representation π of $G(\mathbf{A})$ such that π^* is a quotient of $\pi|_{G^*(\mathbf{A})}$.

(2.13) Automorphic representations of $G(\mathbf{A}) = H(\mathbf{A}_F)$ are closely related to suitably self-dual automorphic representations (Π, ψ) of $H_0(\mathbf{A}_{F_c}) = G_0(\mathbf{A})$, where $H_0 = GL(n)_{F_c} \times GL(1)_{F_c}$ and $G_0 = R_{F_c/\mathbf{Q}}(H_0)$. For unitary groups of a special kind, very precise results in this direction were proved in [Cl1], [Cl2], [ClL], [ClHL], [HL], [HT], [L1], [L2], [Mr] and [Sh]. Recently, endoscopic transfer of discrete automorphic representations of the unitary group $\mathrm{Ker}(\nu)(\mathbf{A}_F)$ to automorphic representations of $GL(n, \mathbf{A}_{F_c})$ has been completely described in the quasi-split case [Mk] and in full generality in [KMSW] (with some of the proofs relegated to subsequent articles). As in [HT, Thm. VI.2.1], this yields a transfer of discrete automorphic representations of $G(\mathbf{A})$ to $G_0(\mathbf{A})$. However, our main result in the form 2.20 and 2.25 below requires only a fairly weak form of this transfer, as formulated in Definition 2.14 below.

The discussion in 2.3 together with (2.5.1) give an isomorphism

$$(G_0)_{\mathbf{C}} \xrightarrow{\sim} \prod_{x \in X} ((GL(n)_{F_c} \times \mathbf{G}_{m,F_c}) \otimes_{F_c,\sigma_x} \mathbf{C} \times (GL(n)_{F_c} \times \mathbf{G}_{m,F_c}) \otimes_{F_c, c\sigma_x} \mathbf{C}),$$

where c denotes the complex conjugation.

(2.14) Definition (cf. [HT, Thm. VI.2.1]). *Let π be as in 2.12. We say that an automorphic representation (Π, ψ) of $GL_n(\mathbf{A}_{F_c}) \times \mathbf{A}_{F_c}^\times$ is a weak transfer of π if the following conditions are satisfied.*
(1) $\Pi^\vee \simeq \Pi^c$;
(2) the restriction of Π_∞ to $SL_n(\mathbf{C})^X \subset (GL_n(\mathbf{C}) \times \mathbf{C}^\times)^X = G_0(\mathbf{R})$ is cohomological with respect to the restriction of $\xi^* \otimes (\xi^*)^c$ to $(SL(n)_\mathbf{C} \times SL(n)_\mathbf{C})^X \subset (G_0)_\mathbf{C}$;
(3) the central characters are related as follows: $\omega_\pi = \psi^c$, $\omega_\Pi = \psi^c/\psi$;
(4) for all but finitely many finite primes v of F at which F_c/F and C are split [(1)] *($v = ww^c$ in F_c), the local representation π_v of $H(F_v) = H((F_c)_w)$ is isomorphic, via $\alpha : H((F_c)_w) \xrightarrow{\sim} (C \otimes_{F_c} (F_c)_w)^\times \times ((F_c)_w)^\times = H_0((F_c)_w)$, to the local representation $(\Pi, \psi)_w$.*

(2.15) Condition 2.14(2) implies that Π_∞ is cohomological with respect to a certain algebraic representation of $(GL(n)_\mathbf{C} \times GL(n)_\mathbf{C})^X$, since the infinity type

$$\omega_{\Pi_\infty} : (F_c \otimes \mathbf{R})^\times \simeq (\mathbf{C}^\times)^X \longrightarrow \mathbf{C}^\times, \qquad z = (z_x) \mapsto \omega_\pi(z)/\omega(\overline{z}) = \prod_{x \in X} (z_x/\overline{z_x})^{u_x}$$

of the central character of Π is algebraic, thanks to 2.14(3) combined with (2.11.1).

According to the discussion in 2.11, the Hecke character $\psi \| \cdot \|_{F_c}^{-s} : \mathbf{A}_{F_c}^\times / F_c^\times \longrightarrow \mathbf{C}^\times$ is algebraic for some $s \in \mathbf{R} \iff \xi^*$ is as in Proposition 2.7(3).

(2.16) The key assumption. Form now on until the end of §2 we assume that the representation π from 2.12 admits a weak transfer (Π, ψ) as in 2.14, with Π a **cuspidal** automorphic representation of $GL_n(\mathbf{A}_{F_c})$.

This cuspidality condition together with 2.14(1) and 2.15 imply that there exists an ℓ-adic Galois representation

$$\rho_{\Pi,\ell} : \Gamma_{F_c} \longrightarrow GL_n(\overline{\mathbf{Q}}_\ell),$$

which is suitably self-dual ($\rho_{\Pi,\ell}^\vee \simeq \rho_{\Pi,\ell}^c(n-1)$, but we are not going to use this fact) and is compatible with Π in a very strong sense [CH, Thm. 3.2.3], [BLGGT1, Thm. 2.1.1], [BLGGT2, Thm. A]. These results are an outcome of a collective effort of many mathematicians (Clozel, Harris, Taylor, Labesse, Shin,...). We need only the following properties of $\rho_{\Pi,\ell}$.

(2.16.1) $\rho_{\Pi,\ell}$ is semisimple [(2)];

(2.16.2) for all but finitely many finite primes $w \nmid \ell$ of F_c there is an equality in $\mathbf{C}[(Nw)^{-s}] \simeq \overline{\mathbf{Q}}_\ell[(Nw)^{-s}]$

$$L_w(\Pi, \mathrm{Std}_n, s)^{-1} = L_w(\rho_{\Pi,\ell}, s + (n-1)/2)^{-1},$$

where Std_n denotes the standard representation of $GL(n)$. In particular, $\rho_{\Pi,\ell}$ is unramified outside a finite set of primes of F_c;

(2.16.3) for each embedding $\tau : F_c \hookrightarrow \overline{\mathbf{Q}}_\ell$, the restriction of $\rho_{\Pi,\ell}$ to $\Gamma_{(F_c)_\tau} = \mathrm{Gal}(\overline{\mathbf{Q}}_\ell/(F_c)_\tau)$ is a Hodge-Tate representation with n distinct Hodge-Tate weights.

(2.16.4) for all but finitely many finite primes $w \nmid \ell$ of F_c all eigenvalues of $\rho_{\Pi,\ell}(\mathrm{Fr}(w))$ in $\overline{\mathbf{Q}}_\ell \simeq \mathbf{C}$ have absolute value $(Nw)^{(n-1)/2}$.

(2.17) Proposition. *Let $\overline{\mathfrak{g}} = \overline{\mathbf{Q}}_\ell \cdot \mathrm{Lie}(\rho_{\Pi,\ell}(\Gamma_{F_c})) \subset gl(n, \overline{\mathbf{Q}}_\ell)$ be the $\overline{\mathbf{Q}}_\ell$-Lie algebra generated by the image of $\rho_{\Pi,\ell}$.*
(1) $\overline{\mathfrak{g}}$ is reductive and its centre acts semisimply on $\overline{\mathbf{Q}}_\ell^n$.
(2) $\overline{\mathbf{Q}}_\ell^n$ is a simple $\overline{\mathfrak{g}}$-module $\iff$ $\rho_{\Pi,\ell}$ is strongly irreducible.
(3) Any Cartan subalgebra $\overline{\mathfrak{h}} \subset \overline{\mathfrak{g}}$ acts on $\overline{\mathbf{Q}}_\ell^n$ by n distinct weights (each with multiplicity one).

Proof. The statement (1) follows from (2.16.1), while (2) is standard. It is enough to prove (3) for a single Cartan subalgebra after extending the scalars from $\overline{\mathbf{Q}}_\ell$ to $\mathbf{C}_\ell$. According to Sen's theory [Se, Thm. 1],

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(1) We are going to use this property only for primes w of degree one, i.e., such that $(F_c)_w = \mathbf{Q}_{p(w)}$, where $p(w)$ is the residue characteristic of w.

(2) In fact, $\rho_{\Pi,\ell}$ is expected to be irreducible. Hopefully, this will be established in a foreseeable future. At the time of writing, the irreducibility is known only if ℓ belongs to a set of primes of density one depending on Π [PT, Thm. D]. See also [CG] for results on irreducibility over F.

$\overline{\mathfrak{g}} \otimes_{\overline{\mathbf{Q}}_\ell} \mathbf{C}_\ell$ contains a semisimple element whose eigenvalues are the n distinct Hodge-Tate weights of the restriction of $\rho_{\Pi,\ell}$ to $\Gamma_{(F_c)_\tau}$ (for any $\tau : F_c \hookrightarrow \overline{\mathbf{Q}}_\ell$). This element, being semisimple, lies in some Cartan subalgebra of $\overline{\mathfrak{g}} \otimes_{\overline{\mathbf{Q}}_\ell} \mathbf{C}_\ell$, which then has the required property.

(2.18) Corollary. *Let $\overline{\mathfrak{h}} \subset \overline{\mathfrak{g}}$ be any Cartan subalgebra.*
(1) *$\overline{\mathfrak{h}}$ acts without multiplicities on $\bigwedge^i \overline{\mathbf{Q}}_\ell^n$ for $i = 1, n-1$.*
(2) *If $\overline{\mathfrak{g}} \supset sl(n, \overline{\mathbf{Q}}_\ell)$ (which is equivalent to $\overline{\mathfrak{g}} = gl(n, \overline{\mathbf{Q}}_\ell)$, since $\det(\rho_{\Pi,\ell}) : \Gamma_{F_c} \longrightarrow \mathbf{Q}_\ell^\times$ is a character of infinite order), then $\overline{\mathfrak{h}}$ acts without multiplicities on $\bigwedge^i \overline{\mathbf{Q}}_\ell^n$, for all $i = 1, \ldots, n-1$.*

(2.19) We are now ready to state the main result of this article.

(2.20) Theorem. *Let ξ^* be an irreducible algebraic representation of $G^*_{\mathbf{C}}$ and π^* a cuspidal automorphic representation of $G^*(\mathbf{A})$ such that π^*_∞ is cohomological with respect to ξ^* in degree $i = \dim = \sum_{x \in X} a_x b_x$. Assume that π^* is a quotient of the restriction to $G^*(\mathbf{A})$ of a cuspidal automorphic representation π of $G(\mathbf{A})$ which admits a weak transfer (in the sense of 2.14) to a* **cuspidal** *automorphic representation (Π, ψ) of $GL_n(\mathbf{A}_{F_c}) \times \mathbf{A}_{F_c}^\times$. If the ℓ-adic Galois representation $\rho_{\Pi,\ell} : \Gamma_{F_c} \longrightarrow GL_n(\overline{\mathbf{Q}}_\ell)$ attached to Π satisfies the following two conditions*
(1) *$\rho_{\Pi,\ell}$ is strongly irreducible;*
(2) *either (2a) $\overline{\mathbf{Q}}_\ell \cdot \mathrm{Lie}(\rho_{\Pi,\ell}(\Gamma_{F_c})) \supset sl(n, \overline{\mathbf{Q}}_\ell)$, or (2b) $G(\mathbf{R}) = \prod_{x \in X} GU(a_x, b_x)$ with $\forall x \in X \quad (a_x, b_x) \in \{(n, 0), (n-1, 1), (1, n-1), (0, n)\}$,*
then the $\overline{\mathbf{Q}}_\ell[\Gamma_E]$-module

$$\mathrm{Hom}_{G^*(\widehat{\mathbf{Q}})}((\pi^*)^\infty, H^{\dim}_{et}(Sh(G^*, \mathscr{X}^*) \otimes_E \overline{\mathbf{Q}}, \mathscr{L}_{\xi^*,\ell})) = \mathrm{Hom}_{G^*(\widehat{\mathbf{Q}})}((\pi^*)^\infty, H^{\dim}_*)$$

(which is finite-dimensional over $\overline{\mathbf{Q}}_\ell$) is semisimple.

(2.21) If we admit a variant of the main global result of [KMSW] for the unitary similitude group H, then the following Proposition allows us to deduce from Theorem 2.20 a semisimplicity result for the intersection cohomology.

(2.22) Proposition. *If ξ^*, π^*, π and (Π, ψ) are as in 2.6, 2.11, 2.12 and 2.16, respectively, then:*
(1) *If π'^* is a discrete automorphic representation of $G^*(\mathbf{A})$ such that $(\pi'^*)^\infty = (\pi^*)^\infty$, then π'^* is cuspidal.*
(2) *The map $V^i((\pi^*)^\infty) \otimes (\pi^*)^\infty \hookrightarrow H^i_{*,BB} \longrightarrow H^i_* = \varinjlim_{K^*} H^i_{et}(Sh_{K^*}(G^*, \mathscr{X}^*) \otimes_E \overline{\mathbf{Q}}, \mathscr{L}_{\xi^*,\ell})$ is injective.*
(3) *If $V^i((\pi^*)^\infty) \neq 0$, then $i = \dim = \sum_{x \in X} a_x b_x$.*

Proof. (1) By a variant of [KMSW, Thm. 1.7.1] for H (rather than for $\mathrm{Ker}(\nu)$), π and π' have the same transfer (Π, ψ) to $(GL(n) \times GL(1))_{F_c}$. Cuspidality of Π then implies cuspidality of π'.
(2) This is a consequence of (1) and [B, Cor. 5.5].
(3) Let $K^* \subset G^*(\widehat{\mathbf{Q}})$ be an open compact subgroup such that $((\pi^*)^\infty)^{K^*} \neq 0$. The sheaf $\mathscr{L}_{\xi^*,\ell}$ can be constructed using the universal abelian variety on the PEL Shimura variety $Sh_{K^*}(G^*, \mathscr{X}^*)$, as in [HT, III.2]. It is pure of weight $w^* = 2u + \sum_{x \in X} u_x$, in the notation of 2.6 and 2.11. This implies that, for all but finitely many finite primes $\widetilde{P}$ of F_c^{gal}, all eigenvalues in $\overline{\mathbf{Q}}_\ell \simeq \mathbf{C}$ of $\mathrm{Fr}(\widetilde{P}) \in \Gamma_{F_c^{gal}} \subset \Gamma_E$ acting on $V^i((\pi^*)^\infty) \otimes (\pi^*)^\infty$ have absolute value $(N\widetilde{P})^{(i+w^*)/2}$. On the other hand, if $\widetilde{P}$ is the restriction of a prime P_S as in (3.3.1) below, then $N\widetilde{P} = p$ and $i + w^* = \dim + w^*$, by (2) and Proposition 3.12(3) below (the reader is invited to check that our arguments are not circular).

(2.23) Corollary. *Under the assumptions 2.20(1) and 2.20(2), the $\overline{\mathbf{Q}}_\ell[\Gamma_E]$-module $V^{\dim}((\pi^*)^\infty)$ (which is isomorphic to its image in $\mathrm{Hom}_{G^*(\widehat{\mathbf{Q}})}((\pi^*)^\infty, H^{\dim}_*)$) is semisimple.*

(2.24) Similar results hold for the Shimura variety $Sh(G, \mathscr{X})$. In the compact case (when the derived group G^{der} is anisotropic) the following variant of Theorem 2.20 was proved in [F].

(2.25) Theorem. *Let ξ be an irreducible algebraic representation of $G_{\mathbf{C}}$ satisfying the conditions from Proposition 2.7(2) and π a cuspidal automorphic representation of $G(\mathbf{A})$ such that π_∞ is cohomological in degree $i = \dim$ with respect to ξ. If π admits a weak transfer to a* **cuspidal** *automorphic representation*

(Π, ψ) *of* $GL_n(\mathbf{A}_{F_c}) \times \mathbf{A}_{F_c}^\times$ *and if the* ℓ*-adic Galois representation* $\rho_{\Pi,\ell} : \Gamma_{F_c} \longrightarrow GL_n(\overline{\mathbf{Q}}_\ell)$ *attached to* Π *satisfies 2.20(1) and 2.20(2), then the* $\overline{\mathbf{Q}}_\ell[\Gamma_E]$*-module*

$$\mathrm{Hom}_{G(\widehat{\mathbf{Q}})}(\pi^\infty, H_{et}^{\dim}(Sh(G, \mathscr{X}) \otimes_E \overline{\mathbf{Q}}, \mathscr{L}_{\xi,\ell})) = \mathrm{Hom}_{G(\widehat{\mathbf{Q}})}(\pi^\infty, H^{\dim})$$

is semisimple.

(2.26) Corollary. *Under the assumptions of Theorem 2.25, the* $\overline{\mathbf{Q}}_\ell[\Gamma_E]$*-module* $V^{\dim}(\pi^\infty)$ *(which is isomorphic to its image in* $\mathrm{Hom}_{G(\widehat{\mathbf{Q}})}(\pi^\infty, H^{\dim})$*) is semisimple.*

3. Proof of Theorems 2.20 and 2.25

(3.1) The proof will combine Theorem 1.7 with arguments along the lines of [N, §6]. Let ξ^*, π^*, π, (Π, ψ) and $\rho_{\Pi,\ell}$ be as in Theorem 2.20.

(3.2) Assume that $V^{\dim}((\pi^*)^\infty) \neq 0$. Fix open compact subgroups $K \subset G(\widehat{\mathbf{Q}})$ and $K^* \subset G^*(\widehat{\mathbf{Q}}) \cap K$ such that $(\pi^\infty)^K \neq 0 \neq ((\pi^*)^\infty)^{K^*}$. Let S be a sufficiently large finite set of places of $\mathbf{Q}$ containing $\{2, \ell, \infty\}$, all primes where 2.14(4), (2.16.2) or (2.16.4) fails, all primes at which $F_c/\mathbf{Q}$ and B ramify, and such that $K = K_S K^S$ and $K^* = K_S^* K^{*S}$, where $K^S = \prod_{p \notin S} K_p$ and $K^{*S} = \prod_{p \notin S} K_p^*$ with $K_p \subset G(\mathbf{Q}_p)$ and $K_p^* \subset G^*(\mathbf{Q}_p)$ maximal compact subgroups.

(3.3) Denote by $\mathbf{Q}_S/\mathbf{Q}$ the maximal subextension of $\overline{\mathbf{Q}}/\mathbf{Q}$ unramified outside S. The Galois representation $\rho_{\Pi,\ell}$ then factors through $\mathrm{Gal}(\mathbf{Q}_S/F_c)$.

We are going to consider primes P_S of $\mathbf{Q}_S$ which are unramified in $\mathbf{Q}_S/\mathbf{Q}$ and which satisfy

$$\mathrm{Fr}_{\mathbf{Q}_S/\mathbf{Q}}(P_S) \in \mathrm{Gal}(\mathbf{Q}_S/F_c^{gal}). \tag{3.3.1}$$

The rational prime $p = P_S \cap \mathbf{Z} \notin S$ then splits completely in F_c. Extend each element $\sigma_x : F_c \hookrightarrow \overline{\mathbf{Q}} \subset \mathbf{C}$ $(x \in X)$ of the CM type Φ of F_c to an element $s(x) \in \Gamma_{\mathbf{Q}}$. For each $x \in X$, the prime $P_x = s(x)^{-1} P_S \cap O_F$ of F (resp. the prime $P'_x = s(x)^{-1} P_S \cap O_{F_c}$ of F_c) depends only on x (resp. on σ_x). These primes satisfy

$$pO_F = \prod_{x \in X} P_x, \qquad P_x O_{F_c} = P'_x P''_x, \qquad P''_x = c(P'_x), \qquad F_{P_x} = (F_c)_{P'_x} = (F_c)_{P''_x} = \mathbf{Q}_p.$$

(3.4) Let p be as in 3.3. Fix an embedding $\overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_p$; this defines a prime $v \mid p$ of the reflex field $E \subset F_c^{gal} \subset \overline{\mathbf{Q}}$ such that $E_v = \mathbf{Q}_p$.

The map α from 2.3 yields an isomorphism

$$G_{\mathbf{Q}_p} \simeq \prod_{x \in X} (GL(n) \times GL(1))_{(F_c)_{P'_x}} = (GL(n) \times GL(1))_{\mathbf{Q}_p}^X.$$

Therefore the pair $(G(\mathbf{Q}_p), K_p = \prod_{x \in X} K_x)$ is isomorphic to $((GL_n(\mathbf{Q}_p) \times \mathbf{Q}_p^\times)^X, (GL_n(\mathbf{Z}_p) \times \mathbf{Z}_p^\times)^X)$ and $(G^*(\mathbf{Q}_p), K_p^*)$ to $(GL_n(\mathbf{Q}_p)^X \times \mathbf{Q}_p^\times, GL_n(\mathbf{Z}_p)^X \times \mathbf{Z}_p^\times)$, with $\mathbf{Q}_p^\times$ embedded diagonally into $(\mathbf{Q}_p^\times)^X$.

By assumption, there exists a surjection $\pi|_{G^*(\mathbf{A})} \longrightarrow \pi^*$. As $G(\mathbf{Q}_p) \simeq G^*(\mathbf{Q}_p) \times (\mathbf{Q}_p^\times)^{r-1}$, the induced surjection $\pi_p|_{G^*(\mathbf{Q}_p)} \longrightarrow \pi_p^*$ is an isomorphism, since π_p and π_p^* are irreducible and $\mathbf{Q}_p^\times$ is abelian. In particular, it induces an isomorphism $\pi_p^{K_p^*} \xrightarrow{\sim} (\pi_p^*)^{K_p^*}$, hence an isomorphism between the spherical lines

$$\pi_p^{K_p} = \bigotimes_{x \in X} \pi_{P_x}^{K_x} \subseteq \pi_p^{K_p^*}$$

and $(\pi_p^*)^{K_p^*}$. Note that

$$\pi_p = \bigotimes_{x \in X} \pi_{P_x} \simeq \bigotimes_{x \in X} (\Pi, \psi)_{P'_x} = \bigotimes_{x \in X} (\Pi_x, \psi_x),$$

hence

$$\pi_p^*|_{GL_n(\mathbf{Q}_p)^X} \simeq \bigotimes_{x\in X} \Pi_{P'_x}.$$

(3.5) There exists an $O_{F_c}\otimes\mathbf{Z}_p$-lattice $\Lambda\subset V_{\mathbf{Q}_p}$ which is stable under K_p^*. Together with the $O_{F_c}\otimes\mathbf{Z}_{(p)}$-order $O_B=\{b\in B \mid b\Lambda\subset\Lambda\}\subset B$, the pair (O_B,Λ) gives rise to an unramified PEL datum in the sense of [K, §5]. For K_S small enough, the construction in [K, §5] defines a smooth quasi-projective model of $Sh_{K^*}(G^*,\mathscr{X}^*)$ over $O_E\otimes\mathbf{Z}_{(p)}$. We denote by S_{K^*} its base change to $O_{E_v}=\mathbf{Z}_p$ and by $S^\circ_{K^*}$ the special fibre of S_{K^*} (defined over the residue field $k(v)=\mathbf{F}_p$).

(3.6) The sets of conjugacy classes of cocharacters $\mathbf{G}_m\longrightarrow G^*$ defined over the fields $\mathbf{C}\supset\overline{\mathbf{Q}}\hookrightarrow\overline{\mathbf{Q}}_p$ coincide. Viewed over $\overline{\mathbf{Q}}_p$, the conjugacy class $[\mu_h]$ of the cocharacter μ_h attached to the Shimura datum $(G^*,\mathscr{X}^*)$ contains a character defined over $\mathbf{Q}_p$, namely,

$$\mu=(\mu_x)_{x\in X}:\mathbf{G}_{m,\mathbf{Q}_p}\longrightarrow G^*_{\mathbf{Q}_p}=GL(n)^X_{\mathbf{Q}_p}\times\mathbf{G}_{m,\mathbf{Q}_p}\subset G_{\mathbf{Q}_p},\qquad \mu(t)=((tI_{a_x},I_{b_x})_{x\in X},t).$$

The parabolic subgroups $P^*_\mu\subset G^*_{\mathbf{Q}_p}$ and $P_\mu\subset G_{\mathbf{Q}_p}$ attached to μ [N, A1.2] are equal to

$$P^*_\mu=P\times\mathbf{G}_{m,\mathbf{Q}_p}\subset P_\mu=P\times\mathbf{G}^X_{m,\mathbf{Q}_p},\qquad P=\prod_{x\in X}\begin{pmatrix} GL(a_x) & * \\ 0 & GL(b_x)\end{pmatrix}_{\mathbf{Q}_p}.$$

We identify their Levi subgroups $M^*\subset M$ with their respective sets of $\mathbf{Q}_p$-points, namely,

$$M^*=\big(\prod_{x\in X}GL_{a_x}(\mathbf{Q}_p)\times GL_{b_x}(\mathbf{Q}_p)\big)\times\mathbf{Q}_p^\times\subset M=\prod_{x\in X}M_x=\prod_{x\in X}(GL_{a_x}(\mathbf{Q}_p)\times GL_{b_x}(\mathbf{Q}_p)\times\mathbf{Q}_p^\times).$$

The subgroups $L^*=M^*\cap K_p^*$ and $L=M\cap K_p=\prod_{x\in X}L_x$ are given by the same formulas, with $\mathbf{Q}_p$ replaced by $\mathbf{Z}_p$.

(3.7) We use the notation $\mathcal{H}(G^*(\mathbf{Q}_p)//K_p^*,\mathbf{Q})$ for the (commutative) Hecke algebra of locally constant functions with compact support $f:K_p^*\backslash G^*(\mathbf{Q}_p)/K_p^*\longrightarrow\mathbf{Q}$ (and similarly for other Hecke algebras).

The maps $K_p^*gK_p^*\mapsto K_pgK_p$ and $L^*mL^*\mapsto LmL$ define embeddings of Hecke algebras

$$\mathcal{H}^*=\mathcal{H}(G^*(\mathbf{Q}_p)//K_p^*,\mathbf{Q})\hookrightarrow\mathcal{H}(G(\mathbf{Q}_p)//K_p,\mathbf{Q})\simeq\bigotimes_{x\in X}\mathcal{H}_x,\quad \mathcal{H}_x=\mathcal{H}(GL_n(\mathbf{Q}_p)\times\mathbf{Q}_p^\times)//GL_n(\mathbf{Z}_p)\times\mathbf{Z}_p^\times),\mathbf{Q})$$

and

$$\mathcal{H}^*_M=\mathcal{H}(M^*//L^*,\mathbf{Q})\hookrightarrow\mathcal{H}(M//L,\mathbf{Q})\simeq\bigotimes_{x\in X}\mathcal{H}(M_x//L_x,\mathbf{Q}),$$
$$\mathcal{H}(M_x//L_x,\mathbf{Q})=\mathcal{H}(GL_{a_x}(\mathbf{Q}_p)\times GL_{b_x}(\mathbf{Q}_p)\times\mathbf{Q}_p^\times)//GL_{a_x}(\mathbf{Z}_p)\times GL_{b_x}(\mathbf{Z}_p)\times\mathbf{Z}_p^\times),\mathbf{Q})$$

that are compatible with the twisted Satake transforms ([N, A1.4])

$$\overline{S}_\mu:\mathcal{H}^*\longrightarrow\mathcal{H}^*_M,\qquad \overline{S}_{\mu_x}:\mathcal{H}_x\longrightarrow\mathcal{H}(M_x//L_x,\mathbf{Q})$$

given by

$$(\overline{S}_\mu f)(m)=\int_{N_\mu}f(mu)\,du,$$

where N_μ is the common unipotent radical of P^*_μ and P_μ and du is the Haar measure on N_μ giving $N_\mu\cap K_p$ volume 1.

(3.8) These Hecke algebras contain the following important elements.

- $\varphi_x = L_x\mu_x(p)^{-1}L_x = L_x((p^{-1}I_{a_x}, I_{b_x}), p^{-1})L_x \in \mathcal{H}(M_x//L_x, \mathbf{Q})$ (the partial Frobenius at P'_x)
- $\varphi = \prod_{x\in X}\varphi_x = L^*\mu(p)^{-1}L^* \in \mathcal{H}^*_M$ (the total Frobenius)
- $U'_x = K_x(I_n, p^{-1})K_x \in \mathcal{H}_x$
- $U_x = \overline{S}_{\mu_x}(U'_x) = L_x(I_n, p^{-1})L_x \in \mathcal{H}(M_x//L_x, \mathbf{Q})$
- $U' = \prod_{x\in X} U'_x \in \mathcal{H}^*$
- $U = \overline{S}_\mu(U') = \prod_{x\in X} U_x \in \mathcal{H}^*_M$
- $\widetilde{\varphi}_x = \varphi_x U_x^{-1} = \varphi_x U_x^{-1} \otimes \bigotimes_{y\neq x} 1_y \in \mathcal{H}^*_M$ (twisted partial Frobenius)

By definition,

$$U\prod_{x\in X}\widetilde{\varphi}_x = \prod_{x\in X} U_x\widetilde{\varphi}_x = \varphi.$$

(3.9) The element $\varphi_x \in \mathcal{H}(M_x//L_x, \mathbf{Q})$ satisfies an abstract Eichler-Shimura relation of the form

$$\left(\overline{S}_{\mu_x}(Q_x)\right)(\varphi_x) = 0 \in \mathcal{H}(M_x//L_x, \mathbf{Q}), \tag{3.9.1}$$

where $Q_x \in \mathcal{H}_x[Y]$ is an explicit polynomial of degree $\deg(Q_x) = \binom{n}{a_x}$ ([Bu, Prop. 3.4], [W, Prop. 2.9], [N, Prop. A1.7, A5.10]). The image of Q_x under the morphism

$$\mathcal{H}_x \longrightarrow \mathbf{C}, \qquad T \mapsto T|_{(\Pi_x,\psi_x)^{K_x}}$$

given by the action of $\mathcal{H}_x$ on the K_x-spherical line of $(\Pi_x, \psi_x) = (\Pi, \psi)_{P'_x}$ is given by the following formula ([N, A5.10]), in which $(R_\lambda Q)(Y) = \lambda^{\deg(Q)}Q(\lambda^{-1}Y)$:

$$(R_{p^{-(a_xb_x/2)}}\, Q_x)\big|_{(\Pi_x,\psi_x)^{K_x}}(Y) = \prod_{|I|=a_x}(Y - t_I^{-1}\psi_x(p)^{-1}) \qquad (I \subset \{1,\ldots,n\},\ t_I = \prod_{i\in I} t_i),$$

where

$$L_{P'_v}(\rho_{\Pi,\ell}, s + (n-1)/2) = L_{P'_v}(\Pi, \mathrm{Std}_n, s) = \prod_{i=1}^{n}(1 - t_i p^{-s})^{-1}, \qquad \psi_x(p) = \psi(P'_x). \tag{3.9.2}$$

It follows from (3.9.1) that

$$\left(R_{U_x^{-1}}(\overline{S}_{\mu_x}(Q_x))\right)(\widetilde{\varphi}_x) = \left(\overline{S}_{\mu_x}(R_{U_x'^{-1}}(Q_x))\right)(\widetilde{\varphi}_x) = 0 \in \mathcal{H}^*_M. \tag{3.9.3}$$

The polynomial $R_{U_x'^{-1}}(Q_x)$ lies in $\mathcal{H}^*[Y]$ and satisfies

$$(R_{p^{-(a_xb_x/2)}}R_{U_x'^{-1}}\, Q_x)\big|_{(\pi_p^*)^{K_p^*}}(Y) = (R_{p^{-(a_xb_x/2)}\psi(P'_x)}(Q_x|_{(\Pi_x,\psi_x)^{K_x}}))(Y) = \prod_{|I|=a_x}(Y - t_I^{-1}). \tag{3.9.4}$$

For any representation ρ of $\mathrm{Gal}(\mathbf{Q}_S/F_c^{gal})$ and $x \in X$, denote by ${}^{s(x)}\rho$ the representation $({}^{s(x)}\rho)(g) = \rho(s(x)^{-1}gs(x))$ of the same group; then

$$({}^{s(x)}\rho)(\mathrm{Fr}(P_S)) = \rho(s(x)^{-1}\mathrm{Fr}(P_S)s(x)) = \rho(\mathrm{Fr}(P'_x)).$$

In particular, the representation

$$\rho'_x = {}^{s(x)}\left(\left(\bigwedge^{a_x}\rho^\vee_{\Pi,\ell}\right)(a_x(a_x+1)/2 - a_xn)\right) : \mathrm{Gal}(\mathbf{Q}_S/F_c^{gal}) \longrightarrow GL_{n_x}(\overline{\mathbf{Q}}_\ell), \qquad n_x = \binom{n}{a_x}$$

satisfies

$$\rho'_x(\mathrm{Fr}(P_S)) = \left(\bigwedge^{a_x} \rho^\vee_{\Pi,\ell}\right)(\mathrm{Fr}(P'_x))\, p^{a_x b_x/2}\, p^{a_x(n-1)/2},$$

which implies that

$$P_{\rho'_x(\mathrm{Fr}(P_S))}(Y) = R_{U'^{-1}_x}(Q_x)\big|_{(\pi^*_p)^{K^*_p}}(Y), \tag{3.9.5}$$

thanks to (3.9.2) and (3.9.4).

(3.10) Following [FC, VII.3], Wedhorn [W, 3.3] defined a scheme of p-isogenies [3] $p-\mathrm{Isog}_{K^p} \longrightarrow S_{K^*}\times S_{K^*}$ over $O_{E_v} = \mathbf{Z}_p$. The main result of [W] (see also [Mo, Thm. 4.2.13]) is the existence of a commutative diagram of $\mathbf{Q}$-algebras (with the sign conventions corrected according to [N, Appendix])

$$\begin{tikzcd} \mathcal{H}(G^*(\mathbf{Q}_p)_-//K^*_p, \mathbf{Q}) \arrow[r, "h"] \arrow[d, "\overline{S}_\mu"] & \mathbf{Q}[p-\mathrm{Isog}_{K^p}\otimes E_v] \arrow[d, "\sigma"] \\ \mathcal{H}(M^*_-//L^*, \mathbf{Q}) \arrow[r, "\overline{h}"] & \mathbf{Q}[p-\mathrm{Isog}_{K^p}\otimes \mathbf{F}_p] \end{tikzcd} \tag{3.10.1}$$

($E_v = \mathbf{Q}_p$, $k(v) = \mathbf{F}_p$). Above, $\mathbf{Q}[p-\mathrm{Isog}_{K^p}/S]$ denotes the $\mathbf{Q}$-vector space on the set of connected components of $p-\mathrm{Isog}_{K^p}(S)$ (with the ring structure defined as in [FC, p. 252]; see [N, A4.1] for the sign conventions in our context), the map σ is given by specialisation of cycles and

$$G^*(\mathbf{Q}_p)_- = \{g \in G^*(\mathbf{Q}_p) \mid g^{-1}(\Lambda)\subset \Lambda\}, \qquad M^*_- = G^*(\mathbf{Q}_p)_- \cap M^*.$$

The rings $\mathcal{H}(G^*(\mathbf{Q}_p)_-//K^*_p, \mathbf{Q})$ and $\mathbf{Q}[p-\mathrm{Isog}_{K^p}\otimes E_v]$ naturally act on $H^i_{et}(Sh_{K^*}\otimes_E \overline{\mathbf{Q}}, \mathscr{L}_{\xi^*,\ell})$ ([FC, VII.2], [N, A5.5, A5.11]). The $\overline{\mathbf{Q}}_\ell$-sheaf $\mathscr{L}_{\xi^*,\ell}$ extends to a $\overline{\mathbf{Q}}_\ell$-sheaf on S_{K^*}, whose restriction to the special fibre of S_{K^*} will be denoted by $\mathscr{L}^\circ_{\xi^*,\ell}$. There is a canonical isomorphism

$$H^i_{et}(Sh_{K^*}\otimes_E \overline{\mathbf{Q}}, \mathscr{L}_{\xi,\ell}) \simeq H^i_{et}(S^\circ_{K^*}\otimes \overline{\mathbf{F}}_p, \mathscr{L}^\circ_{\xi,\ell}) \tag{3.10.2}$$

([FC, Thm. VI.6.1], [LSt, Thm. 6.1]). The ring $\mathbf{Q}[p-\mathrm{Isog}_{K^p}\otimes \mathbf{F}_p]$ naturally acts on $H^i_{et}(S^\circ_{K^*}\otimes\overline{\mathbf{F}}_p, \mathscr{L}^\circ_{\xi,\ell})$ and all of the above actions are compatible via (3.10.1). Under the isomorphism (3.10.2), the action of $\mathrm{Fr}(P_S)$ on the L.H.S. corresponds to the action of $\varphi \in \mathcal{H}(M_-//L, \mathbf{Q})$ on the R.H.S.

(3.11) Let $i \geq 0$ be arbitrary. According to [N, A5.5, A5.12],

$$H^i_*[(\pi^*)^\infty] := \mathrm{Hom}_{G^*(\widehat{\mathbf{Q}})}((\pi^*)^\infty, H^i_*) \otimes (\pi^*)^\infty$$

is a $\overline{\mathbf{Q}}_\ell[\Gamma_E \times G^*(\widehat{\mathbf{Q}}^{(p)})] \times (\overline{\mathbf{Q}}_\ell \otimes_{\mathbf{Q}} \mathcal{H}^*)$-submodule of H^i_*, stable under the action of $\widetilde{\varphi}_x$, for all $x \in X$. Therefore

$$W^i := H^i_*[(\pi^*)^\infty]^{K^*} = \mathrm{Hom}_{G^*(\widehat{\mathbf{Q}})}((\pi^*)^\infty, H^i_*) \otimes ((\pi^*)^\infty)^{K^*} \subset H^i(Sh_{K^*}(G^*, \mathscr{X}^*)\otimes_E \overline{\mathbf{Q}}, \mathscr{L}_{\xi^*,\ell}) = (H^i_*)^{K^*}$$

is a $\overline{\mathbf{Q}}_\ell[\Gamma_E] \times (\overline{\mathbf{Q}}_\ell \otimes_{\mathbf{Q}} \mathcal{H}^*)$-submodule of $(H^i_*)^{K^*}$, stable under all $\widetilde{\varphi}_x$. In addition, $\prod_{x\in X}\widetilde{\varphi}_x = U\varphi$, with φ acting on $(H^i_*)^{K^*}$ as $\mathrm{Fr}(P_S)$.

The abstract Eichler-Shimura relation (3.9.3) together with the commutative diagram (3.10.1) and the compatibilities discussed in 3.10 imply that

$$R_{U'^{-1}_x}(Q_x)\big|_{(\pi^*_p)^{K^*_p}}(\widetilde{\varphi}_x|_{W^i}) = 0 \in \mathrm{End}_{\overline{\mathbf{Q}}_\ell}(W^i),$$

hence

[3] Wedhorn worked with stacks, but if one imposes (as we do) a level structure K^{*p} outside p with small enough K_S, then one obtains a scheme.

$$\forall x \in X \qquad P_{\rho'_x(\mathrm{Fr}(P_S))}\,(\widetilde{\varphi}_x|_{W^i}) = 0 \in \mathrm{End}_{\overline{\mathbf{Q}}_\ell}(W^i), \tag{3.11.1}$$

by (3.9.5). Moreover,

$$\Big(\prod_{x\in X} \widetilde{\varphi}_x\Big)|_{W^i} = U'|_{W^i} \cdot \varphi|_{W^i} = c(P_S)\cdot \mathrm{Fr}(P_S)|_{W^i}, \qquad c(P_S) = \prod_{x\in X} \psi(P'_x)^{-1}. \tag{3.11.2}$$

(3.12) Proposition. (1) *There exists a character* $\chi : \Gamma_{F_c^{gal}} \longrightarrow \overline{\mathbf{Q}}_\ell^\times$ *such that* $\chi(\mathrm{Fr}(P_S)) = \prod_{x\in X}\psi(P'_x)$, *for every* P_S *satisfying (3.3.1).*
(2) *If* P_S *satisfies (3.3.1), then*

$$P_{(\otimes_x \rho'_x)(\mathrm{Fr}(P_S))}(\mathrm{Fr}(P_S)|_{W^i\otimes\chi}) = 0 \in \mathrm{End}_{\overline{\mathbf{Q}}_\ell}(W^i\otimes\chi).$$

(3) *If* P_S *satisfies (3.3.1), then all eigenvalues of* $\mathrm{Fr}(P_S)|_{W^i}$ *in* $\overline{\mathbf{Q}}_\ell \simeq \mathbf{C}$ *have absolute value* $p^{m/2}$*, where*

$$m = 2u + \sum_{x\in X} u_x + \sum_{x\in X} a_x b_x = w^* + \dim,$$

in the notation of 2.6 and 2.11.

Proof. (1) The character

$$\widetilde{\chi} : \prod_{x\in X}\Big(\psi\circ N_{F_c^{gal}/F_c}\circ s(x)^{-1}\Big) : \mathbf{A}^\times_{F_c^{gal}}/(F_c^{gal})^\times \longrightarrow \mathbf{C}^\times$$

sends a uniformiser at $P_S \cap O_{F_c^{gal}}$ to $\prod_{x\in X}\psi(P'_x)$. The infinity type of $\widetilde{\chi}$ is algebraic: if $\tau : F_c^{gal} \hookrightarrow \mathbf{C}$ extends some $\sigma_x : F_c \hookrightarrow \mathbf{C}$, then

$$\widetilde{\chi}_\tau(z) = \prod_{x\in X}\left(\overline{z}^{u_x}|z|^{t-u_x}\right) = z^a\overline{z}^b, \qquad a = \sum_{x\in X}(t-u_x)/2 = u \in \mathbf{Z}, \qquad b = \sum_{x\in X}(t+u_x)/2 = u + \sum_{x\in X} u_x \in \mathbf{Z},$$

in the notation of (2.11.1). The ℓ-adic character $\chi : \Gamma_{F_c^{gal}} \longrightarrow \overline{\mathbf{Q}}_\ell^\times$ attached to $\widetilde{\chi}$ ([Sc, ch. 0, §5], [HT, p. 20]) then satisfies $\chi(\mathrm{Fr}(P_S)) = \prod_{x\in X}\psi(P'_x)$, as claimed.
(2) This is a consequence of (3.11.1), (3.11.2) and (1).
(3) It follows from (2.16.4) that all eigenvalues of $\rho'_x(\mathrm{Fr}(P_S))$ in $\overline{\mathbf{Q}}_\ell \simeq \mathbf{C}$ have absolute value $p^{m_x/2}$, where $m_x = a_x(1-n) - a_x(a_x+1) + 2a_xn = a_x(n-a_x) = a_xb_x$. By construction, $\chi(\mathrm{Fr}(P_S))$ has absolute value $p^{-(a+b)/2}$, with a and b as in the proof of (1). We deduce from (2) that $m = a+b+\sum_{x\in X} m_x$, as claimed.

(3.13) We are now ready to conclude the proof of Theorem 2.20, by applying Theorem 1.7 to the following objects: $\Gamma = \Gamma' = \mathrm{Gal}(\mathbf{Q}_S/F_c^{gal})$, $\Sigma = \{g = \mathrm{Fr}(P_S) \mid P_S \text{ as in (3.3.1)}\}$, $V = V(g) = W^{\dim}|_\Gamma \otimes \chi$, $\{\rho_1,\ldots,\rho_r\} = \{\rho'_x\}_{x\in X}$, $\{u_1,\ldots,u_r\} = \{\widetilde{\varphi}_x\}_{x\in X}$. We can (and will) disregard the elements $x\in X$ for which $(a_x,b_x) = (n,0)$ or $(0,n)$, since $\dim(\rho'_x) = 1$ then.

For the remaining $x \in X$, assumption (A) in Theorem 1.7 is a consequence of 2.20(1) in the case 2.20(2b) (resp. of the fact that $\bigwedge^a\overline{\mathbf{Q}}_\ell^n$ is an irreducible $sl(n,\overline{\mathbf{Q}}_\ell)$-module, for all a, in the case 2.20(2a)). Assumption (B) follows from Corollary 2.18. Finally, assumption (C) is satisfied by (3.11.1), (3.11.2) and Proposition 3.12(1). Theorem 1.7 tells us that V is a semisimple $\overline{\mathbf{Q}}_\ell[\mathrm{Gal}(\mathbf{Q}_S/F_c^{gal})]$-module, hence so is $\mathrm{Hom}_{G^*(\widehat{\mathbf{Q}})}((\pi^*)^\infty, H_*^{\dim})|_\Gamma$, since $((\pi^*)^\infty)^{K^*} \neq 0$. Theorem 2.20 is proved.

(3.14) In the case 2.20(2a) one can avoid the use of Theorem 1.17 and combine instead Proposition 3.12(2) with [N, Prop. 3.10]. Indeed, all exterior powers $\bigwedge^a\overline{\mathbf{Q}}_\ell^n$ are one-dimensional or minuscule $sl(n,\overline{\mathbf{Q}}_\ell)$-modules, which means that [N, Prop. 3.10(1)] applies: there exists an open subgroup $U\subset\Gamma$ such that $V^{\mathrm{ss}}|_U = (V|_U)^{\mathrm{ss}}$ is isomorphic to a subrepresentation of $(\otimes_x\rho'_x)^{\oplus n}|_U$, for some $n\geq 1$. Semisimplicity of V then follows from [N, Thm. 3.3, 3.4, Thm. 3.6].

(3.15) Theorem 2.25 is proved along the same lines as Theorem 2.20. One works directly with the Shimura variety attached to $(G,\mathscr{X})$ and the contorsions involving $\widetilde{\varphi}_x$ are unnecessary.

One considers $K \subset G(\widehat{\mathbf{Q}})$ such that $(\pi^\infty)^K \neq 0$ and primes P_S as in (3.3.1). The Kottwitz model S_{K^*} is replaced by its variant S_K described in [N, A3.7] and the diagram (3.10.1) by [N, (A5.1.1)].

As ξ satisfies the condition in Proposition 2.7(2), the Hecke character ψ is algebraic. Denote by $\rho_\psi : \mathrm{Gal}(\mathbf{Q}_S/F_c) \longrightarrow \mathbf{Q}_\ell^\times$ the corresponding ℓ-adic Galois representation ([Sc, ch. 0, §5], [HT, p. 20]): $\rho_\psi(\mathrm{Fr}(P'_x)) = \psi_x(p) = \psi(P'_x)$ for all $x \in X$.

The $\overline{\mathbf{Q}}_\ell[\Gamma_E] \times (\overline{\mathbf{Q}}_\ell \otimes_{\mathbf{Q}} \mathcal{H})$-module

$$W^i := H^i[\pi^\infty]^K = \mathrm{Hom}_{G(\widehat{\mathbf{Q}})}(\pi^\infty, H^i) \otimes (\pi^\infty)^K \subset H^i(Sh_K(G, \mathscr{X}) \otimes_E \overline{\mathbf{Q}}, \mathscr{L}_{\xi,\ell}) = (H^i)^K$$

is stable under the action of all φ_x. The Eichler-Shimura relation [N, Thm. A5.3, (A5.5.2)] for φ_x

$$Q_x\big|_{\pi_p^{K_p}} (\varphi_x|_{W^i}) = 0 \in \mathrm{End}_{\overline{\mathbf{Q}}_\ell}(W^i)$$

becomes [N, (A5.10.2)]

$$\forall x \in X \qquad P_{\rho_x(\mathrm{Fr}(P_S))} (\varphi_x|_{W^i}) = 0 \in \mathrm{End}_{\overline{\mathbf{Q}}_\ell}(W^i), \tag{3.15.1}$$

where

$$\rho_x = {}^{s(x)}\Big(\Big(\bigwedge^{a_x} \rho_{\Pi,\ell}^\vee\Big) \otimes \rho_\psi^\vee(a_x(a_x+1)/2 - a_x n)\Big) : \mathrm{Gal}(\mathbf{Q}_S/F_c^{gal}) \longrightarrow GL_{n_x}(\overline{\mathbf{Q}}_\ell), \qquad n_x = \binom{n}{a_x}.$$

By definition,

$$\Big(\prod_{x \in X} \varphi_x\Big)|_{W^i} = \varphi|_{W^i} = \mathrm{Fr}(P_S)|_W. \tag{3.15.2}$$

As in 3.13, we apply Theorem 1.17 to $\Gamma = \Gamma' = \mathrm{Gal}(\mathbf{Q}_S/F_c^{gal})$, $\Sigma = \{g = \mathrm{Fr}(P_S) \mid P_S \text{ as in (3.3.1)}\}$, $V = V(g) = W^{\dim}|_\Gamma$, $\{\rho_1, \ldots, \rho_r\} = \{\rho_x\}_{x \in X}$, $\{u_1, \ldots, u_r\} = \{\varphi_x\}_{x \in X}$. Again, we disregard the elements $x \in X$ for which $(a_x, b_x) = (n, 0)$ or $(0, n)$.

Assumptions (A) and (B) of Theorem 1.17 are checked as in 3.13; assumption (C) is a consequence of (3.15.1) and (3.15.2). We obtain that V is a semisimple $\overline{\mathbf{Q}}_\ell[\mathrm{Gal}(\mathbf{Q}_S/F_c^{gal})]$-module, as claimed.

(3.16) If $\rho_{\Pi,\ell}$ in Theorem 2.25 satisfies 2.20(2a), one can again avoid the use of Theorem 1.17, as in 3.14. The formulas (3.15.1-2) imply that

$$P_{(\otimes_x \rho_x)(\mathrm{Fr}(P_S))}(\mathrm{Fr}(P_S)|_{W^i}) = 0 \in \mathrm{End}_{\overline{\mathbf{Q}}_\ell}(W^i)$$

and one deduces from [N, Prop. 3.10(1)] that there exists an open subgroup $U \subset \Gamma$ such that $V^{\mathrm{ss}}|_U = (V|_U)^{\mathrm{ss}}$ is isomorphic to a subrepresentation of $(\otimes_x \rho_x)^{\oplus n}|_U$, for some $n \geq 1$. Semisimplicity of V then follows from [N, Thm. 3.3, 3.4, Thm. 3.6] (and we obtain as a bonus that $V^{\dim}(\pi^\infty)|_U \subset (\otimes_x \rho_x)^{\oplus n}|_U$).

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